Geometry of Qubits

A picture book

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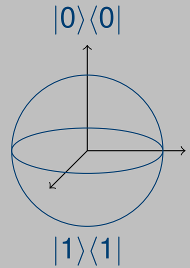
Bloch Sphere

Qubit

- Pure state (Phase removed):

$$|\psi\rangle\langle\psi| = \frac{\mathbb{1} + \boldsymbol{\sigma}\cdot\mathbf{x}}{2}, \quad \underbrace{|\mathbf{x}| = 1}_{\textit{sphere}}$$



- Pauli matrices: $\sigma_\mu, \quad \mu = 0, \ldots, 3$:

$$\sigma_0 = \mathbb{1}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

etc.

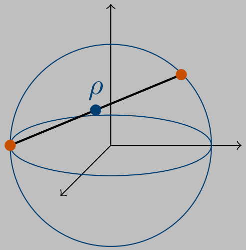
Bloch Ball

Mixed states

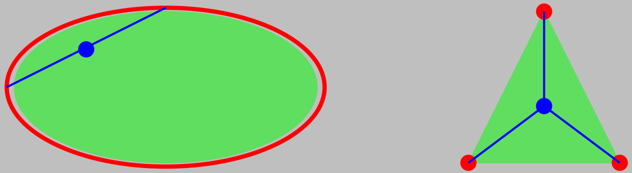
- $\rho = \dfrac{\mathbb{1} + \boldsymbol{\sigma} \cdot \mathbf{x}}{2}$
- $\rho \geq 0 \Longleftrightarrow \underbrace{|\mathbf{x}| \leq 1}_{\textit{ball}}$

Geometry of *n* qubits:

- Live in huge dimensions
- Not n Bloch balls
- Not a high D ball
- Not fully understood

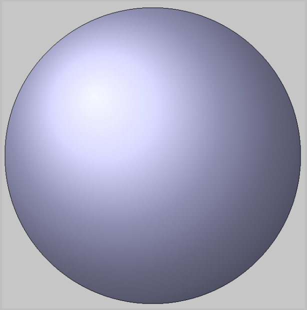


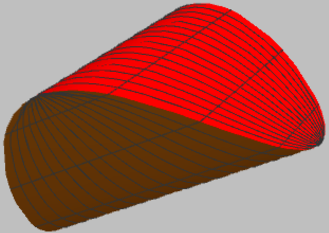
Extreme points and Convex sets



Pure states=Extreme points

$n \geq 2$ vs $n = 1$





Pure states: tiny subset of boundry

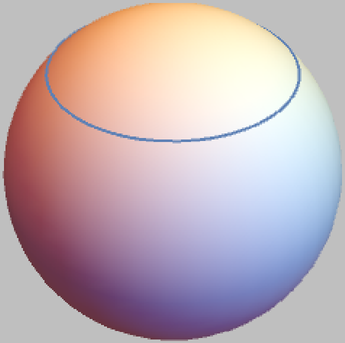
Representation using Pauli matrices

Pure states sit on a high dimensional sphere

- n qubits, $\dim(\mathcal{H}) = N = 2^n$
- Pauli basis: $\sigma_\alpha = \sigma_{\mu_1} \otimes \cdots \otimes \sigma_{\mu_n}$
- $\alpha = 1, \ldots, N^2 - 1$
- $\mathbf{x} \in \mathbb{R}^{N^2-1}$

Pauli representation

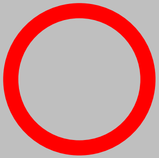
- $\rho = \dfrac{\mathbb{1} + \sqrt{N-1}\,\mathbf{x}\cdot\boldsymbol{\sigma}}{N}, \quad \mathbf{x} \in \mathbb{R}^{N^2-1}$
- Pure states $\Longrightarrow |\mathbf{x}| = 1$ (but not $\Longleftarrow$)
- States $\subset |\mathbf{x}| \leq 1$



