# SUSY Gauge Theories on 5-manifolds (and 6d gravity duals)

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# Motivation

- 5d QFTs are interesting!
- Localization allows exact observables to be computed - partition functions and Wilson loops
- Applied to 5d theories on many curved manifolds
- Through gauge/gravity correspondence it's natural to try to seek gravity duals to the field theory side
- Opens up further tests of AdS/CFT duality

$$\lim_{N\to\infty} -\log Z \;=\; \mathcal{F}_{\text{gravity}}$$

- Not many explicit tests of $\mathrm{CFT}_5/\mathrm{AdS}_6$ correspondence

Describe study of solutions to Euclidean 6d Romans SUGRA. Aim of this is to compute observables of interest in gauge/gravity duality

# G-structures

- Classifying solutions to SUGRA studied using G-structures
- A G-structure is a reduction of the frame bundle group Gl(n) to a subgroup G
- Equivalent to the existence of nowhere vanishing tensors
- SUSY conditions translate into differential equations on various $p$-forms
- Differential equations easier to handle - not matrix eqns
- Provides the general form of the solution in terms of the G-structure data

# Motivation 2

- Much interest in studying rigid SUSY on Riemannian manifolds

$$\nabla\epsilon + a\cdot\Gamma\epsilon \;=\; 0\,, \qquad M\epsilon \;=\; 0$$

- Finding rigid SUSY backgrounds originally an ad-hoc process
- Systematic method - "rigid limit" of SUGRA [Festuccia,Seiberg]
- Also holographic starting point - backgrounds arise as a conformal boundary of some bulk gauged supergravity solution
- General SUSY preserving backgrounds have been studied for 3d and 4d by both "rigid limit" and holographic approaches [Dumitrescu,Festuccia,Seiberg][Klare,Tomasiello,Zaffaroni]
- "Rigid limit" of 5d Poincare SUGRA studied [Pan][Imamura,Matsuno]
- See also talks of Diego and Johannes!

# Outline

1. $CFT_5/AdS_6$

2. Solutions to 6d Romans F(4) SUGRA

3. Applications:

    Squashed Sasaki-Einstein spaces

    Wilson loop gravity duals

    5d background geometry & Lagrangians

4. Summary and outlook

# $\mathrm{CFT}_5/\mathrm{AdS}_6$

# 5d theories with gravity duals

- A class of 5d $\mathcal{N}=1$ SCFTs can be engineered from a system of $N$ D4-branes and $N_f$ D8s in presence of orientifold planes [Seiberg]
- Specifically, $Sp(N)$ gauge group with $N_f$ matter fields in the fundamental and single hypermultiplet in the anti-symmetric rep
- System is expected to have large $N$ description in massive IIA supergravity with near horizon geometry $M_6 \times_w S^4$ [Ferrara,Kehagias,Partouche,Zaffaroni][Brandhuber,Oz]
- Also quiver gauge theories with dual description in massive IIA on $M_6 \times_w S^4/\mathbb{Z}_n$ [Bergman,Rodriguez-Gomez]
- In order to find dual supergravity solutions it is natural to use 6d Romans F(4) gauged SUGRA [Romans]
- This is a consistent truncation of massive IIA SUGRA on $S^4$ to six dimensions [Cvetic,Lu,Pope]
- Round $S^5$/dual AdS$_6$ tested [Jafferis,Pufu][Assel,Estes,Yamazaki]

# Romans F(4) SUGRA

- Bosonic field content is
  - metric
  - dilaton $\phi$, but we use scalar $X = \exp(-\phi/2\sqrt{2})$
  - 2-form potential $B$ with field strength $H = \mathrm{d}B$
  - 1-form potential $A$ with field strength $F = \mathrm{d}A + \frac{2}{3}gB$
  - $SU(2)$ potential $A^i$ ($i = 1,2,3$) with $F^i = \mathrm{d}A^i - \frac{1}{2}g\varepsilon_{ijk}A^j \wedge A^k$
- SUSY preserved if non-trivial $SU(2)$ spinor doublet $\epsilon_I$ satisfies

$$\begin{aligned} D_M\epsilon_I &= \frac{\mathrm{i}}{4\sqrt{2}}(X + \tfrac{1}{3}X^{-3})\Gamma_M\Gamma_7\epsilon_I - \frac{1}{48}X^2 H_{NPQ}\Gamma^{NPQ}\Gamma_M\Gamma_7\epsilon_I \\ &\quad - \frac{\mathrm{i}}{16\sqrt{2}}X^{-1}F_{NP}(\Gamma_M{}^{NP} - 6\delta_M{}^N\Gamma^P)\epsilon_I \\ &\quad + \frac{1}{16\sqrt{2}}X^{-1}F^i_{NP}(\Gamma_M{}^{NP} - 6\delta_M{}^N\Gamma^P)\Gamma_7(\sigma^i)_I{}^J\epsilon_J \end{aligned}$$

$$\begin{aligned} 0 &= -\mathrm{i}X^{-1}\partial_M X\Gamma^M\epsilon_I + \frac{1}{2\sqrt{2}}\left(X - X^{-3}\right)\Gamma_7\epsilon_I + \frac{\mathrm{i}}{24}X^2 H_{MNP}\Gamma^{MNP}\Gamma_7\epsilon_I \\ &\quad - \frac{1}{8\sqrt{2}}X^{-1}F_{MN}\Gamma^{MN}\epsilon_I - \frac{\mathrm{i}}{8\sqrt{2}}X^{-1}F^i_{MN}\Gamma^{MN}\Gamma_7(\sigma^i)_I{}^J\epsilon_J \end{aligned}$$

