Introduction to Zak phase

Fun with periodic 2×2 matrices

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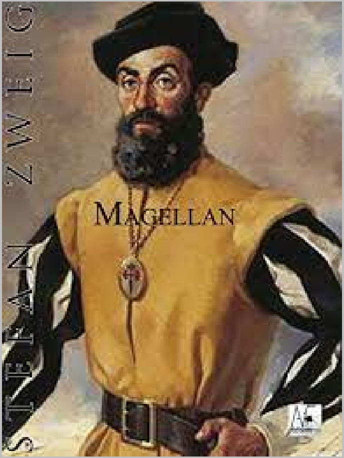
A topological invariant from the 16 century

Magellan & Pigafetta



Joshua Zak

Technion

- Born 1929 (age 93)
- 1940-1945 Labor and death camps
- 1946-1949 Red army
- 1951 Silver medal (physics)
  Gold medal (Kayaking)
- Known for:
  - Zak transfrom
  - Magnetic translations
  - Zak phase

Geometry of 2×2 hermitian matrices

Hamiltonians & states

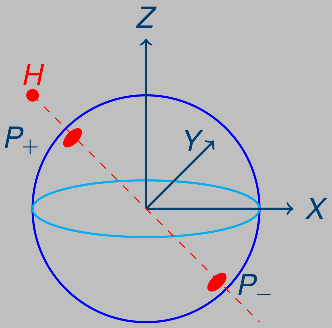
- $H = xX + yY + zZ, \quad x, y, z \in \mathbb{R}^3$
- X, Y, Z Pauli matrices.

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Eigen-projections

$$P_\pm = \frac{\mathbb{1} \pm \hat{H}}{2}, \quad \hat{H} = \frac{H}{|H|}$$

- Smooth away from the origin



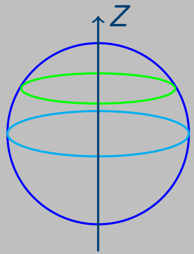
Bloch sphere

Parallel transport

Keeping normalization and phase

Parallel transport $\langle\psi|d\psi\rangle = 0$

- $\mathrm{Re}\langle\psi|d\psi\rangle = 0$
  constant normalization (exact)
- $\mathrm{Im}\langle\psi|d\psi\rangle = 0$
  constant phase (leading order)
- Rigid rotation
  - $|\psi_\theta\rangle = e^{-iZ\theta/2}\,|\psi\rangle$
  - $i\langle\psi_\theta|d\psi_\theta\rangle = \frac{1}{2}\,\langle\psi|\,Z\,|\psi\rangle\,d\theta$



Equator: $\langle\psi|\,Z\,|\psi\rangle = 0$

Berry's phase

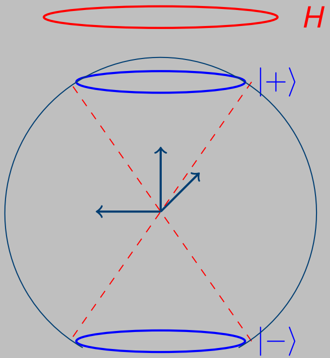
Berry's gauge potential

Berry's 1-form

$$A = -\mathrm{Im}\langle\psi|d\psi\rangle$$

Berry's phase

$$\beta = \left(\oint A\right) \mod 2\pi$$



$\beta = (\text{spherical angle of orbit})/2$

$$\beta_+ + \beta_- = 0 \mod 2\pi$$

What does Berry's phase measure?

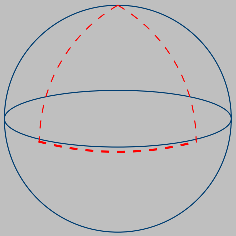
Deviation from parallel transport

- $|\varphi\rangle$ parallel transported
- $|\psi\rangle$ smooth normalized section

$$|\psi\rangle = e^{-i\beta}\,|\varphi\rangle$$

- Berry's connection

$$i\langle\psi|d\psi\rangle = d\beta$$



Berry's phase for a closed path

- 1/2 spherical angle
- Holonomy of parallel transported section

Symmetry protected topological phase

Chiral symmetry

- Chiral symmetry:

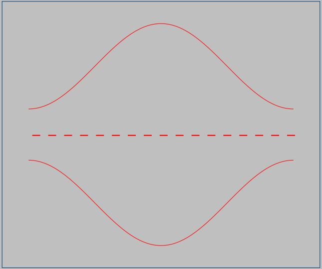
$$H(\theta) = -UH(\theta)U^*$$

- Bands share Berry's gauge

$$\langle \psi_+ | d\psi_+ \rangle = \langle U\psi_- | dU\psi_- \rangle = \langle \psi_- | d\psi_- \rangle$$

