LANDAU HAMILTONIANS ON SYMMETRIC SPACES

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1 INTRODUCTION AND OVERVIEW

A classical particle on a manifold moves on geodesics: Straight lines on the plane, great circles on the sphere and semi-circles on the hyperbolic-plane, represented as the upper-half plane with the usual metric[1].

The corresponding Schrödinger operators are (minus) the Laplacians on the manifolds and their spectra are $[0,\infty)$ for the plane, $\{0,2,6,...,n(n+1),\ldots,\infty\}$ for the sphere, and $[\frac{1}{4},\infty)$ for the hyperbolic plane [see e.g. McKean (1970) or Terras (1985)].

Magnetic fields are 2-forms, and a constant magnetic field is a multiple, B, of the area form, with $B > 0$, the magnetic field strength. Constant magnetic fields are therefore natural for orientable two dimensional manifolds.

In contrast with magnetic fields, constant electric fields are not natural in general: Electric fields are vector fields and on a curved manifold zero is the only constant vector field; Constant electric field are defined only for flat spaces. For this reason the dynamics in constant magnetic fields is the essentially unique generalisation of the free dynamics which preserves the underlying symmetry of the manifold.

In classical mechanics a magnetic field leads to a Lorentz force and the orbit is no longer a geodesic. For example, the orbits in the plane are circles (rather than straight lines). As we shall explain in section 3, for the sphere, the orbits can be described geometrically as the circle of tangency of the sphere, (embedded in three dimensional Euclidean space) with a cone tangent to it, see Fig. 1. The larger the energy of the particle, the thinner the cone. Similarly, for the hyperbolic plane the orbits are the lines of tangency of the hyperboloid, embedded in three dimensional Minkowski space, with cones. If the energy is in $[0,B^2)$, the cone is time-like and the orbit is closed, see Fig. 2; If the energy is in (B^2,∞), the cone is space like, the orbit is open and runs to infinity, see Fig. 3. More on this, in section 3.

We call (minus) the "Magnetic Laplacians" with constant magnetic fields "*Landau Hamiltonians*"[2]. Like the ordinary Laplacians, they represent pure kinetic energy. For the plane and the hyperbolic plane B can be any real number. For the unit sphere $2B$ must be an integer [Dirac (1931), Wu and Yang (1976), Coleman (1982)].

The spectral analysis of the Landau Hamiltonian for the plane goes back to Landau in 1930. The spectrum is

$$\text{Spect}\ (\mathrm{H_P}) = \{\mathrm{B}(2\mathrm{n}+1) \mid \mathrm{n} = 0,1,2,\ldots\infty\}, \tag{1.1}$$

each point is infinitely degenerate[3]. This is, of course, quite unlike the spectrum of the Laplacian.

The Landau Hamiltonian and the Laplacian on the sphere turn out to have related spectral properties. The spectral analysis of the former also goes back to the 30's [Tamm, (1931), Wu and Yang (1976)] and

$$\text{Spect}\ (\mathrm{H_S}) = \{(\mathrm{B}+\mathrm{n})(\mathrm{B}+\mathrm{n}+1) - \mathrm{B}^2 \mid \mathrm{n} = 0,1,\ldots\infty\}, \tag{1.2}$$

with degeneracy $2(n+B)+1$.

Landau Hamiltonians for the Hyperbolic plane were analyzed by [Comtet and Houston (1985), Comtet (1987) , Grosche (1988)] and turn out to have rich spectral properties:

$$\text{Spect}(\mathrm{H_H}) = \left\{\mathrm{B}^2 - (\mathrm{n}-\mathrm{B})(\mathrm{n}-\mathrm{B}+1) \mid \mathrm{n} = 0,1,\ldots,[\mathrm{B}-\frac{1}{2}]\right\} \cup [\frac{1}{4}+\mathrm{B}^2,\infty), \tag{1.3}$$

where $[x]$ is the integer part of x. See Fig. 4. There are no isolated spectral points for $0 \le B \le \frac{1}{2}$. For $B > \frac{1}{2}$, there are $[B+\frac{1}{2}-0]$ isolated spectral points in the interval $(\frac{1}{2}, B^2+\frac{1}{4})$ each infinitely degenerate. In addition, there is continuous spectrum on the semi-infinite line $[\frac{1}{4}+B^2,\infty)$. An elementary derivation of these results is given in section 4.

With the vector spaces associated to the isolated spectral intervals of Schrödinger operators on two dimensional (noncompact) manifolds, one can associate an index defined as follows: Cut the manifold from the origin to infinity[4] and impose $\exp 2\pi i\theta$ boundary conditions across the cut. This gives a circle of operators, $H(\theta)$. Since $H(0) = H(1)$, the spectrum is periodic in θ and in particular, dim $\mathrm{P}(0)$ = dim $\mathrm{P}(1)$ with $\mathrm{P}(\theta)$ the spectral projection on the interval, and we allow dim $\mathrm{P} = \infty$. Define the index by[5] :

$$\text{Index} \equiv \#(\text{Arriving}) - \#(\text{Departing}),$$

where $\#(Arriving)$ is the number of states that joint the spectral interval as θ increases by a period, and similarly for $\#(Departing)$[6].

Finite dimensional projections have index zero. In particular, this is the case for all the spectral points of Landau Hamiltonian on the sphere.

Spectral intervals with infinite dimensional projection can have a nonzero index and the Landau Hamiltonians on the Plane and Hyperbolic plane are examples. The $n-th$ spectral point has:

$$\mathrm{Index}_{\mathrm{P}}(\mathrm{P}_{\mathrm{n}}) = \mathrm{Index}_{\mathrm{H}}(\mathrm{P}_{\mathrm{n}}) = 1. \tag{1.6}$$

The calculation is given in section 5. The results for the plane are known from other considerations that we discuss in the next paragraph. For the Hyperbolic plane the results are new.

The germs of the definition of the index given above go back to Laughlin [Laughlin (1987)] who was interested in the Hall conductance, and in particular wanted to explain why, under appropriate conditions, the Hall conductance assumes integer values. The credit for identifying the Hall conductance with an Index, and saying so in so many words, goes to J. Bellissard [Bellisard (1988)]. He realized this in the context of Connes noncommutative differential geometry [Connes (1986)]. The index defined above is distinct from, although closely related to, the index of certain Fredholm operators that arise in Connes theory [Avron, Seiler and Simon, (1990)]. Connes went part of the way in computing the indices for the plane and the hyperbolic plane.

