Geometry of Qubits

A picture book

J Avron, O. Kenneth

June 18, 2018

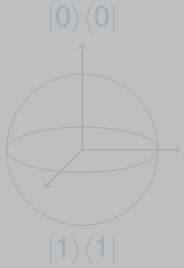
Bloch sphere and ball

Is it a good guide for many qubits?

- Bloch sphere: $\mathbf{x} \in \mathbb{R}^3$

$$\rho = \frac{\mathbb{1} + \boldsymbol{\sigma} \cdot \mathbf{x}}{2}, \quad |\mathbf{x}| = \begin{cases} 1 & \text{pure states} \\ < 1 & \text{mixed} \\ > 1 & \text{not positive} \end{cases}$$

- Pauli matrices: $\sigma_\mu, \quad \mu = 0, \dots, 3$



Geometry of *n* qubits:

- Live in huge dimensions
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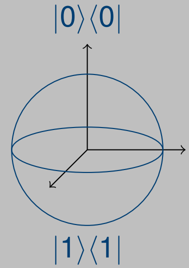

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n-qubits: Separable and Entangled

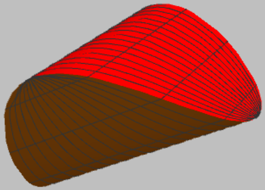
Convex sets and extreme points

Bengtsson & Zyczkowski

- n qubits, $\dim \mathcal{H} = 2^n = N$
- Extreme points=Pure states
- Entangled $\cup$ Separable $=$ States

$\dim(\text{pure}) = O(N)$

	Separable	States
Extreme	$\underbrace{\|\psi\rangle\langle\psi\|}_{n_1-qubits} \otimes \underbrace{\|\phi\rangle\langle\phi\|}_{n_2-qubits}$	$\underbrace{\|\Psi\rangle\langle\Psi\|}_{n-qubits}$
Convex	$\sum p_j \|\psi_j\rangle\langle\psi_j\| \otimes \|\phi_j\rangle\langle\phi_j\|$	$\sum p_j \|\Psi_j\rangle\langle\Psi_j\|$



Extreme points: Tiny part of boundary

Separable $\subset$ States

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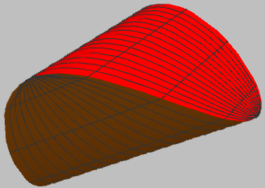

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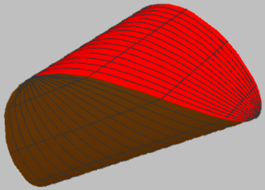

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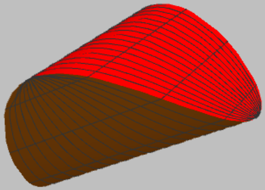

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Separable $\subset$ States

States in Pauli basis

Pure states sit on unit, $O(N^2)$, sphere

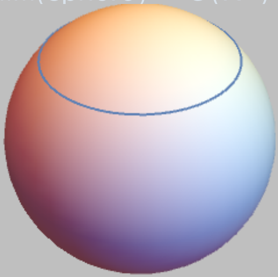
Horodecki, Shor, Zyczkowski, . . .

- Pauli basis: $\sigma_\mu = \sigma_{\mu_1} \otimes \cdots \otimes \sigma_{\mu_n}$

States: Pauli basis

- $\rho = \dfrac{\mathbb{1} + \sqrt{N-1}\,\mathbf{x}\cdot\boldsymbol{\sigma}}{N}, \quad \mathbf{x} \in \mathbb{R}^{N^2-1}$
- Pure states $\Longrightarrow |\mathbf{x}| = 1$ (but not $\Longleftarrow$)
- Fully mixed: $\mathbf{x} = 0$
- States $\subset \{|\mathbf{x}| \le 1\}$

$\dim(\text{sphere}) = O(N^2)$



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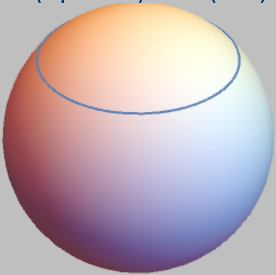


