Supersymmetric gauge theories in six and seven dimensions

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Challenges to QFT in Higher Dimensions

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Conformal field theories in 5 and 6 dimensions cannot be formulated directly in terms of a Lagrangian, and must be studied indirectly by other means. In both cases the theories are defined as UV limits of five-dimensional (non-conformal) gauge theories. Non-perturbative tools

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∗ I will not be considering 5 dimensional gauge theories

∗ Therefore no formulations of CFTs in 5 and 6 dimensions

∗ Instead I will (mainly) discuss 6 and 7 dimensional maximally supersymmetric gauge theories that have nonconformal UV completions

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- Short answer: There exists a corresponding supergroup
- Examples:
  $\mathcal{N}=2$ in 4d: $OSp(2|4)$
  $\mathcal{N}=1$ in 5d: $SU(4|1)$
  $\mathcal{N}=2$ in 5d: $SU(4|1,1)$ (noncompact R-symmetry group)
- Not coincidentally this is related to the existence of a superconformal theory in one less dimension
  $\mathcal{N}=2$ in 3d: $OSp(2|4,\mathbf{R})$
  $\mathcal{N}=1$ in 4d: $SU(2,2|1)$
  $\mathcal{N}=2$ in 4d: $SU(2,2|2)$

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  $\mathcal{N} = 1$ in 4d: $SU(2,2|1)$
  $\mathcal{N} = 2$ in 4d: $SU(2,2|2)$
- Why not spheres in higher dimensions?
  - Superconformal $\mathcal{N} = 1$ in 5d: $F(4) \supset SO(2,5) \times SU(2)$
    $\Longrightarrow$ SUSY $\mathcal{N} = 1$ on S^6: $F(4) \supset SO(7) \times SU(1,1)$
  - Superconformal $\mathcal{N} = 1$ in 6d :$OSp(2,6|2)$
    $\Longrightarrow$ SUSY $\mathcal{N} = 1$ on S^7: $OSp(8|1,1)$

Introduction (continued)

We can study SYM on S^6 and S^7 using localization –

- Used for 4 and 5 dim SYM – We will give a more unified approach in order to generalize to other dimensions (different approach than Festuccia & Seiberg)
- Use Pestun's method by dimensionally reducing 10d SYM to the desired dimension and modify the Lagrangian accordingly.
- The theories naturally have 16 supersymmetries, but these can be straightforwardly reduced for $d \leq 5$.
- In doing this we will reduce to matrix models for 7d and 6d
  They are surprisingly similar to the matrix models for 5d and 4d with only a vector multiplet.
- The 7d and 6d models also have interesting instanton structures.

Outline

SUSY gauge theory in flat space

- 10-dimensional flat-space Lagrangian: Brink, Scherk & Schwarz

$$\mathcal{L} = \frac{1}{g_{10}^2} \mathrm{Tr} \left(\tfrac{1}{2} F_{MN} F^{MN} - \Psi \not{D} \Psi \right) .$$

- Action is invariant under the supersymmetry transformations

$$\begin{aligned} \delta_\epsilon A_M &= \epsilon^\alpha \Gamma_{M\alpha\beta} \Psi^\beta , \qquad M = 0, \ldots 9 \\ \delta_\epsilon \Psi^\alpha &= \tfrac{1}{2} \Gamma^{MN\alpha}{}_\beta F_{MN} \epsilon^\beta , \end{aligned}$$

- Dimensionally reduce to d-dimensional Euclidean gauge theory.

$$A_\mu , \quad \mu = 1 \ldots , d \qquad \phi_I \equiv A_I , \quad I = 0, d+1, \ldots 9 .$$

- Derivatives along compactified directions are zero,

$$F_{\mu I} = [D_\mu, \phi_I] \qquad F_{IJ} = [\phi_I, \phi_J] .$$

- Scalars transform under vector rep. of $SO(1, 9-d)$ R-symmetry in flat Euclidean space. ϕ_0 has wrong-sign kinetic term.
- d-dimensional coupling: $g_{YM}^2 = g_{10}^2 / V_{10-d}$.

The theory on spheres Blau '00

- S^d with radius r.
- $d = 4$: gauge theory is superconformal, $\Longrightarrow$ conformal mass term

$$S_{\phi\phi} = \frac{1}{g_{YM}^2} \int d^4x \sqrt{-g} \left(\frac{2}{r^2} \, \mathrm{Tr} \phi_I \phi^I \right)$$

- $d \neq 4$: not conformal, but we include a similar term:

$$S_{\phi\phi} = \frac{1}{g_{YM}^2} \int d^dx \sqrt{-g} \left(\frac{d\, \Delta_I}{2\, r^2} \, \mathrm{Tr} \phi_I \phi^I \right) , \quad [I \text{ is summed over}]$$

  Δ_I is the analog of the dimension for ϕ_I.
- Need further terms to preserve the supersymmetry.

Conformal Killing spinors

- Supersymmetries defined by conformal Killing spinors (CKS)

$$\nabla_\mu \epsilon^\alpha = \tilde{\Gamma}_\mu^{\alpha\beta} \tilde{\epsilon}_\beta \,, \qquad \nabla_\mu \tilde{\epsilon}_\alpha = -\frac{1}{4r^2} \Gamma_{\mu\alpha\beta} \epsilon^\beta \,. \qquad \Gamma_\mu = e_{\hat{\mu}\mu} \Gamma^{\hat{\mu}}$$

- Sphere metric:

$$ds^2 = \frac{1}{(1+\frac{x^2}{4r^2})^2}\, dx^2 \,,$$

- General solution

$$\epsilon = \frac{1}{(1+\frac{x^2}{4r^2})^{1/2}} \left(\epsilon_s + x \cdot \Gamma \, \tilde{\epsilon}_c\right) \,,$$

ϵ_s and $\tilde{\epsilon}_c$ are arbitrary constant spinors $\Longrightarrow$ 32 independent CKS's.

