A century of adiabatic evolutions

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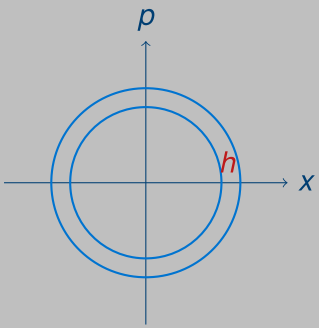
Outline

- History
- A collection of results
- Spectral evolutions
- Pictures
- Why should you care?
- The smooth part of the evolution
- Controlling oscillations

Quantum numbers and adiabatic invariants

Pre QM

- Bohr-Sommerfeld $\oint p\, dq = nh$
- Harmonic Oscillator
  - Adiabatic: $\dot{\omega} \ll \omega^2$
  - Invariant: $\frac{E}{\omega} \approx const$



1880-1933

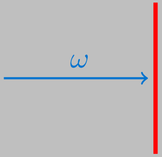
Ehrenfest: Adiabatic Invariants:

Quantum numbers in “old QM"

Photons: $N = E/\hbar\omega$

Einstein 1905

- Light beam hitting a detector
- $\frac{E}{\omega}$ Lorentz invariant
- No role in his photoelectric paper



$$\frac{E}{\omega} = \frac{E'}{\omega'}$$

914 *A. Einstein.*

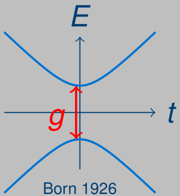
welche Formel für $\varphi = 0$ in die einfachere übergeht:

$$\frac{E'}{E} = \sqrt{\frac{1 - \frac{v}{V}}{1 + \frac{v}{V}}}.$$

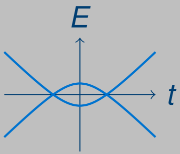
Es ist bemerkenswert, daß die Energie und die Frequenz eines Lichtkomplexes sich nach demselben Gesetze mit dem Bewegungszustande des Beobachters ändern.

Adiabatic theorems with and without crossings

Matrices and the role of the gap



Born 1926



Born and Fock 1928

$\frac{1}{g}$: internal time-scale

wrong folk wisdom:

Adiabaticity fails when $g = 0$

Beweis des Adiabatensatzes.

Von **M. Born** und **V. Fock** in Göttingen.

(Eingegangen am 1. August 1928.)

Der Adiabatensatz in der neuen Quantenmechanik wird für den Fall des Punktspektrums in mathematisch strenger Weise bewiesen, wobei er sich auch bei einer vorübergehenden Entartung des mechanischen Systems als gültig erweist.

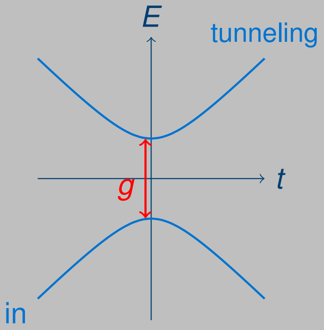
Landau-Majorana-Zener-Stückelberg (1932)

Exact, Exponential tunneling

$$H(t) = \begin{pmatrix} \omega^2 t & g \\ g & -\omega^2 t \end{pmatrix}$$

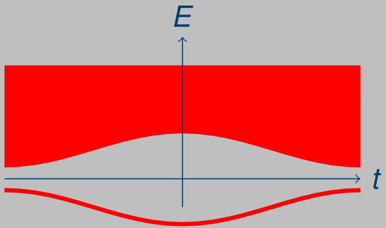
Tunneling$(t = \infty) = e^{-2\pi g^2/\omega^2}$

- Exact
- Exponential
- Adiabaticity $= \omega/g$
- Beyond perturbation in adiabaticity



