Lecture 2

- Introduction to mesoscopic physics
- Transport, transmission and probability of quantum diffusion.
- Mesoscopic limit: characteristic length scales.
- Deviation to classical incoherent transport: quantum crossings.
- Weak localization and Sharvin effect.
- Universal conductance fluctuations.

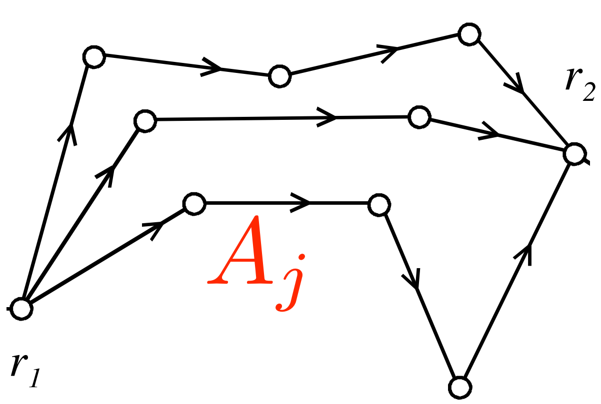
Probability of quantum diffusion

Propagation of a wavepacket centered at energy ϵ between any two points. It is obtained from the probability amplitude (Green's function for the afficionados !) :

$$G_\epsilon(\mathbf{r},\mathbf{r}') = \sum_j A_j(\mathbf{r},\mathbf{r}')$$

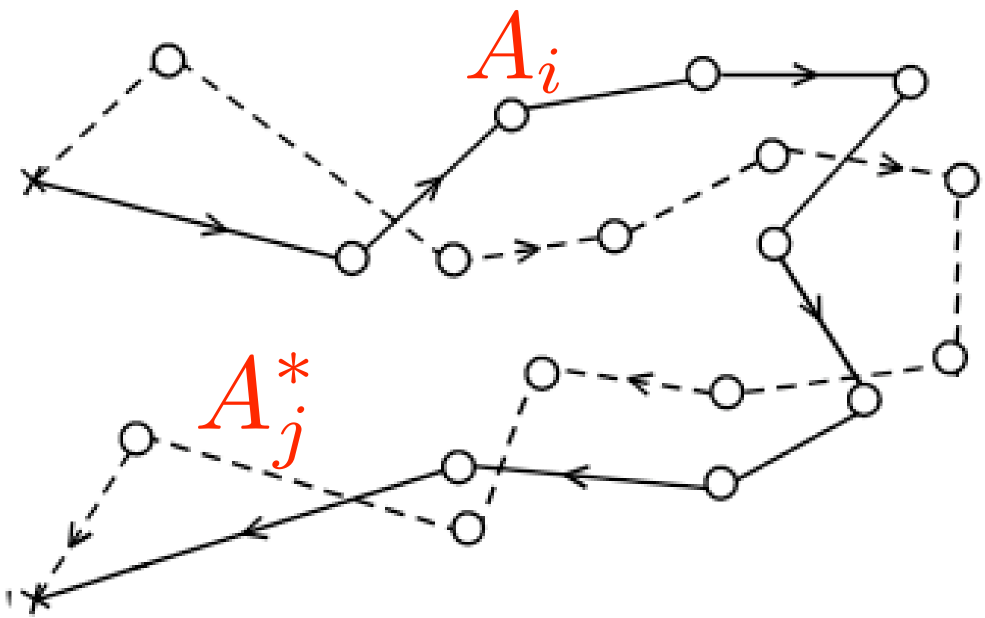
Superposition of amplitudes associated to all multiple scattering trajectories that relate $\mathbf{r}$ and $\mathbf{r}'$. The probability of quantum diffusion averaged over disorder is:

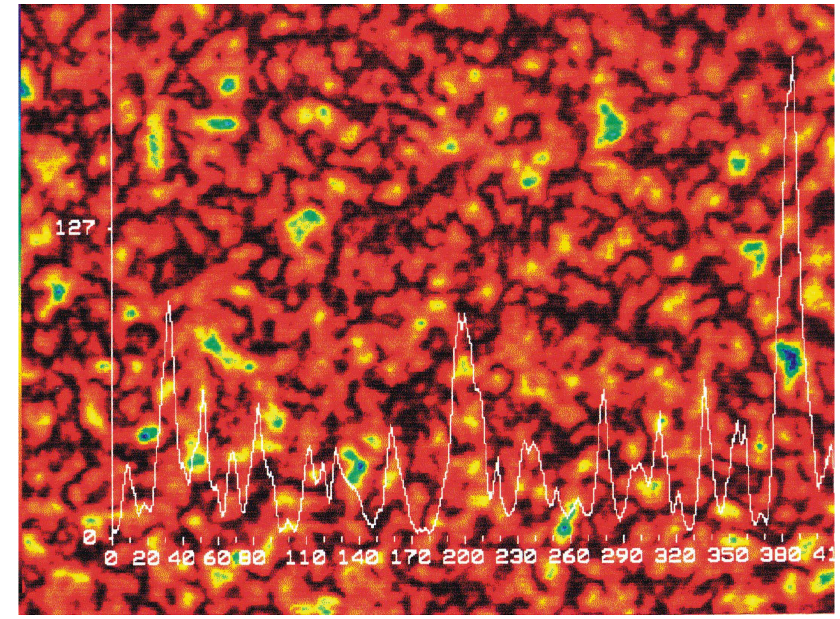
$$P(\mathbf{r},\mathbf{r}') \propto \overline{|G_\epsilon(\mathbf{r},\mathbf{r}')|^2} = \overline{\sum_j |A_j(\mathbf{r},\mathbf{r}')|^2} + \overline{\sum_{i\neq j} A_i^*(\mathbf{r},\mathbf{r}')A_j(\mathbf{r},\mathbf{r}')}$$



classical term

interference between distinct trajectories: vanishes upon averaging

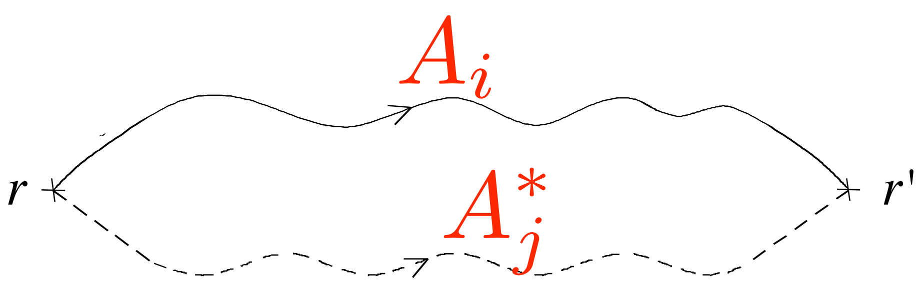




Before averaging : speckle pattern (full coherence)

Configuration average: most of the contributions vanish because of large phase differences.

A new design !



Vanishes upon averaging

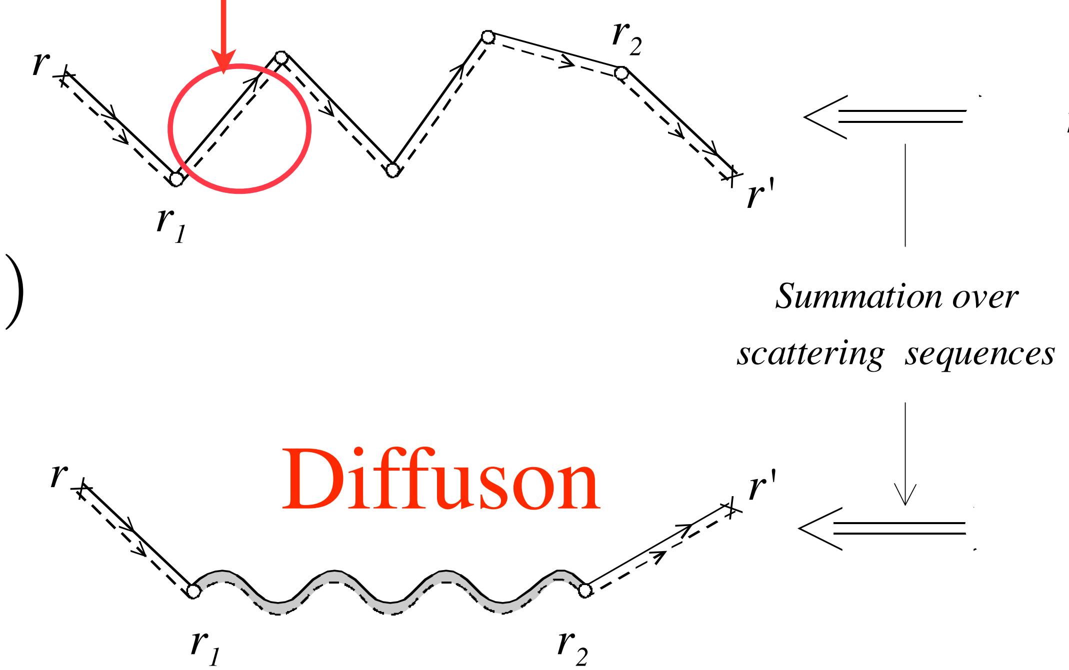
$$P_{cl}(\mathbf{r},\mathbf{r}') = \overline{\sum_j |A_j(\mathbf{r},\mathbf{r}')|^2}$$