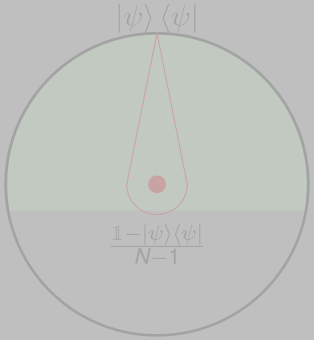
$\dim(\text{Pure}) = O(N)$

Convex but not symmetric

Complicated shape

- $0 \le N\, Tr(\rho\rho') = 1 + (N-1)(\mathbf{x}\cdot\mathbf{x}')$
- $\mathbf{x}\cdot\mathbf{x}' \ge -\dfrac{1}{N-1}$
- Antipode $|\psi\rangle\langle\psi|$

$$\underbrace{\frac{\mathbb{1} - |\psi\rangle\langle\psi|}{N-1}}_{rank=N-1}$$



- $\{|\mathbf{x}| \le O(1/N)\} \subset$ States

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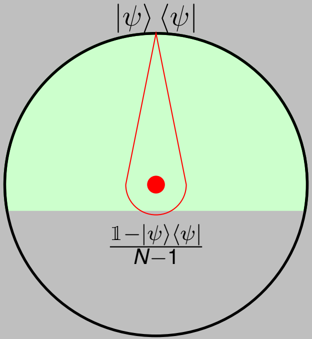

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Orthogonal basis balanced on (N-1)-sphere

States almost lie in a quadrant

- Orthogonal basis

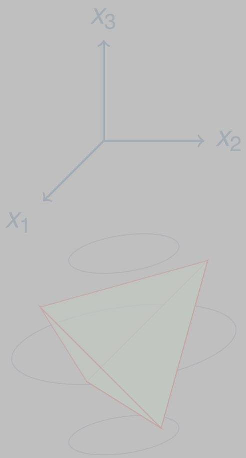
$$\mathbf{x}_j \cdot \mathbf{x}_k = -\frac{1}{N-1} \Longrightarrow \sum \mathbf{x}_j = 0$$

- Lie essentially in a quadrant

$$\mathbf{x} \cdot \mathbf{x}' \geq -\frac{1}{N-1} \approx 0$$

Two qubits: $n = 2, N = 4$

- $(\mathbf{x}_1, \ldots \mathbf{x}_4)$ span tetrahedron



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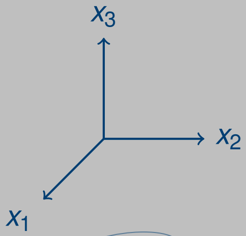

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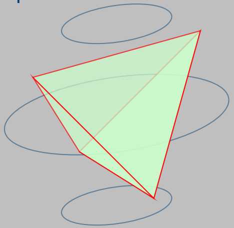


2 Qubits: Generic 2D section in $\mathbb{R}^{15}$

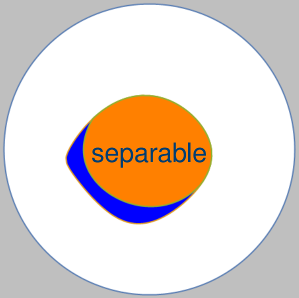
Misses pure states

- Random, orthonormal pair

$$\hat{\mathbf{u}}, \hat{\mathbf{v}} \in \mathbb{R}^{15}$$

$$|\hat{\mathbf{u}}| = |\hat{\mathbf{v}}| = 1, \hat{\mathbf{u}} \cdot \hat{\mathbf{v}} = 0$$

- 2D section: $x\hat{\mathbf{u}} + y\hat{\mathbf{v}} \in \mathbb{R}^{15}$
- Misses pure states



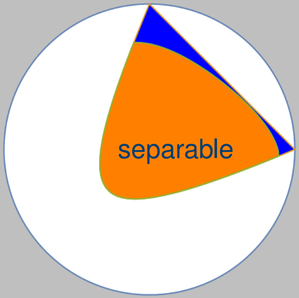
2 Qubits: 2D section in $\mathbb{R}^{15}$

