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# Dissipation in Linear and Nonlinear Models of Quantum Systems 

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# Dissipation in Linear and Nonlinear Models of Quantum Systems 

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## Abstract

In reality, all quantum systems are in fact open systems, that undergo dissipative evolution. One of the main tools in describing the evolution of open quantum systems is the Lindblad equation, which is a general Markovian master equation which is completely positive and trace preserving.

The motivation to this work arises from the SERF effect that occurs in atomic vapor, in which the relaxation caused by interatomic collisions, decreases with the collision rate. Although the effect was explained theoretically in the past, it is difficult to draw an elementary understanding of this counterintuitive effect, as the description of the atomic system is complicated. This motivated us to search for similar effects in a simpler setting, where the effect could be easily understood. The effect also motivated us to study relaxation in different collision-inspired models, which in general are nonlinear.

We start with a linear Lindblad equation that arises from a stochastic term in the Hamiltonian, and gives a discrete set of relaxation rates. We present a simple example of a system, where certain relaxation rates decrease with the noise power, instead of increasing, a result which resembles the SERF effect. We give an interpretation that identifies this result with the quantum Zeno effect. We also show similar phenomenon occurs in general, for degenerate noise operators.

We then investigate nonlinear Lindblad equations, which have not been previously subject to much research. These equations follow from mean field equations describing interacting many body dynamics at the limit of infinitely many particles. These equations can arise, for example, from collisions between particles.

We first discuss a nonlinear equation that describes collision-induced dephasing. The solution to this equation exhibits a continuum of exponential dephasing rates. This result is fundamentally different from the relaxation in linear equations, which takes place only through a discrete set of rates.

Next, we discuss a model for qubits that undergo pairwise decay, where excited qubits are only allowed to decay to the ground state in pairs, and not separately as in usual spontaneous decay. The Lindblad equation that describes an ensemble of such qubits is solved using two different approaches. In the first approach (mean field approach), we take the number of qubits to infinity, and obtain a nonlinear equation for a single qubit. The solution to this equation exhibits a polynomial decay law,
which is essentially different from the exponential decay law that arises from linear equations. In the second approach, given certain initial data, the complexity of the full Lindblad equation reduces from exponential to linear (in the qubit number). Using this, we obtain the solution for a finite number of qubits. We show agreement between the solutions arising from the two approaches, upon taking the number of qubits to infinity.

## Abbreviations and Notations

| $\hbar$ | reduced Planck constant |
| :--- | :--- |
| $c$ | speed of light |
| $\mathbb{1}$ | identity matrix |
| $\sigma_{j}$ | the $j^{\text {th }}$ Pauli matrix |
| $J_{k}$ | the $k^{\text {th }}$ thee dimensional angular momentum operator |
| $\delta(t)$ | Dirac delta function |
| $\delta_{a, b}$ | Kronecker delta |
| $\rho$ | density matrix operator |
| $\mathcal{L}$ | Lindblad superoperator |
| $\|\psi\rangle,\langle\phi\|$ | quantum states in the Dirac notation |
| $\mathbb{E}$ | expected value of a random variable |
| $\langle A\rangle$ | expectation value (ensemble average) |
| $t r$ | trace |
| $t r_{[n, N]}$ | trace over the Hilbert spaces labeled $n, n+1, \ldots, N$ |
| $\operatorname{dim}$ | dimension of a vector space |
| $s p e c$ | spectrum |
| $k e r$ | kernel |
| $\mathcal{O}$ | order of |
| $\mathcal{H}$ | Hilbert space |
| $[A, B]$ | commutator |
| $\{A, B\}$ | anti commutator |
| $\otimes$ | tensor product |
| $A^{\otimes n}$ | tensor power |
| $a^{\dagger}$ and $a$ | raising and lowering operators, respectively |
| $\lambda$ | eigenvalue |
| $P$ | projection operator |

## Chapter 1

## Introduction

In this chapter, we give a preface for this work. In section 1.1 we will present a short theoretical background on open quantum systems and the Lindblad equation. In section 1.2 we will briefly introduce a work on the relaxation properties of atomic vapor, which motivated this work. In section 1.3 we will discuss the objectives of this work.

### 1.1 Open Quantum Systems and the Lindblad Equation

The field of open quantum system deals with quantum mechanical systems that are coupled to their environments, and decohere ${ }^{1}$ as a result of this coupling. The description of decoherence plays an important role in many areas of quantum mechanics, such as quantum computation[1, 2], quantum metrology[3, 4] and studies in phase transitions $[5,6]$. The main objective of describing the dynamics of open quantum systems, is the description of the system of interest (i.e. reduced system) alone, without resorting to a detailed description of the environment degrees of freedom. In general, open systems are in a statistical mixture of quantum states, and thus cannot be described by a wavefunction $|\psi\rangle$ (i.e. vector), but rather by a density matrix $\rho$, which is an operator defined by,

$$
\begin{equation*}
\rho \equiv \sum_{j} p_{j}\left|\psi_{j}\right\rangle\left\langle\psi_{j}\right|, \tag{1.1}
\end{equation*}
$$

where $\left|\psi_{j}\right\rangle$ and $p_{j}$ denote a quantum state and the probability associated with it, respectively. Note that the density matrix satisfies $\rho \geq 0, \operatorname{tr}(\rho)=1$ by definition.

When the environment correlations decay much quicker than the typical time that takes the state of the reduced system to change significantly, the evolution of the reduced system can be regarded as a Markovian process (i.e. a process without memory). The Lindblad equation shown below presents a general form of

[^0]a master equation, describing the state of a reduced system undergoing Markovian evolution[7],
\[

$$
\begin{equation*}
\frac{d}{d t} \rho=-i[H, \rho]+\sum_{j=1}^{(\operatorname{dim} \mathcal{H})^{2}-1} \gamma_{j}\left(L_{j} \rho L_{j}^{\dagger}-\frac{1}{2} L_{j}^{\dagger} L_{j} \rho-\frac{1}{2} \rho L_{j}^{\dagger} L_{j}\right) \equiv \mathcal{L} \rho, \tag{1.2}
\end{equation*}
$$

\]

where $H$ denotes the free Hamiltonian of the reduced system, $\gamma_{j}$ are positive coefficients and $L_{j}$ are operators that characterize the interaction between the reduced system and the environment. $\mathcal{L}$ is a superoperator acting on $\rho$, and it is sometimes referred to as the Lindbladian. The Lindblad equation is the most general form of a Markovian master equation which is completely positive and trace preserving.

### 1.2 Anomalous Relaxation in Atomic Vapor

Happer et al.[8] investigated the precession of atomic spins due to magnetic field (Larmor precession), in which the components of the total atomic spin perpendicular to the precession axis (set here to be the z axis) undergo damped oscillations,

$$
\begin{align*}
& \left\langle S_{x}\right\rangle \propto \cos (\omega t+\varphi) e^{-\Gamma t}  \tag{1.3}\\
& \left\langle S_{y}\right\rangle \propto \sin (\omega t+\varphi) e^{-\Gamma t}
\end{align*}
$$

where $\omega$ is the precession frequency, $\varphi$ is a phase and $\Gamma$ is the precession damping rate (or relaxation rate). The main cause of the damping is collisions between atoms. They measured that at a sufficiently large rate of atomic collisions (caused by a high atomic density), the collisional relaxation rate begins to decrease with the collision rate and approach zero. This observation, named "SERF" (spin exchange relaxation free), is counterintuitive since the collisions are the main cause of relaxation in the first place. The effect was later theoretically reproduced [9, 10] by modeling the dynamics of the atomic vapor, using a Lindblad type master equation that describes the three main interactions of the atom: the hyperfine coupling (between the electron and the nucleus), the Zeeman effect (coupling between the spin and the magnetic field) and the collisions. The term describing the collisions in the resulting Lindblad equation is in fact nonlinear in the density matrix $\rho$, in the sense that the coefficients, denoted $\gamma_{j}$ in equation 1.2, are dependent on $\rho$. This nonlinearity arises from the fact the environment in this case, is an ensemble of subsystems (atoms) represented by the same density matrix $\rho$, as the reduced system.

Happer et al.[9] solved the master equation via linearization ${ }^{2}$ around the fully mixed state (defined by $\rho_{F M} \equiv \frac{1}{\operatorname{dim} \mathcal{H}} \mathbb{1}$ ), and indeed found the atomic spin performs

[^1]decaying Larmor precession. The solution also exhibits two different regimes. when the collision rate (denoted $R$ ) is much smaller than the precession frequency $R \ll \omega$, the relaxation rate is linear in the collision rate $\Gamma \propto R$. In contrast, when the collision rate $R$ is much higher than the precession frequency $R \gg \omega$, the relaxation rate behaves as $\Gamma \propto \frac{\omega^{2}}{R}$, i.e. decreases with the collision rate, in agreement with the experimental results.

The model for SERF considers a detailed description of the atoms, which involves three different interactions, and requires an atomic Hilbert space which is at least six dimensional. Therefore, despite reproducing the experimental results, the model is too complex to offer an elementary understanding of this peculiar effect.

### 1.3 Research Objective

The work described in the previous section demonstrates relaxation that decreases in rate, upon increasing the collision rate, which is the process causing the relaxation. The effect is reproduced by a collisional model that involves a nonlinear master equation. However, it is difficult to characterize the mechanism that generates the effect, as the model is complicated since it describes the realistic system in detail. Inspired by the phenomenon, the goals of this work are:

- Demonstration of a SERF-like effect in a simple quantum system, governed by a linear Lindblad equation. The source of relaxation we will consider is white noise, and the expected dynamics will be damped oscillations, similarly to equation 1.3. In this setting, the goal is to demonstrate a decrease in the relaxation rate due to an increase in the noise power (or amplitude). In chapter 2 we present a model for a system under white noise, and show how it can exhibit a SERF-like effect.
- Investigation of different collision-based models described by nonlinear Lindblad equations. We would like to study the decoherence under such models, with emphasis on finding slow relaxation. A familiar example where many body dynamics alter the behavior of a system is superradiance[11], where the the collective interaction of excited particles with the electric field causes an emission of a high intensity light pulse, which differs from the spontaneous emission of separate particles which obeys an exponential law. In chapter 3 we present other nonlinear models, which lead to different results.


## Chapter 2

## The Dependence of the Relaxation Rates on the Noise Power

Consider the stochastic Hamiltonian,

$$
\begin{equation*}
H(t)=H_{0}+\xi(t) C, \tag{2.1}
\end{equation*}
$$

where $H_{0}, C$ are fixed Hermitian operators that describe the sure and stochastic parts of the Hamiltonian respectively, and $\xi(t)$ is Gaussian white noise ${ }^{1}$,

$$
\begin{equation*}
\mathbb{E}(\xi(t))=0, \quad \mathbb{E}(\xi(t) \xi(s))=\gamma \delta(t-s), \tag{2.2}
\end{equation*}
$$

where $\gamma$ is a positive number, it serves as a measure of the noise power. It has been shown before [12][13] that the noise-averaged system would evolve by a Lindblad type equation of the form,

$$
\begin{equation*}
\frac{d}{d t} \rho=\mathcal{L} \rho=-i\left[H_{0}, \rho\right]+\gamma\left(C \rho C-\frac{1}{2}\left\{C^{2}, \rho\right\}\right)=-i\left[H_{0}, \rho\right]-\frac{1}{2} \gamma[C,[C, \rho]], \tag{2.3}
\end{equation*}
$$

where equation 2.3 holds for any positive value of $\gamma$, and for any set of Hermitian operators $H_{0}, C$. This will be relevant in the following sections, where we will characterize the system's behavior at the limit $\gamma \rightarrow \infty$, i.e. in the strong noise limit.

From general principles, since equation 2.3 is linear, $\rho(t)$ can be decomposed to the sum

$$
\begin{equation*}
\rho(t)=\sum_{j, k} \rho_{j k} t^{k} e^{\lambda_{j} t}, \tag{2.4}
\end{equation*}
$$

Where $\lambda_{j}$ denote the eigenvalues of the Lindbladian $\mathcal{L}$. The imaginary parts of the eigenvalues generate periodic (or quasi-periodic) evolution, and thus can be considered as oscillation frequencies; while the real parts (which are necessarily nonpositive) generate relaxation-type evolution, and can be considered as relaxation

[^2]rates. An eigenvalue that has nonvanishing real and imaginary parts describes a damped oscillation of the eigenvector associated with it.

In the this chapter we will investigate the dependence of the relaxation rates on the noise power $\gamma$. Intuitively, one expects the relaxation rates to increase with the noise power, however we will show how some of them can actually decrease with the noise power and approach zero. First we will present a simple example of a Lindbladian (of the type presented in equation 2.3) that demonstrates this effect. Next we present an alternative explanation to the same result using the quantum Zeno effect. We then generalize the result we found for this example by using a perturbative approach.