Diffuson

The diffusion approximation:

How to calculate $P_{cl}(\mathbf{r},\mathbf{r}')$? It may be obtained as an iteration equation

Iteration of the Drude-Boltzmann term $\boxed{P_0(r,r') = \bar{G}(r,r')\bar{G}^*(r',r) \propto \frac{e^{-R/l}}{R^2}}$



$$P_{cl}(\mathbf{r},\mathbf{r}') = P_0(\mathbf{r},\mathbf{r}') + \frac{1}{\tau}\int d\mathbf{r}'' P_{cl}(\mathbf{r},\mathbf{r}'')P_0(\mathbf{r}'',\mathbf{r}')$$

In the limit of slow spatial and temporal variations, $|\mathbf{r}-\mathbf{r}'| \gg l$ and $t \gg \tau$

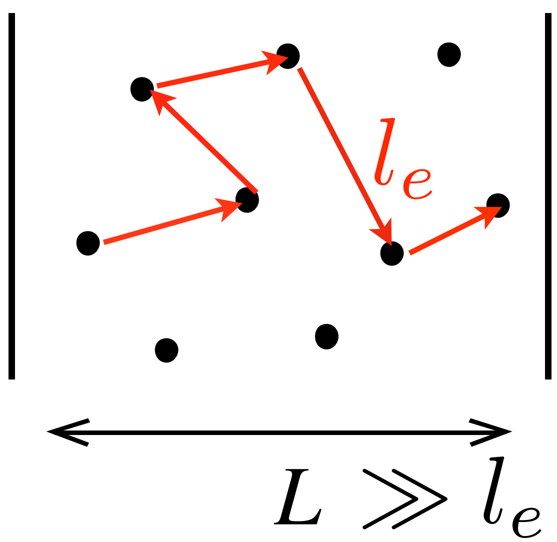
$$\left[\frac{\partial}{\partial t} - D\Delta\right] P_{cl}(\mathbf{r},\mathbf{r}',t) = \delta(\mathbf{r}-\mathbf{r}')\delta(t)$$

with $D = \frac{v_g l}{3}$

(diffusion equation)

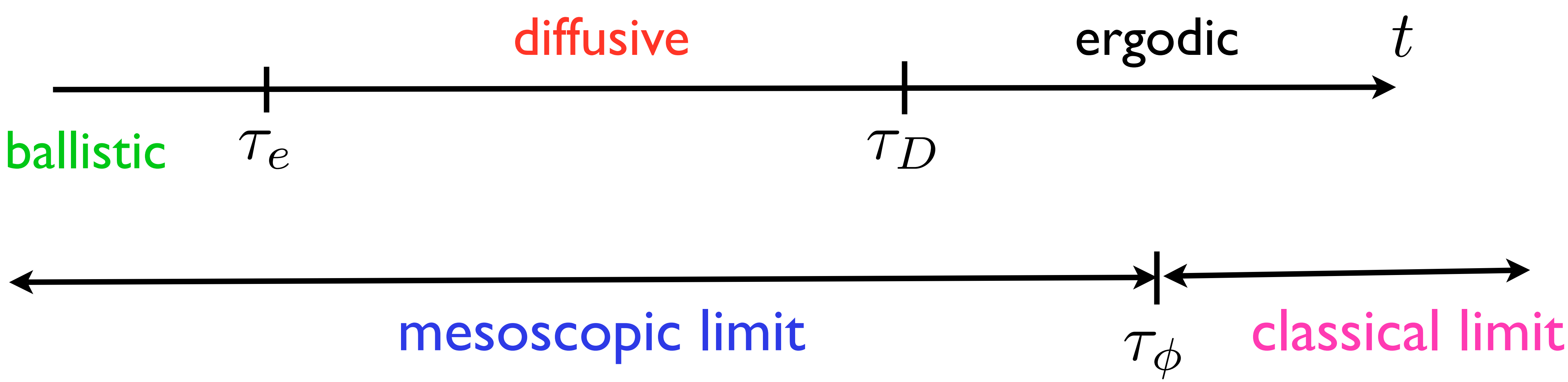
Mesoscopic limit: characteristic length scales