- Reduce to 16 spinors by imposing

$$\tilde{\epsilon} = \beta \Lambda \epsilon \,, \qquad \beta = \frac{1}{2r} \qquad \tilde{\Gamma}^\mu \Lambda = -\tilde{\Lambda} \Gamma^\mu \qquad \tilde{\Lambda}\Lambda = 1$$

$d \neq 4$ also need $\Lambda^T = -\Lambda \quad \Longrightarrow \quad \Lambda = \Gamma^8 \tilde{\Gamma}^9 \Gamma^0$

$\Longrightarrow \quad \tilde{\epsilon}_c = \beta \Lambda \epsilon_s$

- This construction can be used for spheres up to $d = 7$.

Modified SUSY Transformations

- SUSY transfs. need to be modified

$$\begin{aligned} \delta_\epsilon A_M &= \epsilon \Gamma_M \Psi \\ \delta_\epsilon \Psi &= \tfrac{1}{2}\Gamma^{MN} F_{MN} + \frac{\alpha_I}{2}\Gamma^{\mu I}\phi_I \nabla_\mu \epsilon \end{aligned}$$

Split I into two groups

$$\alpha_A = \frac{4(d-3)}{d}\,, \quad A = 8,9,0\,, \qquad \alpha_i = \frac{4}{d}\,, \quad i = d+1,\ldots 7$$

- $\Longrightarrow$ Lagrangian transformation under SUSY:

$$\begin{aligned} g_{YM}^2\, \delta\mathcal{L} &= \mathrm{Tr}\Big[-(d-4)F^{\mu\nu}\tilde{\epsilon}\Gamma_{\mu\nu}\Psi - (d(2-\alpha_I)-4)F^{I\nu}\tilde{\epsilon}\Gamma_{I\nu}\Psi \\ &\quad -d\left(1-\frac{\alpha_I+\alpha_J}{2}\right)F^{IJ}\tilde{\epsilon}\Gamma_{IJ}\Psi - \frac{d}{r^2}\left(\frac{d\alpha_I}{4}-\Delta_I\right)\phi_I\epsilon\Gamma^I\Psi\Big]\,. \end{aligned}$$

- $d=4$: then $\alpha_I = \Delta_I = 1 \Rightarrow \mathcal{L}$ is invariant.
- $d \neq 4$: Need further modifications for $\mathcal{L}$.

Modifying the Lagrangian

- Add term:

$$\mathcal{L}_{\Psi\Psi} = \frac{1}{g_{YM}^2}(d-4)\beta \mathrm{Tr}\Psi\Lambda\Psi \qquad \text{nonzero if } \Lambda^T = -\Lambda$$

- Under SUSY:

$$\begin{aligned} g_{YM}^2\,\delta\mathcal{L} \quad &= \quad \mathrm{Tr}\Big[-(d-4)F^{\mu\nu}\tilde{\epsilon}\Gamma_{\mu\nu}\Psi - (d(2-\alpha_I)-4)F^{I\nu}\tilde{\epsilon}\Gamma_{I\nu}\Psi \\ &+ \quad -d\left(1-\tfrac{1}{2}(\alpha_I+\alpha_J)\right)F^{IJ}\tilde{\epsilon}\Gamma_{IJ}\Psi - \frac{d}{r^2}\left(\tfrac{1}{4}d\alpha_I - \Delta_I\right)\phi_I\epsilon\Gamma^I\Psi \\ g_{YM}^2\delta\mathcal{L}_{\Psi\Psi} \quad &+ \quad -(d-4)F^{\mu\nu}\beta\epsilon\tilde{\Gamma}_{\mu\nu}\Lambda\Psi - 2(d-4)\beta F^{I\nu}\epsilon\Gamma_{\mu I}\Lambda\Psi \\ &\quad\; -(d-4)\beta F^{IJ}\epsilon\Gamma_{IJ}\Lambda\Psi + (d-4)\beta^2\alpha_I d\,\phi_I\epsilon\Lambda\tilde{\Gamma}^I\Lambda\Psi\Big]. \end{aligned}$$

- Using Killing spinor eq. first terms cancel
- Using Killing spinor eqs. and ansatz for α_I second terms cancel.
- Third terms cancel for $i\,j$ and $i\,A$ combos. AB leftover piece:

$$-4(d-4)\beta F^{AB}\epsilon\tilde{\Gamma}_{AB}\Lambda\Psi = 4(d-4)\beta[\phi^A,\phi^B]\epsilon\Gamma^C\Psi\varepsilon_{ABC}\,,$$

- Fourth terms cancel if $\Delta_A = \alpha_A$, $\Delta_i = 2(d-2)/d$.

Complete Lagrangian on the sphere

- Add one more term to cancel the leftover piece:

$$\mathcal{L}_{ABC} = -\frac{1}{g_{YM}^2}\frac{2}{3r}(d-4)\mathrm{Tr}([\phi^A,\phi^B]\phi^C)\varepsilon_{ABC}\,.$$

- Complete SUSY Lagrangian:

$$\begin{aligned}\mathcal{L}_{ss} &= \frac{1}{g_{YM}^2}\mathrm{Tr}\Big[(\tfrac{1}{2}F_{MN}F^{MN} - \Psi\not{D}\Psi + \frac{(d-4)}{2r}\Psi\Lambda\Psi + \frac{2(d-3)}{r^2}\mathrm{Tr}\phi^A\phi_A \\ &\qquad + \frac{(d-2)}{r^2}\mathrm{Tr}\phi^i\phi_i - \frac{2}{3r}(d-4)\mathrm{Tr}([\phi^A,\phi^B]\phi^C)\varepsilon_{ABC}\Big]\,.\end{aligned}$$

- R-symmetry explicitly breaks ($d \neq 4, 7$):

$$SO(1,9-d) \to SO(1,2)\times SO(7-d)$$

$d \leq 5$, 8 supersymmetries

- If $d \leq 5$, $\Psi \to \psi + \chi$

$$\psi = +\Gamma^{6789}\psi \qquad \chi = -\Gamma^{6789}\chi$$

- Vector multiplet: A_μ, ψ, ϕ^I, $I = 0, d+1 \ldots 5$
- Hypermultiplet: χ, ϕ^I, $I = 6 \ldots 9$
- $\epsilon = +\Gamma^{6789}\epsilon$, $\Longrightarrow$ reduces no. of SUSYs to 8.
- Can relax the conditions on the additional terms in $\mathcal{L}$.