2. NOTATIONS AND CONVENTIONS

2.1 Units:

We absorb the charge of the particle and the velocity of light in the definition of the strength of the magnetic field. We choose units of length so that the curvatures of the sphere and the hyperbolic plane are ± 1, and the remaining units so that $\hbar$ and the mass of the particle are unity. Since there is a factor of 2 between (minus) the Laplacian and the (free) Schrödinger operator in these units, we absorb this factor into the definition of the energy scale.

2.2 Spaces:

P, S and H will denote the Euclidean plane, the 2-sphere, and the Hyperbolic plane respectively. E^n denotes Euclidean n-space, and M^3 Minkowski space.

Z is the integers and Z_+ the non-negative integers.

2.3 Metrics:

g denotes the metric 2-form:

$$\mathrm{g} \equiv \pm(\mathrm{dx}^0)^2 + (\mathrm{dx}^1)^2 + (\mathrm{dx}^2)^2 = \pm(\mathrm{dr})^2 + \mathrm{r}^2((\mathrm{d}\tau)^2 + \mathrm{k}^2(\tau)\,(\mathrm{d}\phi)^2), \tag{2.1}$$

where the upper signs are for E^3 and the lower signs for M^3; x^j denote cartesian coordinates and (r, τ, ϕ) spherical coordinates (for the forward time-like cone in M^3), $0 \leq r \leq \infty$, $0 \leq \phi < 2\pi$ and

$$k(\tau) \equiv \begin{cases} \tau, & 0 \leq \tau < \infty, \quad \text{for P;} \\ \sin(\tau), & 0 \leq \tau \leq \pi, \quad \text{for S;} \\ \sinh(\tau), & 0 \leq \tau < \infty, \quad \text{for H.} \end{cases} \tag{2.2}$$

2.4 Signs:
Equation with $\pm$ or $\mp$ signs, the upper sign usually refers to S (or E) and the lower sign to H (or M), except in cases where it is clear from the context that the meaning is otherwise.

Since B and $-B$ are related (by time reversal) we restrict ourselves to $B > 0$.

2.5 Geometry:
We use lower case boldface to denote 2-forms and uppercase boldface to denote 1-forms. $\mathbf{b} = \mathbf{dA}$ stands for the basic relation between the magnetic-field 2-form b and the vector-potential 1-form $\mathbf{A}$. Vectors are denoted by arrows.

$\langle \mathrm{b}|\vec{\mathrm{v}} \otimes \bullet\rangle = \mathbf{W}$ is the contraction of a 2-form b with a vector $\vec{v}$ to yield a 1-form $\mathbf{W}$. We use lower indices for the components of forms and upper indices for the components of vectors. In index notation, the previous equation reads $b_{ij}\ v^j = W_i$.

We write $\mathrm{g}(\vec{\mathrm{v}}) = \mathbf{V}$ and the converse relation $\mathrm{g}^{-1}(\mathbf{V}) = \vec{\mathrm{v}}$, for the correspondence between vectors and forms. Scalar products are denoted by dots, so $\vec{x} \cdot \vec{x} = \langle \mathrm{g}|\vec{\mathrm{x}} \otimes \vec{\mathrm{x}}\rangle$. We set $0 \leq r \equiv \sqrt{|\vec{x} \cdot \vec{x}|}$.

3 THE CLASSICAL ORBITS: A GEOMETRIC DESCRIPTION

In this section we describe the geometry of the classical orbits. Since all three cases, P, S and H are described by (almost) the same set of equations, except that the various symbols mean slightly different things, we start with detailing the notation.

The 2-sphere S and the Hyperbolic-plane H are:

$$\vec{x} \cdot \vec{x} = \pm 1, \tag{3.1}$$

where $\vec{x} \in E^3$ for S and $\vec{x} \in M^3$ for H. The magnetic 2-form

$$\mathbf{b} \equiv \begin{cases} B\ dx^1 \wedge dx^2, & \text{on P;} \\ \frac{B}{r^3}\ (x^0 dx^1 \wedge dx^2 + x^1 dx^2 \wedge dx^0 + x^2 dx^0 \wedge dx^1), & \text{on S, H,} \end{cases} \tag{3.2}$$

satisfies $\mathbf{db} = 0$ (away from $r = 0$). Its restriction to the S and H has $r = 1$.

The Lorentz force is

$$\vec{F} \equiv \mathrm{g}^{-1}(\langle \mathrm{b}|\vec{\mathrm{v}} \otimes \bullet\rangle), \tag{3.3a}$$

and in components

$$F^j = g^{jk} b_{km} v^m, \tag{3.3b}$$

where $\vec{v} \equiv \dot{\vec{x}}$ is the velocity. (Just a fancy way to write $\vec{v} \times \vec{B}$). Newton's equation in the plane is

$$\dot{\vec{v}} = \vec{F}. \tag{3.4a}$$

On S and H the motion is constrained so that $\vec{x} \cdot \vec{v} = 0$ and $\vec{v} \cdot \vec{v} + \vec{x} \cdot \dot{\vec{v}} = 0$. Since

$$\vec{x} \cdot \vec{F} = \langle \mathrm{b}|\vec{\mathrm{v}} \otimes \vec{\mathrm{x}}\rangle = 0, \tag{3.5}$$

the Lorentz force is incompatible with the constraint and the equation of motion (3.4a) is modified to:

$$\dot{\vec{v}} = \mp(\vec{v} \cdot \vec{v})\vec{x} + \vec{F}. \tag{3.4b}$$

The first term on the right hand side is the centripetal force applied by the surface. In index notation, Eq. (3.4b) reads

$$\dot{v}^i = \mp(\vec{v} \cdot \vec{v})\, x^i + g^{il} b_{lj} v^j. \tag{3.4b'}$$

Equations (3.4) have two constants of motion that will play a central role throughout. The first is

$$\vec{c} \equiv \vec{x} + \left(\frac{1}{B}\right)^2 \vec{F}. \tag{3.6}$$

$\dot{\vec{c}} = 0$ is easily verified for the plane by differentiation and substituting the equation of motion. For S and H one needs to use:

$$\langle \mathrm{b}|\vec{\mathrm{x}} \otimes \bullet\rangle = \langle \dot{\mathrm{b}}|\vec{\mathrm{v}} \otimes \bullet\rangle = 0, \tag{3.7}$$

and the geometric fact that a tangent vector to the manifold can be flipped with the help of the area form:

$$B^2\vec{v} = -\mathrm{g}^{-1}\left(\langle \mathrm{b}|\vec{\mathrm{F}} \otimes \bullet\rangle\right). \tag{3.8}$$

Both equations (3.7) and (3.8) rely on the special form of $\mathbf{b}$.

