Quantum games

The Mermin-Peres magic square

J Avron

July 20, 2023

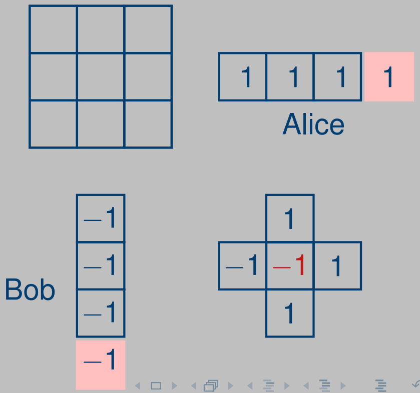
Mermin-Peres game

The House

against

Two cooperating, non-communicating parties

- Alice assigned a random row j
- Fills row $a_{jk} = \pm 1, \prod_k (a_{jk}) = 1$
- Bob assigned a random column k
- Fills column $b_{jk} = \pm 1, \prod_j (b_{jk}) = -1$
- Win if: $a_{jk} = b_{jk}$



Classical strategy: Agree on a common table

- Fill 2×2 green square arbitrariy
- Completes the first 2 rows/columns by constraint
- Disagree on the remaining (red) square
- Win with probability: $8/9$

1	1	1	1
1	1	1	1
-1	-1	± 1	1
-1	-1	-1	

Alice & Bob can't always win

Column and Row constraints can't be satisfied simultaneously

$$1 = (1)^3 = \prod_{jk} a_{jk} \neq \prod_{jk} b_{jk} = (-1)^3 = -1$$

			1
			1
			1
-1	-1	-1	

Randomness does not help

$$P(Win) = \sum_{strategies} P_{strategy} \underbrace{P(Win|strategy)}_{\leq 8/9}$$

The quantum game

Entanglement as resource

- Agree on common of observables
- Binary: $\mathbf{B}_{jk}^2 = \mathbb{1}, \quad Tr\,\mathbf{B}_{jk} = 0$
- Commute in rows/columns
  $[\mathbf{B}_{jk}, \mathbf{B}_{jm}] = 0, \quad [\mathbf{B}_{jk}, \mathbf{B}_{mk}] = 0$
- Satisfy constraints
  $\prod_j \mathbf{B}_{jk} = -\mathbb{1}, \quad \prod_k \mathbf{B}_{j,k} = \mathbb{1}$
- A & B share entangled pairs

$\mathbf{B}_{11}$	$\mathbf{B}_{12}$	$\mathbf{B}_{13}$	$\mathbb{1}$
$\mathbf{B}_{21}$	$\mathbf{B}_{22}$	$\mathbf{B}_{23}$	$\mathbb{1}$
$\mathbf{B}_{31}$	$\mathbf{B}_{32}$	$\mathbf{B}_{33}$	$\mathbb{1}$
$-\mathbb{1}$	$-\mathbb{1}$	$-\mathbb{1}$	

Such tables exist

A & B respond by measuring the binary $\mathbf{B}_{\mathbf{jk}}$

Constraints guaranteed

Filling the table

Binaries as products of Pauli

- $\sigma_\mu = \{\sigma_0 = \mathbb{1}, \sigma_x, \ldots, \sigma_z\}$
- $\sigma_\mu^2 = \mathbb{1},\ \{\sigma_j, \sigma_k\} = 0,\ j \neq k$
- Fill green square:
- Commuting in rows & columns
- Complete by constraints
- Consistent at red square

$\sigma_0 \otimes \sigma_z$	$\sigma_x \otimes \sigma_0$	$\sigma_x \otimes \sigma_z$
$\sigma_z \otimes \sigma_0$	$\sigma_0 \otimes \sigma_x$	$\sigma_z \otimes \sigma_x$
$-\sigma_z \otimes \sigma_z$	$-\sigma_x \otimes \sigma_x$	$-\sigma_y \otimes \sigma_y$

Respond by measuring two qubits

Parity constraint is automatic

Measuring commuting observables

- First row:
  A measures σ_x of 1-st qubit
  A measure σ_z of 2-nd qubit
- First columns:
  B measure σ_z of 1-st qubit
  B measure σ_z of 2-nd qubit

$\sigma_0 \otimes \sigma_z$	$\sigma_x \otimes \sigma_0$	$\sigma_x \otimes \sigma_z$
$\sigma_z \otimes \sigma_0$	$\sigma_0 \otimes \sigma_x$	$\sigma_z \otimes \sigma_x$
$-\sigma_z \otimes \sigma_z$	$-\sigma_x \otimes \sigma_x$	$-\sigma_y \otimes \sigma_y$

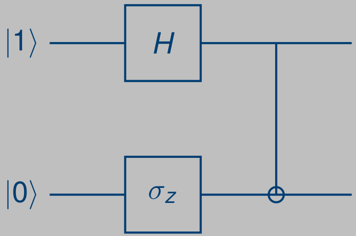
Third row (column) looks difficult

How to measure $\sigma_x \otimes \sigma_z$ with $\sigma_z \otimes \sigma_x$?