- Quantized holonomy

$$\beta_+ + \beta_- = 2\beta_\pm = 0 \mod 2\pi$$

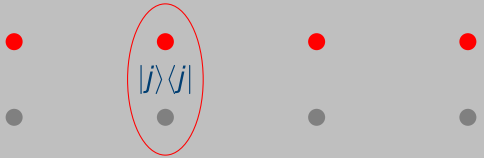


(Chiral) Symmetry protected topological phase

$$\beta_+ = \beta_- = 0 \mod \pi$$

Periodic matrices from periodic Hamiltonians

Unit cell with 2 atoms (or spin)

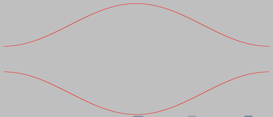


- 1-D Chain: $H = \sum_{j,\ell\in\mathbb{Z}} |j+\ell\rangle\langle j| \otimes H_\ell, \quad H_\ell = H^*_{-\ell} = \begin{pmatrix} \bullet & \bullet \\ \bullet & \bullet \end{pmatrix}$
- Bloch:

$$H(k) = \sum_{\ell\in\mathbb{Z}} e^{ik\ell} H_\ell, \quad H(k) = H^*(k) = \begin{pmatrix} z_k & \zeta_k \\ \bar\zeta_k & -z_k \end{pmatrix}, \quad z_k \in \mathbb{R}$$

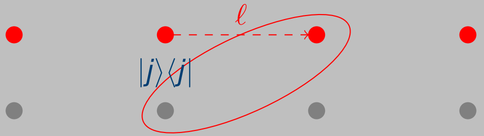
Two bands

$$E^2(k) = z_k^2 + |\zeta_k|^2 > 0$$



Bloch Hamiltonians depend on Choice of Unit cell

k-dependent gauge transformation



- Change of unit cell

$$H(k) \mapsto G_k H(k) G_k^*$$

- Guage transformation:

$$G_k = e^{ik\ell}\,|0\rangle\langle 0| + |1\rangle\langle 1| = \begin{pmatrix} e^{ik\ell} & 0 \\ 0 & 1 \end{pmatrix}$$

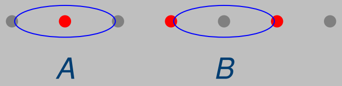
- Berry's phase of Bloch band depends on choice of unit cell

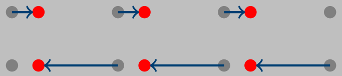
Zak phase: Berry's phase for a 1D Bloch band

Is the dependence on a choice of unit cell a disaster?

- Red atom: At center of unit cell A
  At edges of unit cell B
- Zak phase related to polarization
- Total band charge:

$$N_- = \int_{-\pi}^{\pi} \frac{dk}{2\pi} |\psi_k\rangle\langle\psi_k|$$





Zak phase dependns on unit cell

$$\beta_\ell - \beta_0 \overset{\text{mod } 2\pi}{=} \pi\ell\Big(\langle 1|\, N_- \,|1\rangle - \langle 0|\, N_- \,|0\rangle \Big)$$

Reflection about a bond

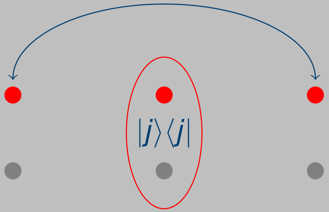
Preserves integrity of cells

- Bloch: $H_k = \begin{pmatrix} z_k & \zeta_k \\ \bar{\zeta}_k & -z_k \end{pmatrix}$
- Bond reflection:

$$H(k) \mapsto X\Big(H(-k)\Big)X$$

- X a k-independent unitary:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



$$R(H(k)) = XH(-k)X$$

$$z_k \mapsto -z_{-k}, \quad \zeta_k \mapsto \bar{\zeta}_{-k}$$

Reflection about an atom

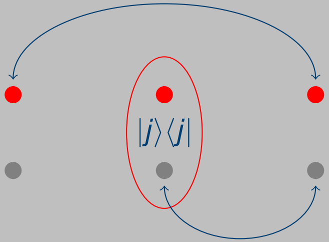
messes up the unit cell

- Bloch: $H_k = \begin{pmatrix} z_k & \zeta_k \\ \bar{\zeta}_k & -z_k \end{pmatrix}$
- Atom reflection:

$$H(k) \mapsto G_k\Big(H(-k)\Big)G_k^*$$

- k-dependent gauge transformation:

$$G_k = \begin{pmatrix} e^{ik} & 0 \\ 0 & 1 \end{pmatrix}$$



Atom reflection:

$$z_k \mapsto z_{-k}, \quad \zeta_k \mapsto e^{ik}\zeta_{-k}$$

Symmetry protected topological phase

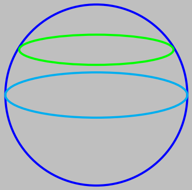
Reflection

- Define

$$\mathcal{A} = \mathrm{Im}\langle\psi|D\psi\rangle, \quad D = d + \frac{1}{2}G^* dG$$