The second constant of motion is (twice) the kinetic energy:

$$E \equiv \vec{v} \cdot \vec{v} = B^2(\vec{c} - \vec{x})^2 \tag{3.9a}$$

$$= B^2(\vec{c} \cdot \vec{c} - \vec{x} \cdot \vec{x}) = B^2(\vec{c} \cdot \vec{c} \mp 1). \tag{3.9b}$$

and Eq. (3.9b) holds for S and H only. The constancy of E follows from the fact that the Lorentz force, and the constraint force for S and H, are both perpendicular to the velocity. (3.9a) comes from (3.6) and (3.8) and (3.9b) from (3.5) and (3.6). The kinetic energy E is non-negative. This is manifest for P and S. For H it can be seen by noting that $\vec{x}$ is time-like, and therefore $\vec{v}$ is space-like.

The two constants give a geometric description of the motion: In the Plane the orbit is a circle with center $\vec{c}$ whose radius increases with the energy. The group of rigid motions of E^2 takes an orbit to an orbit with the same energy, of course. This is the sense in which the classical dynamics in a constant magnetic field retains the symmetry of the manifold.

For S, the positivity of E implies that $\vec{c}$ is outside the unit sphere, and Eq. (3.5) and (3.6) that the orbit is the circle of tangency of a cone, whose apex is at $\vec{c}$, with the unit sphere. This is illustrated in Fig 1. The rotations in E^3 take orbits to orbits with the same energy; the classical dynamics preserves the symmetry of the manifold.

On H, the orbits are the lines of tangency of cones whose apex is at $\vec{c}$ with the hyperboloid. The positivity of E implies that $-1 \leq \vec{c}\cdot\vec{c}$. There are two cases: 1) $\vec{c}$ is time like: $-1 \leq \vec{c}\cdot\vec{c} \leq 0$, which is the case if $0 \leq E < B^2$. The orbits in this case are all closed. 2) $\vec{c}$ is space like, which is the case if the energy is high, $E > B^2$, and the orbits are all open. The two kinds of orbits are illustrated in Figs. 2 and 3. Lorentz transformations in M^3 take orbits to orbits with the same energy, as is the case for the free dynamics.

4. SPECTRA OF LANDAU HAMILTONIANS

Landau Hamiltonians are the self-adjoint operators obtained from the kinetic energy E by replacing the velocity by the operator valued 1-form: $\mathbf{V} \equiv -\mathrm{id} - \mathbf{A}(\vec{\mathrm{x}})$, where $\mathrm{b} = \mathrm{d}\mathbf{A}$. It turns out that much can be said about their spectra using elementary algebraic methods, and solving (at most three) first order ordinary differential equations.

The spectra of Landau Hamiltonians in the plane have been derived in $n+1$ ways by 2^n authors. We reproduce one such derivation here, because it is a prototype of the methods we shall use below for the other cases.

The Landau Hamiltonian is the (self-adjoint, differential) operator corresponding to the energy (3.9a),

$$H_P \equiv \mathbf{V}\cdot\mathbf{V}. \tag{4.1}$$

The relevant commutations are:

$$[V_1, V_2] = iB,\ [\mathbf{V}, \vec{c}] = 0,\ [c^1, c^2] = -i/B. \tag{4.2}$$

The two components of $\mathbf{V}$ satisfy the canonical commutation up to a multiple B. $H = V_+V_- + B$ with $V_\pm \equiv V_1 \mp iV_2$ being creation and annihilation operators. The spectrum is therefore a sequence of points separated by $2B$ whose *vacuum state* solves the first order ordinary differential equation:

$$V_-\ \psi_- = 0. \tag{4.3}$$

In the gauge $\mathbf{A} = \frac{B}{2}(-x^2\, dx_1 + x^1\, dx_2)$, $V_- = i(-\frac{d}{dz} + \frac{B}{2}\bar{z})$, with $z \equiv x^1 + ix^2$, the vacuum is $\psi_-(z) = \exp(-B\ z\bar{z}/2) \in L^2(d^2x)$. The sequence obtained by successive applications of V_+ on ψ_- does not terminate because V_+ has no (square integrable) vacuum. The infinite degeneracy comes from the fact that H_P commutes with the generator of (magnetic) translations[7], $B\vec{c}$.

To determine the spectra of $H_{S,L}$ it is convenient to extend the operators to the embedding space, i.e. as operators on $L^2(E^3) = L^2(S)\otimes L^2(r)$ and $L^2(M^3) = L^2(H)\otimes L^2(r)$. This is done as follows. Let $\vec{J} \equiv \mp B\vec{c}$ on the surface $r = 1$, and extend $\vec{J}$ to all r by *scale invariance*:

$$\begin{aligned} J^0 &= \mp B\ x^0/r + (x^1V_2 - x^2V_1)\,, \\ J^1 &= \mp B\ x^1/r \pm (\pm x^2V_0 - x^0V_2)\,, \\ J^2 &= \mp B\ x^2/r \mp (x^0V_1 \mp x^1V_0)\,. \end{aligned} \tag{4.4}$$

The Landau Hamiltonian, from Eq. (3.8b), is

$$H_{S,H} = \vec{J}\cdot\vec{J} \mp B^2. \tag{4.5}$$

Because of scale invariance the spectra of the operators restricted to the manifold and the operators in the embedding space are the same.