### 2.1 Three Level System Under White Noise

Consider a 3 level system characterized by

$$
H_{0}=b J_{z}+q J_{z}^{2}=\left(\begin{array}{ccc}
q+b & 0 & 0  \tag{2.5}\\
0 & 0 & 0 \\
0 & 0 & q-b
\end{array}\right) \quad ; \quad C=2 J_{x}^{2}-\mathbb{1}=\left(\begin{array}{lll}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{array}\right)
$$

where $b, q$ are free parameters that have dimensions of frequency. In this example $H_{0}$ can be viewed as being composed of a Zeeman term (denoted by $b J_{z}$ ) and a quadrupole splitting term ( $q J_{z}^{2}$ ).

Using the fact: $C^{2}=\mathbb{1}$, the Lindbladian can be simplified to the form

$$
\begin{equation*}
\mathcal{L} \rho=-i\left[H_{0}, \rho\right]+\gamma\left(C \rho C-\frac{1}{2}\{\mathbb{1}, \rho\}\right)=-i\left[H_{0}, \rho\right]+\gamma(C \rho C-\rho) . \tag{2.6}
\end{equation*}
$$

We show that in the case above $\mathcal{L}$ decomposes into smaller blocks, and use this fact to find some of the eigenvalues. Consider the decomposition $\rho=\sum_{j, k} \rho_{j k}|j\rangle\langle k|$, where $\{|j\rangle\}$ is the standard basis of the matrix representation in equation 2.5. Note that excluding the term $C \rho C$, the remainder of the Lindbladian is diagonal, i.e.

$$
\begin{equation*}
-i\left[H_{0},|j\rangle\langle k|\right]-\gamma|j\rangle\langle k| \propto|j\rangle\langle k| . \tag{2.7}
\end{equation*}
$$

Computing the remaining term $C \rho C$ yields,

$$
C \rho C=C\left(\begin{array}{lll}
\rho_{11} & \rho_{12} & \rho_{13}  \tag{2.8}\\
\rho_{21} & \rho_{22} & \rho_{23} \\
\rho_{31} & \rho_{32} & \rho_{33}
\end{array}\right) C=\left(\begin{array}{ccc}
\rho_{33} & \rho_{32} & \rho_{31} \\
\rho_{23} & \rho_{22} & \rho_{21} \\
\rho_{13} & \rho_{12} & \rho_{11}
\end{array}\right)
$$

$C$ swaps the indices $1 \rightleftarrows 3$, Thus the Lindbladian decomposes into $42 \times 2$ blocks that separately act on opposite elements of $\rho$ (i.e. $\left(\rho_{32}, \rho_{12}\right),\left(\rho_{31}, \rho_{13}\right)$ and so on),
and the remaining one dimensional block acts on the element $\rho_{22}$.
Consider the pair $\rho_{21}, \rho_{23}$, by substituting 2.8 in equation 2.6 we find their relevant block in the Lindbladian,

$$
\begin{align*}
\mathcal{L}_{21,23}\binom{\rho_{21}}{\rho_{23}} & =\binom{\gamma\left(\rho_{23}-\rho_{21}\right)+i(q+b) \rho_{21}}{\gamma\left(\rho_{21}-\rho_{23}\right)+i(q-b) \rho_{23}}  \tag{2.9}\\
& =\left(\begin{array}{cc}
-\gamma+i(q+b) & \gamma \\
\gamma & -\gamma+i(q-b)
\end{array}\right)\binom{\rho_{21}}{\rho_{23}} .
\end{align*}
$$

The spectrum of the resulting block $\mathcal{L}_{21,23}$ is,

$$
\begin{equation*}
\lambda_{ \pm}=i q-\gamma \pm \sqrt{\gamma^{2}-b^{2}} \tag{2.10}
\end{equation*}
$$

For $\gamma<b$, the relaxation rate is linear in the noise power, $\Gamma=-\operatorname{Re}\left[\lambda_{ \pm}\right]=\gamma$. Approximating the eigenvalues at $\gamma \gg b$ (i.e. when the noise power is much larger than the analogous magnetic field) yields

$$
\lambda_{ \pm}=i q-\gamma \pm \gamma \sqrt{1-\frac{b^{2}}{\gamma^{2}}} \approx i q-\gamma \pm \gamma\left(1-\frac{b^{2}}{2 \gamma^{2}}\right) \approx\left\{\begin{array}{ll}
i q-\frac{b^{2}}{2 \gamma} & \lambda_{+}  \tag{2.11}\\
i q-2 \gamma & \lambda_{-}
\end{array} .\right.
$$

At $\gamma \gg b$ the relaxation rate associated with $\lambda_{+}$decreases with $\gamma$. The real and imaginary values of $\lambda_{ \pm}$are plotted in figure 2.1 as functions of the noise power $\gamma$, for the parameters: $q=5, b=1$. At $0 \leq \gamma<b$ the real part of $\lambda_{ \pm}$is indeed equal to $-\gamma$, however at $\gamma>b$ it sharply increases and approaches monotonically to 0 . At this range the imaginary part satisfies: $\left.\operatorname{Im}\left[\lambda_{+}\right]\right|_{\gamma>b}=q$. Therefore $\left.\lambda_{+}\right|_{\gamma>b}$ describes damped oscillation with frequency $q$ and a damping rate that decreases with the noise power $\gamma$. In other words, at $\gamma>b$ the quality factor of the oscillation associated with $\lambda_{+}$increases with the noise power.

In order to measure the relaxation decrease, we would like to propose an observable that exhibits this effect. The evolution of observables is governed by the adjoint Lindbladian (this equation corresponds to the Heisenberg picture), which in our case takes the form,

$$
\begin{equation*}
\frac{d}{d t} B=i\left[H_{0}, B\right]+\gamma(C B C-B), \tag{2.12}
\end{equation*}
$$

where $B$ is an arbitrary observable. The difference between equations 2.6 and 2.12 is the in the sign of the commutator with $H_{0}$, thus the adjoint Lindbladian has a spectrum which is complex conjugate to the spectrum of the Lindbladian. An observable that exhibits the behavior of the eigenvalue $\lambda_{+}$should be constructed of the corresponding eigenvector.


Figure 2.1: Real and imaginary values of $\lambda_{+}$(left figure), $\lambda_{-}$(right figure) as functions of the noise power $\gamma$, for the parameters $q=5, b=1$.

One can show that the spectrum of the block corresponding to $\rho_{12}, \rho_{32}$ is $\lambda_{ \pm}^{*}$. At large noise power $\gamma \gg b$, the eigenvectors that correspond to $\lambda_{+}, \lambda_{+}^{*}$ satisfy

$$
\begin{align*}
& \rho_{21}=\rho_{23}+\mathcal{O}\left(\frac{b}{\gamma}\right)  \tag{2.13}\\
& \rho_{12}=\rho_{32}+\mathcal{O}\left(\frac{b}{\gamma}\right) .
\end{align*}
$$

One observable that can be constructed of these eigenvectors is $J_{x}$, which takes the form,

$$
J_{x}=\frac{1}{\sqrt{2}}\left(\begin{array}{lll}
0 & 1 & 0  \tag{2.14}\\
1 & 0 & 1 \\
0 & 1 & 0
\end{array}\right)
$$

Figure 2.2 shows the expectation value $\left\langle J_{x}\right\rangle(t) \equiv \operatorname{tr}\left[J_{x} \rho(t)\right]$ that was calculated numerically for the initial state $\left|j_{x}=+1\right\rangle$, for several values of noise power $\gamma$ (and for the parameters $q=5, b=1$ ). It is seen from the plot that $\left\langle J_{x}\right\rangle(t)$ undergoes damped oscillations (with more that one frequency at $\gamma<b$ ). The plot shows the damping increasing with $\gamma$ up to $\gamma=b$, after which the damping decreases with $\gamma$, making the oscillations more and more long lived.

In conclusion, we presented above an example of a 3 level system that experiences white noise, found an eigenvalue that exhibits a relaxation rate that decreases with the noise power, and demonstrated this effect using a simple observable.


Figure 2.2: The expectation value $\left\langle J_{x}\right\rangle$ as a function of time for several values of noise power $\gamma$, for the parameters $q=5, b=1$.

### 2.2 Quantum Zeno Effect

The quantum Zeno effect is the inhibition of transitions between quantum states due to frequent measurements [14, 15]. The result in the previous example demonstrating decreasing relaxation with the noise power, can be explained through this effect. We do that by showing the stochastic part in Lindblad equation 2.3 can be regarded as a von Neumann measurement.

The operators presented in equation 2.5 can be expressed in an eigenbasis of $C$ in the following manner,

$$
H_{0}=\left(\begin{array}{lll}
q & b & 0  \tag{2.15}\\
b & q & 0 \\
0 & 0 & 0
\end{array}\right) \quad ; \quad C=\left(\begin{array}{ccc}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right) .
$$

Since equation 2.3 is invariant to subtraction of the identity from $H_{0}, C$ and to sign flipping of $C$, it is equivalent to consider the more simple matrix representation,

$$
H_{0}=\left(\begin{array}{ccc}
0 & b & 0  \tag{2.16}\\
b & 0 & 0 \\
0 & 0 & -q
\end{array}\right) \quad ; \quad C=\left(\begin{array}{ccc}
2 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right) .
$$

The operators can be expressed in the Dirac notation,

$$
\begin{equation*}
H_{0}=b(|1\rangle\langle 2|+|2\rangle\langle 1|)-q|3\rangle\langle 3| \quad ; \quad C=2|1\rangle\langle 1| \equiv 2 P_{1}, \tag{2.17}
\end{equation*}
$$

where $P_{1}$ is a projection on the state $|1\rangle$. Computing the stochastic term in equation 2.3 yields

$$
\begin{equation*}
-\frac{1}{2} \gamma[C,[C, \rho]]=-2 \gamma\left[P_{1},\left[P_{1}, \rho\right]\right]=2 \gamma\left(P_{1} \rho P_{1}+P_{1_{\perp}} \rho P_{1_{\perp}}-\rho\right), \tag{2.18}
\end{equation*}
$$

where $P_{1_{\perp}}=\mathbb{1}-P_{1}$, is the complementary projection of $P_{1}$. The expression on the right hand side suggests that the stochastic term on its own, converts the density matrix to a statistical mixture of the form,

$$
\begin{equation*}
\rho \longmapsto P_{1} \rho P_{1}+P_{1_{\perp}} \rho P_{1_{\perp}}, \tag{2.19}
\end{equation*}
$$

which is equivalent to a (continuous) von Neumann measurement of the population in the state $|1\rangle$ (conversely, the population in the subspace spanned by $|2\rangle,|3\rangle$ ).

We decompose $H_{0}$ to the diagonal and off diagonal parts according to the representation in equation 2.16,

$$
\begin{equation*}
H_{0}=H_{1}+H_{2}, \tag{2.20}
\end{equation*}
$$

where,

$$
\begin{equation*}
H_{1}=-q|3\rangle\langle 3| \quad ; \quad H_{2}=b(|1\rangle\langle 2|+|2\rangle\langle 1|) . \tag{2.21}
\end{equation*}
$$

$H_{2}$ causes Rabi oscillation between states $|1\rangle,|2\rangle$ with frequency $b$. Since the stochastic part is equivalent to a measurement of $|1\rangle$ at the typical frequency $\gamma$, we expect the oscillations to be eliminated for $\gamma \gg b$, due to the quantum Zeno effect. At this regime, two protected subspaces (also called Zeno subspace [16]) form, the first corresponds to $|1\rangle$ and the other is the subspace spanned by $|2\rangle,|3\rangle$. The dynamics within the Zeno subspaces is not necessarily stationary, as the diagonal part of the Hamiltonian $H_{1}$ continues to act and generate oscillatory dynamics (called Zeno dynamics [16]). For example, consider an initial pure state $|\psi\rangle(t=0)=\frac{1}{\sqrt{\alpha^{2}+\beta^{2}}}(\alpha|2\rangle+\beta|3\rangle)$. At the Zeno regime, this state would evolve by the $H_{1}$ in the following manner,

$$
\begin{equation*}
\left.|\psi\rangle(t)\right|_{\gamma \gg b}=\frac{1}{\sqrt{\alpha^{2}+\beta^{2}}}\left(\alpha|2\rangle+e^{i q t} \beta|3\rangle\right) . \tag{2.22}
\end{equation*}
$$

The evolution of $|\psi\rangle(t)$ can be described as a rotation in the Bloch sphere of the subspace spanned by the states $|2\rangle,|3\rangle$, this rotation is illustrated in figure 2.3. This motion is related to the evolution of the observable $J_{x}$ presented in the previous section. The expression for the observable in the current basis is

$$
\begin{equation*}
J_{x}=|2\rangle\langle 3|+|3\rangle\langle 2|, \tag{2.23}
\end{equation*}
$$

which represents the x components in the Bloch sphere associated with the states


Figure 2.3: The rotation due to $H_{1}$ (blue trajectory) in the Bloch sphere describing the states $|2\rangle,|3\rangle$.