# Solutions to Romans SUGRA

# Solutions to Romans

- Consider only symplectic-Majorana spinors: $\varepsilon_I{}^J \epsilon_J = \mathcal{C}_6 \epsilon_I^* \equiv \epsilon_I^c$
- Make life simpler: set $A = 0$, $g = 1$ and only turn on $\mathcal{A} = A^3$
- When all fields are real except $B$, which is pure imaginary, then SUSY conditions for $\epsilon_2$ are simply charge conjugates of $\epsilon \equiv \epsilon_1$
- Can form

$$S \;\equiv\; \epsilon^\dagger \epsilon \;, \qquad \tilde{S} \;\equiv\; \epsilon^\dagger \Gamma_7 \epsilon \;, \qquad s \;\equiv\; \epsilon^{\mathrm{T}} \epsilon$$

- Then SUSY conditions give $\mathrm{d}(Xs) = -\mathrm{i}(Xs)\mathcal{A}$. Integrability: $\mathcal{F} = \mathrm{d}\mathcal{A} = 0$ unless $s = 0$. Choosing $s = 0$ allows [Gauntlett,Martelli,Sparks,Waldram]

$$\epsilon \;=\; \epsilon_+ + \epsilon_- \;, \qquad \epsilon_+ \;=\; \sqrt{S}\cos\alpha\,\eta_1 \;, \quad \epsilon_- \;=\; \sqrt{S}\sin\alpha\,\eta_2^*$$

- Here $\eta_1, \eta_2$ are two orthonormal chiral spinors
- Note $S, \tilde{S}$ related: $\tilde{S} \;=\; -S\cos 2\alpha$

# Solutions to Romans

- $\eta_1, \eta_2$ together define a canonical $U(2)$ structure specified by real 1-forms $K_1$,$K_2$, a real 2-form $J$, and a complex 2-form $\Omega$

- The 9 possible non-scalar $\epsilon$ bilinears are related to ($K_1$,$K_2$,$J$, $\Omega$)

$$K \equiv \epsilon^\dagger \Gamma_{(1)} \epsilon = S \sin 2\alpha \, K_1$$

- Can show $K$ is a Killing 1-form, dual, $\xi \equiv K^\#$, is Killing vector

- Allows us to introduce a local coordinate $\psi$ with $\xi = \partial_\psi$ and

$$K_1 = S \sin 2\alpha \, (\mathrm{d}\psi + \sigma)$$

- Romans SUSY conditions applied to remaining bilinears imply differential equations on $U(2)$ structure

- Can show SUSY conditions + EOM are equivalent to differential equations + $\mathrm{i}_K(\mathrm{E}_B)$

# Solutions to Romans

- Further all $U(2)$ and SUGRA fields are annihilated by $\xi = \partial_\psi$, which generates symmetry of full solution

- Partially solve matter fields

$$B = \mathrm{i}K_1 \wedge \left[\tfrac{3}{\sqrt{2}}(S\sin 2\alpha)^{-1}\mathrm{d}(XS) + X^{-2}K_2\right] + B_\perp$$

$$\mathcal{F} = K_1 \wedge \sqrt{2}(S\sin 2\alpha)^{-1}\mathrm{d}(XS\cos 2\alpha) + \mathcal{F}_\perp$$

from which

$$\mathcal{A} = -\sqrt{2}X\cot 2\alpha\, K_1 + \mathcal{A}_\perp$$

- Can complete $K_1$,$K_2$ to an orthonormal frame

$$\mathrm{d}s^2 = S^2\sin^2 2\alpha(\mathrm{d}\psi + \sigma)^2 + K_2^2 + \sum_{a=1}^{4}(e^a)^2$$

$$J = e^1\wedge e^2 + e^3\wedge e^4\ , \quad \Omega = (e^1 + \mathrm{i}e^2)\wedge(e^3 + \mathrm{i}e^4)$$

# Applications

## 1. Squashed SE solutions

# Squashed SE solutions

- Can't solve general equations so make additional assumptions

$$\mathrm{d}s^2 = \delta^2(r)\mathrm{d}r^2 + \gamma^2(r)(\mathrm{d}\psi + \sigma)^2 + \beta^2(r)\mathrm{d}s^2_{\mathrm{KE}}$$

$$B = p(r)\mathrm{d}r \wedge (\mathrm{d}\psi + \sigma) + \tfrac{1}{2}q(r)\mathrm{d}\sigma \qquad \mathrm{d}\sigma = 2\omega_{\mathrm{KE}}$$

$$\mathcal{A} = f(r)(\mathrm{d}\psi + \sigma) - 3\mathrm{d}\psi$$

$$X = X(r)$$

- Constant $r$ hypersurface is a squashed Sasaki-Einstein manifold
- Comparing with general $U(2)$ structure metric and fields gives

$$S \sin 2\alpha = \gamma(r)\ , \qquad K_2 = -\delta(r)\mathrm{d}r$$

$$\Omega = \beta^2(r)\Omega_{\mathrm{KE}}\ , \qquad J = -\beta^2(r)\omega_{\mathrm{KE}}$$

and

$$f(r) = 3 - \sqrt{2}XS\cos 2\alpha\ , \quad \mathcal{F}_\perp = 2f(r)\omega_{\mathrm{KE}}\ , \quad B_\perp = q(r)\omega_{\mathrm{KE}}$$

