Zak phase

Invitation with 2×2 matrices

J Avron & Ari Turner

14 Sept 2022

Joshua Zak

Technion

- Born 1929, Vilnius, (age 93)
- 1940-1945 Ghetto, Labor & death camps
- 1946-1949 Red army
- 1951 Silver medal (physics)
  Gold medal (Kayaking)
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  - Zak transform
  - Magnetic translations
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Geometry of 2×2 hermitian matrices

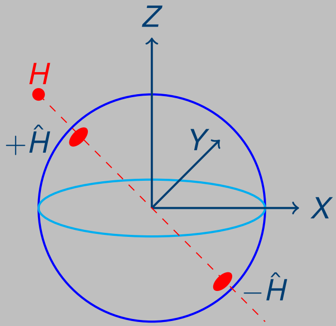
Hamiltonians & states as points in $\mathbb{R}^3$

- $H = xX + yY + zZ, \quad x, y, z \in \mathbb{R}^3$
- X, Y, Z Pauli matrices.

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Eigenstates on the Bloch sphere

$$P_\pm = \frac{\mathbb{1} \pm \hat{H}}{2}, \quad \hat{H} = \frac{H}{\sqrt{x^2 + y^2 + z^2}}$$



Origin: Singular point

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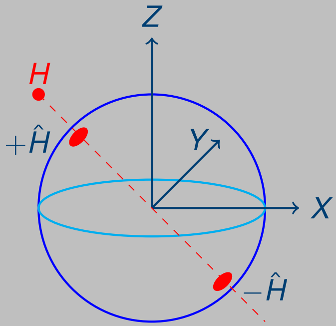


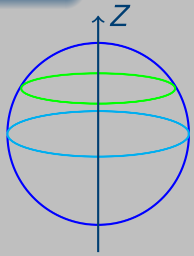
Origin: Singular point

Parallel transport

Keeping normalization and phase

Parallel transport $\langle\psi|d\psi\rangle = 0$

- $\mathrm{Re}\langle\psi|d\psi\rangle = 0$
  constant normalization
- $\mathrm{Im}\langle\psi|d\psi\rangle = 0$
  constant phase



Rigid rotation: $e^{-iZ\phi/2}$

$$\langle\psi|d\psi\rangle = \frac{i}{2}\langle\psi|\,Z\,|\psi\rangle$$

Parallel transport along the equator

Parallel transport

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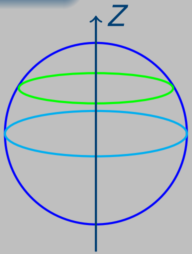

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Parallel transport along the equator

Holonomy of parallel transport

Berry's phase

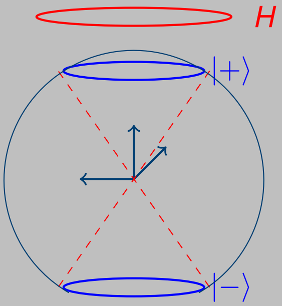
Holnomy of parallel transport

$e^{iZ\pi} = -\mathbb{1}$

Berry:

$$\beta_\pm = \frac{\text{spherical angle}}{2}$$

$$\beta_+ + \beta_- = 0 \mod 2\pi$$



Berry's phase is geometric

Not quantized

Holonomy of parallel transport

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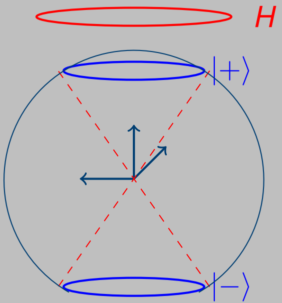

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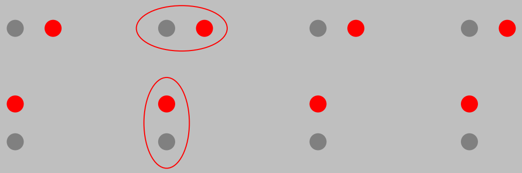
Bloch Hamiltonians as Periodic 2×2 matrices

Unit cell with 2 atoms (or spin)



Bloch Hamiltonian:

- $H(k) = \begin{pmatrix} z_k & x_k + iy_k \\ x_k - iy_k & -z_k \end{pmatrix}$
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- (x_k, y_k, z_k): Closed loop in $\mathbb{R}^3$

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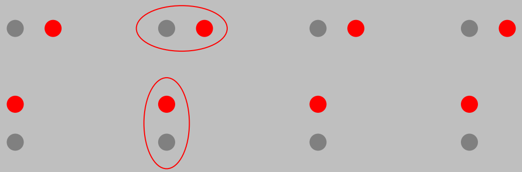