## $|2\rangle,|3\rangle$.

From the Zeno picture, it becomes clear that since any Hermitian operator $C$ can be expressed as a sum of orthogonal projections,

$$
\begin{equation*}
C=\sum_{j} c_{j} P_{j}, \tag{2.24}
\end{equation*}
$$

where $c_{j}$ are real coefficient, the stochastic part in equation 2.3 can always be viewed as a repeated von Neumann measurement, which forms protected Zeno subspaces at a sufficiently high measurement frequency. However, nontrivial Zeno dynamics (i.e. oscillation) can only take place in Zeno subspaces which are at least two dimensional. Thus the occurrence of Zeno dynamics requires the sum in equation 2.24 to contain at least one multidimensional projection. In other words, the formation of Zeno dynamics requires the operator $C$ to be degenerate.

### 2.3 Generalization at the Strong Noise Limit

Consider general Hermitian matrices $H_{0}, C$ at the strong noise limit, i.e. where $H_{0}$ can be considered as a small perturbation to $C$. We will later show that this condition is satisfied whenever the following inequality holds,

$$
\begin{equation*}
\min \left(\gamma\left(\lambda_{C}-\lambda_{C}^{\prime}\right)^{2}\right) \gg \max \left(\lambda_{H}-\lambda_{H}^{\prime}\right), \tag{2.25}
\end{equation*}
$$

Where $\lambda_{C}, \lambda_{C}^{\prime} \in \operatorname{spec}(C), \lambda_{C} \neq \lambda_{C}^{\prime}, \lambda_{H}, \lambda_{H}^{\prime} \in \operatorname{spec}\left(H_{0}\right)$. Clearly, this condition is satisfied for a sufficiently large $\gamma$, i.e. at large noise powers (in a finite dimensional system).

Define the unperturbed Lindbladian $\mathcal{L}_{0}$ and the perturbation $\mathcal{L}_{p}$,

$$
\begin{align*}
\mathcal{L}_{0} \rho & =-\frac{1}{2} \gamma[C,[C, \rho]]  \tag{2.26}\\
\mathcal{L}_{p} \rho & =-i\left[H_{0}, \rho\right]
\end{align*}
$$

Denote by $c_{j}$ and $|j\rangle$ the eigenvalues and eigenstates of $C$, respectively. $\left\{c_{j}\right\}$ are real since $C$ is Hermitian. We propose the following ansatz for the eigenvectors of the unperturbed Lindbladian,

$$
\begin{equation*}
\chi_{j k}=|j\rangle\langle k| . \tag{2.27}
\end{equation*}
$$

Substituting the ansatz in equation 2.26 yields their respective eigenvalues,

$$
\begin{equation*}
\mathcal{L}_{0} \chi_{j k}=-\frac{1}{2} \gamma[C,[C,|j\rangle\langle k|]]=-\frac{1}{2} \gamma\left(c_{j}-c_{k}\right)[C,|j\rangle\langle k|]=-\frac{1}{2} \gamma\left(c_{j}-c_{k}\right)^{2} \chi_{j k} \tag{2.28}
\end{equation*}
$$

The unperturbed eigenvalues are thus,

$$
\begin{equation*}
\lambda_{j k}^{(0)}=-\frac{1}{2} \gamma\left(c_{j}-c_{k}\right)^{2} . \tag{2.29}
\end{equation*}
$$

The unperturbed eigenvalues are all real, thus describing pure relaxation (with no oscillation). Similarly we find the spectrum of the perturbation to be,

$$
\begin{equation*}
\operatorname{spec}\left(\mathcal{L}_{p}\right)=\left\{-i\left(h_{j}-h_{k}\right)\right\}_{j k} \tag{2.30}
\end{equation*}
$$

Perturbation theory is valid when the all the nonzero elements in $\operatorname{spec}\left(\mathcal{L}_{0}\right)$ are much larger than all the elements in $\operatorname{spec}\left(\mathcal{L}_{p}\right)$, which results in the condition expressed in equation 2.25 .

From the expression presented in 2.28 , it is clear that,

$$
\begin{equation*}
\chi_{j k} \in \operatorname{ker}\left(\mathcal{L}_{0}\right) \Longleftrightarrow c_{j}=c_{k} \tag{2.31}
\end{equation*}
$$

in particular, $\chi_{j j}=|j\rangle\langle j| \in \operatorname{ker}\left(\mathcal{L}_{0}\right)$. Since $\operatorname{ker}\left(\mathcal{L}_{0}\right)$ is a degenerate subspace, in order to compute the first order correction we must use degenerate perturbation theory, in which the perturbation matrix is computed and projected onto the degenerate subspace. The eigenvectors of the resulting matrix redefine the unperturbed eigenvectors, and its eigenvalues are the first order corrections. First we define the projection superoperator onto $\operatorname{ker}\left(\mathcal{L}_{0}\right)$,

$$
\begin{equation*}
\mathcal{P}_{\mathcal{K}}(W) \equiv \sum_{j, k=1}^{\operatorname{dim} \mathcal{H}} \delta_{c_{j}, c_{k}}|j\rangle\langle j| W|k\rangle\langle k| \tag{2.32}
\end{equation*}
$$

where $W$ denotes an arbitrary projected operator.

Lemma 2.1. For eigenvectors inside $\operatorname{ker}\left(\mathcal{L}_{0}\right)$,

$$
\left.\mathcal{P}_{\mathcal{K}}\left(-i\left[H_{0}, \chi_{j k}\right]\right)\right|_{\chi_{j k} \in \operatorname{ker}\left(\mathcal{L}_{0}\right)}=-i\left[\mathcal{P}_{\mathcal{K}}\left(H_{0}\right), \chi_{j k}\right]
$$

Proof. Compute the projection of the term,

$$
\begin{align*}
\left.\mathcal{P}_{\mathcal{K}}\left(H_{0} \chi_{m n}\right)\right|_{\chi_{m n} \in \operatorname{ker}\left(\mathcal{L}_{0}\right)} & =\sum_{j, k} \delta_{c_{j}, c_{k}}|j\rangle\langle j| H_{0}|m\rangle\langle n||k\rangle\langle k|  \tag{2.33}\\
& =\sum_{j} \delta_{c_{j}, c_{n}}|j\rangle\langle j| H_{0}|m\rangle\langle n| \\
& =\sum_{j, l} \delta_{c_{j}, c_{n}}|j\rangle\langle j| H_{0}|l\rangle\langle l||m\rangle\langle n| .
\end{align*}
$$

Recall that $c_{m}=c_{n}$ in $\operatorname{ker}\left(\mathcal{L}_{0}\right)$. Also, the term $\langle l \mid m\rangle$ ensures that the expression in the sum vanishes for $l \neq m$, thus we may replace $\delta_{c_{j}, c_{n}}$ with $\delta_{c_{j}, c_{l}}$,

$$
\begin{equation*}
\left.\mathcal{P}_{\mathcal{K}}\left(H_{0} \chi_{m n}\right)\right|_{\chi_{m n} \in \operatorname{ker}\left(\mathcal{L}_{0}\right)}=\left(\sum_{j, l} \delta_{c_{j}, c_{l}}|j\rangle\langle j| H_{0}|l\rangle\langle l|\right)|m\rangle\langle n|=\mathcal{P}_{\mathcal{K}}\left(H_{0}\right) \chi_{m n} . \tag{2.34}
\end{equation*}
$$

By applying adjoint to both sides of equation 2.34 , we get

$$
\begin{equation*}
\left.\mathcal{P}_{\mathcal{K}}\left(\chi_{m n} H_{0}\right)\right|_{\chi_{m n} \in \operatorname{ker}\left(\mathcal{L}_{0}\right)}=\chi_{m n} \mathcal{P}_{\mathcal{K}}\left(H_{0}\right), \tag{2.35}
\end{equation*}
$$

thus,

$$
\begin{equation*}
\left.\mathcal{P}_{\mathcal{K}}\left(-i\left[H_{0}, \chi_{j k}\right]\right)\right|_{\chi_{j k} \in \operatorname{ker}\left(\mathcal{L}_{0}\right)}=-i\left(\mathcal{P}_{\mathcal{K}}\left(H_{0}\right) \chi_{j k}-\chi_{j k} \mathcal{P}_{\mathcal{K}}\left(H_{0}\right)\right)=-i\left[\mathcal{P}_{\mathcal{K}}\left(H_{0}\right), \chi_{j k}\right] \tag{2.36}
\end{equation*}
$$

The perturbation matrix acting on $\operatorname{ker}\left(\mathcal{L}_{0}\right)$ is given by,

$$
\begin{equation*}
\mathcal{P}_{\mathcal{K}}\left(\mathcal{L}_{p} \chi_{j k}\right)=\mathcal{P}_{\mathcal{K}}\left(-i\left[H_{0}, \chi_{j k}\right]\right)=-i\left[\mathcal{P}_{\mathcal{K}}\left(H_{0}\right), \chi_{j k}\right] \equiv \mathcal{L}^{(1)} \chi_{j k}, \tag{2.37}
\end{equation*}
$$

and the first order corrections to eigenvalues associated with $\operatorname{ker}\left(\mathcal{L}_{0}\right)$ are the eigenvalues of the superoperator $\mathcal{L}^{(1)}$,

$$
\begin{equation*}
\lambda_{\text {ker }}^{(1)}=\operatorname{spec}\left(\mathcal{L}^{(1)}\right) \tag{2.38}
\end{equation*}
$$

According to equation 2.32, the projection of the Hamiltonian $\mathcal{P}_{\mathcal{K}}\left(H_{0}\right)$ is a Hermitian operator, thus $\lambda_{\text {ker }}^{(1)}$ are imaginary, and describe pure oscillations without relaxation. Within the kernel, this correction is the dominant correction at the strong noise limit, as it is $\mathcal{O}(1)$ while the higher order corrections are $\mathcal{O}\left(\frac{1}{\gamma}\right), \mathcal{O}\left(\frac{1}{\gamma^{2}}\right)$ and so on (i.e. they approach zero as $\gamma \rightarrow \infty$ ). Therefore, these first order corrections describe the oscillation frequencies corresponding to $\operatorname{ker}\left(\mathcal{L}_{0}\right)$, at the strong noise
limit.
When $C$ is nondegenerate, $\operatorname{ker}\left(\mathcal{L}_{0}\right)$ is spanned by the eigenvectors $\chi_{j j}=|j\rangle\langle j|$, and the perturbation matrix $\mathcal{L}^{(1)}$ vanishes,

$$
\begin{equation*}
-i\left[\mathcal{P}_{\mathcal{K}}\left(H_{0}\right), \chi_{j j}\right]=-i \sum_{k}\left[|k\rangle\langle k| H_{0}|k\rangle\langle k|,|j\rangle\langle j|\right]=-i \sum_{k} H_{0_{k k}}[|k\rangle\langle k|,|j\rangle\langle j|]=0 . \tag{2.39}
\end{equation*}
$$

Thus in agreement with the previous section, we have shown that the appearance of oscillations at the strong noise limit (i.e. the Zeno regime) requires $C$ to be degenerate.

The higher order correction (which will not be computed explicitly) is of order $\mathcal{O}\left(\frac{1}{\gamma}\right)$. This correction scales as $\sim \mathcal{L}_{p}^{2}$, which implies it's real and describes relaxation. This means that the leading order of relaxation in $\operatorname{ker}\left(\mathcal{L}_{0}\right)$ decreases as the inverse of the noise power $\gamma$.

In conclusion, using a perturbative approach for the strong noise limit, we find the decrease in relaxation with the noise power is in fact a general effect, occurring at the kernel of the noise. We also obtained an expression for the oscillation frequencies at this limit. This expression shows that protected oscillations appear only when the noise operator $C$ is degenerate, in agreement with the Zeno interpretation discussed in the previous section.

## Chapter 3

## Anomalous Relaxation in Nonlinear Mean Field Lindblad Equations

This chapter discusses two relaxation features that emerge due to nonlinearity in the Lindblad equation. The first phenomenon shown in this chapter is the emergence of a continuum of relaxation rates (as opposed to a discrete set of rates observed in the linear Lindblad equation). The other phenomenon is polynomial decay (as opposed to exponential decay in the linear equation).

In this chapter we discuss the mean field class of nonlinear Lindblad equations. It describes single particles in an ensemble of identical particles, where the nonlinearity stems from interaction between particles, that can be associated with collisions between particles. The single particle Lindbladian can be obtained under certain conditions via a mean field approximation.