# Squashed SE solutions

- Take $S = S(r), \alpha = \alpha(r)$. Parametrization inv, free choice $\beta(r)$
- Substituting into $U(2)$ equations and additional flux component gives 6 coupled ODEs for $(X, S, \alpha, \delta, p, q)$. Still too hard!
- Remarkably all KE metric information drops out
- Take KE$=\mathbb{CP}_2$. Have constructed 2-parameter family of 1/4-BPS solutions as series expansions around $r = \infty$ and AdS$_6$ [Alday,Fluder,Matte-Gregory,PR,Sparks]

$$f_0 \ \equiv \ f(r)|_{\text{boundary}} \ , \qquad s \ \equiv \ \beta(r)\gamma(r)^{-1}|_{\text{boundary}}$$

- Can compute gravity free energy for generic squashed SE using holographically renormalised Romans SUGRA [Alday,Fluder,Matte-Gregory,PR,Sparks]

$$\mathcal{F}_{\text{gravity}} \ = \ -\frac{27}{4\pi G_N} \cdot \text{vol(SE)}$$

- Volume is unsquashed SE, $\mathcal{F}_{\text{gravity}}$ independent of $f_0$, $s$

# Squashed SE solutions

- Want to compare $\mathcal{F}_{\text{gravity}}$ to large $N$ field theory computation
- Exact perturbative partition function of arbitrary $\mathcal{N}=1$ gauge theories on toric SE manifolds already computed [Qiu,Tizzano,Winding,Zabzine]
- Uses generalization of triple sine function, helpfully asymptotics of this function also given by [Qiu,Tizzano,Winding,Zabzine]
- Tune to the class of $Sp(N)$ gauge theories with $\text{AdS}_6$ duals
- Use standard saddle point technique [Herzog,Klebanov,Pufu,Tesileanu]

$$\mathcal{F}_{\text{gauge theory}} \;=\; -\frac{9\sqrt{2}}{5\pi^2\sqrt{8-N_f}}\text{vol(SE)}N^{5/2}+\cdots$$

- Agrees with gravity computation as $G_N \;=\; \frac{15\pi\sqrt{8-N_f}}{4\sqrt{2}N^{5/2}}$

# Applications

## 2. Wilson loop duals

# Wilson loop duals

- Can compute regularised string action dual to Wilson loops for any Romans solution with ball topology and $U(1)^3$ symmetry

- Fundamental string sits at north pole of internal $S^4$ and wraps $K_1 - K_2$ direction. Boundary Wilson loop wraps orbit of $\xi$

- Regularised action is [Alday,Fluder,Matte-Gregory,PR,Sparks]

$$S_{\text{string}} = \frac{\sqrt{2}N^{1/2}}{3\sqrt{8-N_f}} \int_{\Sigma_2} X^{-2}\text{vol}_2 + \mathrm{i}B - \frac{3}{\sqrt{2}}\text{length}(\partial\Sigma_2)$$

- From general Romans $U(2)$ eqns find

$$X^{-2}K_1 \wedge K_2 + \mathrm{i}(B - B_\perp) = \tfrac{3}{\sqrt{2}}\mathrm{d}\rho \wedge (\mathrm{d}\psi + \sigma)$$

$$\rho = XS$$

- String action reduces to

$$S_{\text{string}} = -\frac{3}{\sqrt{2}}\rho_{\text{origin}} \frac{\sqrt{2}N^{1/2}}{3\sqrt{8-N_f}} \int_{S^1} \mathrm{d}\psi$$

# Wilson loops

- For solution with ball topology and $U(1)^3$ symmetry

$$\rho_{\text{origin}} \;=\; (XS)\,|_{\text{origin}} = \;\tfrac{1}{\sqrt{2}}(b_1+b_2+b_3)$$

- Here $\partial_\psi \;=\; \sum_{i=1}^{3} b_i \partial_{\varphi_i}$ and $\varphi_i$ are $2\pi$ periodic coordinates
- Again this follows for $U(2)$ structure equations
- Hence for a Wilson loop wrapping the $\varphi_i$ circle

$$S_{\text{string}} \;=\; -9\pi \frac{b_1+b_2+b_3}{3b_i}\,\frac{\sqrt{2}N^{1/2}}{3\sqrt{8-N_f}}$$