Pure states: tiny subset of sphere

Ball is mostly empty

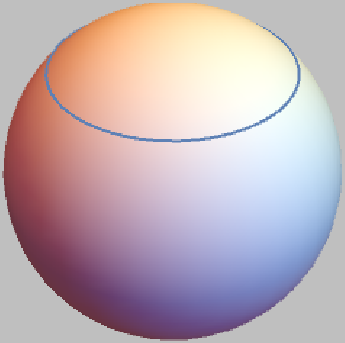
Typical sections miss extreme pts



Most of the orange is its skin:

$(1-\varepsilon)^N \leq e^{-N\varepsilon}$

$\dim(\text{sphere}) = O(N^2)$

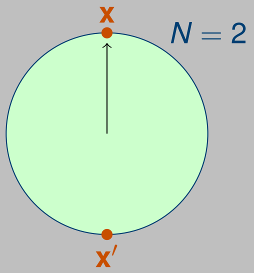


$\dim(\text{Pure}) = O(N)$

Much of the ball empty

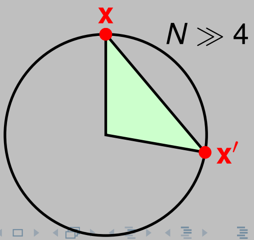
Orthogonal, antipodal no inversion

- $0 \leq N\, Tr(\rho\rho') = 1 + (N-1)(\mathbf{x}\cdot\mathbf{x}')$
- $\underbrace{\mathbf{x}\cdot\mathbf{x}' \geq -\frac{1}{N-1}}_{Cone}$



Two or more qubits

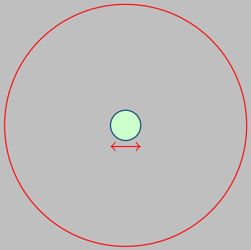
- Ball mostly empty
- Inside a cone
- Orthogonal vs antipodal

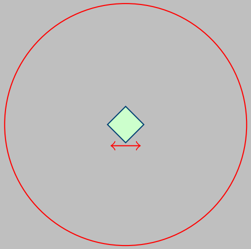


Two dimensional sections

Tiny squares and circles

- $0 \le \mathbb{1} + \sqrt{N-1}\,\mathbf{x}\cdot\boldsymbol{\sigma}$
- 2-d section $\mathbf{x}\cdot\boldsymbol{\sigma} = x\sigma_\alpha + y\sigma_\beta$
- Two cases $\sigma_\alpha\sigma_\beta = \pm\sigma_\beta\sigma_\alpha$
- $$Spec(x\sigma_\alpha + y\sigma_\beta) = \begin{cases} \pm\sqrt{x^2+y^2} & - \\ \{\pm(x \pm y)\} & + \end{cases}$$





Cross sections exponentially small

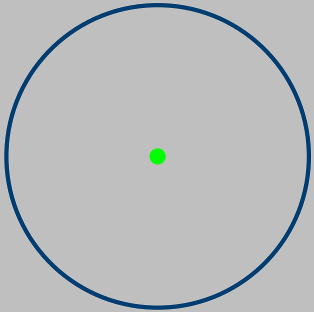
$$O\left(\frac{1}{\sqrt{N}}\right), \quad \text{tiny if } N \gg 1$$

n dim section

Clifford: $\{\sigma_\alpha, \sigma_\beta\} = 2\delta_{\alpha\beta}$

- $\rho \geq 0 \Longrightarrow \mathbb{1} + \sqrt{N-1}\,\mathbf{x}\cdot\boldsymbol{\sigma} \geq 0$
- Spectrum $(\mathbf{x}\cdot\boldsymbol{\sigma}) = \pm|\mathbf{x}|$

$$(\mathbf{x}\cdot\boldsymbol{\sigma})^2 = \sum x_j x_k \sigma_j \sigma_k = \mathbf{x}^2\,\mathbb{1}$$



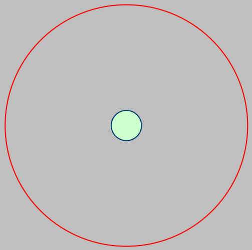
Clifford section: exponentially tiny ball

$$|\mathbf{x}| \leq \frac{1}{\sqrt{N-1}}, \quad \text{tiny if } N \gg 1$$

Generic section

Application of Random matrix theory

- $\rho \geq 0 \Longrightarrow \mathbb{1} + \sqrt{N-1}\,\mathbf{x}\cdot\boldsymbol{\sigma} \geq 0$
- $\mathbf{x}$ vector in random direction
- $\mathbf{x}\cdot\boldsymbol{\sigma}$ =Random matrix
- Wigner semi-circle law implies:



Cross sections exponentially small

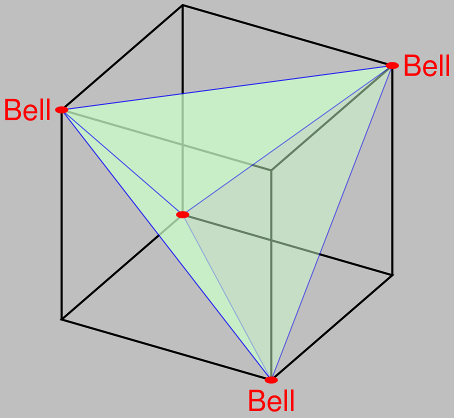
$$O\left(\frac{1}{\sqrt{N}}\right), \quad N = 2^n$$

2 Qubits: 15 dimensional space

3-d section through 4 Bell states, $n = 2$

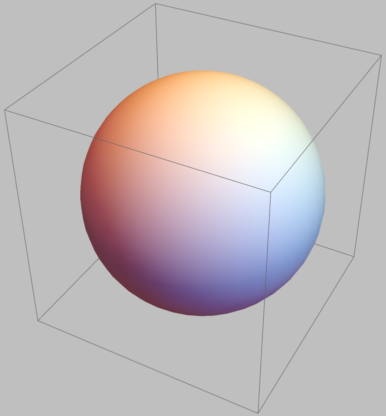
- $\rho \geq 0 \implies 0 \leq \mathbb{1} + \sqrt{3}\, \mathbf{x} \cdot \boldsymbol{\sigma}$
- $\mathbf{x} \cdot \boldsymbol{\sigma}^{(2)} = \sum_{j=1}^{3} x_j \underbrace{\sigma_j \otimes \sigma_j}_{commuting}$
- Tetrahedron

$$\pm x_1 \pm x_2 \pm x_3 \leq \frac{1}{\sqrt{3}},$$

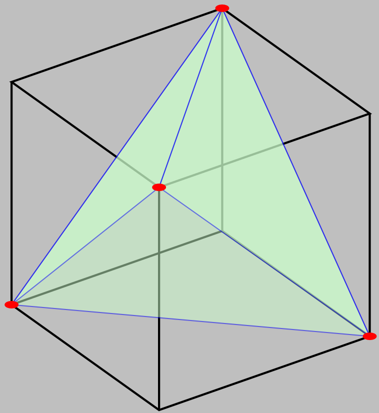


Antipode of Bell states are planes

No inversion symmetry



Qubit, $N = 2$



Two qubits $N = 4$

Separable states

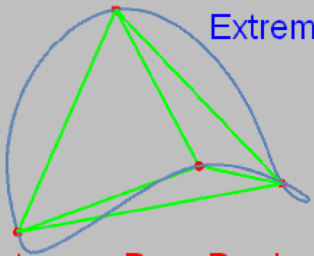
Extreme points: Product states

- Extreme points: Product states

$$|\psi\rangle\langle\psi|\otimes|\phi\rangle\langle\phi| \in S^2 \times S^2$$

- separable $\subset$ all states

$$\rho_s = \sum p_j\,|\psi_j\rangle\langle\psi_j|\otimes|\phi_j\rangle\langle\phi_j|$$