The diffusion motion is characterized by its elementary step, the elastic mean free path l_e related to the elastic collision time by $l_e = v_g \tau_e$



$$\langle R^2 \rangle = Dt \quad \text{with } D = v_g l_e / 3$$

traversal time (Thouless time) : $\boxed{L^2 = D\tau_D}$



Normalization of the probability

The probability of quantum diffusion must be *normalized*,

$$\int dr' \, P(r,r',t) = 1 \quad \forall t \Leftrightarrow \quad P(q=0,\omega) = \frac{i}{\omega}$$

At the approximation of the Diffuson, we have from the iteration eq.

$$P_{cl}(q,\omega) = \frac{P_0(q,\omega)}{1 - \frac{P_0(q,\omega)}{\tau_e}}$$

since

$$P_0(q,\omega) = \frac{\tau_e}{1 - i\omega\tau_e} \rightarrow P_{cl}(q=0,\omega) = \frac{i}{\omega}$$

The Diffuson provides a normalized approx. to the probability of Quantum diffusion ! Missing terms ?

Reciprocity theorem

For time reversal invariant systems, Green's functions have the property:

$$G(r,r',t) = G(r',r,t)$$

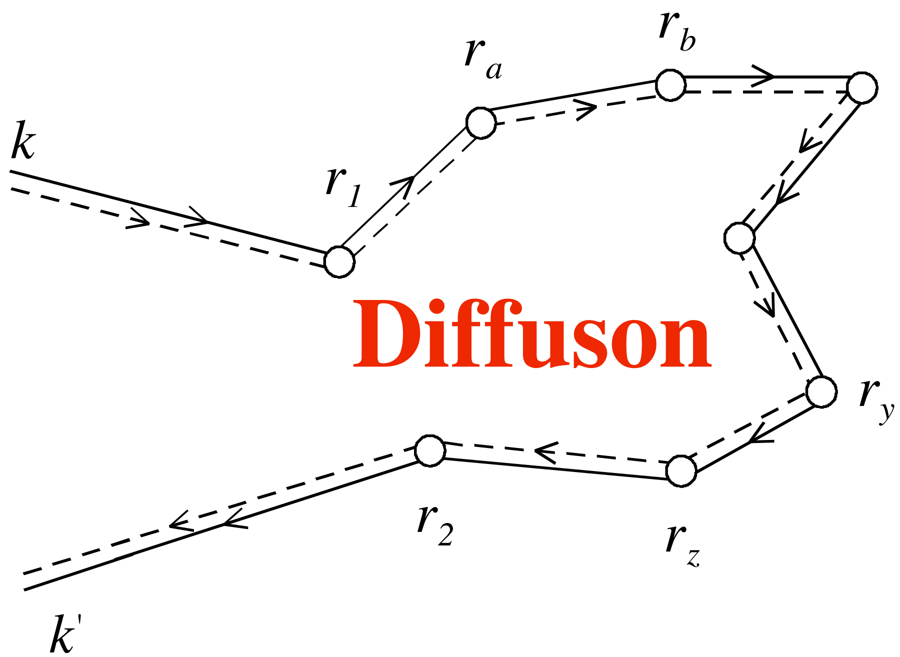
Reciprocity thm. states that complex amplitudes associated to a multiple scattering sequence and its time reversed are equal.

By reversing the two amplitudes of $P_{cl}(r,r')$ gives $P_{cl}(r',r)$

Reversing only ONE of the two amplitudes should also give a contribution to the probability, but it is not anymore a Diffuson!

