Quantum games

GHZ and the Mermin-Peres magic square

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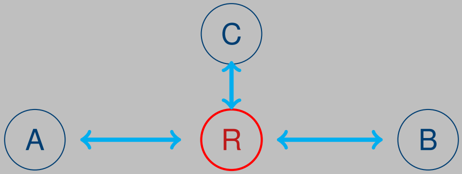
Quantum games

Face lifted Bell inequalities

Non-communicating players

A referee picks random questions from a known set



Team wins if answers satisfy a global constraint

Classical players sometimes lose

Quantum players never lose

GHZ game

Random questions X or Y, response ± 1

	A	B	C	Win
Q_1	X	X	X	1
Q_2	Y	Y	X	-1
Q_3	X	Y	Y	-1
Q_4	Y	X	Y	-1

No (classical) winning strategy

No satisfying table

	A	B	C	Win
Q_1	X	X	X	1
Q_2	Y	Y	X	-1
Q_3	X	Y	Y	-1
Q_4	Y	X	Y	-1
	1	1	1	

Team must fail on (at least) one question

Winning probability 3/4 (at most)

The quantum game

Binary observables: Pauli Matrices

- Binary observables: Measurement outcomes: ± 1
- Example: The 3 Pauli matrices

$$X^2 = Y^2 = Z^2 = \mathbb{1}, \quad \text{Tr}\, X = \text{Tr}\, Y = \text{Tr}\, Z = 0$$

- A, B & C prepare an (entangled) state
- A, B & C report the measurement result

Consistent table

Pauli

- Replace the entries in the table by Pauli
- Anti-commuting

$$XY = -YX, \quad X^2 = Y^2 = \mathbb{1}$$

	A	B	C	Win
Q_1	X	X	X	1
Q_2	Y	Y	X	-1
Q_3	X	Y	Y	-1
Q_4	Y	X	Y	-1
	$-\mathbb{1}$	$\mathbb{1}$	$\mathbb{1}$	

Choosing the entangles sate

$|GHZ\rangle$

	A	B	C	
Q_1	X	X	X	1
Q_2	Y	Y	X	-1
Q_3	X	Y	Y	-1
Q_4	Y	X	Y	-1

- Commuting questions

$$[Q_j, Q_k] = 0, \quad Q_1 = X \otimes X \otimes X, \dots, Q_4 = Y \otimes X \otimes Y$$

- Eigenvector that guarantees winning:

$$Q_1 |GHZ\rangle = |GHZ\rangle, \quad Q_{2,3,4} |GHZ\rangle = -|GHZ\rangle$$

$$|GHZ\rangle = |\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle$$

Mermin-Peres game

Magic square

A

1	1	1	1

B

-1		
-1		
-1		
-1		

Easy

Making the game hard

Consistency constraint

- A & B must agree on the intersect

	1		
1	1	1	1
	-1		
	-1		

Winning strategy: Agree on a satisfying table

No satisfying table

Frustrating constraints

			1
			1
			1
-1	-1	-1	

Can't satisfy table with ± 1

The best one can do

Satisfy 8 of 9

1	1	1	1
1	1	1	1
-1	-1	± 1	1
-1	-1	-1	

Randomness does not help

$$P(Win) = \sum_{strategies} P_{strategy} P(Win|strategy) \leq 8/9$$

A & B agree on a satisfying table with Pauli

- Fill green square:

  commuting in rows & columns

  anti-commuting diagonals

- Complete by constraints

- Consistent X:

$$\underbrace{AB\ CD}_{Bob} = -\underbrace{AC\ BD}_{Alice}$$

- $[AB, CD] = 0 \implies AD = -DA$

A	B	AB
C	D	CD
$-AC$	$-BD$	X

Example

Pauli: $\sigma_\mu \otimes \sigma_\nu$

$\sigma_0 \otimes \sigma_z$	$\sigma_x \otimes \sigma_0$	$\sigma_x \otimes \sigma_z$
$\sigma_z \otimes \sigma_0$	$\sigma_0 \otimes \sigma_x$	$\sigma_z \otimes \sigma_x$
$-\sigma_z \otimes \sigma_z$	$-\sigma_x \otimes \sigma_x$	$-\sigma_y \otimes \sigma_y$