In this chapter we first present the mean field nonlinear Lindblad equations. Then we present two examples of mean field Lindbladians, each demonstrating one of the two phenomena mentioned above.

### 3.1 Nonlinear Lindblad Equations

Nonlinear Lindblad equations emerge as an approximation for interacting many body systems. In these systems, the particles act as a bath that depends on their quantum state, giving rise to nonlinearity in the density matrix. When the interactions between particles are symmetric and scale as the inverse of the number of particles, the system becomes tractable. In that case, upon taking the number of particles to infinity, the interacting particles evolve effectively by a single particle Lindblad equation which is nonlinear in general.

Mean field nonlinear Lindblad equations are discussed in [17, 18]. Consider an ensemble of $N$ identical particles with the following Hamiltonian,

$$
\begin{equation*}
\mathrm{H}_{N}=\sum_{j=1}^{N} H_{j}+\frac{1}{N} \sum_{j \neq k=1}^{N} V_{j k} \tag{3.1}
\end{equation*}
$$

Where $H_{j}$ acts on $\mathcal{H}_{j}$ (the ${ }^{\prime}$ 'th Hilbert space, where $\mathcal{H}=\otimes_{j} \mathcal{H}_{j}$ ) and is an identical copy of $H_{1}$, which is the Hamiltonian of the single particle; and $V_{j k}$ acts on $\mathcal{H}_{j} \otimes \mathcal{H}_{k}$ and describes an interaction term between two particles, where $V_{j k}$ is an identical copy of $V_{12}$ (setting $V_{j k}=V_{k j}$ ).

The partial trace over the Hilbert spaces labeled $n, n+1, \ldots, N$ is denoted by $\operatorname{tr}_{[n, N]}$. It is proven in [17] that given a symmetric product state as an initial state $\rho_{N}(t=0)=\rho_{0}^{\otimes N}$, then at the infinitely many particles limit, the partial trace over the evolved state also becomes a product state,

$$
\begin{equation*}
\lim _{N \rightarrow \infty} \operatorname{tr}_{[n+1, N]}\left(e^{-i \mathrm{H}_{N} t} \rho_{N}(t=0) e^{i \mathrm{H}_{N} t}\right)=(\rho(t))^{\otimes n} \tag{3.2}
\end{equation*}
$$

and moreover $\rho(t)$ satisfies the equation

$$
\begin{equation*}
\frac{d}{d t} \rho(t)=-i\left[H_{1}, \rho(t)\right]+\operatorname{tr}_{2}\left(-i\left[V_{12}+V_{21}, \rho(t) \otimes \rho(t)\right]\right) \tag{3.3}
\end{equation*}
$$

Equation 3.3 is the mean field equation, and is nonlinear in general. We shall now show that this equation can be expressed as the Heisenberg equation, with an effective Hamiltonian that depends on $\rho$,

$$
\begin{equation*}
\frac{d}{d t} \rho=-i\left[H_{e f f}(\rho), \rho\right], \tag{3.4}
\end{equation*}
$$

and thus cannot describe decoherence.

Proof. Decompose the interaction terms into a sum of tensor products

$$
\begin{equation*}
V_{12}+V_{21}=\sum_{j, k} h_{j k} W_{j} \otimes W_{k}, \tag{3.5}
\end{equation*}
$$

and substitute in equation 3.3 to yield

$$
\begin{align*}
\frac{d}{d t} \rho & =-i\left[H_{1}, \rho\right]-i \operatorname{tr}_{2}\left[\sum_{j, k} h_{j k} W_{j} \otimes W_{k}, \rho \otimes \rho\right]  \tag{3.6}\\
& =-i\left[H_{1}, \rho\right]-i \sum_{j, k} h_{j k} t r_{2}\left(W_{j} \rho \otimes W_{k} \rho-\rho W_{j} \otimes \rho W_{k}\right) \\
& =-i\left[H_{1}, \rho\right]-i \sum_{j} \operatorname{tr}\left(\sum_{k} h_{j k} W_{k} \rho\right)\left[W_{j}, \rho\right] \\
& \equiv-i\left[H_{1}, \rho\right]-i \sum_{j} w_{j}(\rho)\left[W_{j}, \rho\right] \equiv-i\left[H_{e f f}(\rho), \rho\right] .
\end{align*}
$$

The theorem expressed in 3.2 was extended in [18] to cover many body Lindbladians of the form,

$$
\begin{equation*}
\frac{d}{d t} \rho_{N}(t)=-\sum_{j} i\left[H_{j}, \rho(t)\right]+\frac{1}{2 N} \sum_{j, k=1}^{N} \mathcal{L}_{j k} \rho_{N}(t) \tag{3.7}
\end{equation*}
$$

where $H_{j}$ is the same as before. $\mathcal{L}_{j k} \forall j \neq k$ is a Lindblad superoperator that acts on states that lie in $\mathcal{H}_{j} \otimes \mathcal{H}_{k}$ and describes a, possibly dissipative, interaction term between two particles. $\mathcal{L}_{j k}$ is an identical copy of $\mathcal{L}_{12}$ (setting $\mathcal{L}_{j k}=\mathcal{L}_{k j}$ ). $\mathcal{L}_{j j}$ is a Lindblad superoperator that acts on states that lie in $\mathcal{H}_{j}$. In particular, the mean field theorem is proven, i.e. given a symmetric product state $\rho_{N}(t=0)=\rho_{0}^{\otimes N}$, there exists $\rho(t)$ such that

$$
\begin{equation*}
\lim _{N \rightarrow \infty} \operatorname{tr}_{[n+1, N]} \rho_{N}(t)=(\rho(t))^{\otimes n} \tag{3.8}
\end{equation*}
$$

where $\rho(t)$ satisfies the nonlinear equation,

$$
\begin{equation*}
\frac{d}{d t} \rho(t)=-i\left[H_{1}, \rho(t)\right]+\frac{1}{2} \operatorname{tr}_{2}\left[\left(\mathcal{L}_{12}+\mathcal{L}_{21}\right) \rho(t) \otimes \rho(t)\right] \tag{3.9}
\end{equation*}
$$

In the following sections we will use equation 3.9 in various cases.

### 3.2 Continuum of Dephasing Rates

Consider an ensemble of $N$ particles that evolve by the Lindblad equation,

$$
\begin{equation*}
\frac{d}{d t} \rho_{N}(t)=\frac{1}{2 N} \sum_{j \neq k=1}^{N} \mathcal{L}_{j k} \rho_{N}(t) \tag{3.10}
\end{equation*}
$$

Where $\mathcal{L}_{j k}$ is defined by

$$
\begin{equation*}
\mathcal{L}_{j k} \rho_{N}=4\left[P_{0}^{j k} \rho_{N} P_{0}^{j k}-\frac{1}{2}\left(P_{0}^{j k}\right)^{2} \rho_{N}-\frac{1}{2} \rho_{N}\left(P_{0}^{j k}\right)^{2}\right] \tag{3.11}
\end{equation*}
$$

$P_{0}^{j k}$ denotes a projection on the state $|0\rangle_{j} \otimes|0\rangle_{k}$. The operator $P_{0}^{j k}$ is Hermitian, thus the Lindbladian above conforms to the general form of stochastic Lindbladians, presented in equation 2.3.

The Lindbladian above can be interpreted as describing a pair of particles coupled by a Hamiltonian $\xi(t) P_{0} \otimes P_{0} \quad\left(P_{0} \equiv|0\rangle\langle 0|\right)$, where $\xi(t)$ denotes a stochastic coupling strength which may be a result of random collisions between the particles (where the distance between them vary stochastically with time). According to this interpretation, equation 3.10 describes an ensemble of particles that undergo pairwise collisions associated with the interaction Hamiltonian $H_{\text {int }}=P_{0} \otimes P_{0}$.

Taking the number of particles $N$ to infinity, we obtain the single particle master equation,

$$
\begin{equation*}
\frac{d}{d t} \rho(t)=4 t r_{2}\left[P_{0}^{\otimes 2} \rho(t)^{\otimes 2} P_{0}^{\otimes 2}-\frac{1}{2}\left(P_{0}^{\otimes 2}\right)^{2} \rho(t)^{\otimes 2}-\frac{1}{2} \rho(t)^{\otimes 2}\left(P_{0}^{\otimes 2}\right)^{2}\right] . \tag{3.12}
\end{equation*}
$$

Computing the partial trace yields,

$$
\begin{align*}
& \operatorname{tr}_{2}\left[P_{0}^{\otimes 2} \rho^{\otimes 2} P_{0}^{\otimes 2}-\frac{1}{2}\left(P_{0}^{\otimes 2}\right)^{2} \rho^{\otimes 2}-\frac{1}{2} \rho^{\otimes 2}\left(P_{0}^{\otimes 2}\right)^{2}\right]  \tag{3.13}\\
& =P_{0} \rho P_{0} \operatorname{tr}\left(P_{0} \rho P_{0}\right)-\frac{1}{2} P_{0}^{2} \rho \operatorname{tr}\left(P_{0}^{2} \rho\right)-\frac{1}{2} \rho P_{0}^{2} \operatorname{tr}\left(\rho P_{0}^{2}\right) \\
& =\left(P_{0} \rho P_{0}-\frac{1}{2} P_{0}^{2} \rho-\frac{1}{2} \rho P_{0}^{2}\right) \operatorname{tr}\left(P_{0} \rho\right) \\
& =-\frac{1}{2} \operatorname{tr}\left(P_{0} \rho\right)\left[P_{0},\left[P_{0}, \rho\right]\right] .
\end{align*}
$$

In conclusion, the single particle master equation takes the form,

$$
\begin{equation*}
\frac{d}{d t} \rho(t)=-2 \operatorname{tr}\left(P_{0} \rho\right)\left[P_{0},\left[P_{0}, \rho\right]\right] . \tag{3.14}
\end{equation*}
$$

Clearly, equation 3.14 is trace preserving. This equation describes dephasing without decay.

When the particles in question are qubits ${ }^{1}$, the projection $P_{0}$ can be expressed as: $P_{0}=\frac{1+\sigma_{z}}{2}$, and equation 3.14 reduces to,

$$
\begin{equation*}
\frac{d}{d t} \rho=-\frac{1}{4}\left[1+\operatorname{tr}\left(\sigma_{z} \rho\right)\right]\left[\sigma_{z},\left[\sigma_{z}, \rho\right]\right], \tag{3.15}
\end{equation*}
$$

where we used the fact $\operatorname{tr}(\rho)=1$.
In order to solve the equation, we use the Bloch vector representation for the qubit state,

$$
\begin{equation*}
\rho=\frac{\mathbb{1}+\mathbf{u} \cdot \boldsymbol{\sigma}}{2} . \tag{3.16}
\end{equation*}
$$

Clearly,

$$
\begin{equation*}
\operatorname{tr}\left(\sigma_{z} \rho\right)=\operatorname{tr}\left(\sigma_{z} \frac{u_{z} \sigma_{z}}{2}\right)=u_{z} . \tag{3.17}
\end{equation*}
$$

Also,

$$
\begin{equation*}
\left[\sigma_{z},\left[\sigma_{z}, \frac{\sigma_{z}}{2}\right]\right]=0, \quad\left[\sigma_{z},\left[\sigma_{z}, \frac{\sigma_{x, y}}{2}\right]\right]=2 \sigma_{x, y} \tag{3.18}
\end{equation*}
$$

Thus the equations of motion for the three components of Bloch vector are,

[^3]\[

$$
\begin{align*}
& \frac{d}{d t} u_{x, y}=-\frac{1}{2}\left(1+u_{z}\right) u_{x, y}  \tag{3.19}\\
& \frac{d}{d t} u_{z}=0
\end{align*}
$$
\]

The z component is constant in time, thus the equations are decoupled and the solution for the x and y component is,

$$
\begin{equation*}
u_{x, y}(t)=u_{x, y}^{i} e^{-\frac{1+u_{z}}{2} t} . \tag{3.20}
\end{equation*}
$$

The solution describes a Bloch vector that approaches the z axis as $t \rightarrow \infty$ along a straight line, perpendicular to the z axis. This motion is exponential in time, with a rate that depends linearly on the height of the trajectory from the bottom of the Bloch sphere (see figure 3.1). Depending on the height, the exponential rate may take any value in the continuous range $[0,1]$. A continuum of dephasing rates cannot emerge from a linear Lindblad equation, which can only give a discrete set of rates.


Figure 3.1: Visualization of the solutions expressed in 3.20 through a vector field in the Bloch sphere. The directions of the arrows represent the direction of motion (i.e. towards the z axis), while their sizes represent the exponential rate of the motion.