- Reflection symmetry: $G_k\,|\psi_k\rangle = \lambda_k\,|\psi_{-k}\rangle$
- Relates Relates $\mathcal{A}_k$ to $\mathcal{A}_{-k}$

$$\mathcal{A}_{-k} + \mathcal{A}_k = id \log \lambda$$



Green: Berry's phase
Cyan: STP phase

(Reflection) Symmetry protected topological phase

$$2\int_{-\pi}^{\pi} \mathcal{A}\, dk \in 2\pi\mathbb{Z} \implies \int_{-\pi}^{\pi} \mathcal{A}\, dk \in \pi\mathbb{Z}$$

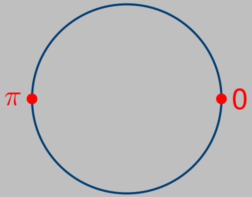
- Reflection Symmetry:

$$G_k H_k = H_{-k} G_k$$

- $G_{0,\pi}$ and $H_{0,\pi}$ share eigenvectors:

$$[H_0, G_0] = [H_\pi, G_\pi] = 0$$

- Involution: $G_0^2 = G_\pi^2 = \mathbb{1}$



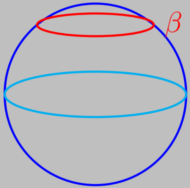
Index of symmetry points

$$\frac{\langle\psi_0|\, G_0\, |\psi_0\rangle}{\langle\psi_\pi|\, G_\pi\, |\psi_\pi\rangle} \in \pm 1$$

Index-phase correspondence

Deformation argument

- Take: $G_k = e^{ikg}$
- $G_\pi = G_{-\pi} \Longrightarrow \mathrm{spect}(g) \in \mathbb{Z}$
- Covariant connection $D = d + \frac{i}{2} g$
- Deform $z_k \to -\infty$



$$H_k = \begin{pmatrix} z_k & \zeta_k \\ \bar{\zeta}_k & -z_k \end{pmatrix} \Longrightarrow |\psi_k\rangle = |0\rangle \Longrightarrow \langle \psi | \partial \psi \rangle = 0$$

Index and phase

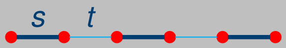
$$\int_{-\pi}^{\pi} \mathcal{A}\, dk \overset{\mathrm{mod}\ 2\pi}{=} -\pi \langle 0 | g | 0 \rangle = 0 \mod \pi$$

SSH model

Su Schriefer Heeger

- SSH Hamiltonian

$$H(k) = \begin{pmatrix} 0 & s + te^{ik} \\ s + te^{-ik} & 0 \end{pmatrix}$$



Staggered hopping

- Reflection symmetry $H(-k) = XH(k)X$
- Chiral symmetry $H(k) = -ZH(k)Z$
- Gapped: $|s| \neq |t|$
- $H_{0,\pi} = (s \pm t)X$

$$\text{Index} = \frac{\langle \psi_0 | X | \psi_0 \rangle}{\langle \psi_\pi | X | \psi_\pi \rangle} = \text{sgn}(|s| - |t|)$$

Bloch direct measurement of Zak phase

The challenge

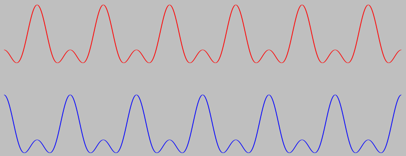
- Interference
- Subdominant

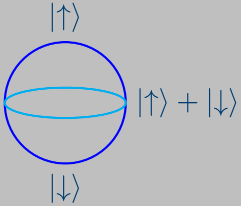
$$\varphi = \underbrace{\frac{1}{\varepsilon}\int E(t)dt}_{\text{Dynamical}} + \underbrace{\int i\langle\psi|\dot{\psi}\rangle dt}_{\text{Berry-Zak}} + \text{Noise}$$

I. Bloch et. al. experiment

Cold atoms in Optical lattice

- Optical Switching $s \Longleftrightarrow t$
- Spin superposition
  $\pi/2$ pulse
- Spin reversal
  π pulse
- Adiabatic magnetic driving
  $\varepsilon = \partial_x B_z$





$$H(k) \otimes \mathbb{1} + \varepsilon\, i\partial_k \otimes Z \Longleftrightarrow H(k + \varepsilon t) \otimes Z$$