Besides scale invariance, the $\vec{J}$'s have additional interesting properties: They satisfy $su(2)$ algebra for S, and $su(1,1)$ algebra for H, and operate on vectors like the generators of rotation for S and like the generators of Lorentz transformation for H. The relevant commutation relations are:

$$\begin{gathered} [V_0, V_1] = i\ b_{01},\ [V_1, V_2] = i\ b_{12},\ [V_2, V_0] = i\ b_{20} \\ [J^0, J^1] = i\ J^2,\ [J^1, J^2] = \pm i\ J^0,\ [J^2, J^0] = i\ J^1 \\ [J^1, x^1] = 0,\ [J^1, x^2] = \pm i\ x^0,\ [J^1, x^0] = i\ x^2 \end{gathered} \tag{4.6}$$

The representations of these groups are labeled by (j, ℓ), with ℓ the eigenvalues of J^0 and $\pm j(j+1)$ the eigenvalues of $\vec{J}\cdot\vec{J}$ [V. Bargmann (1947), N. Ja. Vilenkin (1968), Wybourne (1974)]. $J_\pm \equiv J^1 \pm iJ^2$ are ladder operators on ℓ.

The point spectra can be determined by solving an eigenvalue problem associated with the first order differential operator J^0 and two first order ordinary differential equations for *the vacuum states* $J_\pm\, \psi_\pm = 0$.

We choose a gauge (in spherical coordinates)

$$\mathbf{A} = \mathrm{B\ K}(\tau)\ \mathbf{d}\phi, \quad \mathrm{K}'(\tau) = \mathrm{k}(\tau), \quad \mathrm{K}(0) = 0, \tag{4.7}$$

$k(\tau)$ given in 2.2. This gauge field is smooth on H and on S it is smooth away from the south pole. (One can not do better on S if $B \neq 0$). One finds

$$J^0 = -i\partial_\phi \mp B. \tag{4.8}$$

From this it follows that

$$\mathrm{Spect}\ (\mathrm{J}^0) = \Big\{\mathrm{m} \mp \mathrm{B} \mid \mathrm{m} \in\ \mathrm{Z}\Big\}. \tag{4.9}$$

4.1 $SU(2)$:
The unitary representations of $SU(2)$ are such that $m \in \{-j, -j+1, \ldots, j-1, j\}$, with $2j$ a (non-negative) integer. It follows that $2B$ must be integer, which gives an easy way of seeing Dirac quantization. Furthermore,

$$\mathrm{Spect}_\mathrm{S}(\vec{\mathrm{J}}\cdot\vec{\mathrm{J}})\ \subseteq\ \left\{\mathrm{j}(\mathrm{j}+1) \mid\ \mathrm{j} = \left\{\begin{array}{ll} 0, 1, \ldots, \infty, & \text{if } B = 0\ (\mathrm{mod}1) \\ \frac{1}{2}, \frac{3}{2}, \ldots, \infty, & \text{if } B = \frac{1}{2}\ (\mathrm{mod}1) \end{array}\right\}\right\}.$$

The positivity of H_S restricts the spectrum[8] to that given in Eq. (1.2).

4.2 $SU(1,1)$:
Fix $m \in Z$. The vacua equations have the solutions:

$$\psi_{m,\pm}(\tau,\phi) = \Big(\exp im\phi\Big)\Big(\sinh(\tau/2)\Big)^{\pm m}\ \Big(\cosh(\tau/2)\Big)^{\pm(m+2B)}. \tag{4.10}$$

Smoothness at $\tau = 0$ imposes $\pm m > 0$. $\int d\tau\ k(\tau) \mid \psi_{m,\pm}(\tau,\phi)|^2 < \infty$ forces $\pm 2(m+B) < -1$. These inequalities rule out the $\psi_{m,+}$ for all m (recall that $B \geq 0$), and give the eigenfunctions $\psi_{m,-}$ with m a *negative integer* and $M > \frac{1}{2} - B$. From $\vec{J}\cdot\vec{J} = -J^0(J^0-1) + 2J_+J_-$ the vacuum states are square integrable eigenfunctions of $\vec{J}\cdot\vec{J}$ and J^0. On each of these one builds a semi-infinite tower of states by applying J_+. As a consequence:

$$\mathrm{Spect}_\mathrm{H}(\vec{\mathrm{J}}\cdot\vec{\mathrm{J}}) \supseteq \Big\{-(\mathrm{n}-\mathrm{B})(\mathrm{n}-\mathrm{B}+1) \mid \mathrm{n} \in\ \mathrm{Z}_+,\ \mathrm{n} < \mathrm{B} - \frac{1}{2}\Big\}, \tag{4.11}$$

and each point in the set is infinitely degenerate. This gives the isolated points in the spectrum.

We can not get complete handle on the continuous part of the spectrum in these elementary ways. Since anyway it is for the isolated points that we define and compute an index for, we leave this at that.

That the spectrum has no continuous spectrum at low energies, (in the interval $[\frac{1}{4}, \frac{1}{4}+B^2]$), is in agreement with the fact that the orbits at low energies (i.e. $B^2[0,1]$) are all closed. Conversely, that there is continuous spectrum at high energies is in agreement with the fact that at high energies classical orbits go to infinity.

5. THE INDEX

5.1 A circle of Landau Hamiltonians:

Landau Hamiltonians are defined directly on P, S and H, by the quadratic form:

$$\langle\psi|H|\psi\rangle = \langle\mathbf{V}\psi|\cdot|\mathbf{V}\psi\rangle. \tag{5.1}$$

This gives, in index notation, the formal differential operator:

$$H = \frac{1}{\sqrt{g}}(-i\partial_i - A_i)\sqrt{g}g^{ij}(-i\partial_j - A_j), \tag{5.2}$$

with

$$\mathrm{g} = (\mathrm{d}\tau)^2 + \mathrm{k}^2(\tau)\ (\mathrm{d}\phi)^2, \tag{5.3}$$

$k(\tau)$ given in Eq. (2.2). We define a circle of Landau Hamiltonians, $H(\theta)$, by the boundary condition

$$\Psi(\tau,\phi=0) = (\exp 2\pi i\theta)\ \Psi(\tau,\phi=2\pi). \tag{5.4}$$

5.2 A circle of eigenvalue problems:

Associated with the circle of Landau Hamiltonians, $H(\theta)$, $0<\theta<1$, we have a circle of eigenvalue problems that we now proceed to analyze by separation of variables. Take the vector potential as in Eq. (4.7). Let,

$$\Psi(\tau,\phi) = \left(\exp im\phi\right)\ \psi_m(\tau),\ m\in\{-\infty,\ldots,-1+\theta,\theta,\theta+1,\ldots\infty\}. \tag{5.5}$$