### 3.3 Polynomial Decay

In this section we shall again consider an ensemble of $N$ interacting qubits, where we will examine a process that generates decay. Decay processes emerge under
interaction with an external reservoir which absorbs the excess energy. A convenient choice for such a reservoir is a quantized field of massless bosons (e.g. photons).

We shall consider an interaction that causes the qubits to decay in pairs, rather than separately. This decay may be a result of binary collisions between pairs of particles ${ }^{2}$. First we will present the master equation for the ensemble of qubits, then present two approaches for its solution. The first approach takes the number of qubits to infinity and solves the emerging single particle nonlinear equation; while the second approach solves the full equation for any number of qubits, but is restricted to symmetric states. We will discuss the two solutions, their features and the relations between them.

## The Master Equation

In order to construct an $N$ qubit Lindblad equation that describes pairwise decay between the qubits, we shall consider a mutual interaction of a single pair of qubits with a quantized field of massless bosons, held at zero temperature. We rely on a previous derivation presented in [7], which considered an interaction with a bath of quantized radiation. By allowing transitions only between the excited-excited state (i.e. a state where both qubits are excited) and the ground state, we obtain the Lindbladian for a single pair of qubits (the full derivation is given in appendix A),

$$
\begin{equation*}
\mathcal{L}_{12} \rho=a^{\otimes 2} \rho\left(a^{\dagger}\right)^{\otimes 2}-\frac{1}{2}\left(a^{\dagger}\right)^{\otimes 2} a^{\otimes 2} \rho-\frac{1}{2} \rho\left(a^{\dagger}\right)^{\otimes 2} a^{\otimes 2}, \tag{3.21}
\end{equation*}
$$

where $a \equiv|0\rangle\langle 1|$ is the lowering operator. It is may be verified that the states $|00\rangle,|01\rangle,|10\rangle$ are the stationary states of $\mathcal{L}_{12}$, while $|11\rangle$ is unstable and decays to $|00\rangle$. This means that the qubits cannot decay separately, but they must do it as a pair, as initially desired.

Now we consider an ensemble of qubits, each interacting with all the other qubits via $\mathcal{L}_{12}$ expressed in equation 3.21 , with a coupling strength that scales as $\frac{1}{2 N}$. The master equation that governs such an ensemble is,

$$
\begin{align*}
\frac{d}{d t} \rho_{N}(t) & =\frac{1}{2 N} \sum_{k \neq j=1}^{N} \mathcal{L}_{j k} \rho_{N}(t)  \tag{3.22}\\
& =\frac{1}{N} \sum_{k>j=1}^{N}\left(a_{j k} \rho_{N} a_{j k}^{\dagger}-\frac{1}{2} a_{j k}^{\dagger} a_{j k} \rho_{N}-\frac{1}{2} \rho_{N} a_{j k}^{\dagger} a_{j k}\right),
\end{align*}
$$

where, as in equation 3.7, $\mathcal{L}_{j k}$ are identical copies of $\mathcal{L}_{12}$, and act on states in $\mathcal{H}_{j} \otimes \mathcal{H}_{k}$; $\rho_{N}$ denotes the $N$ qubit state and $a_{j k} \equiv a_{j} \otimes a_{k}$, which act on $\mathcal{H}_{j} \otimes \mathcal{H}_{k}$.

[^4]
## Solution in the Infinite Particle Limit

At the limit of an infinite number of qubits $(N \rightarrow \infty)$, given an initial product state, the single qubits evolve separately according to the nonlinear master equation,

$$
\begin{align*}
\frac{d}{d t} \rho & =\operatorname{tr}_{2}\left(\mathcal{L}_{12} \rho^{\otimes 2}\right)  \tag{3.23}\\
& =\operatorname{tr}_{2}\left[\left(a \rho a^{\dagger}\right) \otimes\left(a \rho a^{\dagger}\right)-\frac{1}{2}\left(a^{\dagger} a \rho\right) \otimes\left(a^{\dagger} a \rho\right)-\frac{1}{2}\left(\rho a^{\dagger} a\right) \otimes\left(\rho a^{\dagger} a\right)\right]
\end{align*}
$$

where $\rho$ is a single qubit state. Further simplifying the equation yields,

$$
\begin{align*}
\frac{d}{d t} \rho & =\operatorname{tr}\left(a^{\dagger} a \rho\right)\left(a \rho a^{\dagger}-\frac{1}{2} a^{\dagger} a \rho-\frac{1}{2} \rho a^{\dagger} a\right)  \tag{3.24}\\
& =\operatorname{tr}\left(P_{1} \rho\right)\left(a \rho a^{\dagger}-\frac{1}{2} a^{\dagger} a \rho-\frac{1}{2} \rho a^{\dagger} a\right) \equiv \operatorname{tr}\left(P_{1} \rho\right) \mathcal{L}_{s} \rho,
\end{align*}
$$

where $P_{1}=a^{\dagger} a$ is the projection onto the state $|1\rangle$, and $\mathcal{L}_{s}$ is a linear Lindbladian describing regular spontaneous decay (we derive the spontaneous decay Lindbladian in A). We characterize $\mathcal{L}_{s}$ by finding its eigenvectors.

Consider $P_{0} \equiv|0\rangle\langle 0|$, and recall that $a^{2}=\left(a^{\dagger}\right)^{2}=0, P_{0}=a a^{\dagger}$,

$$
\begin{equation*}
\mathcal{L}_{s} P_{0}=\mathcal{L}_{s}\left(a a^{\dagger}\right)=a^{2}\left(a^{\dagger}\right)^{2}-\frac{1}{2} a^{\dagger} a a a^{\dagger}-\frac{1}{2} a a^{\dagger} a^{\dagger} a=0 . \tag{3.25}
\end{equation*}
$$

Next consider the Pauli matrix $\sigma_{z}$, which can be expressed as

$$
\begin{gather*}
\sigma_{z}=P_{0}-P_{1}  \tag{3.26}\\
\mathcal{L}_{s} \sigma_{z}=\mathcal{L}_{s}\left(P_{0}-P_{1}\right)=\mathcal{L}_{s} P_{0}-\mathcal{L}_{s}\left(a^{\dagger} a\right)  \tag{3.27}\\
=0-\left(a a^{\dagger} a a^{\dagger}-\frac{1}{2} a^{\dagger} a a^{\dagger} a-\frac{1}{2} a^{\dagger} a a^{\dagger} a\right)=-\left(P_{1}^{2}-P_{0}^{2}\right)=-\sigma_{z} .
\end{gather*}
$$

Next $\sigma_{x}$, which can be expressed as,

$$
\begin{equation*}
\sigma_{x}=a+a^{\dagger} \tag{3.28}
\end{equation*}
$$

$$
\begin{align*}
\mathcal{L}_{s} \sigma_{x} & =\mathcal{L}_{s}\left(a+a^{\dagger}\right)=a\left(a+a^{\dagger}\right) a^{\dagger}-\frac{1}{2} a^{\dagger} a\left(a+a^{\dagger}\right)-\frac{1}{2}\left(a+a^{\dagger}\right) a^{\dagger} a  \tag{3.29}\\
& =0-\frac{1}{2} P_{1} a^{\dagger}-\frac{1}{2} a P_{1}=-\frac{1}{2}\left(a+a^{\dagger}\right)=-\frac{1}{2} \sigma_{x} .
\end{align*}
$$

Finally $\sigma_{y}$, which behaves similarly to $\sigma_{x}$,

$$
\begin{gather*}
\sigma_{y}=i a^{\dagger}-i a  \tag{3.30}\\
\mathcal{L}_{s} \sigma_{y}=\mathcal{L}_{s}\left(i a^{\dagger}-i a\right)=0-\frac{1}{2} i a^{\dagger}+\frac{1}{2} i a=-\frac{1}{2} \sigma_{y} . \tag{3.31}
\end{gather*}
$$

Table 3.1 gives the four eigenvectors and eigenvalues obtained above. We see that the ground state $|0\rangle$ is the only stationary state of $\mathcal{L}_{s}$, which is consistent with the fact it describes spontaneous decay.

| Eigenvector | Eigenvalue |
| :---: | :---: |
| $P_{0}$ | 0 |
| $\sigma_{x}$ | $-\frac{1}{2}$ |
| $\sigma_{y}$ | $-\frac{1}{2}$ |
| $\sigma_{z}$ | -1 |

Table 3.1: The eigenvectors and corresponding eigenvalues of the Lindbladian $\mathcal{L}_{s}$.

Using the eigenvectors obtained for $\mathcal{L}_{s}$, we choose a useful representation of $\rho$,

$$
\begin{equation*}
\rho(t)=P_{0}+\frac{1}{2} \mathbf{u}(t) \cdot \boldsymbol{\sigma} . \tag{3.32}
\end{equation*}
$$

$\mathbf{u}(t)$ describes a vector in the Bloch sphere anchored at the north pole (i.e. the $|0\rangle$ state, unlike the Bloch vector which is rooted at the center of the sphere). The representation satisfies the normalization condition $\operatorname{tr}(\rho)=1$ and it describes a positive state as long as the following condition for $\mathbf{u}$ is satisfied,

$$
\begin{equation*}
|\mathbf{u}+\hat{z}| \leq 1 \Rightarrow-2 \leq u_{3} \leq 0 . \tag{3.33}
\end{equation*}
$$

Substituting the representation above in the master equation (equation 3.24),

$$
\begin{align*}
\frac{d}{d t} \rho & =\frac{1}{2} \frac{d \mathbf{u}}{d t} \cdot \boldsymbol{\sigma}=\operatorname{tr}\left(P_{1} \rho\right) \mathcal{L}_{s} \rho=\operatorname{tr}\left(P_{1} \frac{1}{2} u_{3} \sigma_{z}\right) \mathcal{L}_{s}\left(\frac{1}{2} \mathbf{u} \cdot \boldsymbol{\sigma}\right)  \tag{3.34}\\
& =-\frac{1}{4} u_{3}\left(-\frac{1}{2} u_{1} \sigma_{x}-\frac{1}{2} u_{2} \sigma_{y}-u_{3} \sigma_{z}\right)
\end{align*}
$$

Equating the coefficients of the different Pauli matrices on both sides yields three coupled, nonlinear differential equations,

$$
\begin{align*}
\frac{d}{d t} u_{1} & =\frac{1}{4} u_{3} u_{1}  \tag{3.35}\\
\frac{d}{d t} u_{2} & =\frac{1}{4} u_{3} u_{2} \\
\frac{d}{d t} u_{3} & =\frac{1}{2} u_{3}^{2}
\end{align*}
$$

The equation for $u_{3}$ is decoupled from the rest, and can be solved separately. Its solution is,

$$
\begin{equation*}
u_{3}(t)=\frac{u_{3 i}}{1-\frac{1}{2} u_{3 i} t}=\mathcal{O}\left(\frac{1}{t}\right) \tag{3.36}
\end{equation*}
$$

where $u_{3 i}$ denotes the initial value $u_{3}(t=0)$. The solution exhibits a polynomial decay of the $z$ component (note $u_{3}$ is negative by equation 3.33 ), we may use it to compute the probability to find the qubit in the $|1\rangle$ state,

$$
\begin{equation*}
p_{|1\rangle}(t)=\left\langle P_{1}\right\rangle=\operatorname{tr}\left(P_{1} \rho\right)=-\frac{1}{2} u_{3}(t)=\frac{p_{|1\rangle i}}{1+p_{|1\rangle i}}, \tag{3.37}
\end{equation*}
$$

where $p_{|1\rangle i}$ denoted the initial probability $p_{|1\rangle i} \equiv p_{|1\rangle}(t=0)$. We thus found that the survival probability of the $|1\rangle$ state decays polynomially with time, rather than the standard exponential decay, typical to linear Lindbladians. This feature is a result of the nonlinearity of the master equation.

Remark 3.1. It is known that the decay of states under Hamiltonian evolution (given the Hamiltonian is bounded from below), is described by an exponential decay law $e^{-\Gamma t}$ for intermediate times, and by a powerlaw decay for long times (after a large number of decay times $1 / \Gamma$ have passed) $[19,20,21,22,23,24]$. The powerlaw decay appears at a late time, after the amplitude of the state has managed to decrease in many orders of magnitude, making the effect practically undetectable in most cases. The polynomial decay we found above expressed in equation 3.37 (which is approximately powerlaw at $t \gg 1$ ), is different (it is a direct result of the nonlinearity introduced by the many body interaction), and it behaves non-exponentially right from the beginning, allowing the effect to be detected relatively easily.