From matrices to operators

Kato 1950



1917-1999

Motivation

- Degenerate ground state
- Tunneling to continuum

Byproducts

- New techniques
- Geometric picture

Spectral evolutions

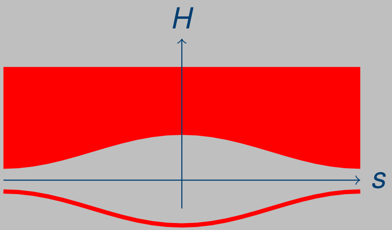
aka quasi-adiabaticity

- P: Spectral projection
- V: spectral evolution

$$V_s P_0 = P_s V_s$$

- Generator:

$$dK = i\,(dV)\,V^* = (dK)^*$$



$$V_s : P_0 \longrightarrow P_s$$

Commutator equation

$$[dK, P] = i\, dP$$

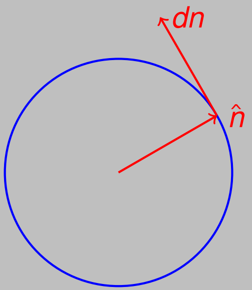
A useful identity for projections

Kato's generator

- $P^2 = P$
- $P(dP) + (dP)P = dP$

Useful identity

$$P(dP)P = 0$$

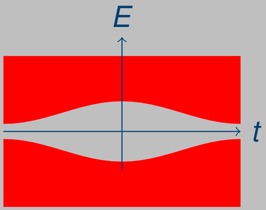


Kato's spectral generator

- $dK = i[dP, P]$ solves $[dK, P] = idP$
- Mother of the "Adiabatic curvature"

dK: Non-uniqueness

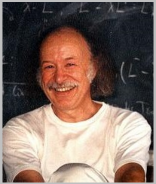
Band spectra and shortcuts to adiabaticity



$$\dot{K} = H + \underbrace{i[\dot{P}, P]}_{\text{kato}}$$

also solves $[\dot{K}, P] = i\dot{P}$

- Physical close to spectral

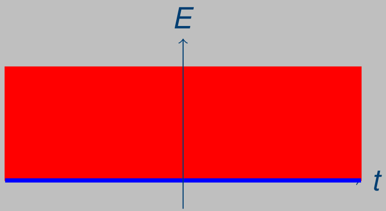


A. Seiler, Yaffe (1987)

Berry (2009)

Adiabatic theorem without a gap condition

Atom in radiation field



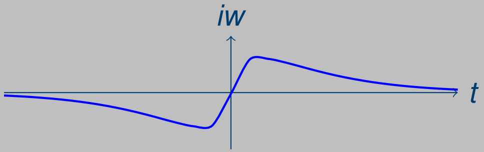
No effective error bound

- Non-commutative analog of

$$f \in L^1 \Longrightarrow \tilde{f}(\infty) = 0$$

A. Elgart (1999)

- dK local e.g. $\left(\sum a_j^* a_k\right), |j-k| < con$
- w has rapid decay



Matt Hastings (2015)

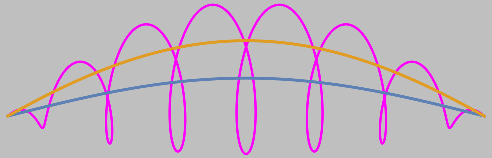
Hastings spectral generator

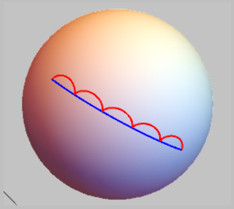
$$dK = i \int dt\, w(t) e^{iHt} (dH) e^{-iHt}$$

- dK inherits the localization of H
- Strategy: Lieb-Robinson propagation estimates