We get the family of eigenvalue problems associated with ordinary differential operators on $L^2(k(\tau)d\tau)$ with Dirichlet boundary conditions at $\tau=0$ (and $\tau=\pi$ for S) if $m\neq 0$:

$$H_m = -k^{-1}(\tau)\ \frac{d}{d\tau}\ k(\tau)\frac{d}{d\tau} + \left(\frac{m-BK(\tau)}{k(\tau)}\right)^2. \tag{5.6}$$

Let:

$$x \equiv \begin{cases} \tau^2/2; \\ -\cos(\tau); \\ \cosh(\tau), \end{cases} \quad p(x)\equiv k^2(\tau); \quad q_m(x) \equiv \begin{cases} m-Bx; \\ m-Bx-B; \\ m-Bx+B, \end{cases} \tag{5.7}$$

for P, S and H respectively. We find

$$H_m = -\frac{d}{dx}p(x)\frac{d}{dx} + \frac{q_m^2(x)}{p(x)}, \tag{5.8}$$

as an operator on (an appropriate domain in) $L^2(dx)$. This gives an hypergeometric eigenvalue problem, and it can be solved by a machine which we outline in the next section:

5.3 A Hypergeometric Machine [Nikiforov and Uvarov (1988)]:
One starts with the observation that the ordinary differential equation (in the complex):

$$p(x)y''(x) + \ell(x)y'(x) + \lambda y(x) = 0, \tag{5.9}$$

with $p(x)$ a quadratic polynomial, $\ell(x)$ linear and λ a number, is solved by:

$$\begin{aligned} \lambda_n &= -n\ell' - \frac{n(n-1)}{2}p'' \\ y_n(x) &= \frac{1}{\rho^2(x)}\frac{d^n}{dx^n}(p^n(x)\rho^2(x)), \end{aligned} \tag{5.10}$$

where $\rho(x)$ is a solution of:

$$(p(x)\rho^2(x))' = \ell(x)\rho^2(x), \tag{5.11}$$

and $n \in Z_+$. $y_n(x)$ are polynomials[9]. (5.10) is known as a generalized Rodrigues formula.

The L^2 solutions of the differential equation $H_m\psi = E\psi$ reduces to the study of solutions of 5.9 by the following procedure: Choose μ so that

$$q_m^2(x) - \mu p(x) \equiv P^2(x), \tag{5.12}$$

with $P(x)$ *a linear function* of x. There are, in general, two complex values, μ_1 and μ_2 that will do that. Set $\psi(x) = \rho(x)y(x)$. $y(x)$ solves 5.9 with

$$\lambda = P'(x) + E - \mu, \quad \ell(x) = p'(x) + 2P(x), \quad \rho(x) = \exp \pm \int^x dx' \left(\frac{P(x')}{p(x')}\right). \tag{5.13a}$$

In particular, the (generalised) eigenvalues, E_n, of (5.8) are

$$E_n = -(2n+1)P'(x) - \frac{n(n+1)}{2}p''(x) + \mu. \tag{5.13b}$$

Not all the solutions satisfy the boundary conditions. This has to be checked separately by examining ρ (recall that $y_n(x)$ are polynomials). The solutions that do, solve the eigenvalue problem[10].

5.4 Stable and Moving States

The solutions to the circle of eigenvalue problems are of two kinds. Most of the states are such that their eigenvalues are independent of θ. We call them the stable states; *Ipso facto* they do not contribute to the index. On the other hand, the few eigenstates whose eigenvalues do depend on θ, are those we shall have to follow in order to determine the index. We call these the moving states. The hypergeometric machine separates the stable from the moving states according to whether $\mu = 0$ or not. The stable states all have $\mu = 0$ and we describe them first, even though, as we have said, for our purpose these states are redundant. The interesting, moving states are analyzed in the following subsection.

5.5 The Stable States

Eq. (5.12) has the immediate solution

$$\mu_1 = 0, \quad P_{1\pm}(x) = \pm q_m(x), \tag{5.14}$$

Plugging (5.14) in Eq. (5.13), one finds:

$$E^{1,\pm}_{nm} = \pm B\ (2n+1),\ \rho(x) = \exp \mp (Bx/2)\ x^{\pm m/2}, \text{ for } P; \tag{5.15a}$$

$$E^{1,\pm}_{nm} = n(n+1) \pm B\ (2n+1),\ \rho(x) = (1+x)^{\pm m/2}\ (1-x)^{\pm(B-m/2)}, \text{ for } S; \tag{5.15b}$$

$$E^{1,\pm}_{nm} = -\ n(n+1) \pm B\ (2n+1),\ \rho(x) = (x-1)^{\pm m/2}\ (1+x)^{\mp(B+m/2)}, \text{ for } H. \tag{5.15c}$$

The $\pm$ signs correspond to those in 5.14.

5.5.1 The Plane

From 5.15a and the behavior of ρ at $x = 0$ and $x = \infty$, the eigenvalues are:

$$E^{1,+}_{nm} = B\ (2n+1),\ n \in Z_+,\ m \in \{\theta, \theta+1, \ldots, \infty\}. \tag{5.16a}$$

There are *infinitely many* such eigenstates whose eigenvalue is independent of θ. They do not go anywhere as θ varies and are irrelevant for the index. The stability of infinitely many states is a consequence of the stability of the essential spectrum [Weyl (1909), Kato (1967)].

5.5.2 The Sphere

For the regularity of ρ at $x = \pm 1$ we need $\pm m \geq 0,\ \pm(2B - m) \geq 0$. The lower sign in 5.15b is rejected for all m (recall $B \geq 0$) and the upper sign is constrained by $0 \leq m \leq 2B$. Hence the eigenvalues are:

$$\begin{gathered} E^{1,+}_{nm} = n(n+1) + B\ (2n+1), \\ n \in Z_+,\ m \in \{\theta, 1+\theta, \ldots, \}, m \leq 2B. \end{gathered} \tag{5.16b}$$

With each spectral point ($\theta \neq 0$) there are $[2B]$ eigenstates whose energy is θ independent and are irrelevant for the index.