$d \leq 5$, 8 supersymmetries

- $\Psi \to \psi$

$$
\begin{aligned}
g_{YM}^2\, \delta\mathcal{L} &= -(d-4)F^{\mu\nu}\tilde{\epsilon}\Gamma_{\mu\nu}\psi - (d(2-\alpha_I)-4)F^{I\nu}\tilde{\epsilon}\Gamma_{I\nu}\psi \\
+ &\quad -d\left(1-\tfrac{1}{2}(\alpha_I+\alpha_J)\right)F^{IJ}\tilde{\epsilon}\Gamma_{IJ}\psi - \frac{d}{r^2}\left(\tfrac{1}{4}d\alpha_I - \Delta_I\right)\phi_I\epsilon\Gamma^I\psi \\
g_{YM}^2\delta\mathcal{L}_{\psi\psi} &+ -(d-4)F^{\mu\nu}\beta\epsilon\tilde{\Gamma}_{\mu\nu}\Lambda\psi - 2(d-4)\beta F^{I\nu}\epsilon\Gamma_{\mu I}\Lambda\psi \\
&\quad -(d-4)\beta F^{IJ}\epsilon\Gamma_{IJ}\Lambda\psi + (d-4)\beta^2\alpha_I d\,\phi_I\epsilon\Lambda\tilde{\Gamma}^I\Lambda\psi
\end{aligned}
$$

  - Vector scalars in last three terms $\Longrightarrow$ α_I unchanged.
  - Hypermultiplet scalars in third terms $\Longrightarrow$ modify with $\alpha_I + \alpha_J$ unchanged for certain I, J.

$$
\begin{aligned}
\alpha_I &= \frac{2(d-2)}{d} + \frac{4i\sigma_I\, m\, r}{d} \qquad I = 6\ldots 9 \\
\sigma_I &= +1\,(-1) \qquad\qquad I = 6,7\ (8,9)
\end{aligned}
$$

- Extra terms canceled by adding to $\mathcal{L}$

$$
\frac{1}{g_{YM}^2}\left(\left(\frac{2(d-4)}{r} + 4i\,m\right)\mathrm{Tr}(\phi^0[\phi^6,\phi^7]) - \left(\frac{2(d-4)}{r} - 4i\,m\right)\mathrm{Tr}(\phi^0[\phi^8,\phi^9])\right)
$$

- $d = 5$, Maximal SUSY at $m = i/(2r)$, matching Kim & Kim result

$d \leq 5$, 8 supersymmetries

- $\Psi \to \chi$; $(d-4)\beta \to i\, m$

$$\mathcal{L}_{\chi\chi} = \frac{1}{g_{YM}^2} i\, m \,\mathrm{Tr}\, \chi \Lambda \chi$$

$$\begin{aligned} g_{YM}^2\, \delta\mathcal{L} \quad &= \quad -(d-4) F^{\mu\nu} \cancel{\tilde{\epsilon}} \Gamma_{\mu\nu}\chi \; - (d(2-\alpha_I)-4) F^{I\nu} \tilde{\epsilon}\Gamma_{I\nu}\chi \\ + \quad & \quad -d\left(1-\tfrac{1}{2}(\alpha_I+\alpha_J)\right) F^{IJ}\tilde{\epsilon}\Gamma_{IJ}\chi - \frac{d}{r^2}\left(\tfrac{1}{4} d\alpha_I - \Delta_I\right)\phi_I \epsilon \Gamma^I \chi \\ g_{YM}^2 \delta\mathcal{L}_{\chi\chi} \quad + & \quad -imF^{\mu\nu}\epsilon \cancel{\tilde{\Gamma}}_{\mu\nu}\Lambda\chi \; - 2imF^{I\nu}\epsilon\Gamma_{\mu I}\Lambda\chi \\ & \quad -imF^{IJ}\epsilon\Gamma_{IJ}\Lambda\chi + im\beta\alpha_I d\, \phi_I \epsilon\Lambda\tilde{\Gamma}^I\Lambda\chi\,. \end{aligned}$$

- Previous α_I cancel second and third terms.
- To cancel fourth terms

$$\Delta_I = \frac{2}{d}\left(mr(mr+i\sigma_I) + \frac{d(d-2)}{4}\right) \qquad I = 6,7,8,9\,.$$

Off-shell formulation

see also Fujitsuka et. al. '12

- Does not exist for all 16 SUSYs simultaneously
- Choose an ϵ to pick a convenient vector field $v^M \equiv \epsilon \Gamma^M \epsilon$.
- 7 bosonic pure-spinors ν_m and auxilliary fields K^m, m=1...7.

$$\epsilon \Gamma^M \nu_m = 0 \qquad \nu_m \Gamma^M \nu_n = \delta_{nm} v^M \,.$$

- Off-shell SUSY transformations:

$$\begin{aligned} \delta_\epsilon A_M &= \epsilon \, \Gamma_M \Psi \,, \\ \delta_\epsilon \Psi &= \tfrac{1}{2} \Gamma^{MN} F_{MN} \epsilon + \frac{\alpha_I}{2} \Gamma^{\mu I} \phi_I \nabla_\mu \, \epsilon + K^m \nu_m \\ \delta_\epsilon K^m &= -\nu^m (\not{D} \Psi - (d-4) \beta \Lambda \Psi) \,, \end{aligned}$$

- α_I are same as before. $\delta_\epsilon K^m = 0$ on-shell.