# Applications

## 3. 5d background geometry & Lagrangians

# AlAdS Solutions

- Asymptotically locally AdS solutions admit a radial coord $r$ in which fields have a series expansion [Fefferman,Graham]

$$\mathrm{d}s^2 \;=\; \frac{9}{2}\frac{\mathrm{d}r^2}{r^2} + r^2\left[g_{\mu\nu} + \frac{1}{r^2}g^{(2)}_{\mu\nu} + \cdots\right]\mathrm{d}x^\mu \mathrm{d}x^\nu$$

- The 6d spin connection expands as

$$\Omega_\mu{}^{0\nu} \;=\; -\tfrac{\sqrt{2}}{3}\delta_\mu{}^\nu + \tfrac{1}{r^2}\,\omega_\mu{}^\nu + \cdots\,, \qquad \Omega_0{}^{MN} \;=\; 0$$

- Non-metric expansions introduce 5d fields $X_2$, $a$, $b$, $A^{(0)}$

- Solve $r$ direction bulk KSE:

$$\epsilon \;=\; \sqrt{r}\begin{pmatrix}\chi \\ -\mathrm{i}\chi\end{pmatrix} + \frac{1}{\sqrt{r}}\begin{pmatrix}\varphi \\ \mathrm{i}\varphi\end{pmatrix} + O(r^{-3/2})$$

- 6d spinor bilinears

$$\epsilon^\dagger\Gamma_7\epsilon \;=\; 4\alpha S + \cdots \qquad\qquad S \;\equiv\; \chi^\dagger\chi$$

$$\mathrm{i}\epsilon^\dagger\Gamma_7\Gamma_{(1)}\epsilon \;=\; 2SrK_2 - 3\sqrt{2}S\mathrm{d}r + \cdots \qquad\qquad \varphi \;=\; -\alpha\chi - \tfrac{\mathrm{i}}{2}(K_2)_\nu\gamma^\nu\chi$$

# AlAdS Solutions

- Expanding the non-$r$ direction bulk KSE leads to

$$\left(\nabla_\mu + \tfrac{\mathrm{i}}{2}a_\mu\right)\chi \;=\; -\tfrac{\sqrt{2}}{3}\mathrm{i}\gamma_\mu\varphi - \tfrac{\mathrm{i}}{12\sqrt{2}}b_{\nu\sigma}\gamma_\mu{}^{\nu\sigma}\chi + \tfrac{\mathrm{i}}{3\sqrt{2}}b_{\mu\nu}\gamma^\nu\chi$$

$$\begin{aligned}\left(\nabla_\mu + \tfrac{\mathrm{i}}{2}a_\mu\right)\varphi \;=\; & -\tfrac{\mathrm{i}}{6\sqrt{2}}b_{\mu\nu}\gamma^\nu\varphi + \tfrac{1}{16\sqrt{2}}f_{\nu\sigma}\gamma_\mu{}^{\nu\sigma}\chi - \tfrac{3}{8\sqrt{2}}f_{\mu\nu}\gamma^\nu\chi \\ & + \tfrac{1}{48}(\mathrm{d}b)_{\nu\rho\sigma}\gamma^{\nu\rho\sigma}\gamma_\mu\chi - \tfrac{1}{36}A^{(0)}_\nu\gamma_\mu{}^\nu\chi + \tfrac{1}{12}A^{(0)}_\mu\chi \\ & + \tfrac{\mathrm{i}}{2}\omega_\mu{}^\nu\gamma_\nu\chi\end{aligned}$$

with $f = \mathrm{d}a$

- Bulk dilatino equation gives

$$\begin{aligned}0 \;=\; & -\tfrac{1}{6\sqrt{2}}b_{\mu\nu}\gamma^{\mu\nu}\varphi - \tfrac{\sqrt{2}}{3}X_2\chi + \tfrac{\mathrm{i}}{8\sqrt{2}}f_{\mu\nu}\gamma^{\mu\nu}\chi \\ & + \tfrac{\mathrm{i}}{24}(\mathrm{d}b)_{\mu\nu\sigma}\gamma^{\mu\nu\sigma}\chi - \tfrac{\mathrm{i}}{18}A^{(0)}_\mu\gamma^\mu\chi\end{aligned}$$

# Setup

- What constraints do 5d SUSY conditions for $(\chi, \varphi)$ place on $(\mathcal{M}_5, g)$ and background fields $(a, X_2, \omega, \alpha, K_2, A^{(0)}, b)$
- Take $\chi \in \mathrm{Spin}^c(5)$ and $\chi \neq 0$ on $\mathcal{M}_5$, we can construct bilinear forms

$$S \equiv \chi^\dagger \chi \,, \qquad K_1 \equiv \frac{1}{S}\chi^\dagger \gamma_{(1)} \chi$$

$$J \equiv -\frac{\mathrm{i}}{S}\chi^\dagger \gamma_{(2)} \chi \,, \qquad \Omega \equiv -\frac{1}{S}(\chi^c)^\dagger \gamma_{(2)} \chi$$

- This defines a 5d global $U(2)$-structure
- Introduce a local orthonormal frame $e^a, \; a = 1, \ldots, 5$

$$K_1 = e^5 \,, \quad J = e^1 \wedge e^2 + e^3 \wedge e^4 \,, \quad \Omega = (e^1 + \mathrm{i} e^2) \wedge (e^3 + \mathrm{i} e^4)$$