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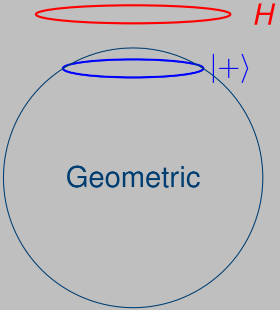
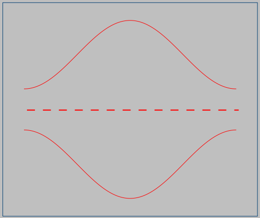
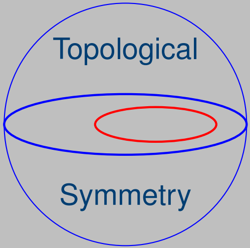
Zak phase: Berry's phase for a 1D Bloch band

Geometric/Topological characterization of 1D insulators







Mother of Symmetry protected topological insulators

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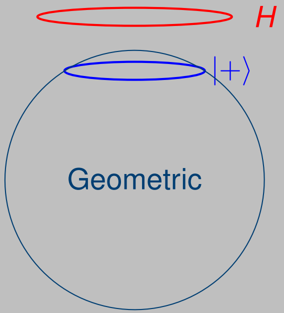
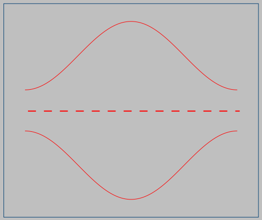
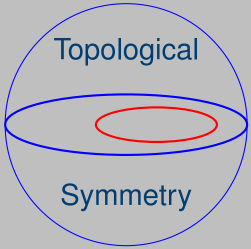






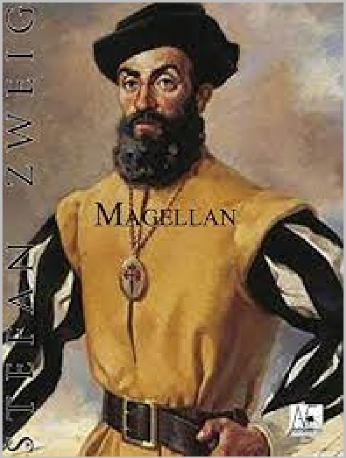
Mother of Symmetry protected topological insulators

A topological invariant from the 16 century

Magellan circumnavigating earth

Lose a day traveling west

Gain a day traveling east

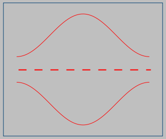


Topological Zak phase

Topological insulators in 1-D

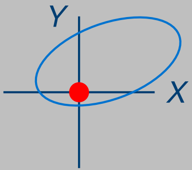
- Chiral symmetry:

$$H(k) = -ZH(k)Z, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



- Forces

$$H(k) = \begin{pmatrix} 0 & x_k + iy_k \\ x_k - iy_k & 0 \end{pmatrix}$$



- Winding of $x_k + iy_k$

(Chiral) Symmetry protected topological phase

$$\beta = 0 \mod \pi$$

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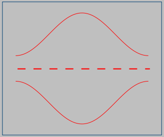

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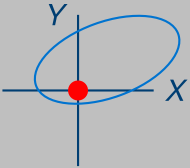

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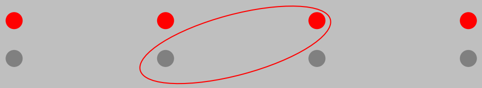
Choice of unit cell =Gauge freedom

Zak phase is relative to unit cell



Change of unit cell=Gauge transformation

$$H(k) \mapsto G_k H(k) G_k^* \quad G_k = \begin{pmatrix} e^{ik\ell} & 0 \\ 0 & 1 \end{pmatrix}$$

Quantized Zak phase dependence on unit cell

$$\pi \iff 0$$

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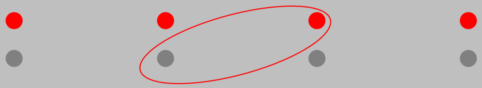

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Quantized Zak phase dependence on unit cell

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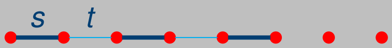
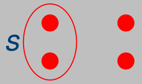
Example: SSH model

Su-Schrieffer-Heeger : Staggered hopping



- s-unit cell: $\begin{pmatrix} 0 & s+te^{ik} \\ s+te^{-ik} & 0 \end{pmatrix}$



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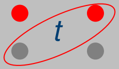

Zak phase

cell	$t > s$	t<s
s	π	0
t	0	π

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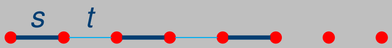