5.5.3 The Hyperbolic Plane

The regularity of ρ at $x = 1$ and square integrability at infinity require $\pm m \geq 0$ and $0 \leq n < -\frac{1}{2} \pm B$ respectively. The lower sign is always rejected (since $B \geq 0$), and the upper sign survives for $n < B - \frac{1}{2}$ and m non-negative. The eigenvalues are:

$$E^{1+}_{nm} = -\,n(n+1) + B\,(2n+1),$$
$$m \in \{\theta, 1+\theta, \ldots, \infty\},\; n \in Z_+,\; n < B - \frac{1}{2}. \tag{5.16c}$$

At each spectral point, there are *infinitely many* eigenstates whose eigenvalues are independent of θ that do not go anywhere and are irrelevant for the index. The stability of these states is a consequence of the stability of the essential spectrum, as was the case for the plane.

5.6 The Moving States

The states that are relevant for the index, are those states whose energies depend on θ. They arise from the solutions of Eq. (5.12) with non-zero μ's, which we denote μ_2. There is a finite number of such states for each spectral point of the ($\theta = 0$) Landau Hamiltonian.

5.6.1 The Plane:

From 5.12 and 5.13:

$$\mu_2 = -2mB;\; P_2(x) = \pm(Bx + m);\; \rho_{2\pm}(x) = \exp \pm(Bx/2)\; x^{\pm m/2}. \tag{5.17a}$$

The upper sign is rejected because of bad behavior at infinity, and the lower sign gives admissible eigenfunctions for $m \leq 0$. Since $P'(x) = -B$, $\ell'(x) = -2B$, Eq. (5.10) and (5.13) give

$$E^{2-}_{nm} = B(1 + 2n - 2m),\quad n \in Z_+,\quad m \in \{-\infty, \ldots, -1+\theta\}. \tag{5.18a}$$

The lowest spectral point $E = B$ has no moving states, the next one, $E = 3B$, has one moving state *etc.* The energy of each moving state is a decreasing linear function of θ with slope $2B$. As a consequence, as θ increases from zero to 1, n states from the the (n+1)-st spectral point move down to the spectral point just below. For each point in the Landau spectrum, there is one more state arriving (from above) than departing (below) and the index is unity. A flow diagram is shown in Fig. 5.

5.6.2 The Sphere:
Set $j \equiv m - B$, m given in 5.5. From 5.12 and 5.13 we get:

$$\mu_2 = j^2 - B^2;\ P_2(x) = \pm(jx - B);\ \rho_{2\pm}(x) = (1-x)^{\pm\frac{B-j}{2}}\ (1+x)^{\mp\frac{B+j}{2}}. \qquad (5.17b)$$

Regularity of the solutions at $x = \pm 1$ gives: $\pm(B-j) \geq 0,\ \mp(B+j) \geq 0$. The upper sign holds if $j \leq -B$ and the lower sign if $j \geq B$

$$E^{\pm 2}_{n,m} = (n + |j|)(n + 1 + |j|) - B^2, for\ |j| \geq |B|. \qquad (5.18b)$$

Figure 6 show the flow of the states with θ. The index is zero, as it should be.

5.6.3 The Hyperbolic Plane:
From 5.12 and 5.13 we get:

$$\begin{aligned}\mu_2 = -(m+B)^2 + B^2;\ P_2(x) = \pm\big((m+B)x - B\big);\\ \rho_{2\pm}(x) = (x-1)^{\pm\frac{m}{2}}\ (1+x)^{\pm(B+\frac{m}{2})}\end{aligned}. \qquad (5.17c)$$

Regularity of the solutions at $x = 1$ requires $\pm m \geq 0$, and square integrability requires $n + \frac{1}{2} < \mp(B+m)$. The two inequalities rule out the upper sign for non-negative B. The lower sign holds provided $n - B + \frac{1}{2} < m \leq 0$. We therefore find for the moving states, from 5.13:

$$\begin{aligned}&E^{2,-}_{n,m} = -(n - m - B)\ (n-m-B+1) + B^2,\\ &m \in \{\ldots, -2+\theta - 1 + \theta\};\ n \in Z_+, n < m + B - \frac{1}{2}.\end{aligned} \qquad (5.18c)$$

The energy is constant on the lines $n - m = const$. On such a line there are $[n-m]$ (degenerate) moving states, all moving together with θ. The lowest spectral point has no moving state, and the next spectral point has one moving states, etc. until the highest spectral point just below the continuum has $[B - \frac{1}{2}]$ moving states. As θ increases, the states all move down to the spectral point just below. For each spectral point there is always one more state arriving from above than departing below. The index is therefore 1. (The spectral point just below the continuum too has index 1, and the states that arrive all have their birth at the bottom of the continuum). A flow diagram is shown in Fig. 7.

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FOOTNOTES

1. That is: $y^{-2}\big((dx)^2+(dy)^2\big)$.

2. Schrödinger operators with magnetic fields in R^n have been extensively studied and a good part of the mathematical results is reviewed in [Cycon, Froese, Kirsch and Simon, (1986)]. Schrödinger operators with magnetic fields on general Riemanian manifolds have not attracted a lot of attention, for obvious reasons.

3. Landau computed the density of states associated with the infinite degeneracy. A bare hand computation of this density of states is given in [Avron and Seiler (1983)].

4. For the sphere we take a cut from the north to the south pole. This is an *ad-hoc* definition, because there appears to be no natural notion of this index for compact manifolds.

5. In order to follow the eigenvalues one needs to assume some mild continuity properties.

6. *Index* $=+\infty$ means that there are infinitely many arriving and finitely many departing states (similarly for Index $=-\infty$). Infinite indices however, do not occur for the specific operators we study here.

7. The Magnetic translation group is due to [Zak (1964)] who applied it in a solid state physics context. In the context of separation of center of mass in atomic physics see [Avron, Herbst and Simon, (1978)] and [Johnson, Hirschfelder and Yang (1983)].

8. In order to show that one gets *all* the spectrum this way one needs to solve the vacuum equations in $L^2(S)$, for all j's in Eq. (1.2). We shall not do that here.

9. $y_0(x)=1,\ y_1(x)=\ell(x)$, etc.

10. It is not clear from this outline that one gets a complete spectral resolution in this way. For a discussion of this point see [Nikiforov and Uvarov (1988)].