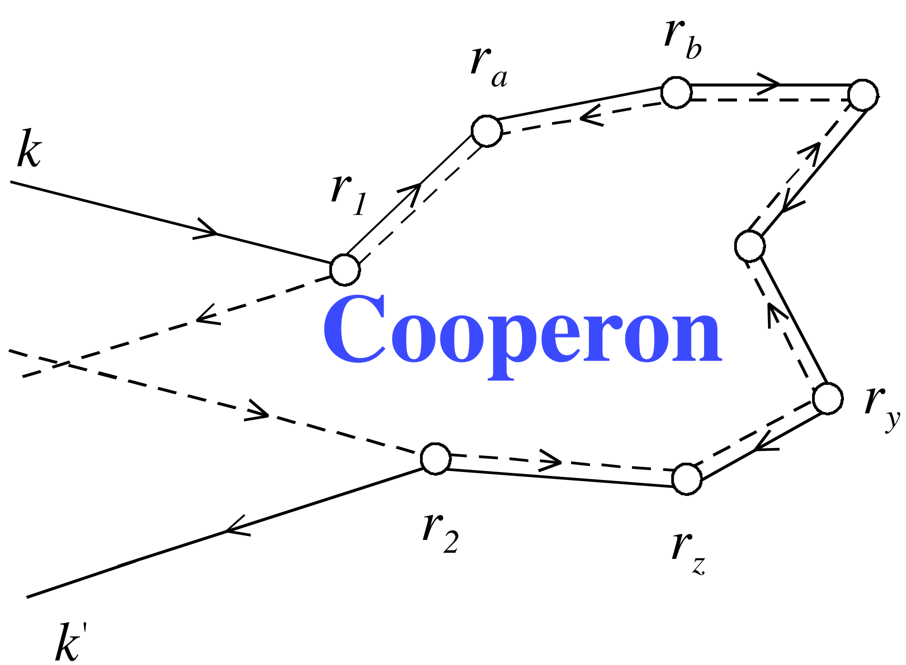
The Diffuson approx. does not take into account all contributions to the probability.

(a)



$$\mathbf{r_1} \longrightarrow \mathbf{r_a} \longrightarrow \mathbf{r_b} \cdots \longrightarrow \mathbf{r_y} \longrightarrow \mathbf{r_z} \longrightarrow \mathbf{r_2}$$

(b)



$$\mathbf{r_2} \longrightarrow \mathbf{r_z} \longrightarrow \mathbf{r_y} \cdots \longrightarrow \mathbf{r_b} \longrightarrow \mathbf{r_a} \longrightarrow \mathbf{r_1}$$

incoherent classical term

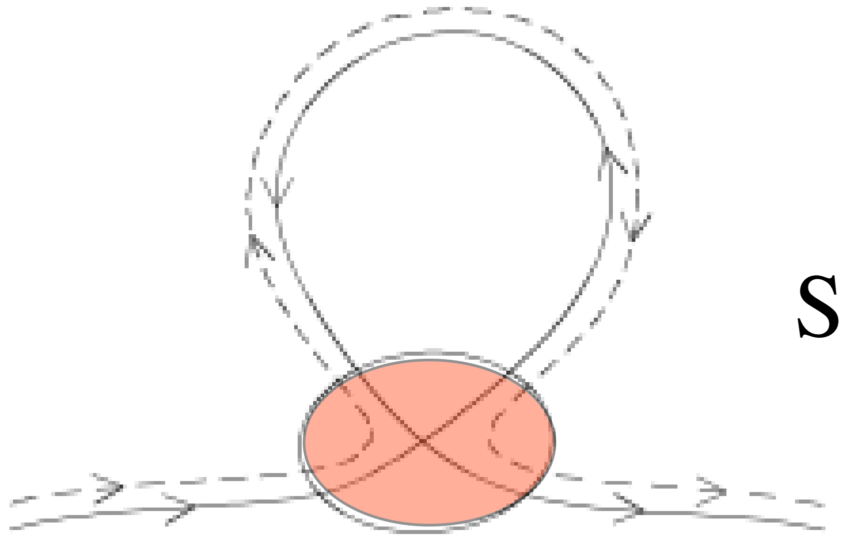
interference term

The total average intensity is:

$$\overline{|A(\mathbf{k},\mathbf{k}')|^2} = \overline{\sum_{\mathbf{r_1},\mathbf{r_2}} |f(\mathbf{r_1},\mathbf{r_2})|^2 \left[1 + e^{i(\mathbf{k}+\mathbf{k}').(\mathbf{r_1}-\mathbf{r_2})}\right]}$$

Quantum crossings

A diffuson is the product of 2 complex amplitudes: it can be viewed as a" diffusive trajectory with a phase". Coherent effects result from the Cooperon which can be viewed as a self-crossing



Crossing mixes the amplitudes and pair them differently $\Rightarrow$ phase shift.