- By Fierz $J_a{}^b$ defines an almost contact structure on $\mathcal{M}_5$

$$J_a{}^c J_c{}^b = -1_a{}^b + (K_1)^b (K_1)_a$$

# Differential conditions

- The 1-form $SK_1 \;=\; \chi^\dagger\gamma_{(1)}\chi$ is a conformal Killing 1-form

$$\nabla_{(\mu}(SK_1)_{\nu)} \;=\; \mathcal{L}_\xi(\log S)g_{\mu\nu} \qquad\qquad \xi^\mu = g^{\mu\nu}(SK_1)_\nu$$

- The 5d KSEs and dilatino imply

$$S\alpha = -\tfrac{1}{2}(\chi^\dagger\varphi + \varphi^\dagger\chi)$$

$$SK_2 = \mathrm{i}(\chi^\dagger\gamma_{(1)}\varphi - \varphi^\dagger\gamma_{(1)}\chi)$$

$$\mathrm{d}S \;=\; -\,\tfrac{\sqrt{2}}{3}\left(SK_2 + \mathrm{i}i_\xi b\right)$$

$$\mathrm{d}(S\alpha) \;=\; -\,\tfrac{1}{2\sqrt{2}}i_\xi\mathrm{d}a$$

$$\mathrm{d}(SK_1) \;=\; \tfrac{2\sqrt{2}}{3}\left[2\alpha SJ + SK_1\wedge K_2 + \mathrm{i}Sb - \tfrac{\mathrm{i}}{2}i_\xi(*b)\right]$$

$$\mathrm{d}(SK_2) \;=\mathrm{i}i_\xi\mathrm{d}b - \mathrm{i}\mathcal{L}_\xi(\log S)b$$

$$\mathrm{d}(SJ) \;=\; -\sqrt{2}K_2\wedge(SJ)$$

$$\mathrm{d}(S\Omega) \;=\; -\,\mathrm{i}\left(a - 2\sqrt{2}\alpha K_1 - \mathrm{i}\sqrt{2}K_2\right)\wedge(S\Omega)$$

# Solving

- After a suitable gauge choice for $a$, find $\mathcal{L}_\xi(\,\cdot\,) \;=\; 0$ on

$$\alpha S\,, \quad a\,, \quad S^{-1}b\,, \quad S^{-2}J\,, \quad S^{-2}\Omega$$

- Exception

$$\mathcal{L}_\xi(\log S) \;=\; -\tfrac{\sqrt{2}}{3} i_\xi K_2$$

- Convenient to impose $\mathcal{L}_\xi S \;=\; 0$
- Then introduce local coordinate $\psi$ through $\xi \;=\; \partial_\psi$
- As $\xi$ is nowhere zero its orbits define a foliation of $\mathcal{M}_5$
- Dual Killing 1-form is $K_1 \;=\; S(\mathrm{d}\psi + \rho) \;=\; S\eta$ $\qquad \mathrm{i}_\xi \rho \;=\; 0$
- 5d metric takes the form

$$\mathrm{d}s^2_{\mathcal{M}_5} \;=\; S^2(\mathrm{d}\psi + \rho)^2 + \mathrm{d}s^2_4$$

- 4d metric is almost Hermitian with $J$ it's fundamental 2-form

# Solving

- Two scalar differential equations give

$$a \;=\; 2\sqrt{2}S\alpha\eta + a_\perp\,, \qquad b \;=\; \mathrm{i}S\eta\wedge\left(K_2 + \tfrac{3}{\sqrt{2}}\mathrm{d}\log S\right) + b_\perp$$

- Whilst $\mathrm{d}(SK_1)$ gives $\mathrm{d}\rho \;=\; \frac{\sqrt{2}}{3}S^{-1}\left(-\mathrm{i}\ast_4 b_\perp + 2\mathrm{i}b_\perp + 4\alpha J\right)$ or

$$b^+ \;=\; \mathrm{i}\left(4\alpha J - \tfrac{3}{\sqrt{2}}S\mathrm{d}\rho^+\right)\,, \qquad b^- \;=\; -\tfrac{\mathrm{i}}{\sqrt{2}}S\mathrm{d}\rho^-$$

- $\mathrm{d}(SK_2)$ gives no new information
- $\mathrm{d}(SJ)$ identifies $\theta \;\equiv\; J\lrcorner\,\mathrm{d}J \;=\; -\sqrt{2}K_2 - \mathrm{d}\log S$
- Finally $\mathrm{d}(S\Omega)$ gives $\mathrm{d}\Omega \;=\; (\theta - \mathrm{i}a_\perp)\wedge\Omega$
- Implies the almost complex structure is integrable i.e. complex structure

# Solving

- This defines a transversely holomorphic foliation - odd dimension equivalent of complex structure
- The transverse metric is Hermitian
- Can introduce local coords $(\psi, z_1, z_2)$ so that

$$J \ = \ \frac{\mathrm{i}}{2} g^{(4)}_{\alpha\bar{\beta}}\, \mathrm{d}z^{\alpha} \wedge \mathrm{d}\bar{z}^{\bar{\beta}}\,, \qquad \Omega \ = \ \sqrt{\det g^{(4)}}\, \mathrm{d}z^1 \wedge \mathrm{d}z^2$$