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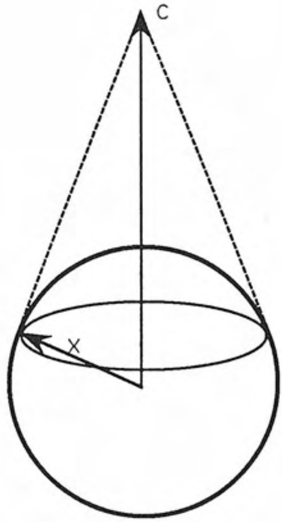
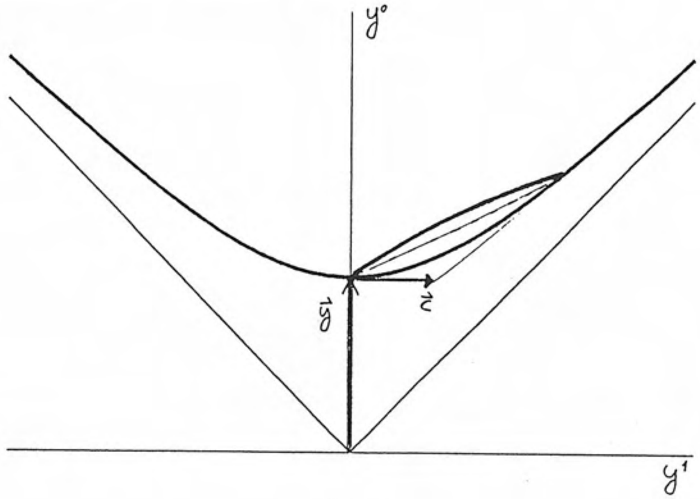


fig. 1



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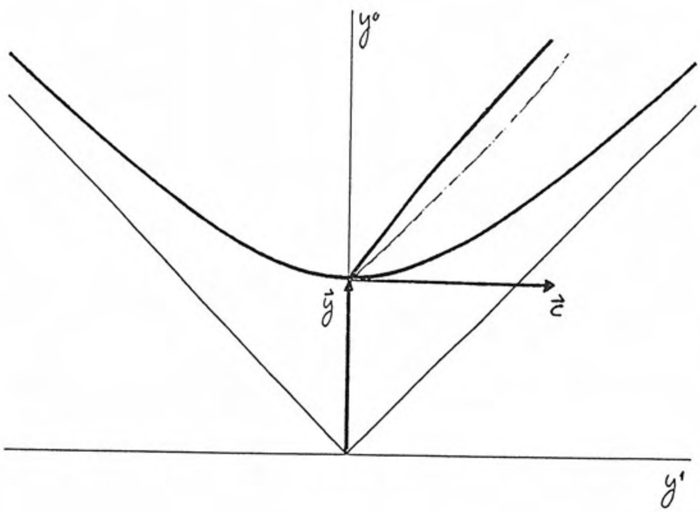

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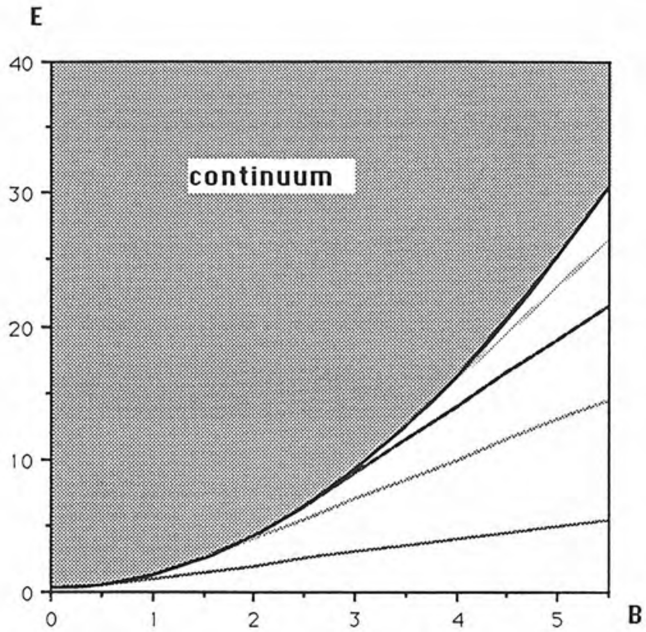

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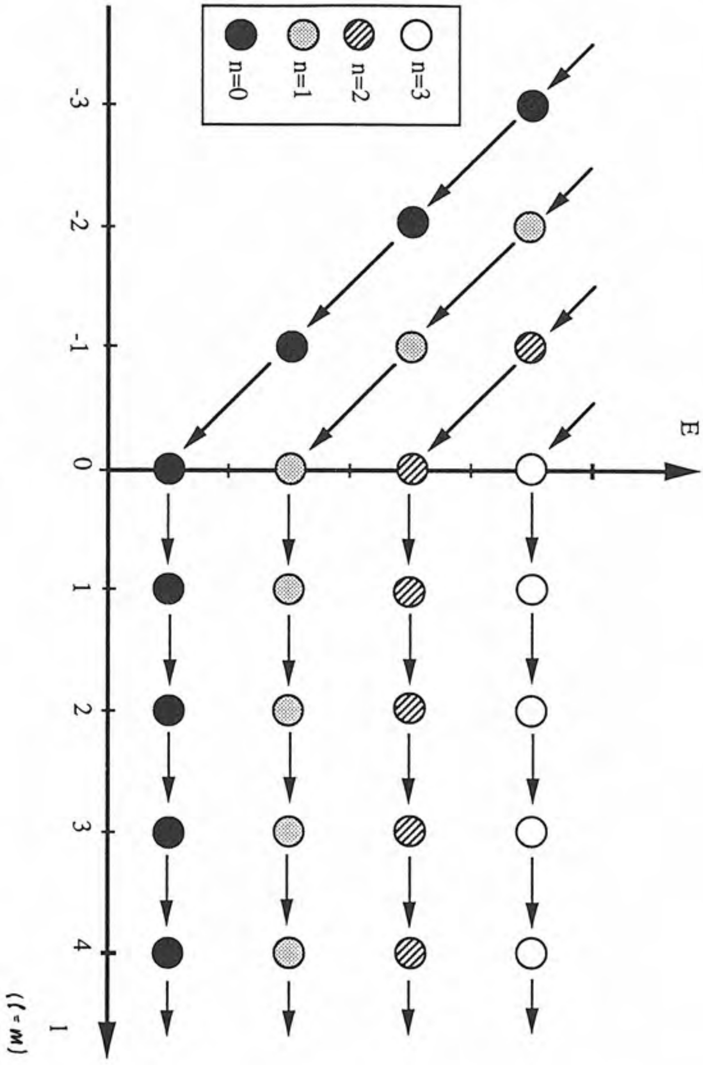