Closure of algebra

- SUSY transfs. must close in the algebra

$$\begin{aligned}\delta_\epsilon^2 A_\mu &= \delta_\epsilon(\epsilon\Gamma_\mu\Psi) = \frac{1}{2}F^{MN}\epsilon\Gamma_\mu\Gamma_{MN}\epsilon - \frac{1}{2}\beta d\alpha_I\phi_I\epsilon\Gamma_\mu\tilde{\Gamma}^I\Lambda\epsilon + K^m\epsilon\Gamma_\mu\nu_m \\ &= -v^\nu F_{\nu\mu} + [D_\mu, v^I\phi_I]\,.\end{aligned}$$

  Lie derivative + gauge transf.

- Similiarly,

$$\delta_\epsilon^2\phi_I = -v^\nu D_\nu\phi_I - [v^J\phi_J, \phi_I] - \frac{1}{2}\alpha_I\beta d\,\epsilon\tilde{\Gamma}_{IJ}\Lambda\epsilon\,\phi^J\,,$$

  Lie derivative + gauge transformation + R-symmetry

- And

$$\begin{aligned}\delta_\epsilon^2\Psi &= -v^N D_N\Psi - \frac{1}{4}(\nabla_{[\mu}v_{\nu]})\Gamma^{\mu\nu}\Psi \\ &\qquad -\frac{1}{2}(d-3)\beta(\epsilon\tilde{\Gamma}^{AB}\Lambda\epsilon)\Gamma_{AB}\Psi - \frac{1}{2}\beta(\epsilon\tilde{\Gamma}^{ij}\Lambda\epsilon)\Gamma_{ij}\Psi\,.\end{aligned}$$

  Lie derivative + gauge transformation + R-symmetry

- $$\delta_\epsilon^2 K^m = -v^M D_M K^m - (\nu^{[m}\Gamma^\mu\nabla_\mu\nu^{n]})K_n + (d-4)\beta(\nu^{[m}\Lambda\nu^{n]})K_n\,.$$

Off-shell Lagrangian

- Look for the Lagrangian invariant under these transformations.

$$\mathcal{L}_{aux} = -\frac{1}{g_{YM}^2}\,\mathrm{Tr} K^m K_m\,,$$

invariant under the internal $SO(7)$ symmetry.

- Check terms linear in K^m. Verify

$$\delta_\epsilon^K\left(-\Psi \not{D}\Psi + (d-4)\beta\Psi\Lambda\Psi\right) - \delta_\epsilon(K^m K_m) = 0\,.$$

Clearly true based on transformations, hence $\mathcal{L}_{ss} + \mathcal{L}_{aux}$ is invariant.

Localization

- Localizing the off-shell action. Modify the path integral to

$$Z = \int \mathcal{D}\Phi e^{-S - tQV} \,,$$

Q is a fermionic symmetry generator. QV positive definite.

- Take $t \to \infty$ so fields localize onto fixed points of V under Q.
- For Q choose δ_ϵ, while

$$V = \int d^d x \sqrt{-g}\, \Psi\, \overline{\delta_\epsilon \Psi} \,.$$

$$\overline{\delta_\epsilon \Psi} = \tfrac{1}{2}\Gamma^{MN} F_{MN} \Gamma^0 \epsilon + \frac{\alpha_I}{2} \Gamma^{\mu I} \phi_I \Gamma^0 \nabla_\mu \epsilon - K^m \Gamma^0 \nu_m \,.$$

Bosonic part of $\delta_\epsilon V$

$$\delta_\epsilon V \Big|_{bos} = \int d^d x \sqrt{-g}\, \mathrm{Tr}(\delta_\epsilon \Psi\, \overline{\delta_\epsilon \Psi}) \,.$$

Localization (continued)

- Many terms are zero. Left-over terms (assume $v^0 = 1$, $v^{8,9} = 0$)

$$\begin{aligned}\delta_\epsilon \Psi\, \overline{\delta_\epsilon \Psi} \quad = \quad & \frac{1}{2} F_{MN} F^{MN} - \frac{1}{4} F_{MNM'N'} (\epsilon \Gamma^{MNM'N'0} \epsilon) \\ & + \frac{\beta d \alpha_I}{4} F_{MN} \phi_I (\epsilon \Lambda (\tilde{\Gamma}^I \tilde{\Gamma}^{MN} \Gamma^0 - \tilde{\Gamma}^0 \Gamma^I \Gamma^{MN}) \epsilon) \\ & - [K^m + 2\beta(d-3)\phi_0 (\nu_m \Lambda \epsilon)]^2 + \frac{\beta^2 d^2}{4} \sum_{J \neq 0} (\alpha_J)^2 \phi_J \phi^J \,.\end{aligned}$$

- Fixed-point locus (after analytically continuing K^m and ϕ_0):

$$K^m = -2\beta(d-3)\phi_0(\nu_m \Lambda \epsilon)\,, \qquad \phi_J = 0 \quad J \neq 0\,.$$

- $F_{MN} F^{MN}$ has kinetic terms for ϕ_0, $\Rightarrow$ ϕ_0 constant on the sphere.
- The rest of the fixed point conditions allow for instantons.
- Substitute the fixed point into $\mathcal{L}$, (zero instanton sector)

$$\mathcal{L}_{fp} = -\frac{1}{g_{YM}^2} \frac{(d-1)(d-3)}{r^2} \mathrm{Tr}(\phi_0 \phi_0)\,.$$

Localization (continued)

- Define dimensionless variable: $\sigma = r\phi_0$. Hence the action is

$$S_{fp} = V_d \mathcal{L}_{fp} = -\frac{4\pi^2 r^{d-4} S_{d-4}}{g_{YM}^2} \mathrm{Tr}\sigma^2 \,,$$

- Analytically continue $\sigma \to i\,\sigma$.