Quantum circuits

Quantum gates

- $\sigma_z |0\rangle = |0\rangle \, , \; \sigma_z |1\rangle = - |1\rangle$
- $H\sigma_z = \sigma_x H$
- $H = \frac{\sigma_z + \sigma_x}{\sqrt{2}}$
- CNOT$= |0\rangle\langle 0| \otimes \sigma_0 + |1\rangle\langle 1| \otimes \sigma_x$

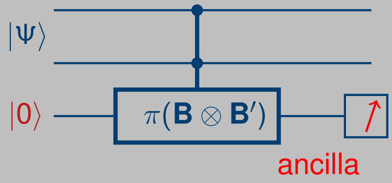


Recording parity product on ancilla

B binary observable

- Binary observables

$$\mathbf{B} = \mathbf{B}_+ - \mathbf{B}_-, \ \underbrace{\mathbf{B}_\pm^2 = \mathbf{B}_\pm}_{\textit{projection}}$$



- Unitary recording joint parity on ancilla

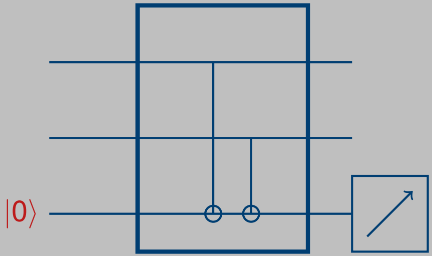
$$\textit{Unitary} = \underbrace{(\mathbf{B}_+ \otimes \mathbf{B}'_+ + \mathbf{B}_- \otimes \mathbf{B}'_-)}_{\textit{positive}} \otimes \mathbb{1} + \underbrace{(\mathbf{B}_+ \otimes \mathbf{B}'_- + \mathbf{B}_- \otimes \mathbf{B}'_+)}_{\textit{negative}} \otimes \sigma_x$$

$$|\textit{ancilla}\rangle = |0\rangle \, \delta_{\textit{positive}} + |1\rangle \, \delta_{\textit{negative}}$$

No information on individual parities

Joint parity boxes

$z \otimes z$-parity



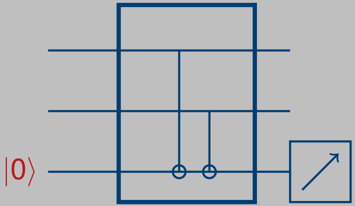
$\pi(\sigma_z \otimes \sigma_z)$

$$(\alpha |0\rangle \otimes |1\rangle + \beta |1\rangle \otimes |0\rangle) \otimes |0\rangle \mapsto (\alpha |0\rangle \otimes |1\rangle + \beta |0\rangle \otimes |1\rangle) \otimes |1\rangle$$

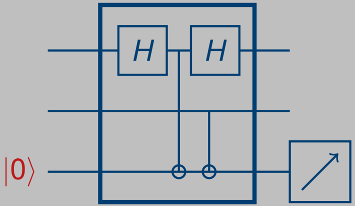
$$(\alpha |0\rangle \otimes |0\rangle + \beta |1\rangle \otimes |1\rangle) \otimes |0\rangle \mapsto (\alpha |0\rangle \otimes |0\rangle + \beta |1\rangle \otimes |1\rangle) \otimes |0\rangle$$

Joint parity boxes

$z \otimes x$-parity



$\pi(\sigma_z \otimes \sigma_z)$



$\pi(\sigma_x \otimes \sigma_z)$

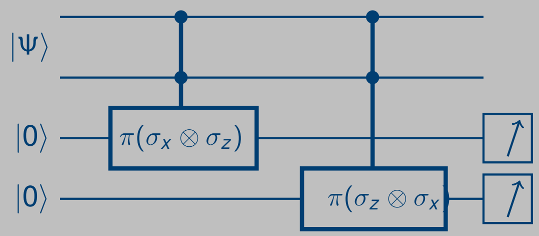
H intertwines σ_x and σ_z

$$H\sigma_x = \sigma_z H$$

Measuring ancilla projects on joint parity subspace

Measuring $\sigma_x \otimes \sigma_z$ and $\sigma_z \otimes \sigma_x$

Two ancillas and two parity boxes



$\sigma_x \otimes \sigma_z$	$\sigma_z \otimes \sigma_x$	$\sigma_y \otimes \sigma_y$

Measure Calculate

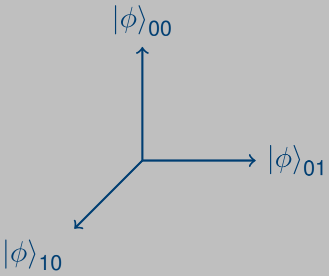
What about the back action on $|\Psi\rangle$?

$$\pi(\sigma_x \otimes \sigma_z)\pi(\sigma_z \otimes \sigma_x) \overset{?}{=} \pi(\sigma_y \otimes \sigma_y)$$

Commuting measurement projects on a common basis

Constraint guaranteed

- $\sigma_x \otimes \sigma_z \left|\phi\right\rangle_{ab} = (-)^a \left|\phi\right\rangle_{ab}$
- $\sigma_z \otimes \sigma_x \left|\phi\right\rangle_{ab} = (-)^b \left|\phi\right\rangle_{ab}$
- $\left|\phi\right\rangle_{ab}$ joint parity basis.