Next we solve the equations for the other two components, $u_{1,2}$ by first substi-
tuting the solution 3.36 in equations 3.35 ,

$$
\begin{equation*}
\frac{d}{d t} u_{1,2}=\frac{1}{4} \frac{u_{3 i}}{1-\frac{1}{2} u_{3 i} t} u_{1,2} . \tag{3.38}
\end{equation*}
$$

Equation 3.38 may be solved via separation of variables, the resulting solution is,

$$
\begin{equation*}
u_{1,2}(t)=\frac{u_{1,2 i}}{\sqrt{1-\frac{1}{2} u_{3 i} t}}, \tag{3.39}
\end{equation*}
$$

where $u_{1,2 i}$ denote the initial values for the transverse components $u_{1,2}(t=0)$. $u_{1,2}(t)$ decay as a square root of the time, and the resulting trajectory in the Bloch sphere is the parabola,

$$
\begin{equation*}
\frac{u_{3}(t)}{u_{1,2}^{2}(t)}=\frac{u_{3 i}}{u_{1,2 i}^{2}} \equiv-\alpha_{1,2}, \tag{3.40}
\end{equation*}
$$

where $\alpha_{1,2}$ denote two constants that characterizes the trajectory. At long timescales $t \gg 2\left|u_{3 i}^{-1}\right|$, equation 3.39 becomes,

$$
\begin{equation*}
u_{1,2}(t) \approx\left(-\frac{u_{3 i}}{2 u_{1,2 i}} t\right)^{-\frac{1}{2}}=\left(\frac{1}{2} \alpha_{1,2} t\right)^{-\frac{1}{2}} . \tag{3.41}
\end{equation*}
$$

Note that the expression on the right hand side of equation 3.41 depends solely on $\alpha_{1,2}$, and not only on the initial values $u_{1,2}$. This behavior causes a Mpemba-like effect $^{3}$ [25], where comparing two trajectories each with a different initial $r$ component ( $r$ is the cylindrical radial coordinate), we get that in some cases, the outermost trajectory approaches quicker to the $z$ axis and overtakes the other trajectory. This effect is shown in figure 3.2, which compares different trajectories initialized at the surface of the Bloch sphere.

## Solution of the Lindblad Equation For the Entire Ensemble

Consider the symmetric $N$ qubit states of the form,

$$
\begin{equation*}
R_{n}=\mathcal{S}\left(P_{1}^{\otimes 2 n} \otimes P_{0}^{\otimes(N-2 n)}\right), \tag{3.42}
\end{equation*}
$$

where $\mathcal{S}$ denotes a symmetrization which is symmetric under all possible permutations of qubit labels, and $P_{0,1}$ are the projections on states $|0\rangle,|1\rangle$ respectively. For instance, for $N=3$ the possible $R_{n}$ states are

[^5]

Figure 3.2: An illustration of Mpemba in two trajectories in the Bloch sphere. The green trajectory starts at the equator, while the blue trajectory starts above it, with a lower initial $y$ component. The red dots represent the position of the states on both trajectories at a certain time. It is evident that the position on the green trajectory is closer to the $z$ axis (i.e. has a lower $y$ components) than the position on the blue trajectory, despite their initial positions. Unlike the radial component, the $z$ component does not experience a Mpemba effect, as it is seen that the position on the blue trajectory is higher in the sphere compared with the green trajectory, in accordance with their initial positions.

$$
\begin{align*}
R_{0} & =P_{0}^{\otimes 3}  \tag{3.43}\\
R_{1} & =\frac{1}{3}\left(P_{1}^{\otimes 2} \otimes P_{0}+P_{1} \otimes P_{0} \otimes P_{1}+P_{0} \otimes P_{1}^{\otimes 2}\right) .
\end{align*}
$$

These states are similar to the Dicke states, defined by $\mathcal{S}\left(|1\rangle^{\otimes 2 n} \otimes|0\rangle^{\otimes(N-2 n)}\right)$, however they differ in the fact that the Dicke states are pure states by definition, while the states $R_{n}$ are mixed in general.

To make the notations simple, from here and for the rest of the chapter we will restrict the discussion to cases where $N$ is even, though the solution presented below may be adapted for odd numbers of qubits. When $N$ is even, $n \in\left\{0,1, \ldots, \frac{N}{2}\right\}$.

Since the Lindblad equation 3.22 is symmetric under swapping of qubits, an initially symmetric state of the form,

$$
\begin{equation*}
\rho_{N}(t=0)=\sum_{n=0}^{N / 2} c_{n}(t=0) R_{n}, \tag{3.44}
\end{equation*}
$$

( $c_{n}$ are general positive coefficients) will remain symmetric, and with an even num-
ber of excitations throughout its evolution,

$$
\begin{equation*}
\rho_{N}(t)=\sum_{n=0}^{N / 2} c_{n}(t) R_{n} . \tag{3.45}
\end{equation*}
$$

$c_{n}(t)$ is the time dependent probability to have $2 n$ excitations.
Since: $a^{\dagger} a=P_{1}$ the two-qubit Lindbladian can be rewritten as,

$$
\begin{equation*}
\mathcal{L}_{12} \rho_{S}=a^{\otimes 2} \rho_{S}\left(a^{\dagger}\right)^{\otimes 2}-\frac{1}{2}\left\{P_{1}^{\otimes 2}, \rho_{S}\right\} . \tag{3.46}
\end{equation*}
$$

We shall now make use of the fact,

$$
\mathcal{L}_{12}\left(P_{a} \otimes P_{b}\right)=\left\{\begin{array}{ll}
P_{0} \otimes P_{0}-P_{1} \otimes P_{1} & a=b=1  \tag{3.47}\\
0 & \text { otherwise }
\end{array} .\right.
$$

According to the equation above, substitution of the state expressed in 3.42 in the $N$ qubit Lindbladian in equation 3.22 yields

$$
\frac{1}{N} \sum_{k>j=1}^{N} \mathcal{L}_{j k} R_{n}= \begin{cases}\frac{1}{N}\binom{2 n}{2}\left(R_{n-1}-R_{n}\right) & n \geq 1  \tag{3.48}\\ 0 & n=0\end{cases}
$$

Where the factor $\binom{2 n}{2}$ is the number of nonvanishing terms in the sum, which is the number of choices of $P_{1}$ pairs present in the state. Thus substituting the state expressed in 3.45 in equation 3.22 (the master equation for the ensemble) yields,

$$
\begin{align*}
\frac{d}{d t} \sum_{n} c_{n}(t) R_{n} & =\frac{1}{N} \sum_{n=1}^{N / 2} c_{n}(t)\binom{2 n}{2}\left(R_{n-1}-R_{n}\right)  \tag{3.49}\\
& =\frac{1}{N} \sum_{n=0}^{N / 2-1}\binom{2(n+1)}{2} c_{n+1}(t) R_{n}-\frac{1}{N} \sum_{n=1}^{N / 2} c_{n}(t)\binom{2 n}{2} R_{n}
\end{align*}
$$

We can now obtain a set of coupled ordinary differential equations for the probabilities $c_{n}(t)$,

$$
\frac{d}{d t} c_{n}(t)= \begin{cases}\frac{1}{N} c_{1}(t) & n=0  \tag{3.50}\\ \frac{(n+1)(2 n+1)}{N} c_{n+1}(t)-\frac{n(2 n-1)}{N} c_{n}(t) & 0<n<N / 2 \\ -\frac{N-1}{2} c_{N / 2}(t) & n=N / 2\end{cases}
$$

This is a set of first order linear differential equations, which can be rewritten as,

$$
\begin{equation*}
\frac{d}{d t} c_{n}(t)=\sum_{m} \Gamma_{n m} c_{m}(t) \Rightarrow \mathbf{c}(t)=e^{\Gamma t} \mathbf{c}(t=0) \tag{3.51}
\end{equation*}
$$

where $\Gamma$ is a matrix of coefficients that has nonzero elements only on the diagonal and the superdiagonal, which are given by equation 3.50. The explicit form of the solution in equation 3.51 was calculated numerically for several values of $N$, and the respective probabilties of a single particle to remain in the excited state were extracted. These probabilities are plotted in figure 3.3.

Since $\Gamma$ is a triangular matrix, its eigenvalues are given by its diagonal,

$$
\begin{equation*}
\operatorname{spec}(\Gamma)=-\frac{n(2 n-1)}{N} \forall n \in\left\{0,1, \ldots, \frac{N}{2}\right\}, \tag{3.52}
\end{equation*}
$$

thus the solution can be expressed as the sum of exponentials,

$$
\begin{equation*}
c_{n}(t)=\sum_{m=0}^{N / 2} \beta_{m} e^{-\frac{m(2 m-1)}{N} t}, \tag{3.53}
\end{equation*}
$$

where $\beta_{m}$ are general coefficients. Also, the diagonalizing matrix (which consists of the eigenvectors of $\Gamma$ ) is also triangular, which suggests the existence of a recursive formula for the eigenvectors (we will not explicitly obtain this formula). The smallest (in absolute value) nonzero eigenvalue in 3.52 is $-\frac{1}{N}$, which means that at large times $c_{n}(t) \sim e^{-\frac{t}{N}}$.

PDE for the Probability Distribution in the Limit $N \rightarrow \infty$
In the limit where the number of particles $N$ approaches infinity, it is possible to consider the variable $x$ defined by $x \equiv \frac{2 n}{N}$, as a continuous variable in the range $0 \leq x \leq 1$. the definition of $x$ suggests that the differential of $x$ should be $\delta x=$ $\frac{2}{N}$. Substituting these two definitions into the equation 3.50 (the middle equation) yields,

$$
\begin{align*}
\frac{\partial}{\partial t} c(x, t) & =\frac{\delta x}{2}\left(\frac{x}{\delta x}+1\right)\left(2 \frac{x}{\delta x}+1\right) c(x+\delta x, t)-\frac{\delta x}{2} \frac{x}{\delta x}\left(2 \frac{x}{\delta x}-1\right) c(x, t)  \tag{3.54}\\
& =x^{2} \frac{c(x+\delta x, t)-c(x, t)}{\delta x}+\frac{3 x}{2} c(x+\delta x, t)+\frac{x}{2} c(x, t)+\frac{\delta x}{2} c(x+\delta x, t) \\
& \approx x^{2} \frac{c(x+\delta x, t)-c(x, t)}{\delta x}+2 x c(x, t) .
\end{align*}
$$

Using the definition for the derivative: $\frac{c(x+\delta x, t)-c(x, t)}{\delta x} \equiv \frac{\partial}{\partial x} c(x, t)$ results in a partial differential equation for $c(x, t)$,

$$
\begin{equation*}
\frac{\partial}{\partial t} c(x, t)-x^{2} \frac{\partial}{\partial x} c(x, t)=2 x c(x, t) . \tag{3.55}
\end{equation*}
$$

## Solving the PDE Using the Method of Characteristics

Equation 3.55 may be solved using the method of characteristics [26], which reduces solving the partial differential equation to solving a set of ordinary differential equations. This is done by introducing a curve in the $x, t$ plane along which the partial differential equation transforms into an ordinary differential equation,

$$
\begin{equation*}
\frac{d}{d s} c(x(s), t(s))=F(c, x(s), t(s)), \tag{3.56}
\end{equation*}
$$

where $s$ is a variable associated with the curve, and $F$ is a some function of $c, x, t$.
The left hand side of the equation above may be rewritten as,

$$
\begin{equation*}
\frac{d}{d s} c(x(s), t(s))=\frac{\partial c}{\partial t} \frac{d t}{d s}+\frac{\partial c}{\partial x} \frac{d x}{d s} . \tag{3.57}
\end{equation*}
$$

Equating the coefficients between equations 3.57 and 3.55 yields a set of three ODEs,

$$
\begin{align*}
\frac{d t}{d s} & =1  \tag{3.58}\\
\frac{d x}{d s} & =-x^{2} \\
F(c, x, t) & =2 x c
\end{align*}
$$

The solution for $t(s)$ is

$$
\begin{equation*}
t(s)=s \tag{3.59}
\end{equation*}
$$

The solution for $x(s)$ is

$$
\begin{equation*}
x(s)=\frac{1}{s+1 / x_{0}}=\frac{1}{t+1 / x_{0}} \Leftrightarrow x_{0}=(1 / x-t)^{-1} . \tag{3.60}
\end{equation*}
$$

Finally, the solution for $c(s)$ is,

$$
\begin{equation*}
\frac{d}{d s} c=2 \frac{1}{s+1 / x_{0}} c \Rightarrow \ln c=2 \ln \left(s+1 / x_{0}\right)+\ln \left(f\left(x_{0}\right)\right)=\ln \left(\frac{1}{x^{2}} f\left((1 / x-t)^{-1}\right)\right) \tag{3.61}
\end{equation*}
$$

$f\left(x_{0}\right)$ is an integration constant. Therefore the solution to the partial differential equation for $c(x, t)$ (equation 3.55) is,

$$
\begin{equation*}
c(x, t)=\frac{1}{x^{2}} f\left((1 / x-t)^{-1}\right) . \tag{3.62}
\end{equation*}
$$