Orbits in projective space

- 2-level system lives on Bloch sphere
- Physical orbit
- Instantaneous orbit





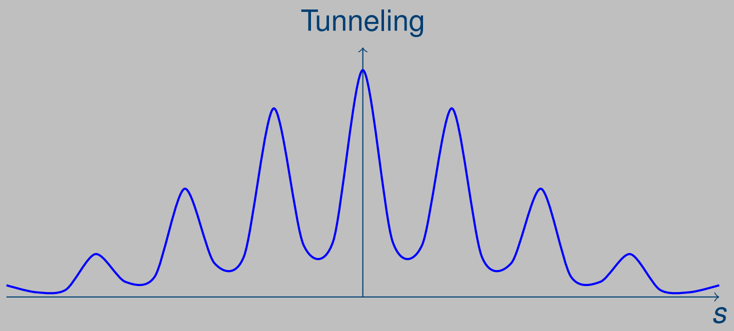
Physical
Instantaneous
Adiabatic

Adiabatic limit

- $\#$ oscillations $\to \infty$
- Wilder and tighter

Tunneling: Distance of state from spectral projection

Wildly oscillatory



Tunneling reversible

Fine tuning

Fubini-Study speed: Driving time scale

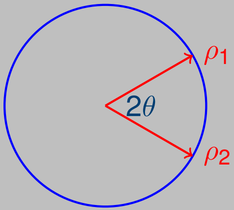
Gap: Internal time scale

- Pure state (=projection) $\rho^2 = \rho$
- Fubini-Study metric

$$(d\theta)^2 = \frac{1}{2} Tr(d\rho)^2,$$

- Fubini-Study speed $\dot\theta^2 = \frac{1}{2} Tr(\dot\rho^2)$
- If $i\, d\rho = [H, \rho]\, dt$

$$(\dot\theta)^2 = \underbrace{Tr(H^2\rho) - (Tr(\rho H))^2}_{Uncertainty}$$



Time scales

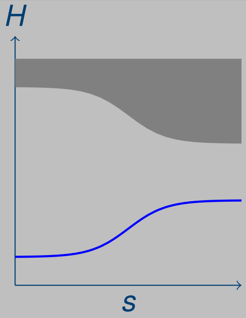
- Gap: internal
- $\dot P$: external
- $\dot E$: gauge freedom

What makes the adiabatic limit interesting

Fast and slow time: $t = \tau s$

$$i\,|d_t\psi\rangle = H(t/\tau)\,|\psi\rangle \Longleftrightarrow i\,|d_s\psi\rangle = \tau H(s)\,|\psi\rangle$$

- Long time $t \in [0, \tau]$
- $\delta H = O(1)$
- Singular limit



Adiabatic expansion

Motions in kernel and co-kernel have different character

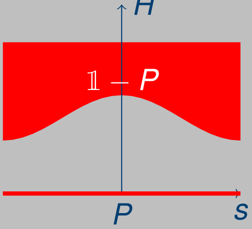
- Asymptotic: $i|\dot{\Psi}\rangle = \tau H(s)\,|\Psi\rangle\,,\ \tau \gg 1$
- $|\Psi\rangle = \underbrace{\sum \tau^{-n}\,|\Psi_n\rangle}_{\text{smooth part}} + O(e^{-i\tau})$
- $|\Psi\rangle = \underbrace{P\,|\Psi\rangle}_{|\alpha\rangle} + \underbrace{(\mathbb{1} - P)\,|\Psi\rangle}_{|\beta\rangle}$

Recursion

Linear: $\beta_n = M(\dot{\beta}_{n-1}, \dot{\alpha}_{n-1})$

ODE: $\dot{\alpha}_n = F(\dot{\beta}_n, \alpha_n)$

G.M. Graf



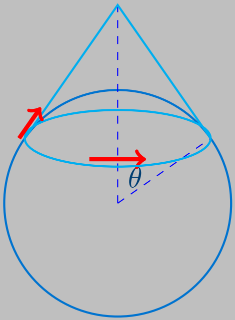
$n = 0$

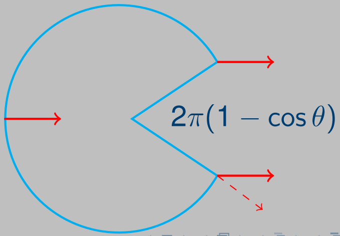
Berry's phase=Holonomy of parallel transport