Generic section through pure states

- Random pure $|\psi\rangle, |\phi\rangle \in \mathbb{C}^4$
- Orthonormal:

$$\langle\psi|\psi\rangle = \langle\phi|\phi\rangle = 1, \quad \langle\psi|\phi\rangle = 0$$

- 2D section: $x\,|\psi\rangle\langle\psi| + y\,|\phi\rangle\langle\phi|$



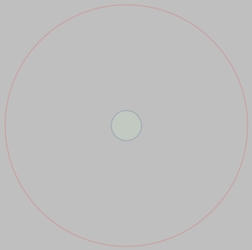
Two dimensional sections: $n \gg 1$

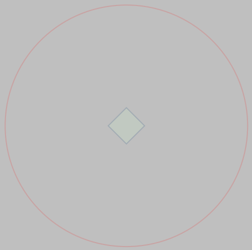
Tiny squares and circles

- $0 \leq \mathbb{1} + \sqrt{N-1}\,\mathbf{x}\cdot\boldsymbol{\sigma}$
- 2-d section $\mathbf{x}\cdot\boldsymbol{\sigma} = x\sigma_\mu + y\sigma_\beta$
- $\sigma_\mu\sigma_\beta = \pm\sigma_\beta\sigma_\mu$
- $$Spec(x\sigma_\mu + y\sigma_\beta) = \begin{cases} \pm\sqrt{x^2+y^2} & - \\ \{\pm(x \pm y)\} & + \end{cases}$$

Tiny cross section

$$O\left(\frac{1}{\sqrt{N}}\right)$$





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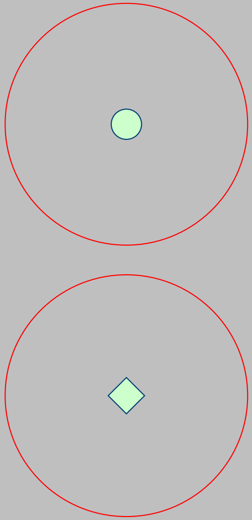


3D sections: 2 Qubits

Section through 4 Bell states

- Section through Bell

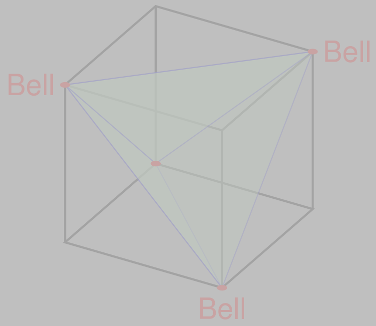
$$\mathbf{x}\cdot\boldsymbol{\sigma}^{\otimes 2}=\sum_{j=1}^{3} x_j\sigma_j\otimes\sigma_j$$

- Positivity:

$$\mathbb{1}+\sqrt{3}\,\mathbf{x}\cdot\boldsymbol{\sigma}^{\otimes 2}\geq 0$$

- $\sigma_j^{\otimes 2}$: Commuting with relation

$$\sigma_1^{\otimes 2}\sigma_2^{\otimes 2}\sigma_3^{\otimes 2}=-\mathbb{1}$$



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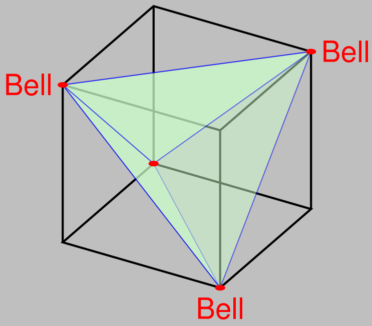

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3D Entanglement: Separable states