The function $f\left((1 / x-t)^{-1}\right)$ is determined by the initial condition.
In the case where the ensemble is initially fully excited, the appropriate initial
probability distribution is,

$$
c_{n}(t=0)=\left\{\begin{array}{ll}
1 & n=N / 2  \tag{3.63}\\
0 & \text { otherwise }
\end{array},\right.
$$

and at the limit $N \rightarrow \infty$, this corresponds to the initial condition,

$$
\begin{equation*}
c(x, t=0)=\frac{1}{x^{2}} f(x)=\delta(x-1) \Rightarrow f(x)=x^{2} \delta(x-1)=\delta(x-1), \tag{3.64}
\end{equation*}
$$

substituting back in equation 3.62 yields,

$$
\begin{equation*}
c(x, t)=\frac{1}{x^{2}} \delta\left((1 / x-t)^{-1}-1\right)=\frac{(1-x t)^{2}}{x^{2}} \delta\left(x-\frac{1}{1+t}\right)=\delta\left(x-\frac{1}{1+t}\right) . \tag{3.65}
\end{equation*}
$$

Note that the distribution above is normalized, as expected.
We may now compute the single particle survival probability (the probability of a particle to remain in the excited state) as a function of time,

$$
\begin{align*}
p_{1}(t) & =\left\langle P_{1}\right\rangle(t)=\sum_{n} c_{n}(t) \operatorname{tr}\left(P_{1} \otimes \mathbb{1}^{N-1} R_{n}\right)=\sum_{n} \frac{2 n}{N} c_{n}(t)  \tag{3.66}\\
\lim _{N \rightarrow \infty} p_{1}(t) & =\int_{0}^{1} d x x c(x, t)==\int_{0}^{1} d x x \delta\left(x-\frac{1}{1+t}\right)=\frac{1}{1+t} .
\end{align*}
$$

This solution agrees with the mean field solution presented in equation 3.37, in the case $p_{1}(t=0)=1$.

In the analysis above, this non exponential decay arises from the infinite dimensional Lindbladian, which gives a continuum of rates from 0 to infinity. For comparison, when we previously obtained the nonlinear mean field equation describing the single particle, the nonlinearity gave rise to the non exponential decay.

Figure 3.3 shows how the solution in equation 3.51 for a finite number of qubits, approaches the limiting solution (appearing in equation 3.66) as the number of qubits in the ensemble increases. Each curve computed for finite $N$, approximately follow the limiting solution $(N \rightarrow \infty)$ up to a point in time from which it continues an exponential law which is tangent to the local slope. This point in time scales as $N$, which may be seen by equating the logarithmic derivatives of the limiting solution with that of the slowest exponent in equation 3.53,

$$
\begin{equation*}
\frac{d}{d t} \log \left(\frac{1}{1+t}\right) \sim \frac{d}{d t} \log \left(e^{-\frac{1}{N} t}\right) \Rightarrow t \sim N . \tag{3.67}
\end{equation*}
$$

Indeed, as the number of qubits $N$ increases, the exponential law appears later in time and with an accordingly slower rate (this point is more clearly seen in the log plot shown in figure 3.3).


Figure 3.3: The single particle survival probability as a function of time. The probability is shown in linear scale in the upper figure, and in a logarithmic scale in the lower figure. The probabilities are computed for an initially excited state that evolves according to equations 3.50 , for several values of the particle number $N$. The analytic solution obtained for the limit of infinite number of particles (equation 3.66 ) is also plotted. We notice a trend where probability curves tend to the analytic solution as the number of particles increases, as expected (the trend is shown more clearly in the logarithmic scale).

Finally, we would like to propose an intuitive interpretation to the polynomial law obtained. Recall that the Lindblad equation discussed (equation 3.22) describes an ensemble of qubits that undergo pairwise decay. Upon initializing the ensemble at the fully excited state, a single qubit may perform pairwise decay with any of the other $N-1$ qubits, with a certain probability per qubit. As the ensemble decays with time, a still-excited qubit has less and less potential partners with which it can decay. Thus, the decay rate decreases as the ensemble decays, and approaches zero as the ensemble approaches the ground state. An analogy from relationships is that if one does not marry sufficiently early, he may struggle to find a spouse later in life (although this analogy breaks when considering divorce, which reintroduced people back into the market).

## Chapter 4

## Summary and Conclusion

In this work we studied relaxation in quantum systems obeying the Lindblad master equation. Both linear and nonlinear Lindblad equations were considered. The new results found in the work are bulleted below.

We discussed linear Lindblad equations that arise from a noise term in the Hamiltonian.

- We presented a simple example where certain relaxation rates decrease with increasing noise power.
- We have shown how to construct an oscillatory observable, whose damping rate decreases with the noise power.
- We have shown this effect occurs in general, whenever the noise term is represented by a degenerate matrix.

This phenomenon was explained using the quantum Zeno effect, where the noise plays the role of a continuous measurement. This in turn causes the state to be kept inside a protected subspace.

Nonlinear Lindblad equations are a mean field description of the evolution of many interacting identical particles.

- We constructed a nonlinear equation describing dephasing that arises from collisions between the particles. In this case, we found a continuum of dephasing rates.
- We constructed a nonlinear equation describing pairwise decay of interacting qubits. this decay can be, for example, a result of collisions between particles.
- We solved this equation in the mean field limit, and found the excited particles population decay as $\sim \frac{1}{t}$.
- We have shown that given symmetric initial data, the complexity of the problem reduces from exponential to linear (in the number of particles).
- At the infinitely many particles limit, we derived a PDE describing the probability distribution for the number of excited qubits. We solved it using the method of characteristics and confirmed the $\frac{1}{t}$ decay obtained by the mean field solution.


## Appendix A

## Construction of The Pairwise Decay Lindbladian

In this appendix we shall show the construction of a two-qubit Lindbladian describing pairwise decay. We shall consider a mutual interaction between a single pair of qubits and a quantized massless boson field, held at zero temperature. We rely on a previous derivation presented in [7], which considers an interaction with a bath of quantized radiation (photons). This derivation gives a general master equation (often referred as the quantum optical master equation) at the weak coupling limit. The Hamiltonian presented there for the system and bath combined is divided into three terms: the free Hamiltonians of the reduced system and the bath, denoted $H_{S}, H_{B}$ respectively, and the interaction term between them, denoted $H_{I}$. The three terms are expressed as,

$$
\begin{align*}
H_{S} & =\sum_{j} e_{j} P_{j}  \tag{A.1}\\
H_{B} & =\sum_{\mathbf{k}} \sum_{\lambda \in\{1,2\}} \omega_{k} b_{\lambda}^{\dagger}(\mathbf{k}) b_{\lambda}(\mathbf{k}) \\
H_{I} & =-\mathbf{D} \cdot \mathbf{E}
\end{align*}
$$

where $H_{S}$ is expressed as a sum of its eigenvalues $e_{j}$ multiplied by the projections $P_{j}$ onto their corresponding eigenspaces (which make up a complete and orthonormal set of projections). $b_{\lambda}(\mathbf{k})$ is the photon annihilation operator associated with the wave vector $\mathbf{k}$ and polarization $\lambda, \omega_{k}=c|\mathbf{k}|$ is the photon energy, $\mathbf{D}$ is the dipole operator of the qubit and $\mathbf{E}$ is the electric field operator in the Schrödinger picture, which under the dipole approximation takes the form

$$
\begin{equation*}
\mathbf{E}=i \sum_{\mathbf{k}} \sum_{\lambda \in\{1,2\}} \sqrt{\frac{2 \pi \omega_{k}}{V}} \mathbf{e}_{\lambda}(\mathbf{k})\left(b_{\lambda}(\mathbf{k})-b_{\lambda}^{\dagger}(\mathbf{k})\right), \tag{A.2}
\end{equation*}
$$

where $\mathbf{e}_{\lambda}$ are two unit polarization vectors, and $V$ is the volume where the photon modes are assumed to lie (later taken to infinity). Introduce the ladder operators,

$$
\begin{equation*}
\mathbf{A}(\omega)=\sum_{e_{k}-e_{j}=\omega} P_{j} \mathbf{D} P_{k} \tag{A.3}
\end{equation*}
$$

note that the resulting ladder operators are eigenoperators of the system Hamiltonian $H_{S}$, i.e. $\left[H_{S}, \mathbf{A}(\omega)\right]=-\omega \mathbf{A}(\omega)$. Using the definitions above, the master equation for the reduced system $\rho$, sometimes referred as "the quantum optical master equation", is then derived (the full derivation is shown in [7], we will only use the final equation) at the weak coupling limit ${ }^{1}$,

$$
\begin{align*}
\frac{d}{d t} \rho & =-i\left[H_{S}+H_{L S}, \rho\right]  \tag{A.4}\\
& +\sum_{\omega>0} \frac{4 \omega^{3}}{3 c^{3}}(1+n(\omega))\left(\mathbf{A}(\omega) \rho \mathbf{A}^{\dagger}(\omega)-\frac{1}{2}\left\{\mathbf{A}^{\dagger}(\omega) \mathbf{A}(\omega), \rho\right\}\right) \\
& +\sum_{\omega>0} \frac{4 \omega^{3}}{3 c^{3}} n(\omega)\left(\mathbf{A}(\omega) \rho \mathbf{A}^{\dagger}(\omega)-\frac{1}{2}\left\{\mathbf{A}^{\dagger}(\omega) \mathbf{A}(\omega), \rho\right\}\right)
\end{align*}
$$

$c$ denotes the speed of light in vacuum, $n(\omega)$ denotes the Bose-Einstein distribution (the mean photon number at frequency $\omega$ ) describing the photon bath (the temperature of the bath will be introduced through this distribution) and $H_{L S}$ denotes a Lamb shift induced by the coupling to the bath, which takes the form,

$$
\begin{equation*}
H_{L S}=\sum_{\omega} S(\omega) \mathbf{A}^{\dagger}(\omega) \cdot \mathbf{A}(\omega) \tag{A.5}
\end{equation*}
$$

where $S(\omega)$ is a c-number function which is related to the properties of the the bath (the explicit expression is not presented since it is irrelevant for the derivation of the master equation).

As mentioned above, equation A. 4 is valid when the coupling between the system and the bath is sufficiently weak, such that it has a negligible influence on the state of the bath, which can thus be regarded to be constant ${ }^{2}$. This condition is met provided D is sufficiently small.

As a simple demonstration, we will first use equation A. 4 to obtain the master equation for the case of a single qubit in a photon bath at zero temperature. Consider

[^6]the following Hamiltonian and dipole operator,
\[

$$
\begin{align*}
H_{S} & =\omega_{0}|1\rangle\langle 1|  \tag{A.6}\\
\mathbf{D} & =\mathbf{d}|0\rangle\langle 1|+\mathbf{d}^{*}|1\rangle\langle 0|,
\end{align*}
$$
\]

which give rise to the eigenoperator $\mathbf{A}(\omega)$ and Lamb shift $H_{L S}$,

$$
\begin{align*}
\mathbf{A}\left(\omega_{0}\right) & =|0\rangle\langle 0| \mathbf{D}|1\rangle\langle 1|=\mathbf{d}|0\rangle\langle 1| \equiv \mathbf{d} a  \tag{A.7}\\
H_{L S} & \propto a^{\dagger} a=|1\rangle\langle 1| \propto H_{S},
\end{align*}
$$

where $\omega_{0}$ is the qubit energy splitting, $\mathbf{d}$ describes the transition dipole moment between the states $|0\rangle,|1\rangle$, and $a$ is the lowering operator. Since $\mathbf{A}\left(-\omega_{0}\right)$ gives no contribution to equation A.4, we shall not compute it. We would like to compute the master equation in the rotating frame associated with the unitary transformation $U=e^{i\left(H_{S}+H_{L S}\right) t}$. In this frame, the Hamiltonian part in equation A. 4 (i.e. the first term in the right hand side) vanish, however the operator $\mathbf{A}\left(\omega_{0}\right)$ becomes time dependent and takes the form,

$$
\begin{equation*}
e^{i\left(H_{S}+H_{L S}\right) t} \mathbf{A}\left(\omega_{0}\right) e^{-i\left(H_{S}+H_{L S}\right) t}=\mathbf{d} e^{i S t|1\rangle\langle 1|} a e^{-i S t|1\rangle\langle 1|} \tag{A.8}
\end{equation*}
$$

Where $\Omega$ is defined by $H_{S}+H_{L S}=\Omega|1\rangle\langle 1|$. The exponent takes the form

$$
\begin{equation*}
e^{i \Omega t|1\rangle\langle 1|}=|0\rangle\langle 0|+e^{i \Omega t}|1\rangle\langle 1|, \tag{A.9}
\end{equation*}
$$

thus the transformed operator becomes,

$$
\begin{align*}
& e^{i\left(H_{S}+H_{L S}\right) t} \mathbf{A}\left(\omega_{0}\right) e^{-i\left(H_{S}+H_{L S}\right) t}  \tag{A.10}\\
& =\mathbf{d}\left(|0\rangle\langle 0|+e^{i \Omega t}|1\rangle\langle 1|\right) a\left(|0\rangle\langle 0|+e^{-i \Omega t}|1\rangle\langle 1|\right)=\mathbf{d} a e^{i \Omega t} .
\end{align*}
$$

We found that upon rotation, the ladder operator is simply multiplied by a time dependent phase. Clearly, upon substituting the rotated ladder operators in equation A.4, these phases cancel out, and the non-Hamiltonian parts of the equation appear to be invariant to the transformation $U$. Substituting the ladder operator and setting the bath temperature to zero (i.e. $n(\omega)=0$ ) yields the quantum optical master equation for the qubit in the rotating frame, which is up to a constant,

$$
\begin{equation*}
\frac{d}{d t} \rho \propto \mathcal{L}_{s} \rho=a \rho a^{\dagger}-\frac{1}{2}\left\{a^{\dagger} a, \rho\right\} \tag{A.11}
\end{equation*}
$$

Using the identities $a a^{\dagger}=|0\rangle\langle 0|, a^{\dagger} a=|1\rangle\langle 1|, a^{2}=0$, it is possible to verify that the $|0\rangle$ state is a stationary state of equation A.11, while the $|1\rangle$ state decays
exponentially to $|0\rangle$. This behavior resembles spontaneous decay, a process expected to emerge at the current setting.