$$d=4: \quad S_{fp} = \frac{8\pi^2}{g_{YM}^2}\mathrm{Tr}\sigma^2 \qquad d=5: \quad S_{fp} = \frac{8\pi^3 r}{g_{YM}^2}\mathrm{Tr}\sigma^2$$

$$d=6: \quad S_{fp} = \frac{16\pi^3 r^2}{g_{YM}^2}\mathrm{Tr}\sigma^2 \qquad d=7: \quad S_{fp} = \frac{8\pi^4 r^3}{g_{YM}^2}\mathrm{Tr}\sigma^2$$

$$d=3: \quad S_{fp} = 0$$

One-loop determinants

- When restricting to the fixed point locus, need the 1-loop fluctuations
- $1/t$ is an effective coupling:

$$
\begin{aligned}
S_{eff} \sim \quad & t\, Q(\Psi\overline{Q\Psi})\Big|_{fp} + S_{fp} + (\text{ 1-loop}) + t^{-1}(\text{2-loop}) + \ldots \\
& \qquad 0 \qquad\quad + S_{fp} + (\text{ 1-loop}) + 0
\end{aligned}
$$

- Need to gauge fix:

$$
Q = \delta_\epsilon + \delta_{BRST}
$$

- Introduce Faddeev-Popov ghost fields c, $\tilde{c}$.

$$
\text{Odd } d\text{:} \quad \mathcal{L}_{gh} \quad = \quad Q(c\nabla^2 v^\mu A_\mu + \tilde{c}\nabla^\mu \tilde{A}_\mu)\,,
$$

$\Longrightarrow$ c ghost only couples to $v^\mu A_\mu$, $\tilde{c}$ only to $\tilde{A}_\mu$, $v^\mu \tilde{A}_\mu = 0$.

- c and $v^\mu A_\mu$ cancel each other out in the determinant.

Some cohomology (7d)

- Choose $v^I = 0,\ I \neq 0 \Rightarrow v^\mu v_\mu = 1$. v^μ gens. $U(1)$ action G on S^7.
  $\Longrightarrow \mathcal{M} = S^7 \backslash G \simeq CP^3$.
- v_μ is a contact 1-form $\kappa_\mu \Rightarrow$ Use to construct a two-form,

$$\nabla_{[\mu}\kappa_{\nu]} = -\frac{1}{r}\epsilon\tilde{\Gamma}_{\mu\nu}\Lambda\epsilon \equiv -\frac{1}{r}\omega_{\mu\nu}\,,$$

- $v^\mu \omega_{\mu\nu} = 0$ and $\nabla_{[\mu}\omega_{\nu\lambda]} = 0$
- $\omega_{\mu\nu}$ is the Kähler form in the 6 dimensional horiz. space $\perp$ to v^μ.

$$\Psi_M = \epsilon\Gamma_M\Psi\,, \qquad \begin{aligned} \Upsilon_{\mu\nu} &= \tfrac{1}{2}\epsilon\Gamma^{0\nu\mu}\Psi + \tfrac{1}{2}\epsilon\Gamma^{\lambda\nu\mu}\Psi\, v_\lambda \quad \Longrightarrow \quad v^\mu\Upsilon_{\mu\nu} = 0\,. \\ \Phi_{\mu\nu\lambda} &= \tfrac{1}{2}\phi_A(\epsilon\Gamma_{\mu\nu\lambda}\Gamma^{A0}\epsilon)\,, \quad \Longrightarrow \quad v^\mu\phi_{\mu\nu\lambda} = 0 \end{aligned}$$

- Can show $\Upsilon_{\mu\nu} = \hat{\Upsilon}_{\mu\nu} + \tilde{\Upsilon}\omega_{\mu\nu}\,, \quad \omega^{\mu\sigma}\hat{\Upsilon}_{\sigma\nu} = 0$
  $\hat{\Upsilon}_{\mu\nu}$ horizontal $(2,0)+(0,2)$ forms
- $\Phi_{\mu\nu\lambda}$ horizontal $(3,0)+(0,3)$ form.

One-loop determinants: 7d

- Up to quadratic terms near the fixed point:

$$\begin{aligned}\Psi\overline{Q\Psi} \quad = \quad & -\Psi_\nu(\mathcal{L}_v + [\phi_0, \bullet])A^\nu - \tfrac{2}{15}\Psi_{\mu\nu\lambda}(\mathcal{L}_v + [\phi_0, \bullet])\Phi^{\mu\nu\lambda} \\ & -\tfrac{1}{6}\Upsilon_{\mu\nu}H^{\mu\nu} + \left[\Upsilon_{\mu\nu}(2\nabla^\mu A^\nu - \tfrac{1}{3}\nabla_\lambda\Phi^{\mu\nu\lambda}) + \tilde{c}\nabla^\mu\tilde{A}_\mu\right]\end{aligned}$$

- Combined SUSY and BRST transformations

$$\begin{aligned} QA_\mu &= \Psi_\mu\,, & Q\Psi_\mu &= -(\mathcal{L}_v + [\phi_0, \bullet])A_\mu \\ Q\Phi_{\mu\nu\lambda} &= \Psi_{\mu\nu\lambda}\,, & Q\Psi_{\mu\nu\lambda} &= -(\mathcal{L}_v + [\phi_0, \bullet])\Phi_{\mu\nu\lambda} \\ Q\Upsilon_{\mu\nu} &= H_{\mu\nu} & QH_{\mu\nu} &= -(\mathcal{L}_v + [\phi_0, \bullet])\Upsilon_{\mu\nu}\,, \\ Q\tilde{c} &= b\,, & Qb &= -(\mathcal{L}_v + [\phi_0, \bullet])b \end{aligned}$$

$$Q\Theta_\alpha = \Theta'_\alpha,\ Q\Theta'_\alpha = \mathcal{L}\Theta_\alpha \equiv -(\mathcal{L}_v + [\phi_0, \bullet])\Theta_\alpha,\ \alpha = 0, 1$$

$$Q(\Psi\overline{Q\Psi}) = \begin{pmatrix}\Theta_0\\ \Theta'_1\end{pmatrix}^T \begin{pmatrix}\mathcal{L} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}D_{00} & D_{01}\\ D_{10} & D_{00}\end{pmatrix}\begin{pmatrix}\Theta_0\\ \Theta'_1\end{pmatrix} + \begin{pmatrix}\Theta'_0\\ \Theta_1\end{pmatrix}^T \begin{pmatrix}D_{00} & D_{01}\\ D_{10} & D_{00}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & \mathcal{L}\end{pmatrix}\begin{pmatrix}\Theta'_0\\ \Theta_1\end{pmatrix}$$