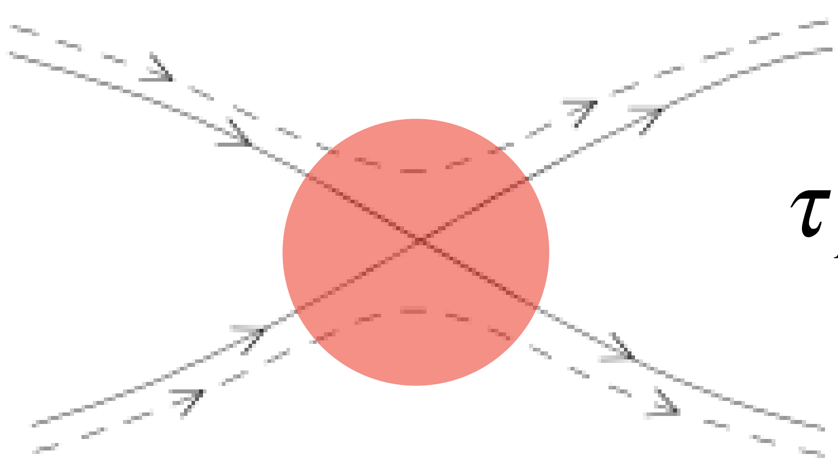
Small phase shift $\leq 2\pi \Rightarrow$ localized crossing

volume of a crossing

$\lambda^{d-1} l_e$

Crossing probability of 2 diffusons:

$$p_\times(t) = \int_0^{\tau_D} \frac{\lambda^{d-1} v_g dt}{L^d} = \frac{1}{g}$$



$\tau_D = L^2 / D$

$$g = \frac{l_e}{3\lambda^{d-1}} L^{d-2} \gg 1$$

Electrical conductance of a metal

A metal can be modeled as a quantum gas of electrons scattered by an elastic disorder.

At T=0 and in the absence of decoherence, it is a complex quantum system.

Classically, the conductance of a cubic sample of size L^d is given by Ohm's law: $G = \sigma L^{d-2}$ where σ is the conductivity.

$$g = \frac{l_e}{3\lambda^{d-1}} L^{d-2} = G_{cl}/(e^2/h)$$

where G_{cl} is the classical electrical conductance s.t.

$$G_{cl}/(e^2/h) \gg 1$$

Weak disorder physics

Weak disorder limit: $\lambda \ll l \Rightarrow g \gg 1$

Probability of a crossing $\left(\propto 1/g\right)$ is small: phase coherent corrections to the classical limit are small

Quantum crossings modify the classical probability (*i.e*. the Diffuson) but it remains normalized

The long range behavior of the Diffuson propagates (localized) coherent effects over large distances.

Quantum crossings are independently distributed :

We can generate higher order corrections to the Diffuson as an expansion in powers of $1/g$

A direct consequence: corrections to electrical transport

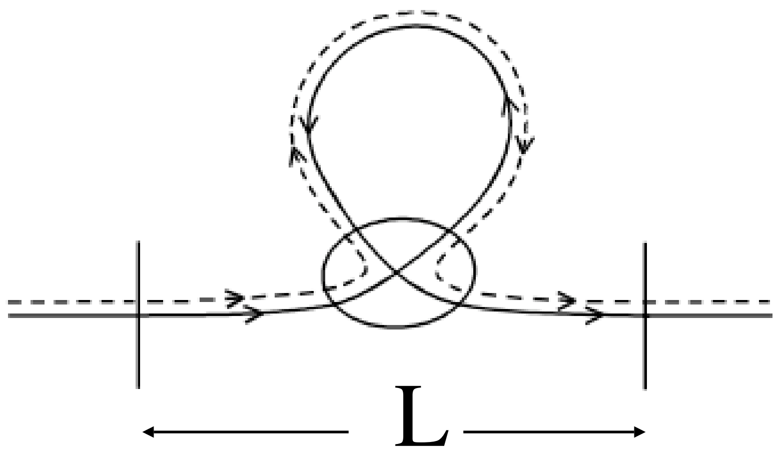
Classical transport : $G_{cl} = g \times \frac{e^2}{h}\,\mathrm{with}\, g \gg 1$