Table satisfied for any (2-qubits) state

Alice & Bob agree on a satisfying table

3 commuting binary measurements

$\mathbf{B}_{11}$	$\mathbf{B}_{12}$	$\mathbf{B}_{13}$	$\mathbb{1}$
$\mathbf{B}_{21}$	$\mathbf{B}_{22}$	$\mathbf{B}_{23}$	$\mathbb{1}$
$\mathbf{B}_{31}$	$\mathbf{B}_{32}$	$\mathbf{B}_{33}$	$\mathbb{1}$
$-\mathbb{1}$	$-\mathbb{1}$	$-\mathbb{1}$	

$\mathbf{B}_{jk}$ in a row/column commute

Otherwise anti-commute

Pseudo-telepathy

Alice & Bob agree on the common square

	-1	
-1	1	-1
	1	

Pseudo-telepathy needs $|\psi\rangle_{AB}$ such that

$$\underbrace{\mathbf{B}_{jk}}_{Alice} \otimes \underbrace{\mathbf{B}_{jk}}_{Bob} |\psi\rangle_{AB} = |\psi\rangle_{AB}\,, \quad j,k \in \{1,2,3\}$$

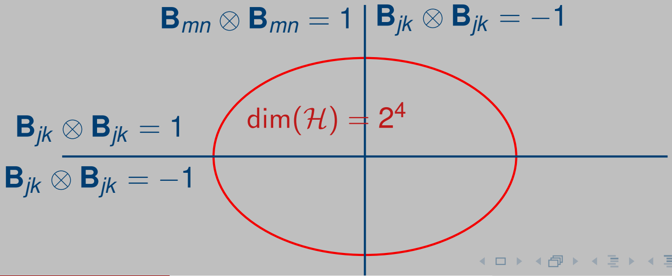
- $\mathbf{B}_{jk}$ either commute or anti-commute
- $\mathbf{B}_{jk} \otimes \mathbf{B}_{jk}$ commute

How to choose $|\psi\rangle_{AB}$

$|\psi\rangle_{AB} \in \mathbb{C}^{2^4}$ satisfies 9 binary constraints

- Green square–Independent.
- $\mathbf{B}_{jk} \otimes \mathbf{B}_{jk} |\psi\rangle_{AB} = |\psi\rangle_{AB}, \quad j,k = 1,2$
- Determine $|\psi\rangle_{AB} \in \mathbb{C}^{2^4}$

$\mathbf{B}_{11}$	$\mathbf{B}_{12}$	
$\mathbf{B}_{21}$	$\mathbf{B}_{22}$	



$|\psi\rangle_{AB}$

Two Bell pairs

- $\mathbf{B} \otimes \mathbf{B} \, |\psi_{AB}\rangle = |\psi_{AB}\rangle \,, \quad \mathbf{B} = \sigma_\mu \otimes \sigma_\nu$
- $(\sigma_\mu^A \otimes \sigma_\mu^B) \otimes (\sigma_\nu^A \otimes \sigma_\nu^B) \, |\psi\rangle_{AB} = |\psi\rangle_{AB} \,, \quad \sigma_\mu \in \{\sigma_0 = \mathbb{1}, \sigma_x, \sigma_z\}$
- Write

$$|\psi\rangle_{AB} = |\Phi\rangle_{AB} \otimes |\Phi\rangle_{AB}$$

$$\sigma_\mu^A \otimes \sigma_\mu^B \, |\Phi\rangle_{AB} = |\Phi\rangle_{AB} \quad \sigma_\mu \in \{\mathbb{1}, \sigma_x, \sigma_z\}$$

Determine $|\Phi\rangle_{AB} = |Bell\rangle$

$$|Bell\rangle = |0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B$$

Measuring parity

The easy cases

- First row: Alice measures

$$\sigma_x \otimes \sigma_z$$

- Suppose Alice finds $(1, -1)$

$$1 \cdot (-1) = -1$$
$$1 \cdot 1 = +1$$
$$1 \cdot (-1) = -1$$

$\sigma_0 \otimes \sigma_z$ -1	$\sigma_x \otimes \sigma_0$ $+1$	$\sigma_x \otimes \sigma_z$ -1
$\sigma_z \otimes \sigma_0$	$\sigma_0 \otimes \sigma_x$	$\sigma_z \otimes \sigma_x$
$-\sigma_z \otimes \sigma_z$	$-\sigma_x \otimes \sigma_x$	$-\sigma_y \otimes \sigma_y$