Hence, determinant coming from $\int Q(\Psi\overline{Q\Psi})$

$$H_7(\phi_0) \equiv \left(\frac{\det_{\mathrm{Ker}D_{10}}(\mathcal{L}_v + [\phi_0, \bullet])}{\det_{\mathrm{Coker}D_{10}}(\mathcal{L}_v + [\phi_0, \bullet])}\right)^{-1/2}$$

One-loop determinants: 7d (continued)

- Hence, determinant coming from $\int Q(\Psi \overline{Q\Psi})$

$$H_7(\phi_0) \equiv \left(\frac{\det_{\mathrm{Ker}_{D_{10}}}(\mathcal{L}_v + [\phi_0, \bullet])}{\det_{\mathrm{Coker}_{D_{10}}}(\mathcal{L}_v + [\phi_0, \bullet])} \right)^{-1/2}$$

- This can be computed using the Atiyah-Singer index theorem
  - 2 fermionic (0,0) forms: $\omega^{\mu\nu}\Upsilon_{\mu\nu}$, $\tilde{c}$
  - 3 bosonic (1,0) and 3 (0,1) forms: $\tilde{A}_\mu$
  - 3 fermionic (2,0) and 3 (0,2) forms: $\hat{\Upsilon}_{\mu\nu}$
  - 1 bosonic (3,0) and 1 (0,3) form: $\Phi_{\mu\nu\lambda}$
- Index is the same as for Dolbeault complex twisted by a $U(1)$ bundle

$$\Omega^{(0,0)} \xrightarrow{D_{10}} \Omega^{(1,0)} \longrightarrow \Omega^{(2,0)} \longrightarrow \Omega^{(3,0)}$$

$$H_7(\sigma) = \prod_{\beta \in \Lambda} \prod_{n \neq 0} (n + \langle \beta, \sigma \rangle)^{1+n^2}$$

- Similar to $d = 5$ w/ vector multiplet (twisted Dolbeault on CP^2)

$$H_{5V}(\sigma) = \prod_{\beta \in \Lambda} \prod_{n \neq 0} (n + \langle \beta, \sigma \rangle)^{1+n^2/2}$$

Källén & Zabzine

One-loop determinants: 6d

- Likewise, the $d = 6$ determinant is very similar to the $d = 4$ determinant with a vector multiplet

$$H_6(\sigma) = \prod_{\beta \in \Lambda} \prod_{n \neq 0} (n + \langle \beta, \sigma \rangle)^{3|n|}$$

$$H_{4,V}(\sigma) = \prod_{\beta \in \Lambda} \prod_{n \neq 0} (n + \langle \beta, \sigma \rangle)^{2|n|}$$

Pestun

Instantons: 6d

$$v^\mu v_\mu + {v_7}^2 = 1, \qquad v_\mu = 0 \text{ at poles}$$

$$\delta_\epsilon\Psi\,\overline{\delta_\epsilon\Psi} =$$

$$\left(D_\mu\phi_7 - \tfrac{1}{3}F_{\lambda\rho}(\epsilon\Gamma_\mu{}^{7\lambda\rho0}\epsilon)\right)^2 + \tfrac{4}{9}\,v^\lambda v^\sigma F_{\mu\lambda}F^\mu{}_\sigma$$

$$+\cos^2\tfrac{1}{2}\theta\left(\left(\phi_7 - \tfrac{1}{6}f^{\mathbf{N}}\right)^2 + \tfrac{2}{9}\cos^2\tfrac{1}{2}\theta(f^{\mathbf{N}})^2 + \left(1-\tfrac{8}{9}\sin^2\tfrac{1}{2}\theta\right)F^{\mathbf{N}+}_{\mu\nu}F^{\mathbf{N}+\mu\nu}\right)$$

$$+\sin^2\tfrac{1}{2}\theta\left(\left(\phi_7 - \tfrac{1}{6}f^{\mathbf{S}}\right)^2 + \tfrac{2}{9}\sin^2\tfrac{1}{2}\theta(f^{\mathbf{S}})^2 + \left(1-\tfrac{8}{9}\cos^2\tfrac{1}{2}\theta\right)F^{\mathbf{S}+}_{\mu\nu}F^{\mathbf{S}+\mu\nu}\right)$$

$$+D_{\hat{p}}\phi_A D^{\hat{p}}\phi^A + (v^p D_p\phi_A + \beta\phi^B\epsilon_{AB})^2 + 35\beta^2\phi_A\phi^A$$

$$+D_p\phi_0 D^p\phi_0 + (K^m + 6\beta\phi_0(\nu_m\Lambda\epsilon))^2 + [\phi_0,\phi_A]^2$$

$$\tilde{\omega}_{\sigma\kappa} \equiv \epsilon\Gamma_{\sigma\kappa}\Gamma^{789}\epsilon = \cos^2\tfrac{1}{2}\theta\,\omega^{\mathbf{N}}_{\sigma\kappa} + \sin^2\tfrac{1}{2}\theta\,\omega^{\mathbf{S}}_{\sigma\kappa}$$

