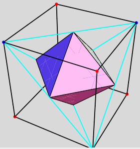
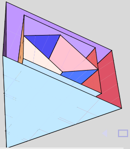
Visualizing 2 qubits

Oded Kenneth, Yosi Avron

December 13, 2010

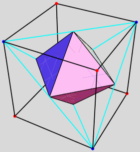
Related and overlapping with:
Horodeckies; Verstraete Dehaene, and De Moor;
Leinaas, Myrheim, and Ovrum; Zyczkowski and Bengtsson,
Wootters, and many others

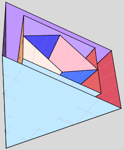


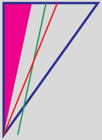


Outline

1. Visualizing a qubit
   - Bloch sphere
2. The problem of visualizing 2 qubits
   - What is entanglment
   - Witnesses
   - Equivalence classes
   - Local operations
3. Visualizing 2 qubits
   - Tetrahedron of states
4. Applications
   - Peres test
   - Measures of entanglement







Geometry of states

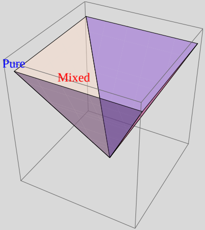
- $|\psi\rangle$ (Normalized): Sphere $dim = 2^{n+1} - 1 = 3, 7, \ldots$
- Pure state $\rho = |\psi\rangle\langle\psi|$; (forget phase)

  $dim = 2^{n+1} - 2 = 2, 6, \ldots$
- Mixed (unnormalized) state

  $\rho = \sum p_j |\psi_j\rangle\langle\psi_j|, \quad p_j \geq 0$; convex cone
- Normalized state $Tr(\rho) = 1$; capped cone

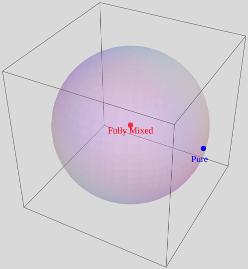
  Convex body, pure states on bdry

$$dim[states] = (2^n)^2 - 1 = 3, 15, 63, \ldots$$

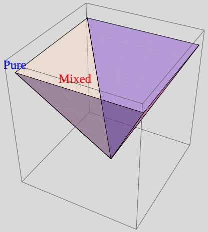


Bloch (Poincare) ball

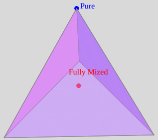
- State of qubit: $\rho = \frac{1+\vec{n}\cdot\vec{\sigma}}{2}, \quad |\vec{n}| \leq 1$
- Pauli matrices: $\sigma^{\mu} = \{\mathbb{I}, \sigma_x, \sigma_y, \sigma_z\}$,
  $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, etc.
- Convex; $dim[ball] = 3$
- All pure states on bdry–Generally true
- All points on bdry pure –Generally false



2 qubits



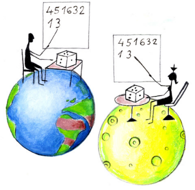
- Entanglement, Separable, Witnesses
- Two qubits are not 2 Bloch Balls
- Separable Mom—Pure product: $|00\rangle$
- Entangled Mom — 4 Bell $\sqrt{2}\,|\beta_0\rangle = |00\rangle + |11\rangle$
- Pure $|\psi\rangle = \sum x_\mu\,|\beta_\mu\rangle$, $\sum_0^3\,|x_\mu|^2 = 1,\ x \in S^7$
- Statesl: $\rho_2 = 4 \times 4$ positive matrix; unit trace; Convex body; 15 dimensions



- All pure states on bdry
- Few, Codim=8, states on bdry, pure

What is entanglement?

- Rosen's envelopes
- Classical independence $P(x, y) = P_A(x)P_B(y)$
- Classical correlations: $P(x, y) = \sum p_j P_A^j(x)P_B^j(y)$
- Quantum to classical: $P \Leftrightarrow \rho$
- Separable states $\rho = \sum p_j \rho_A^j \otimes \rho_B^j, \quad p_j \geq 0$
- Not all q-states are separable–these are entangled



States and witnesses

- Unnormalized states = cone of positive matrices

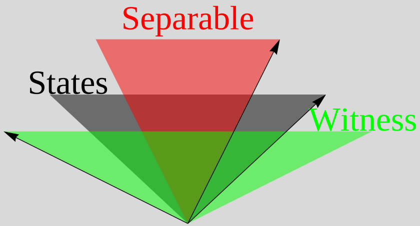
$$\rho = \sum p_j \left|\psi_j\right\rangle \left\langle\psi_j\right|, \quad p_j \geq 0$$