Measuring parity

The hard case: In praise of modesty

$\sigma_0 \otimes \sigma_z$ -1	$\sigma_x \otimes \sigma_0$ $+1$	$\sigma_x \otimes \sigma_z$ -1
$\sigma_z \otimes \sigma_0$	$\sigma_0 \otimes \sigma_x$	$\sigma_z \otimes \sigma_x$
$-\sigma_z \otimes \sigma_z$	$-\sigma_x \otimes \sigma_x$	$-\sigma_y \otimes \sigma_y$

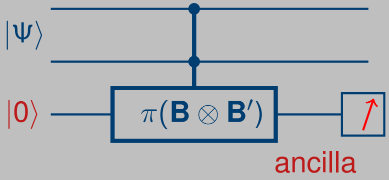
$$\pi(\sigma_j \otimes \sigma_k) = 1 \iff \pi(\sigma_j) = \pi(\sigma_k) = \pm 1$$

Measuring parity

Ancilla

- Binary observables

$$\mathbf{B} = \mathbf{B}_+ - \mathbf{B}_-, \ \underbrace{\mathbf{B}_\pm^2 = \mathbf{B}_\pm}_{\textit{projection}}$$



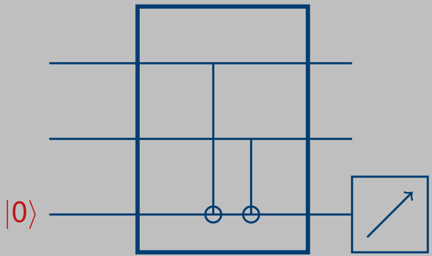
- Unitary map

$$\underbrace{(\mathbf{B}_+ \otimes \mathbf{B}'_+ + \mathbf{B}_- \otimes \mathbf{B}'_-)}_{\textit{positive}} \otimes \mathbb{1} + \underbrace{(\mathbf{B}_+ \otimes \mathbf{B}'_- + \mathbf{B}_- \otimes \mathbf{B}'_+)}_{\textit{negative}} \otimes \sigma_x$$

Measuring the ancilla
projects $|\Psi\rangle$ on a (joint) parity eigenstate

Example I

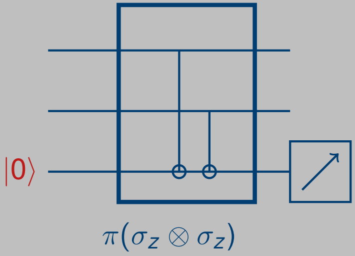
$z \otimes z$ parity boxes

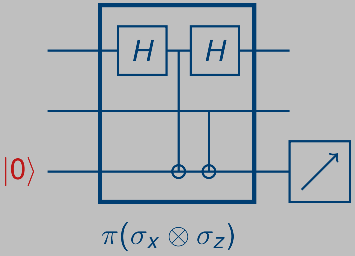


$\pi(\sigma_z \otimes \sigma_z)$

Example II

$\sigma_x \otimes \sigma_z$ parity box





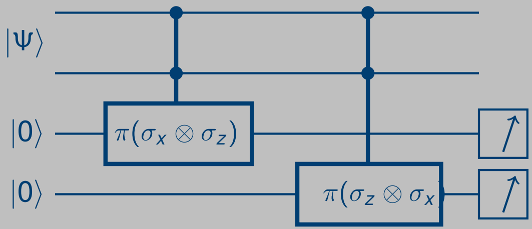
H intertwines σ_x and σ_z

$$H\sigma_x = \sigma_z H$$

Measuring ancilla projects on joint parity subspace

Measuring $\sigma_x \otimes \sigma_z$ and $\sigma_z \otimes \sigma_x$

Two ancillas and two parity boxes



$\sigma_x \otimes \sigma_z$	$\sigma_z \otimes \sigma_x$	$\sigma_y \otimes \sigma_y$

Measure Calculate

Swapping entanglement

- $\sigma_x \otimes \sigma_z |\phi_{ab}\rangle_A = (-)^a |\phi_{ab}\rangle_A$
- $\sigma_z \otimes \sigma_x |\phi_{ab}\rangle_A = (-)^b |\phi_{ab}\rangle_A$

Example: $a = b = 0$

$$|\phi_{00}\rangle = |+\rangle \otimes |0\rangle + |-\rangle \otimes |1\rangle$$

Concluding remarks

- Hidden Variables
- Locality vs contextuality
- Kochen-Specker
- Cirelson problem
- Reviews:
  - CS: Brassard, Broadbent and Tapp
  - RMP: Buhrman, Cleve, Masar and de Wolf
- Thomas Vidick
- Applications to quantum communication: Wigderson et. al.