Octahedron=Intersection of tetrahedra

- Partial Transpose

$$\sigma_j \otimes \sigma_k \mapsto \sigma_j \otimes \sigma_k^t$$

- Peres test

$$\rho_s \in Separable \Longrightarrow \rho_s^{PT} \geq 0$$

Platonic

- States=Tetrahedron,
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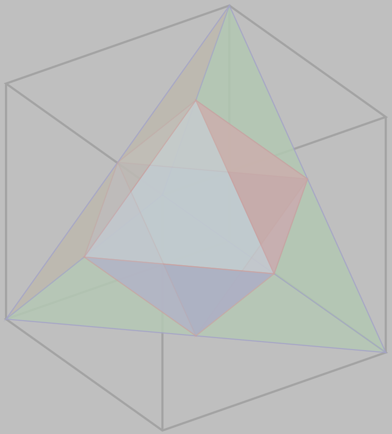
- Witnesses=Cube



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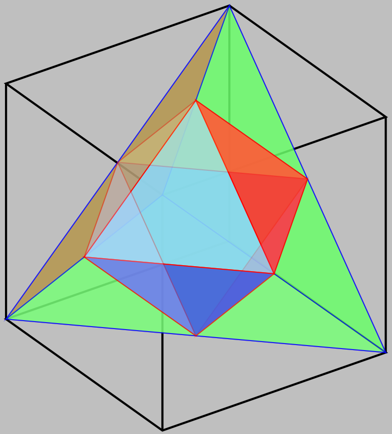

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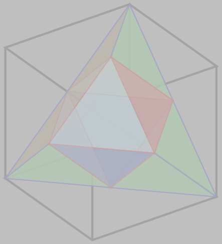
Cross section vs equivalence classes

Relation to work of Horodecki

- LOCC: States/$SU(2) \otimes SU(2)$
- Fully mixed subsystems

$$Tr_A \rho = Tr_B \rho = \frac{1}{2}\mathbb{1}$$

- $|00\rangle\langle 00|$ out of diagram



Bloch ball: $SU(2)$ equivalent to interval

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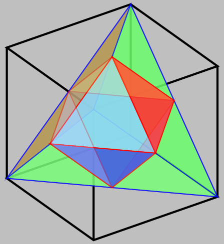

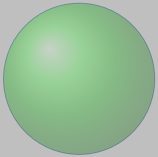

$\equiv$

Bloch ball: $SU(2)$ equivalent to interval

n dim Clifford: section

$\{\sigma_\mu, \sigma_\beta\} = 2\delta_{\alpha\beta}$

- $\rho \geq 0 \Longrightarrow \mathbb{1} + \sqrt{N-1}\, \mathbf{x}\cdot\boldsymbol{\sigma} \geq 0$
- $$(\mathbf{x}\cdot\boldsymbol{\sigma})^2 = \sum x_j x_k \sigma_j \sigma_k = \mathbf{x}^2 \mathbb{1}$$
- Spectrum $(\mathbf{x}\cdot\boldsymbol{\sigma}) = \pm|\mathbf{x}|$

Clifford section: Tiny n-ball

$$|\mathbf{x}| \leq \frac{1}{\sqrt{N-1}}$$

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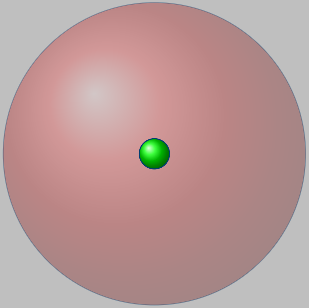
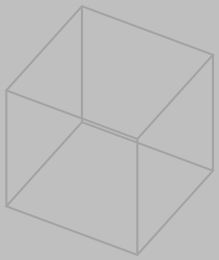


Dvoretzki Milman theorem

Theorem

A generic $O(n)$ dimensional section through 0 of a convex body in $\mathbb{R}^N$, containing 0 is approximately the n-ball