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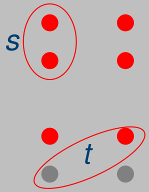

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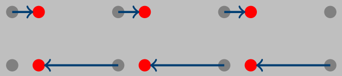
Physics of the Zak phase

Polarization

Depend on choice of unit cell



Polarization=Zak phase

Resta & Vanderbilt

Physics of the Zak phase

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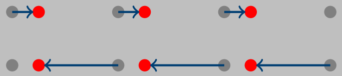

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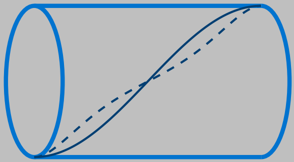
Cold atoms Bloch^2 interferometer

How to measure quantized Zak phase

Bloch^2 interferometer:

- Immanuel and Felix
- Split path in k-space



Interference

- Zak: Constructive
- Dynamical phase & Noise: Destructive

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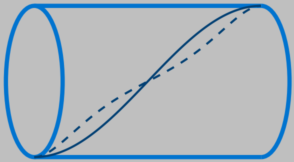

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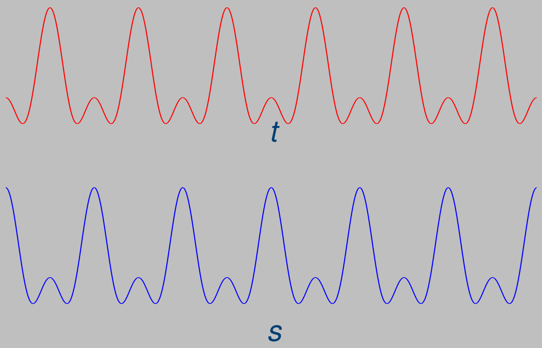
Optical lattice

Gauge invariant measurement

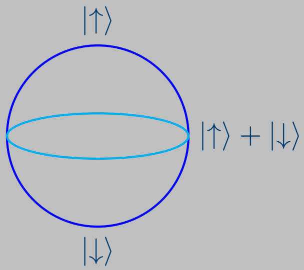


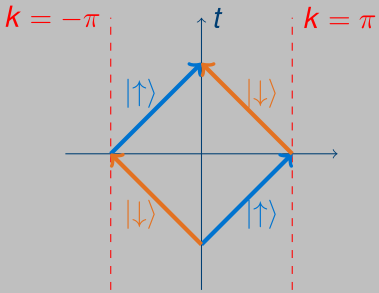
Gauge invariant observable

$$(Zak)_{st} - (Zak)_{ts} = \pm\pi, \quad e^{\pm i\pi} = -1$$

Spin induced Schizophrenia: Split path in $k-$space

Spin dependent forcing





Magnetic acceleration $\varepsilon = \partial_x B_z$

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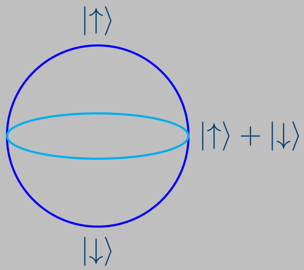

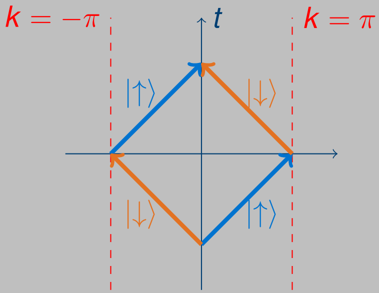

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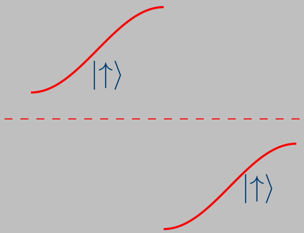
Getting rid of the dynamical phase

SSH is chiral symmetric



Band switching gets rid of the dynamical phase

$$\left(\int_0^{\pi}+\int_{-\pi}^{0}\right)E_{\uparrow}(k)dk=0$$

Short history of Pancharatnam-Berry-Zak phase

Nothing is ever discovered for the first time

- 1956 Pancharatnam, Polarization
- 1979: Mead and Truhlar: Jahn-Teller
- 1982: TKNN: Topology for 2-D bands
- 1983: B. Simon: "Berry's phase", "Chern bundle"
- 1984: M. Berry: Phase & curvature
- 1989: J. Zak, Geometry & topology of 1-D bands
- 1993: Resta & Vanderbilt, Zak phase=Polarization,
- 2013: Bloch: Measurement of Zak phase