Quantum corrections: $\Delta G = G_{cl} \times \frac{1}{g}$

so that $\boxed{\Delta G \simeq \frac{e^2}{h}}$

Weak localization- Electronic transport

To the classical probability corresponds the Drude conductance G_{cl}



First correction $(\propto 1/g)$ involves one quantum crossing and the probability to have a closed loop:

$$\boxed{\frac{\Delta G}{G_{cl}} \sim -p_o(\tau_D)}$$

$$\tau_D = L^2/D$$

$$\boxed{p_o(\tau_D) \sim \frac{1}{g}\int_0^{\tau_D} Z(t)\frac{dt}{\tau_D}}$$

quantum correction decreases the conductance: weak localization

Return probability $Z(t) = \int dr P_{int}(r,r,t) = \left(\frac{\tau_D}{4\pi t}\right)^{d/2}$

Time reversal invariance

Note that the 2 trajectories involved in a loop evolve in opposite directions. If there is time reversal invariance, amplitudes associated to j and to its time reversed j^T are equal so that their product is the return probability of a classical diffusion process:

$$Z(t) = \int dr P_{int}(r,r,t) = \left(\frac{\tau_D}{4\pi t}\right)^{d/2}$$

$P_{int}(r,r',t)$ is solution of a diffusion equation with $r = r'$

$$P_{int}(r,r,t) = P_{cl}(r,r,t)$$

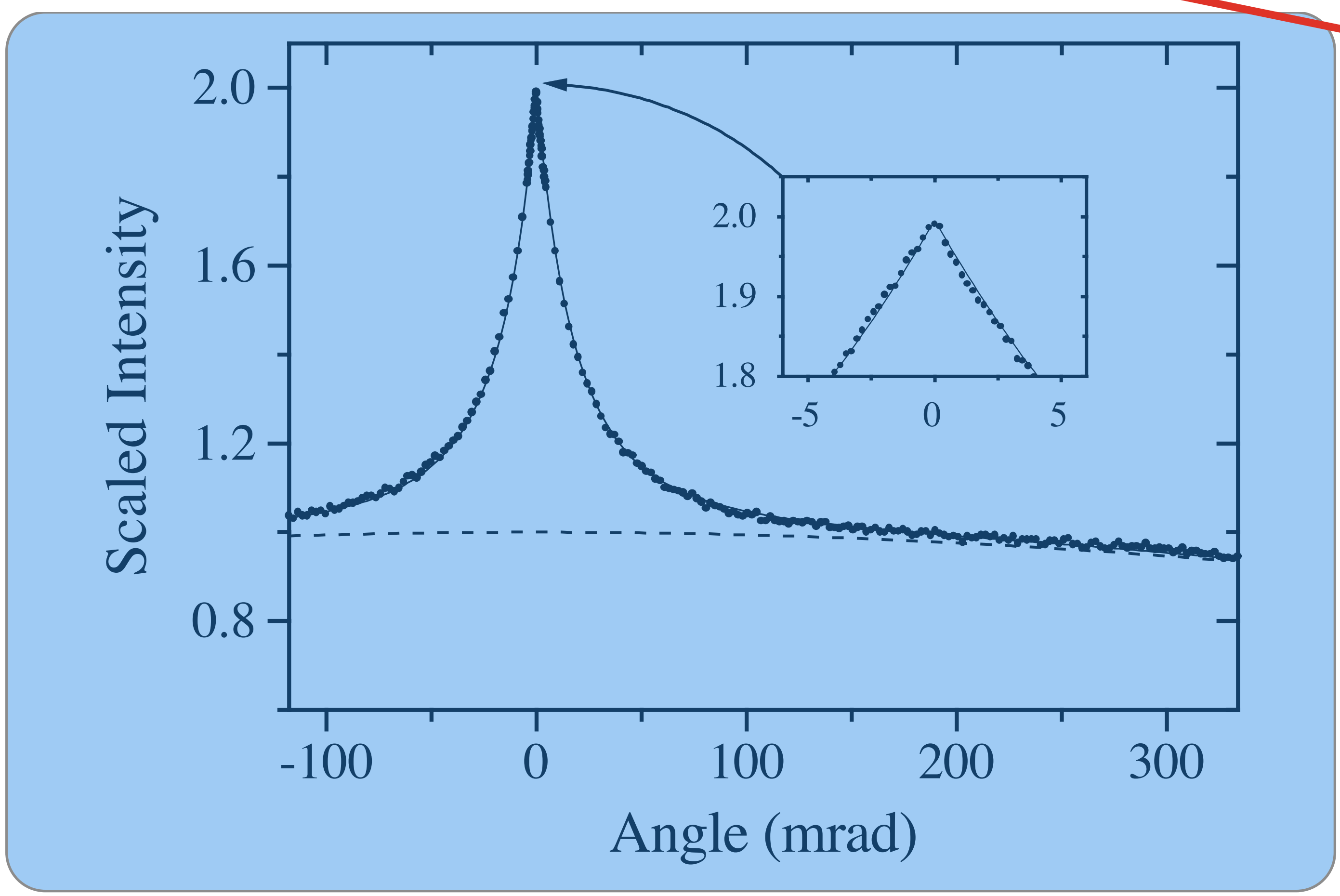
The return probability to the origin is doubled compared to incoherent processes.