- Example: Random 2D sections of unit cube in $\mathbb{R}^N$



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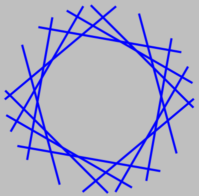
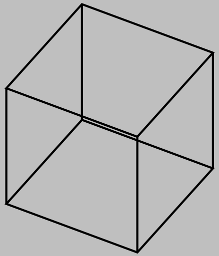
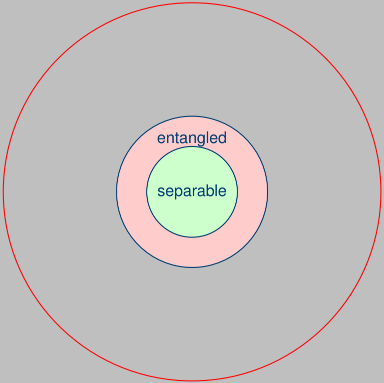




Application to quantum states

Szarek

- $N = 2^n \gg n \gg 1$.
- Can we say more
- Bare hands

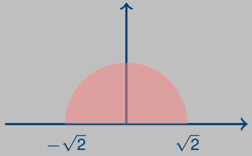


Generic ray

Connection with Random matrix theory

- $\mathbb{1} + \sqrt{N-1}\,\mathbf{x}\cdot\boldsymbol{\sigma} \geq 0$
- $\hat{\mathbf{x}} \in \mathbb{R}^{N^2-1}$ vector in random direction
- $\hat{\mathbf{x}}\cdot\boldsymbol{\sigma} =$Random matrix
- Wigner semi-circle law
- Radius= $O\left(\frac{1}{\sqrt{N}}\right)$,

Spectrum$(\hat{\mathbf{x}}\cdot\boldsymbol{\sigma})$



Cross sections with $D \ll N$; States: a tiny ball

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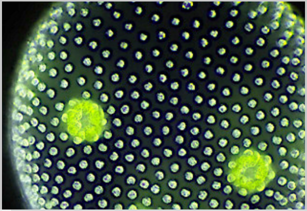

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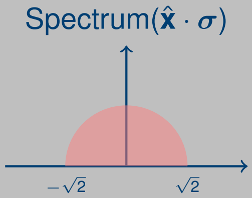

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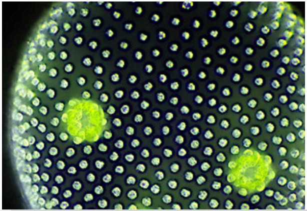
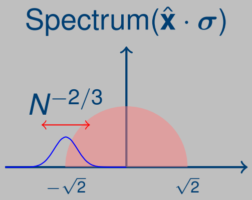


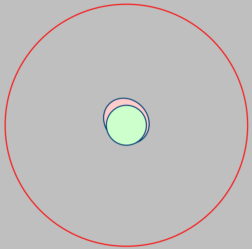
Unbound entangled states

Connection to Tracy-Widom

- Peres: Reflection symmetry test
- Balls: Reflection symmetric
- Statistics of extreme eigenvalues:
- Tracy-Widom
- Small relative width $O(N^{-2/3})$

Unbound Entanglement needs fine tuning



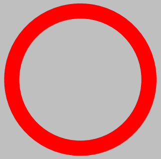


Ball is mostly empty

Most of the orange is its skin

Euclidean volume:

super-exponentially small



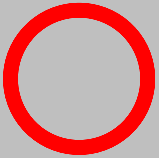
Available volume unbound entanglement

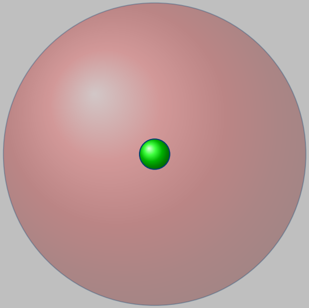
$(1+N^{-2/3})^N \approx e^{N^{1/3}}$

Ball is mostly empty

Most of the orange is its skin

Euclidean volume:
super-exponentially small





Available volume unbound entanglement

$(1 + N^{-2/3})^N \approx e^{N^{1/3}}$

States and Witnesses

- Witnesses

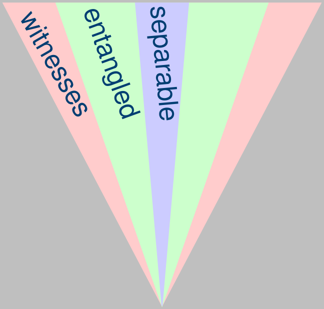
$$Tr\left(W\rho_{separabale}\right) \geq 0$$