$n = 0$

CoKernel: $|\beta_0\rangle = 0$

Kernel: $\underbrace{Pd\,|\alpha_0\rangle = 0}_{\text{parallel transport}}$



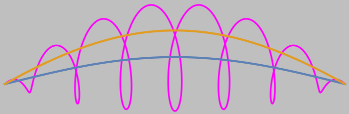


Tunneling is reversible and memory-less

$n = 1$

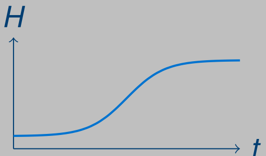
- $|\beta_1\rangle = \frac{i}{H}\,\dot{P}\,|\alpha_0\rangle$
- Tunneling probability

$$\approx \langle\beta_1|\beta_1\rangle = \langle\alpha_0|\,\dot{P}\,\frac{1}{H^2}\,\dot{P}\,|\alpha_0\rangle \geq 0$$



Remarkably:

- $|\beta(0)\rangle$ not free initial data!
- Tunneling memory-less
- No jerk: $\dot{P}(T) = 0 \Rightarrow |\beta_1(T)\rangle = 0$

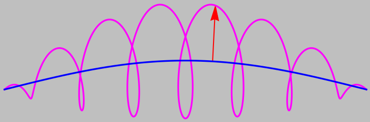


Confronting the oscillations

The adiabatic thm: Non-commutative integration by parts

- Physical evolution:

$$i\dot{U} = \tau H(s)U, \quad U(0) = \mathbb{1}$$



- Spectral evolution V
- Comparison $\Omega = V^*U$

$$\Omega - \mathbb{1} = -\int_0^s \underbrace{[\dot{P}, P]_s \Omega}_{\text{Oscillatory}} \, ds', \quad X_{s'} = V^* X V$$

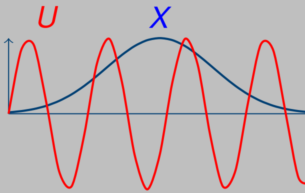
MP: When is $\|\Omega - \mathbb{1}\| = O(\tau^{-n}) \, \forall n \geq 1$

CS: When is $\|\Omega - \mathbb{1}\| < 1/3$ guaranteed?

Self-averaging commutators

Hamilton equation

- Rapid oscillations: $i\dot{U} = \tau HU$
- $X_s = U^* X U = O(1), \ (\dot{X_s}) = O(\tau)$



Hamilton equation

$$(\dot{X_s}) = i\,\tau\,[H, X]_s + (\dot{X})_s$$

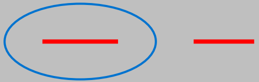
Self-averaging commutators

$$\int ds\,[H, X]_s = O(1/\tau)$$

The gap condition

Trade P for H in commutators

- $\Omega - \mathbb{1} = -\int [\dot{P}, P]_s \Omega \, ds$
- $[\dot{P}, P]_s \Omega = [X, H]_s \Omega$



$$R = \frac{1}{H - z}$$

$$X = \frac{1}{2\pi i} \oint dz \, R \, \dot{P} \, R = O\left(\frac{\dot{P}}{gap}\right)$$

Proof:

$$[X, H] = \frac{1}{2\pi i} \oint dz \left[\frac{1}{H - z} \dot{P} \frac{1}{H - z}, H - z\right] = [\dot{P}, P]$$

What controls the remainder

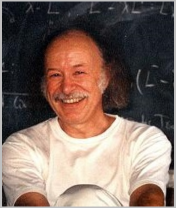
Jensen, Beth-Ruskay, Seiler; Reichradt

$$\|\Omega - \mathbb{1}\| = \frac{1}{\tau}\left(O(\dot{X}) + (\dot{P}X)\right), \quad X = \widetilde{\dot{P}}$$

Where is the gap?

$$\dot{P} = O\left(\frac{\dot{H}}{g}\right), \quad \tilde{X} = O\left(\frac{X}{g}\right)$$

Acknowledgment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