Now, in order to obtain a Lindbladian describing pairwise decay, we shall construct the master equation for a pair of qubits that mutually interact with an (empty) photon bath. Consider the Hamiltonian and the dipole operator for the two qubits,

$$
\begin{align*}
H_{S} & =\frac{1}{2} \omega_{0}(|1\rangle\langle 1| \otimes \mathbb{1}+\mathbb{1} \otimes|1\rangle\langle 1|)=\frac{1}{2} \omega_{0}(|10\rangle\langle 10|+|01\rangle\langle 01|)+\omega_{0}|11\rangle\langle 11|  \tag{A.12}\\
& =0 \cdot P_{00}+\frac{1}{2} \omega_{0}\left(P_{10}+P_{01}\right)+\omega_{0} P_{11} \\
\mathbf{D} & =\mathbf{d}|00\rangle\langle 11|+\mathbf{d}^{*}|11\rangle\langle 00|=\mathbf{d} a^{\otimes 2}+\mathbf{d}^{*}\left(a^{\dagger}\right)^{\otimes 2}
\end{align*}
$$

where $P_{a b}$ denotes a projection onto the two qubit state $|a b\rangle$ and $\mathbf{d}$ again denotes a transition dipole moment, this time between the states $|00\rangle$ and $|11\rangle$. Note that the Hamiltonian $H_{S}$ is a sum of two single qubit Hamiltonians, and generates independent evolution of the two qubits (i.e. it does not couple between the qubits). However, the dipole operator $\mathbf{D}$ does not act separately on single qubits, and may therefore generate coupling between them. Computing the eigenoperators $\mathbf{A}(\omega)$ yields (as before, we will suffice in positive frequencies only),

$$
\begin{align*}
\mathbf{A}\left(\frac{1}{2} \omega_{0}\right) & =P_{00} \mathbf{D}\left(P_{10}+P_{01}\right)+\left(P_{10}+P_{01}\right) \mathbf{D} P_{11}=0  \tag{A.13}\\
\mathbf{A}\left(\omega_{0}\right) & =P_{00} \mathbf{D} P_{11}=\mathbf{d}|00\rangle\langle 11|=\mathbf{d} a^{\otimes 2}
\end{align*}
$$

The Lamb shift Hamiltonian is thus,

$$
\begin{equation*}
H_{L S} \propto\left(a^{\dagger}\right)^{\otimes 2} a^{\otimes 2}=P_{11} \tag{A.14}
\end{equation*}
$$

We now move to a rotating frame in which the Hamiltonian part in equation A. 4 vanishes, and similarly to the previous example we find that the ladder operators A ( $\omega$ ) transform as,

$$
\begin{equation*}
e^{i\left(H_{S}+H_{L S}\right) t} a^{\otimes 2} e^{-i\left(H_{S}+H_{L S}\right) t}=e^{i \Omega t} a^{\otimes 2} \tag{A.15}
\end{equation*}
$$

which means that the non-Hamiltonian parts of equation A. 4 remain invariant to the rotation. Substituting the ladder operator in equation A. 4 and imposing zero temperature for the bath yields the master equation for the qubit pair in the rotating frame, which is up to a constant,

$$
\begin{equation*}
\frac{d}{d t} \rho \propto \mathcal{L}_{12} \rho=a^{\otimes 2} \rho\left(a^{\dagger}\right)^{\otimes 2}-\frac{1}{2}\left(a^{\dagger}\right)^{\otimes 2} a^{\otimes 2} \rho-\frac{1}{2} \rho\left(a^{\dagger}\right)^{\otimes 2} a^{\otimes 2} . \tag{A.16}
\end{equation*}
$$

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 תוצאה זו שונה באופן עקרוני מרלקסציה במשוואות ליניאריות, שבאה לידי ביטוי אך ורק דרך סדרה בדידה של קצבים.
המודל הבא מתאר קיוביטים שחווים דעיכה בזוגות. במודל זה, קיוביטים מעוררים

 אותה בשתי גישות שונות. הגישה הראשונה הינה גישת שדה ממוצע, שבה נלקח מות מספר אינסופי של חלקיקים, ואז משוואת לינדלבאר שנבנתה עבור הצבר מצטמצעמת למשו לוח פוואת שרה ממוצע לא-ליניארית, המתארת קיוביט בודר מתוך הצבר. אנו מוּ מציגיגים פתרון עבור
 שונה באופן מהותי מהרעיכה האקספוננציאלית המתקבלת כפתורון למוּ למשוואות לינדבלאד ליניאריות. בגישה השנייה לפתרון המשוואה עבור הצבר, אנו מראים שבהינתן מצב
 המלאה, מסיבוכיות אקספוננציאלית לסיבוכיות ליניארית (במספר הקיובי פליטים פלים). אנו
 מראים שבנבול שבו מספר החלקיקים שושים שואף לאינסוף, מתקבלת הסכמה בין הפתרונות המתקבלים לפי שתי הגישות.

## תקציר


#### Abstract

במציאות, כל המערכות הקוונטיות הינן למעשה מערכות פתוחות, שמתפתחות באופן לא הפיך, תוך איבוד אינפורמציה על המצב ההתחלתי. אחר הכלים העיקריים שמאפשרים לתאר את התפתחותן של מערכות קוונטיות פתוחות, הוא משוואת לינדבלאד. משוואה זו מהווה צורה כללית למשוואה מרקובית, דהיינו משוואה חסרת זיכרון, שהינה חיובית לחלוטין ומשמרת עקבה.


המוטיבציה לעבודה זו עולה מאפקט שנקרא SERF שמתרחש באדי אטומים, שם הרלקסציה, כלומר תהליך הריטת המצב הקוונטי, הנגרמת ע"י התנגשויות בין אטומים, פוחתת עם הגדלת קצב ההתנושויות. כיוון זה מנוגד לאינטואיציה, שגורסת שקצב הרלקסציה צריך לעלות עם קצב ההתנגשויות. לאפקט זה ניתן הסבר תיאורטי בעבר, המערב בנייה ופתרון של משוואת לינדבלאד עבור האטומים. אולם, היות ומשוואה זו כוללת תיאור מפורט של האטום והאינטראקציה שהוא מבצע, קשה לקבל ממנה הבנה בסיסית עבור האפקט הלא-אינטואיטיבי הנ"ל מחמת המורכבות. עובדה זו הניעה אותנו לחפש את האפקט במערכת פשוטה יותר, בה יהיה ניתן להבין את מקור האפקט באופן פשוט. בנוסף, האפקט הניע אותנו לחקור רלקסציה במודלים אחרים המתארים התנגשויות. שבאופן כללי מתוארים ע״י משוואת לינדבלאד לא-ליניארית.

אנו מתחילים במשוואת לינדבלאר ליניארית שמתארת מערכת שבהנילטוניאן שלה קיים איבר רועש (לפי רעש לבן). ממשוואה זו מתקבלת סדרה בדידה של קצבי רלקסציה. אנו מציגים דוגמא פשוטה למערכת שכזו, בה מתקבלים קצבי דעיכה מסויצים שיורדים עם עוצמת הרעש, במקום לעלות. תוצאה זו דומה לאפקט ה-SERF שהוזכר לעיל בהקשר מערכות אטוניות. אנו מדגיטים כיצד ניתן לבנות גודל מדיד שיציג את האפקט, בכך שערך התצפית מבצע תנודות מרוסנות, שריסונן מאט עם הגדלת עוצמת הרעש. אנו נותנים פרשנות לתוצאה זו, שמזהה אותה עם אפקט זנון הקוונטי. אפקט זה מתאר כיצד מדידה חזרתית ותדירה של מערכת יכול למנוע ממנה להשתנות בזמן. בנוסף, אנו מכלילים את התוצאה שהרגמנו, כתופעה שמתרחשת עבור כל אופרטור רעש מנוון.

בהמשך אנו בוחנים משוואות לינבלאד לא-ליניאריות, נושא שלא היה מושא למחקר קודם רב. המשוואות הלא-ליניאריות הנדונות בעבודה זו הינן משוואות מסוג שדה ממוצע. שמתארות מערכות רב-גופיות עם אינטראקציות בגבול של מספר אינסופי של חלקיקים. משוואות אלה יכולות לנבוע, למשל, כתוצאה מהתנגשויות בין החלקיקים. תחילה אנו דנים בדוגמא למשוואה המתארת איבוד פאזה (סוג של רלקסציה) כתוצאה
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# המחקר בוצע בהנחייתו של פרופסור יוסף אברון. בפקולטה לפיטיקה. 

תודות
אני מורה לפרופסור יוסף אברון על החדרכה ועל המסירות שלו למחקר זה. ארצה להודות לד"ר עודד קנת על הרבה דיונים פוריים ועל תרומתו המשמעותחת לעבודה. אני מודרה לד"ר אור פלג. אור כץ ויונתן רוזמרין על דיונים מועילים רבים לאורך
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דיסיפציה במודלים ליניאריים ולא-ליניאריים של מערכות קוונטיות

## חיבור על מחקר

## לשם מילוי חלקי של הדרישות לקבלת התואר מגיסטר למרעים בפיסיקה

## עמית צברי

> הוגש לסנט הטכניון - מכון טכנולווי לישראל ניסן ה'תשע"ז חיפה מרץ 2017
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# דיסיפציה במודלים ליניאריים ולא-ליניאריים של מערכות קוונטיות 

עמית צברי


[^0]:    ${ }^{1}$ Throughout the work, by decoherence we shall mean either dissipation or dephasing.

[^1]:    ${ }^{2}$ It should be noted that the linearization does not result in a Lindblad equation, which raises the question whether a similar effect may be obtained within a linear Lindblad equation.

[^2]:    ${ }^{1}$ We set $H_{0}$ and $\xi(t)$ to have dimensions of frequency, and $C$ to be dimensionless.

[^3]:    ${ }^{1}$ We will use the word "qubit" throughout the work to describe any two level system or particle.

[^4]:    ${ }^{2}$ An example for a scenario where collisional decay is a dominant decay process, is a particle in which spontaneous decay is forbidden due to selection rules.

[^5]:    ${ }^{3}$ The Mpemba effect is the phenomenon in which a system initiated at a high temperature cools quicker compared to the same system initiated at a lower temperature [25]. The most common realization of this effect via freezing water.

[^6]:    ${ }^{1}$ It should be noted equation A. 4 relies on the secular approximation, which involved discarding of terms of the form $e^{i\left(e_{j}-e_{k}\right) t}, \forall e_{j} \neq e_{k}$. This elimination is justified when the frequencies determined by the system Hamiltonian, $\left|e_{j}-e_{k}\right|$ are much higher that the relaxation rate of the system. This condition is met provided the dipole operator $\mathbf{D}$ is sufficiently small.
    ${ }^{2}$ More precisely, the weak coupling condition does not require the bath to be free of excitations due to the coupling. It does require the state of the bath to be approximately constant in the coarse grained time of the system. This means that excitations in the bath, should they occur, are assumed to vanish due to the bath thermalization before the the state of the system manages to change significantly.