Single atom interferometer

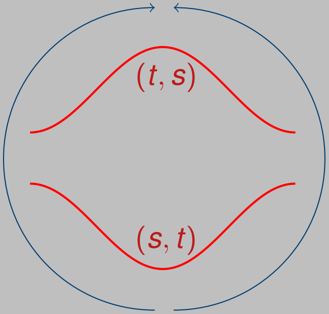
Beam splitter for spinning atoms

Interference

$\varphi_\uparrow = -Z \qquad \varphi_\downarrow = Z$

$|\downarrow\rangle \mapsto |\uparrow\rangle$

$\varphi = D - Z + N$



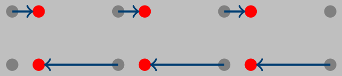
$|\uparrow\rangle \mapsto |\downarrow\rangle$

$\varphi_\uparrow = D + Z + N$

Beam splitter

The physics of Zak phase

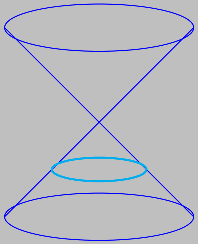
- Practical theory of polarization of dielectrics
  Resta and Vanderbilt
- Zero modes in Poly-acetylene
  Symmetry protected topological phases in 1D



Short history of Zak phase

Mother of SPT

- 1979: Mead and Truhlar: Conics crossing, molecules
- 1982: TKNN: Topology for 2-D bands
- 1984: M. Berry: adiabatic connection & curvature
- 1983: B. Simon: "Berry's phase", "Chern bundle"
- 1989: J. Zak, Geometry & topology of 1-D bands
- 1993: Resta & Vanderbilt, Zak phase=Polarization,
- 2013: Bloch: Measurement of Zak phase



Reference

Review

- JK Asbóth, L Oroszlány, A Pályi, A short course on topological insulators
- Kitaev, Gu & Wen, Pollmann
- Graf, Shapiro, Schultz-Baldes, Ogata

Kitaev chain

A model without particle conservation

- Periodic Kitaev Chain:

$$H(k) = \varepsilon_k a_k^* a_k + \underbrace{\left(e^{ik} a^*{}_k a^*{}_{-k} + h.c.\right)}_{\text{particle non-conserving}}, \quad \varepsilon_k = \mu + t\cos k$$

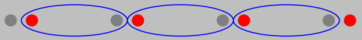
- Majorana:

$$\gamma_{2j+1} = a_j + a_j^*, \quad \gamma_{2j+2} = -i(a_j - a_j^*)$$

- μ term: $2a_j^* a_j = i\gamma_{2j+1}\gamma_{2j+2}$

- Hopping & super-conductivity:

$$i\gamma_{2j+2}\gamma_{2j+3} = a_j a_{j+1} + a_j a_{j+1}^* + h.c.$$



Reduction to 2 × 2 matrices

Fermionic Bloch-Fock space

- $$H(k)+H(-k)=\varepsilon_k\Big({a_k}^*a_k+{a_{-k}}^*a_{-k}\Big)+2i\sin k\Big({a^*}_k{a^*}_{-k}-a_{-k}a_k\Big)$$
- Fock-Bloch basis

$$|0\rangle\,,\ {a^*}_k\,|0\rangle\,,\ {a^*}_{-k}\,|0\rangle\,,\ {a^*}_k{a^*}_{-k}\,|0\rangle$$

- 4 × 4 matrix

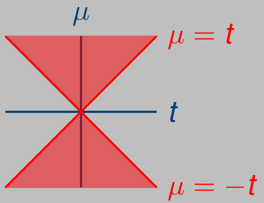
$$H_k+H_{-k}-\varepsilon_k=\begin{pmatrix}-\varepsilon_k & 0 & 0 & -2i\sin k\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 2i\sin k & 0 & 0 & \varepsilon_k\end{pmatrix}$$

2 × 2: Span $|0\rangle$ and $a_k^*{a^*}_{-k}\,|0\rangle$

Bands in the vacuum-Cooper pairs sector

$\mathbb{Z}_2$ Index

- 2-Bands
  $H(k) = Z\varepsilon_k + 2Y\sin k$
- $H_{0,\pi} = (\mu \pm t)Z$
- Gapped: $\mu \neq \pm t$
- Symmetry: $H(-k) = ZH(k)Z$



Topological phase diagram

$$\text{Index} = \frac{\langle\psi_0|\, Z\, |\psi_0\rangle}{\langle\psi_\pi|\, Z\, |\psi_\pi\rangle} = \text{sgn}(\,|\mu| - |t|)$$