- Cone of separable

$$\rho_{sep} = \sum p_j \, \rho_j \otimes \rho_j, \quad p_j \geq 0$$

- Witnesses: Dual cone to seperable

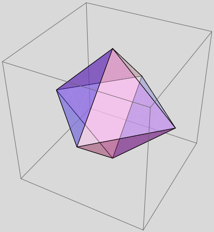
$$Tr(W\rho_{sep}) \geq 0$$



Swap: Mother of all witnesses

- $S\,|a\rangle \otimes |b\rangle = |b\rangle \otimes |a\rangle\,, \quad a, b = 0, 1$
- Positive on pure products $\langle ab|\, S\, |ab\rangle = \langle ab|ba\rangle = |\langle a|b\rangle|^2 \geq 0$
- Bell singlet: $\sqrt{2}\,|\beta_2\rangle = |01\rangle - |10\rangle$
- $S\,|\beta_2\rangle = -\,|\beta_2\rangle$
- Singlet is entangled: $\langle\beta_2|\, S\, |\beta_2\rangle = -1$
- Cube (witnesses) Octahedron (separable) are dual

Platonic	Faces	Vertices	Edges
Cube	6	8	12
Octahedron	8	6	12
Tetrahedron	4	4	6

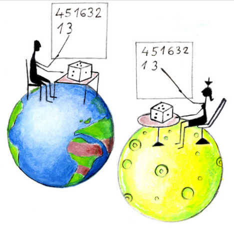


Singular values

- Singular values represent equivalence class of matrix W by $spectrum(\sqrt{W^\dagger W})$

  Singular values useless for entanglement: Separable–$|00\rangle$–and entangled –$|\beta_\mu\rangle$– share singular values (1,0,0,0)

- Need introduce locality: Alice/Bob

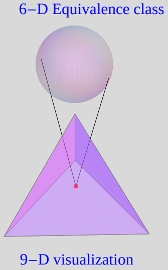


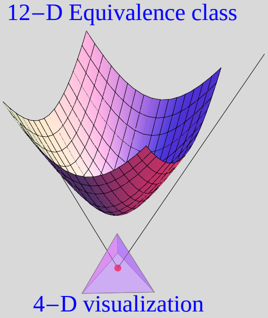
Q-operations

- Quantum operations $\rho \to M\rho M^*$
- Positivity preserving $\rho \geq 0 \Rightarrow M\rho M^* \geq 0$
- Examples : Evolution–M unitary
  Measurment—M projection
- Q-Operations: Partial order vs equivalence
- Equivalence if M invertible
- Examples : Evolution–Equivalence
  Measurment—Partial order
- Local operation $M = M_A \otimes M_B$; Respects separability
- Local operations can't create Entanglement

Local Equivalence operations

- $M_{A,B} \in SU(2)$
- $dim[SU(2)] = 3$
- Visualized in 9 dimensions

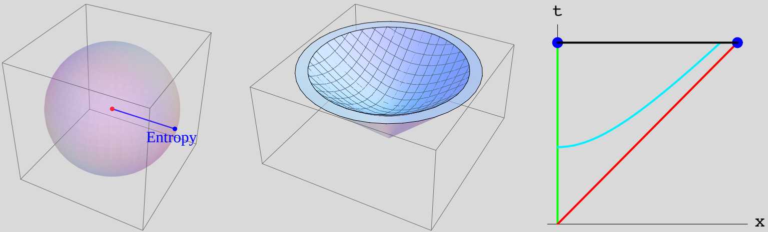




- $M_{A,B} \in SL(2,\mathbb{C}) \Rightarrow \det M_{AB} = 1$
- $dim[SL(2,\mathbb{C})] = 6$
- Visualized in 4 dimensions
- Reversible filtering

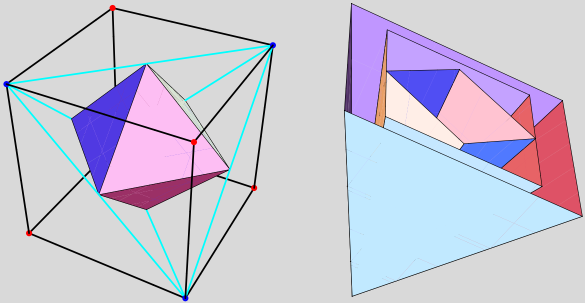
Equivalence classes: Single qubit

- $SU(2)$ trace preserving
- Entropy labels equivalence classes
- $\rho = x^\mu \sigma_\mu,\ x^\mu$ transforms like a Lorentz vector under $SL(2,\mathbb{C})$
- Rest frame is the trace minimizer
- Entropy preserved