fig. 5

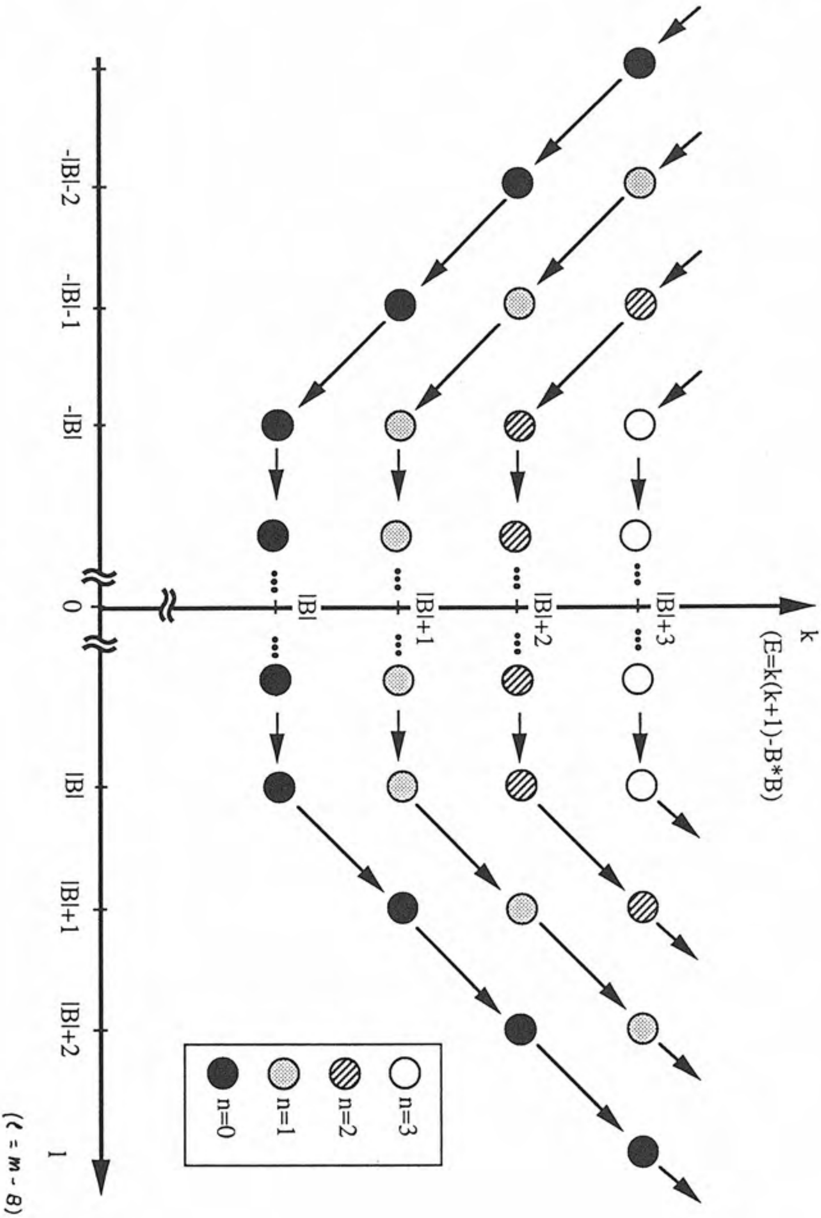

fig. 6

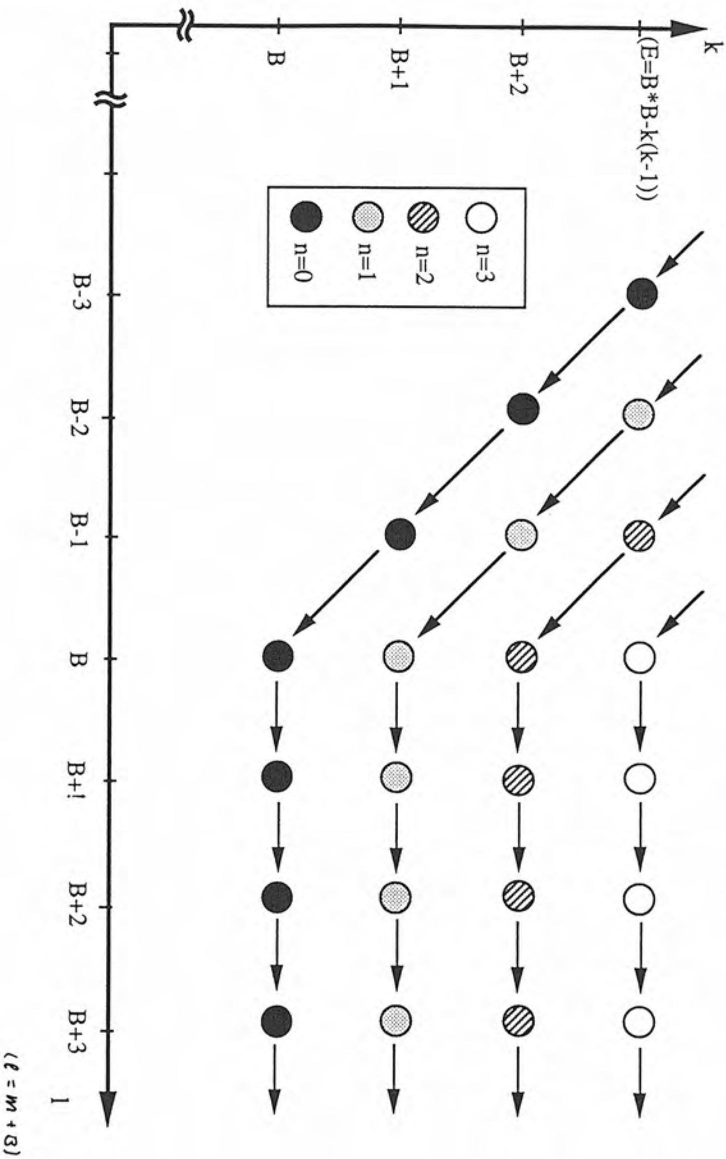

fig. 7