- Can define

$$a_{\mathrm{Chern}} \ \equiv \ a_{\perp} + i_{\theta^{\#}} J \ = \ -\tfrac{1}{2} I \circ \mathrm{d} \log \det g^{(4)}$$

- Background fields are then

$$\begin{aligned} a \ &= 2\sqrt{2} S \alpha \eta + a_{\mathrm{Chern}} + i_{\theta^{\#}} J \\ b \ &= \ -\tfrac{\mathrm{i}}{\sqrt{2}} S \eta \wedge (\theta - 2\, \mathrm{d} \log S) + 4\mathrm{i}\alpha J - \tfrac{\mathrm{i}}{\sqrt{2}} S (3\mathrm{d}\rho^{+} + \mathrm{d}\rho^{-}) \\ K_2 \ &= \ -\tfrac{1}{\sqrt{2}} (\theta - \mathrm{d} \log S) \end{aligned}$$

# Solving

- Given the differential equations hold, the SUSY conditions are satisfied provided

$$
\begin{aligned}
A^{(0)} &= -\tfrac{9}{4} * \left( \mathrm{d} * b - \tfrac{\mathrm{i}\sqrt{2}}{3} b \wedge b \right) \\
X_2 &= -4\alpha^2 - \tfrac{1}{4}\langle K_2, K_2 \rangle - \tfrac{\mathrm{i}}{6\sqrt{2}} S \langle \eta, A^{(0)} \rangle \\
&\quad - \tfrac{3}{16}\langle \mathrm{d}a_\perp, J \rangle - \tfrac{3}{4\sqrt{2}} \langle K_2, \mathrm{d}\log S \rangle \\
\omega_{\mu\nu} &= \text{horrible mess}
\end{aligned}
$$

- KSE + dilatino are equivalent to the form differential conditions

- Everything determined in terms of the almost contact one-form $\eta$, the Hermitian metric and functions $S$,$\alpha$

$$
\begin{aligned}
\omega_\mu{}^\mu &= 2\sqrt{2}\alpha^2 + \tfrac{\sqrt{2}}{3} X_2 - \langle K_2, \tfrac{1}{\sqrt{2}} K_2 + \mathrm{d}\log S \rangle \\
&\quad + \tfrac{1}{2\sqrt{2}} \langle \mathrm{d}a_\perp, J \rangle - \nabla^{(4)}_m K_2^m
\end{aligned}
$$

# Subclasses

- Product metrics:
  - Set $\rho = 0$ and $S = 1$: $\mathcal{M}_5 = \mathbb{R} \times \mathcal{M}_4$ or $\mathcal{M}_5 = S^1 \times \mathcal{M}_4$
  - Here metric on $\mathcal{M}_4$ is any Hermitian manifold
- Circle bundle over Riemann surfaces: $S^1 \hookrightarrow \Sigma_1 \times \Sigma_2$
  - Fibre over only one RS: $\mathcal{M}_3 \times \Sigma_2$, where $\mathcal{M}_3$ is a Siefert manifold
- If $\mathrm{d}\theta = 0$ then transverse space is locally conformally Kahler
  - When $\theta = 0$, we get 4d Kahler metric
  - For $\theta = 0$ and $\mathrm{d}\rho = \mathbb{R}^+ \times J$ get conformally Sasakian
    - Sasaki-Einstein are a further subclass ($S = 1$)
- Do find conformally flat $S^1 \times S^4$ of [Kim,Kim,Lee] for which $S \neq$ const (cf [Pini,Rodriguez-Gomez,Schmude])

# SUSY gauge theories

- Want to build $\mathcal{N} = 1$ SUSY gauge theories on our backgrounds
- Think of their Lagrangians as being the result of adding a relevant operator in the SCFT and flowing to the IR
- We reinstate the full $SU(2)$ R-symmetry $(\chi, \varphi) \to (\xi_I, \tilde{\xi}_I)$
- These are constructed from vector and hypermultiplets
- The 5d off-shell vector multiplet consists of $(\mathcal{A}, \sigma, \lambda_I, D_{IJ})$
- In 5d an off-shell hypermultiplet does not close unless there are an infinite number of auxiliary fields (Grassmann odd spinors)
- A collection of $r$ on-shell hypermultiplets consists of $(q_I^A, \psi^A)$ with $A = 1,\ldots,2r$
- They are in an arbitrary representation of the gauge group and are coupled to the vector multiplet