Parity measurement selects parity eigenstate

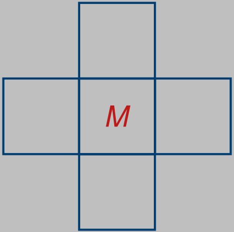
$$\rho_A \mapsto \left|\phi_{ab}\right\rangle\left\langle\phi_{ab}\right|$$

$$\sigma_y \otimes \sigma_y \left|\phi\right\rangle_{ab} = (-)^{a+b} \left|\phi\right\rangle_{ab}$$
$$\pi(\sigma_x \otimes \sigma_z)\pi(\sigma_z \otimes \sigma_x) = \pi(\sigma_y \otimes \sigma_y)$$

Entanglement: Correlations without communication

Guarantee agreement on the intersect

- On intersect both measure M
- e.g. $M = \sigma_z \otimes \sigma_x$
- $M \otimes M \, |\psi_{AB}\rangle = \underbrace{+}_{agree} |\psi_{AB}\rangle$
- Green square–Independent Eq.
- 4 stabilizers determine $|\psi_{AB}\rangle \in \mathbb{C}^{2^4}$



The entangles shared state

$|\psi_{AB}\rangle = |\Phi_{AB}\rangle \otimes |\Phi_{AB}\rangle$

- $M_A \otimes M_B \, |\psi_{AB}\rangle = |\psi_{AB}\rangle \,, \quad M = \sigma_\mu \otimes \sigma_\nu$
- $(\sigma^A_\mu \otimes \sigma^B_\mu) \otimes (\sigma^A_\nu \otimes \sigma^B_\nu) \, |\psi_{AB}\rangle = |\psi_{AB}\rangle \,, \quad \sigma_\mu \in \{\sigma_0 = \mathbb{1}, \sigma_x, \sigma_z\}$
- Solution has product structure

$$|\psi_{AB}\rangle = |\Phi_{AB}\rangle \otimes |\Phi_{AB}\rangle$$

$$\sigma^A_\mu \otimes \sigma^B_\mu \, |\Phi_{AB}\rangle = |\Phi_{AB}\rangle \quad \sigma_\mu \in \{\mathbb{1}, \sigma_x, \sigma_z\}$$

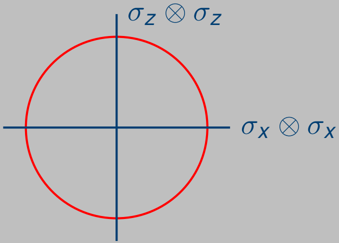
2 stabilizers determine $|\Phi_{AB}\rangle$

Bell state

Syndrome

- $|\Phi_{AB}\rangle \in \mathbb{C}^{2^2}$
- Stabilizers

$$|\Phi_{AB}\rangle = \underbrace{\sigma_x \otimes \sigma_x}_{P_{AB}} |\Phi_{AB}\rangle = \underbrace{\sigma_z \otimes \sigma_z}_{P'_{AB}} |\Phi_{AB}\rangle$$



- $\exists!$ Bell state

$$|\Phi_{AB}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \in \mathbb{C}^4$$

Alice and Bob agree

$$(M_A \otimes M_B) |\Phi_{AB}\rangle = |\Phi_{AB}\rangle\,, \quad M_A = M_B = \sigma_\mu \otimes \sigma_\nu$$

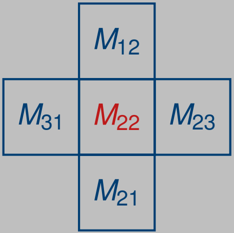
Alice state is Fully mixed

- Alice shares with Bob Bell states
- Partial Trace of Bell states is fully mixed
- Alice measures a fully Mixed
- Good enough to ensure Allice's parity constraint
- Alice's measurment prepare a random vector $|e_{jk\pm}\rangle$

$$M_{jk}\,|e_{jk\pm}\rangle = \pm\,|e_{jk\pm}\rangle\,, \quad j = \text{fixed}, k = 1, 2$$

Contextuality

- Every column gives consistent values
- Every row gives different consistent values
- Values depend on context
- Hidden variables are contextual



Concluding remarks

- Mermin and Peres invented the square to give a simple proof of Kochen Specker theorem
- Multiprover, Interactive proof systems (MPI^*)
- Cleve, Hoyer, Toner and Watrous (2010)