- Swap

$$S\left|\psi\right\rangle \otimes \left|\phi\right\rangle = \left|\phi\right\rangle \otimes \left|\psi\right\rangle$$

Witnesses $\supset$ *States* $\supset$ *Separable*

- Witnesses NOT positive



- Equivalence

$$W^{MN} \cong M \otimes N\, W\, M^\dagger \otimes N^\dagger, \quad M, N \in SL(2,\mathbb{C})$$

- Invariants

$$\det W, \quad Spectrum(W^\star W), \quad W^\star = \sigma_2^{\otimes 2} W^t \sigma_2^{\otimes 2}$$

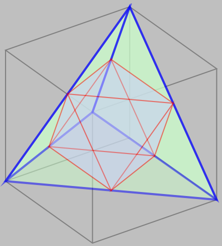
Theorem

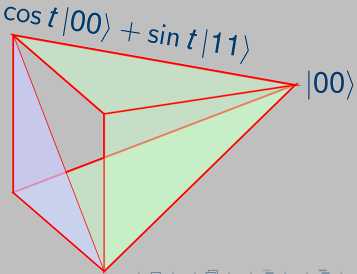
- $\min_{MN} \; Tr\; W^{MN} \; \exists$
- Minimizer: $\sum W_\mu \underbrace{|\sigma_\mu\rangle\langle\sigma_\mu|}_{Bell}$
- $W_0 \geq W_1 \geq W_2 \geq |W_3| \geq 0$

Visualizing 2 qubits

4D representation

- $$\rho_\mu^2 = \text{Eigenvalues}(\rho^\star \rho)$$
- $$1 \geq \rho_0 \geq \dots \rho_3 \geq 0, \quad \sum \rho_\mu \leq 1$$



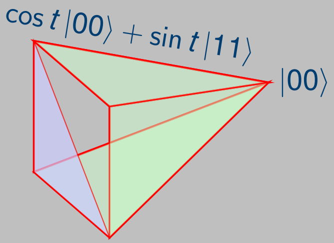


4 Stokes like parameters

Faithful to entanglement

Verstraete et. al. , Wootters

$\sum \rho_\mu = 1$	$\Longleftrightarrow$	Fully mixed subsystems
$\rho_0 - \rho_1 - \rho_2 - \rho_3$	$\Longleftrightarrow$	Entanglement measure
$\rho_0 \neq 0, \quad \rho_1 = \rho_2 = \rho_3 = 0$	$\Longleftrightarrow$	Pure state
$\rho = 0$	$\Longleftrightarrow$	Pure product

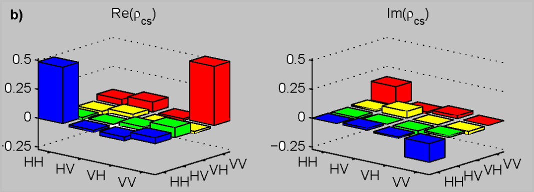


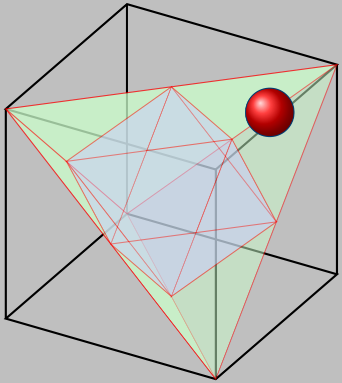
Illustration

Entangled photons from quantum dot — Gershoni et. al.

Tomography: 16 opaque numbers

Stokes: transparent





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- Verstraete, Dehaene and De Moor, PRA
- Wootters, PRL
- Avron Kenneth, AP
- Bengtsson and Zyczkowski *Geometry of quantum states*
- Aubrun and Szarek *Alice and Bob Meet Banach*