Closed form: $\qquad \omega_{\sigma\kappa} = \cos^2\tfrac{1}{2}\theta\,\omega^{\mathbf{N}}_{\sigma\kappa} - \sin^2\tfrac{1}{2}\theta\,\omega^{\mathbf{S}}_{\sigma\kappa}$

$$\begin{aligned} F_{\mu\nu} &= F^{\mathbf{N}+}_{\mu\nu} + F^{\mathbf{N}-}_{\mu\nu} + \tfrac{1}{6}f^{\mathbf{N}}\omega^{\mathbf{N}}_{\mu\nu} \\ &= F^{\mathbf{S}+}_{\mu\nu} + F^{\mathbf{S}-}_{\mu\nu} + \tfrac{1}{6}f^{\mathbf{S}}\omega^{\mathbf{S}}_{\mu\nu} \end{aligned} \qquad F^{\mathbf{N},\mathbf{S}\pm} = \pm*(F^{\mathbf{N},\mathbf{S}\pm}\wedge\omega^{\mathbf{N},\mathbf{S}})$$

Instantons: 6d

$\Longrightarrow$ $\phi_A = v^\mu F_{\mu\nu} = 0$ everywhere, ϕ_0 constant $K^m = -6\beta\phi_0(\nu^m\Lambda\epsilon)$.
$f^{\mathbf{N}} = F^{\mathbf{N}+} = 0$ except South pole, $f^{\mathbf{S}} = F^{\mathbf{S}+} = 0$ except North pole
$\Longrightarrow$ $\phi_7 = 0 \Longrightarrow F_{\lambda\rho}(\epsilon\Gamma_\mu{}^{7\lambda\rho 0}\epsilon) = 0$

Allows point-like anti-instantons on north pole wrt $\omega^{\mathbf{N}}$
and point-like anti-instantons on south pole wrt $\omega^{\mathbf{S}}$.

Between poles need $F = F^{\mathbf{N}-} = F^{\mathbf{S}-}$ $\Longrightarrow$

$$F = - * (F \wedge \tilde{\omega}) \,, \qquad 0 = F \wedge \bar{\omega}$$

where $\bar{\omega} \equiv \frac{1}{2}(\omega^N - \omega^S)$. Then

$$\iota_v \bar{\omega} = \iota_\theta \bar{\omega} = 0 \quad \Rightarrow \quad \bar{\omega} \text{ spans 4d space}$$

Bianchi identity $\Longrightarrow$ $$d_A * F = 0$$

Solves YM eqs. Extended anti-instantons transverse to θ and v^μ.
Spans CP^2. Contribution of little strings

Instantons: 6d

Can add the terms to the action

$$i\,\alpha \int \mathrm{Tr}[F \wedge F] \wedge \omega + i\,\beta \int \mathrm{Tr}[F \wedge F \wedge F]$$

Pt-like anti-instantons don't contribute to YM action nor α term (scaling)
Contribution from β term

$$i\,\beta \int \mathrm{Tr}[F \wedge F \wedge F] = 24\pi^3 i\,\beta\,\mathbb{Z}$$

- Extended anti-instantons contribute to *YM* action $\sim r^2/g_{YM}^2$ but not to the α and β terms.
- No contribution from the α term because ω is odd. Put another way, there is no non-contractible 2-surface to wrap.

Instantons: 7d

Complete fixed point locus:

$$
\begin{aligned}
K^m &= -\frac{4}{r}\phi_0\,(\nu_m \Lambda \epsilon)\,, \qquad D_\mu \phi_0 = 0 \\
\hat{F}^+_{\mu\nu} &= D_\sigma \Phi_{\mu\nu}{}^{\sigma} \\
f &= \tfrac{1}{6}[\Phi_{\mu\nu\lambda}, \Phi^{\mu\nu}{}_{\sigma}]\omega^{\lambda\sigma} \\
v^\mu F_{\mu\nu} &= 0 \\
v^\sigma\, H_{\sigma\mu\nu\lambda} &= 0
\end{aligned}
$$

$$
\begin{aligned}
\hat{F}^+ &= \iota_v * (\hat{F}^+ \wedge \omega) \\
H &\equiv D\Phi
\end{aligned}
$$

- $\hat{F}^+$ is (2,0) plus (0,2) form.
- These eqs are the 7d lift of Hermitian Higgs-Yang-Mills system

Discussion

- We have given a uniform description of maximal SUSY gauge theories on S^d.
- This allows us to find the localized versions for 6d and 7d relatively easily

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- We can use the $d = 6$ theory to explore little string theories, which come from (1,1) gauge theories in six dimensions. $d = 7$ can explore "little m-theory".
- Wilson lines, *etc.*

Thank You!

Other stuff

Matrix Models

- The localized path integral reduces to

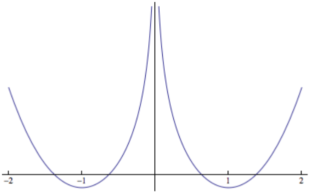
$$\int \prod d\sigma_i \exp\left(\frac{4\pi^2 r^{d-4} S_{d-4}}{r^2 g_{YM}^2} \sum \sigma_i^2\right) \prod_{i<j} (\sigma_i - \sigma_j)^2 H(\sigma_i - \sigma_j)^2$$

- In the large-N limit this can be solved by saddle point
- Wick rotation, $\sigma_i \to i\sigma_i$
- $H(\sigma_i - \sigma_j)$ needs to be regularized. $\Longrightarrow$ Redefinition of coupling

Matrix Models: 7d

- For $d = 7$ (after regularization)

$$
\begin{aligned}
-\log x^2 - \log(H(i\,x)^2) &= V_{int}(x) \\
&= \frac{2\pi}{3}x^3 + 2x^2\log(1-e^{-2\pi x}) - \log\sinh^2(\pi x) \\
&\quad + \frac{2}{\pi}x\mathrm{Li}_2(e^{-2\pi x}) - \frac{1}{\pi^3}\mathrm{Li}_3(e^{-2\pi x})
\end{aligned}
$$



- Saddle point equation:

$$\frac{8\pi^4 r^3}{g_{YM}^2}\sigma_i = 2\pi\sum_{j\neq i}(1-(\sigma_i-\sigma_j)^2)\coth(\pi(\sigma_i-\sigma_j))$$