# SUSY algebra

- Scaling dimensions/Lorentz covariance fix the form of the SUSYs transformations to be

$$
\begin{aligned}
\delta_\xi \sigma &= \mathrm{i}\varepsilon^{IJ}\xi_I\lambda_J \\
\delta_\xi \mathcal{A}_\mu &= \mathrm{i}\varepsilon^{IJ}\xi_I\gamma_\mu\lambda_J \\
\delta_\xi \lambda_I &= -\frac{1}{2}\gamma^{\mu\nu}\xi_I\mathcal{F}_{\mu\nu} + \gamma^\mu\xi_I\mathrm{D}_\mu\sigma - D_{IJ}\xi^J + \frac{\mathrm{i}}{3\sqrt{2}}\gamma^{\mu\nu}\xi_I b_{\mu\nu}\sigma - \frac{2\sqrt{2}\mathrm{i}}{3}\tilde{\xi}_I\sigma \\
\delta_\xi D_{IJ} &= -2\mathrm{i}\xi_{(I}\gamma^\mu\mathrm{D}_\mu\lambda_{J)} + 2[\sigma,\xi_{(I}\lambda_{J)}] + \frac{2\sqrt{2}}{3}\tilde{\xi}_{(I}\lambda_{J)} - \frac{1}{6\sqrt{2}}\xi_{(I}\gamma^{\mu\nu}\lambda_{J)}b_{\mu\nu} \\
\delta_\xi q_I^A &= -2\mathrm{i}\xi_I\psi^A \\
\delta_\xi \psi^A &= \varepsilon^{IJ}\gamma^\mu\xi_I\mathrm{D}_\mu q_J^A + \mathrm{i}\varepsilon^{IJ}\xi_I\sigma q_J^A - \sqrt{2}\mathrm{i}\varepsilon^{IJ}\tilde{\xi}_I q_J^A
\end{aligned}
$$

- These SUSY transformations close onto

$$
[\delta,\delta]\Phi = -\mathrm{i}v^\mu\mathrm{D}_\mu\Phi + \mathcal{G}(\Phi) - \tfrac{\sqrt{2}\mathrm{i}}{3}\left[w_\Phi\varrho + R - \tfrac{1}{4}\Theta^{\alpha\beta}\gamma_{\alpha\beta}\right]\Phi
$$

# HM Lagrangian

- Closure on the off-shell hyper fermion gives

$$E_\psi \equiv \mathrm{i}\gamma^\mu \mathrm{D}_\mu \psi^A + \sigma\psi^A + \varepsilon^{IJ}\lambda_I q_J^A - \frac{1}{4\sqrt{2}}\gamma^{\mu\nu}\psi^A b_{\mu\nu} = 0$$

- We get the bosonic EOM for free by SUSY
- Integrate the EOM to find

$$\begin{aligned}\mathcal{L}_{\mathrm{hm}} = \Omega_{AB}\Big[ & -\tfrac{1}{2}\varepsilon^{IJ}\mathrm{D}^\mu q_I^A \mathrm{D}_\mu q_J^B + \tfrac{1}{2}\varepsilon^{IJ} q_I^A \sigma^2 q_J^B + \tfrac{\mathrm{i}}{2} q_I^A D^{IJ} q_J^B \\ & - 2\varepsilon^{IJ} q_I^A (\psi^B \lambda_J) + \varepsilon^{IJ} q_I^A q_J^B \left(\tfrac{1}{2\sqrt{2}}\omega_\mu{}^\mu - \tfrac{1}{6}X_2\right) \\ & + \mathrm{i}(\psi^A \gamma^\mu \mathrm{D}_\mu \psi^B) + \psi^A \sigma \psi^B - \tfrac{1}{4\sqrt{2}}(\psi^A \gamma^{\mu\nu}\psi^B) b_{\mu\nu}\Big]\end{aligned}$$

# VM Lagrangian

- The vector multiplet action is determined by the prepotential

$$\mathscr{F}(\mathcal{V}) \;=\; \mathrm{Tr}\left[\tfrac{1}{2\mathrm{g}^2}\mathcal{V}^2 + \tfrac{k}{6}\mathcal{V}^3\right]$$

- To find the cubic part in our backgrounds start from known flat space Lagrangian and use Noether method
- A long time later find $\mathcal{L}_{\text{cubic}}$
- Depends on $\mathrm{d}_{abc} \propto \mathrm{Tr}(\mathrm{T}_{(a}\mathrm{T}_b\mathrm{T}_{c)})$ which vanishes for $Sp(N)$
- $\mathcal{L}_{\text{cubic}}$ is not completely useless, can use it to find $\mathcal{L}_{\text{YM}}$

$$\mathcal{L}_{\text{YM}} \;=\; \tfrac{1}{2\mathrm{g}^2}\mathcal{V}^{(1)}\mathrm{Tr}[\mathcal{V}^2]$$

- Here $\mathcal{V}^{(1)}$ is a constant SUSY preserving vector superfield

# VM Lagrangian

- Choose $\lambda^{(1)} = 0, \sigma^{(1)} = 1$ then $\mathcal{V}^{(1)}$ preserves SUSY iff

$$0 = -\tfrac{1}{2}\gamma^{\mu\nu}\xi_I\Big(\mathcal{F}^{(1)}_{\mu\nu} - \tfrac{\mathrm{i}\sqrt{2}}{3}b_{\mu\nu}\Big) - \Big(D^{(1)}_{IJ} + \tfrac{8\sqrt{2}}{3}\alpha_{IJ}\Big)\xi^J - \tfrac{\sqrt{2}}{3}(K_2)_\mu\gamma^\mu\xi_I$$