Optimal characterization

- Find $M \in SL(2,\mathbb{C})^2$ minimizing $tr\, W \to tr(MWM^*)$
- Minimizer "diagonalize" $W \to \sum \omega_\mu \left|\beta_\mu\right\rangle \left\langle\beta_\mu\right|$
- ω_μ: entanglement singular values



Bell states; Swap witnesses; Tetraheder=states; Octaheder=Separable

Spectral characterization

- Replace $W^\dagger W \Longrightarrow \tilde{W} W$;
- The good choice (Wootters)

$$\tilde{W} = \sigma \; W^t \; \sigma, \quad \sigma = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \sigma_y \otimes \sigma_y$$

- Example 1: $W = |00\rangle \langle 00| \,, \to \tilde{W} = |11\rangle \langle 11|$;
- Example 2: $W = |\beta_\mu\rangle \langle \beta_\mu| = \tilde{W}$

Theorem

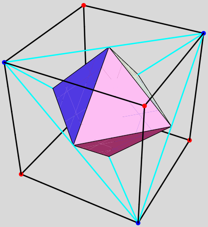
- *For any witness (state)* $\omega_\mu^2 \geq 0$ *eigenvalues of* $\tilde{W} W$

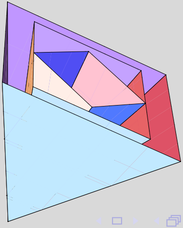
Examples

Table: Selected $SL(2,\mathbb{C})points$

Name	*state*	$\rho\tilde{\rho}$	Point
Product	$\|00\rangle$	0	(0,0,0,0)
Bell	$\|\beta_0\rangle$	$\|\beta_0\rangle\langle\beta_0\|$	(1,0,0,0)
Fully mixed	$\mathbb{I}/4$	$\mathbb{I}/16$	$\frac{1}{4}$(1,1,1,1)
Swap	S	$S^2 = \mathbb{I}$	$\frac{1}{2}$(1,1,1,-1)

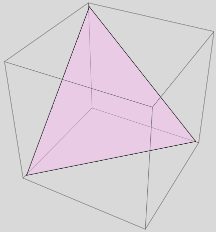
Remark: $\det S = -1$ affects choice of roots (and normalization)

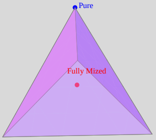




Tetrahederon of states

- Spectrum($\rho\tilde{\rho}$)= Spectrum($\sqrt{\rho}\tilde{\rho}\sqrt{\rho}$) ≥ 0
- Spectrum=Positive quadrant in 4D;
- Cross section: Tetrahederon
- Vertices: $(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)$

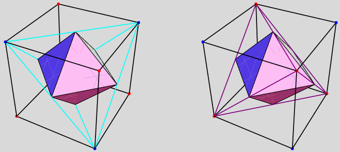


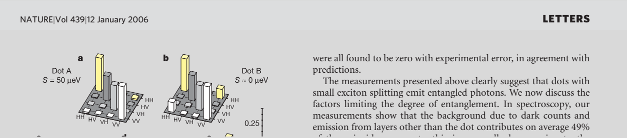


Vertices of triangle: orthogonal unit vectors in 3D   Vertices of tetrahedron: orthogonal unit vectors in 4D

Peres test

- Partial transposition
- $$2\rho = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad 2\rho^P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
- Peres thm: If $\rho \geq 0$ *while* $\rho^P \not\geq 0, \Rightarrow \rho$ entangled
- Geometry of Peres test
- Coherence $\neq$ entanglment





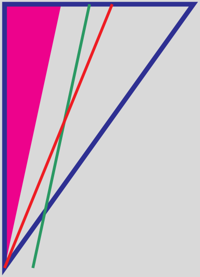
NATURE|Vol 439|12 January 2006 LETTERS

were all found to be zero with experimental error, in agreement with predictions.

The measurements presented above clearly suggest that dots with small exciton splitting emit entangled photons. We now discuss the factors limiting the degree of entanglement. In spectroscopy, our measurements show that the background due to dark counts and emission from layers other than the dot contributes on average 49%

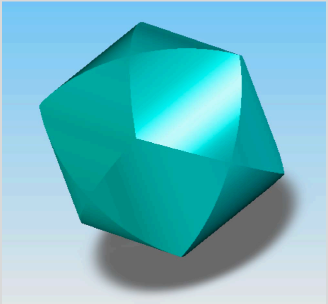
Measure and entanglement distillation

- Entanglement measure (concurrence) : Distance from octahedron; Equi-entangelment line
- diagonalizing map =Optimal entanglement distillation



Concluding remarks

- The Hilbert space of Alice and Bob (i.e. tensor product structure) has beautiful geometry
- What is the physical significance of the entanglement singular values?



Vizaulising the CHSH Bell inequalities (Bisker)