- $\rho(\sigma) = \frac{1}{N}\sum\delta(\sigma-\sigma_i)$, $\int\rho(\sigma)d\sigma = 1$, $\lambda = g^2N/r^3 \Longrightarrow$

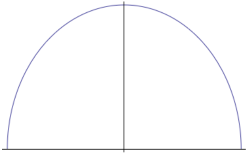
$$\frac{16\pi^4}{\lambda}\sigma = 2\pi \fint (1-(\sigma-\sigma')^2)\coth(\pi(\sigma-\sigma'))\rho(\sigma')d\sigma'$$

Matrix Models: 7d (continued)

- Can't solve the saddle-point equation exactly:
- Weak coupling:
  Saddle point equation:

$$\frac{16\pi^4}{\lambda}\sigma \approx 2 \fint \frac{\rho(\sigma')\, d\sigma'}{\sigma - \sigma'}$$

$$F \sim -N^2 \log \lambda$$



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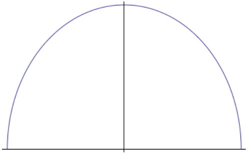
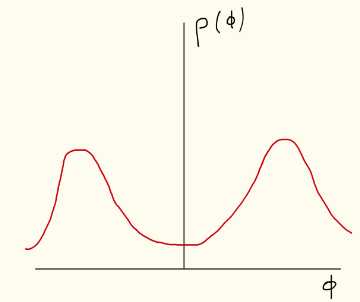


- Strong coupling: Solve numerically



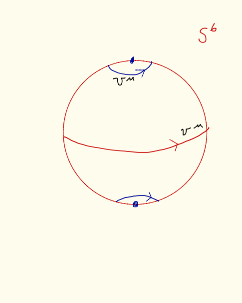
- As $\lambda \to \infty$
  $\rho(\sigma) \to$ fixed distribution
- Free energy scales as N^2.
- Basically same behavior as pure $\mathcal{N} = 1$ in $d = 5$.
- At $\lambda = \infty$,
  "scaling" symmetry,
  $\phi_I \to e^\beta \phi_I$

$$SO(1,2) \to SO(2,3) \supset SO(1,1)\times SO(1,2)$$

Cohomology: 6 dimensions

- 6d does not have a nonvanishing vector field

$$v^7 = \cos\theta\,, \qquad v^\mu v_\mu = \sin\theta^2\,.$$



- Construct $\Upsilon_{\mu\nu}$ and $\Phi_{\mu\nu\lambda}$
- In general

$$v^\mu \Upsilon_{\mu\nu} \neq 0\,, \qquad v^\mu \Phi_{\mu\nu\lambda} = 0$$

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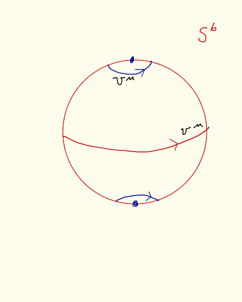


- Construct $\Upsilon_{\mu\nu}$ and $\Phi_{\mu\nu\lambda}$
- In general

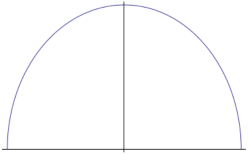
$$v^\mu \Upsilon_{\mu\nu} \neq 0\,, \qquad v^\mu \Phi_{\mu\nu\lambda} = 0$$

- Except at the poles:
  - Fermionic (2,0) and (0,2) forms and Kähler (1,1) on local C^3
  - Bosonic (3,0) and (0,3) forms
  - Bosonic (1,0) and (0,1) forms from A_μ

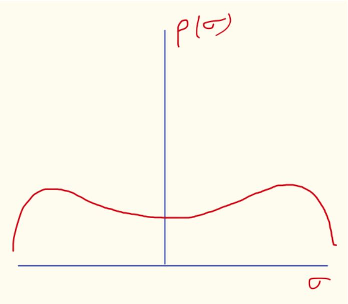
Matrix Models: 6d

- Can't solve the saddle-point equation exactly, but analytic results at strong coupling: Russo & Zarembo
- Weak coupling:
  Saddle point equation:

$$\frac{32\pi^3}{\lambda}\sigma \approx 2 \int \frac{\rho(\sigma')\, d\sigma'}{\sigma - \sigma'}$$



- Strong coupling:



- As $\lambda \to \infty$
  $\rho(\sigma) \to$ fixed distribution
- Free energy scales as N^2.
- Basically same behavior as pure $\mathcal{N} = 2$ in $d = 4$.

Negative couplings?

- In $d = 4$ (Russo & Zarembo) and $d = 5$ (Nedelin) there can be a negative coupling
  Start with adjoint hypermultiplet with mass M:

$$d = 4: \quad \frac{1}{g_{eff}^2} = \frac{1}{g_{YM}^2} - \frac{N}{4\pi^2}\log(Mr); \qquad d = 4: \quad \frac{1}{g_{eff}^2} = \frac{1}{g_{YM}^2} - \frac{N}{8\pi^2}M$$

- For $d = 5$ if $1/\lambda_{eff} \ll -1$

$$\frac{16\pi^3}{\lambda_{eff}}\sigma \approx -\pi \fint (\sigma - \sigma')^2 \mathrm{sign}(\sigma - \sigma'))\rho(\sigma')d\sigma'$$

- Taking two derivatives:

$$0 = \int_{\sigma}^{a} \rho(\sigma)d\sigma - \int_{-a}^{\sigma} \rho(\sigma)d\sigma$$

$$\Longrightarrow \quad \rho(\sigma) = \tfrac{1}{2}(\delta(\sigma - a) + \delta(\sigma + a)) \quad \Longrightarrow \quad a = -\frac{32\pi^2}{\lambda_{eff}}$$

$$F \approx -\frac{\pi}{3}N^2 a^3$$

- We can do the same in $d = 7$ and $d = 6$.