Separable+Entangled=All states

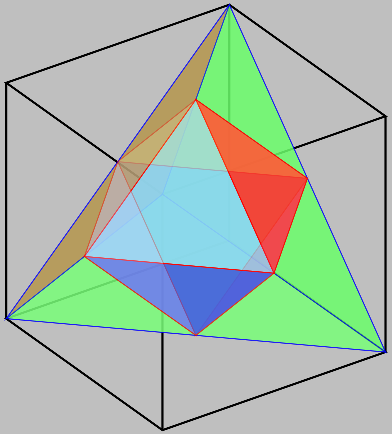
Entanglement

Platonic solids

- Partial Transpose

$$\sigma_j \otimes \sigma_k \mapsto \sigma_j \otimes \sigma_k^t$$

- Peres test
  $\rho_s \geq 0 \Longrightarrow \rho_s^{PT} \geq 0$



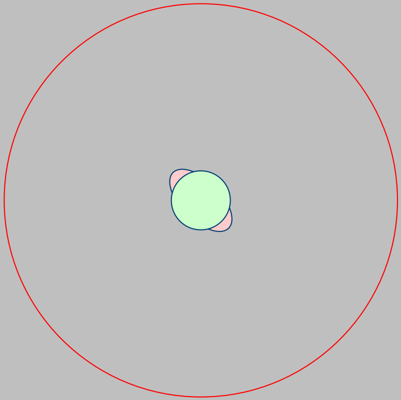
Platonic

- States=Tetrahedron,
- Separable=Octahedron
- Witnesses=Cube

Generic section

Where are the entangled states?

- Peres: Broken symmetry test
- Balls fail test
- Random deviation from spherical: pass
- Tracy-Widom distribution
- Small width $O(N^{-1/3})$



Entanglement

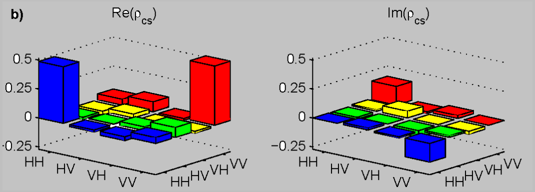
Life on the edge

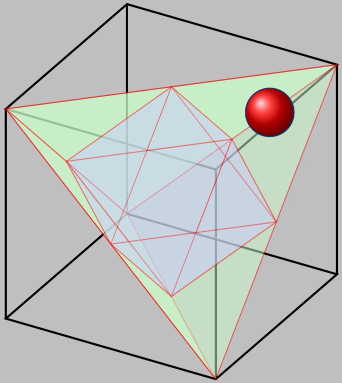
Stokes parameters for 2 qubits

Dudi's Bracha

Full tomography 16 opaque numbers

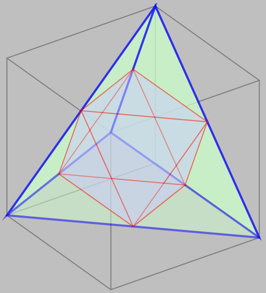
4 coordinates-transparent interpretation

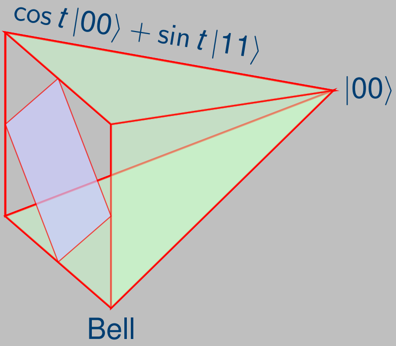




Visualizing 4 d

A cone whose cross section are platonic solids





4 coordinates

- 4 coordinates: $\rho_0 \geq \rho_1 \geq \rho_2 \geq |\rho_3| \geq 0$
- $\rho_\nu^2 = Eigenvalues(\rho^\star \rho), \quad \rho^\star = \sigma_2^{\otimes 2} \rho^t \sigma_2^{\otimes 2}$
- Cone: $1 \geq \sum \rho_\mu \geq 0$
- Entanglement: Distance between tetrahedron and octahedron

$$(\rho_0 - \rho_1 - \rho_2 - \rho_3)_+$$

- Radial motion $\rho_\mu \mapsto \lambda \rho_\mu \Longleftrightarrow$ Local probabilistically reversible operations

References

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- Verstraete, Dehaene and De Moor, PRA
- Wootters, PRL
- Avron Kenneth, AP
- Bengtsson and Zyczkowski *Geometry of quantum states*
- Aubrun and Szarek *Alice and Bob Meet Banach*