- Here we've used $\tilde{\xi}_I = -2\mathrm{i}\alpha_I{}^J\xi_J - \tfrac{\mathrm{i}}{2}(K_2)_\mu\gamma^\mu\xi_I$
- Need to find $D^{(1)}_{IJ}$ and $\mathcal{A}^{(1)}$ with $\mathcal{F}^{(1)} = \mathrm{d}\mathcal{A}^{(1)}$
- Two candidate 1-forms from the geometry: $\mathcal{A}^{(1)} = \{SK_1,\, SK_2\}$
- We know $\mathrm{d}(SK_1)$ and $\mathrm{d}(SK_2)$ so can compare against $\delta\lambda^{(1)} = 0$
- Choosing $\mathcal{A}^{(1)} = SK_2$ doesn't work
- But $\mathcal{A}^{(1)} = SK_1$ does when

$$D^{(1)}_{IJ} = -4\sqrt{2}\alpha_{IJ} \qquad\qquad \mathcal{L}_{K_1}\log S = 0$$

# VM Lagrangian

- This leads to

$$\mathcal{L}_{\mathrm{YM}} = \frac{1}{\mathrm{g}^2}\mathrm{Tr}\Big[\tfrac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \tfrac{1}{2}\mathrm{D}_\mu\sigma\mathrm{D}^\mu\sigma - \tfrac{1}{4}D_{IJ}D^{IJ} + \tfrac{\mathrm{i}}{2}\epsilon^{IJ}(\lambda_I\gamma^\mu\mathrm{D}_\mu\lambda_J) - \tfrac{1}{2}\varepsilon^{IJ}\lambda_I[\lambda_J,\sigma]$$
$$+ \tfrac{1}{8}\epsilon^{\mu\nu\rho\sigma\tau}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}(K_1)_\tau - \tfrac{\mathrm{i}}{\sqrt{2}}\sigma\mathcal{F}_{\mu\nu}b^{\mu\nu} + \tfrac{1}{2}\sigma\mathcal{F}^{\mu\nu}(\mathrm{d}K_1)_{\mu\nu}$$
$$+ 2\sqrt{2}\sigma D^{IJ}\alpha_{IJ} + \sigma^2\left(\tfrac{\sqrt{2}}{3}\omega_\mu{}^\mu + \tfrac{2}{3}X_2 - \tfrac{5}{18}b_{\mu\nu}b^{\mu\nu} - \tfrac{\mathrm{i}}{2\sqrt{2}}(\mathrm{d}K_1)_{\mu\nu}b^{\mu\nu}\right)$$
$$+ \tfrac{\mathrm{i}}{8}\varepsilon^{IJ}(\lambda_I\gamma^{\mu\nu}\lambda_J)(\mathrm{d}K_1)_{\mu\nu} + \tfrac{1}{8\sqrt{2}}\varepsilon^{IJ}(\lambda_I\gamma^{\mu\nu}\lambda_J)b_{\mu\nu} - \sqrt{2}\mathrm{i}(\lambda_I\lambda_J)\alpha^{IJ}\Big]$$

where recall $\omega_\mu{}^\mu, b_{\mu\nu}$, $X_2$ have complicated expressions in terms of the background geometry and

$$\mathrm{d}K_1 = \frac{2\sqrt{2}}{3}\left[4\alpha_{IJ}J^{IJ} + K_1\wedge K_2 + \mathrm{i}b - \tfrac{\mathrm{i}}{2}i_{K_1}(*b)\right]$$

# Summary and outlook

# Summary

- Shown that SUSY solutions to 6d Romans are in 1-to-1 correspondence with differential conditions on a $U(2)$ structure
- The differential conditions allow us to deduce generic 1/4-BPS squashed SE boundary solution
- Successfully compared gravity side to large $N$ field theory
- Also Wilson loops dual for rather general solutions
- The 6d $U(2)$ structure reduces to a $U(2)$ structure on a 5d conformal boundary
- Found the general SUSY preserving 5d $\mathcal{N} = 1$ backgrounds from holography
- The 5-manifold preserves some SUSY iff it is equipped with a CKV which generates a THF
- All the background fields are determined except $\alpha, S$

# and outlook

- The class of gravity solutions with $s \equiv \epsilon^{\mathrm{T}}\epsilon \neq 0$ is non-empty. Can we analyse this case too

- Are all 5d $\mathcal{N} = 1$ backgrounds accounted for?
  - Rigid limit "standard" conformal Weyl multiplet [Pini,Rodriguez-Gomez,Schmude] and related gauge fixed Poincare SUGRA [Pan][Imamura,Matsuno]
  - $\exists$ a second inequivalent (off-shell) "dilaton" Weyl multiplet [Bergshoeff,Cucu,Derix,de Wit,Halbersma,van Proeyen]

- What can we say about background dependence of 5d partition functions?
  - Locally all deformations of [Imamura,Matsuno] backgrounds are Q-exact
  - Given 3d/4d results and our 5d geometry it's natural to conjecture that only the THF enters

Thank you