$$\overline{|A(\mathbf{k},\mathbf{k}')|^2} = \overline{\sum_{\mathbf{r_1},\mathbf{r_2}} |f(\mathbf{r_1},\mathbf{r_2})|^2 \left[1 + e^{i(\mathbf{k}+\mathbf{k}').(\mathbf{r_1}-\mathbf{r_2})}\right]}$$

Generally, the interference term vanishes due to the sum over $\mathbf{r_1}$ and $\mathbf{r_2}$, except for two notable cases:

$\mathbf{k} + \mathbf{k}' \simeq 0$: Coherent backscattering

$\mathbf{r_1} - \mathbf{r_2} \simeq 0$: closed loops, weak localization and $\phi_0/2$ periodicity of the Sharvin effect.

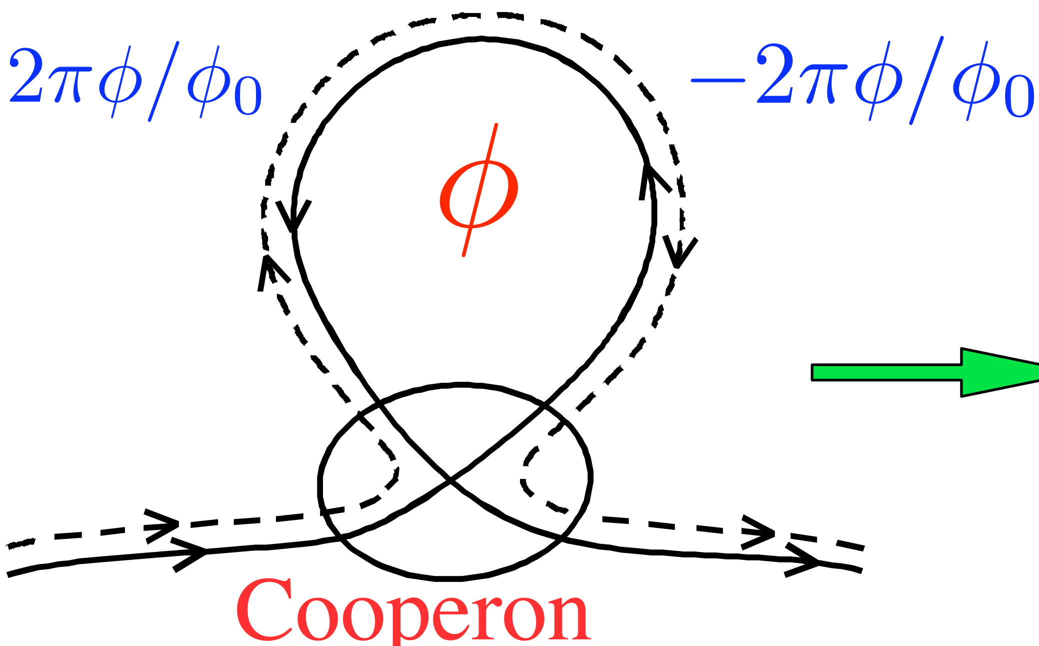


this case

Coherent backscattering

In the presence of a dephasing mechanism that breaks time coherence, only trajectories with $t < \tau_\phi$ contribute.

For instance, in the presence of an Aharonov-Bohm flux, paired amplitudes in the Cooperon acquire opposite phases:



the phase difference becomes: $4\pi\phi/\phi_0$

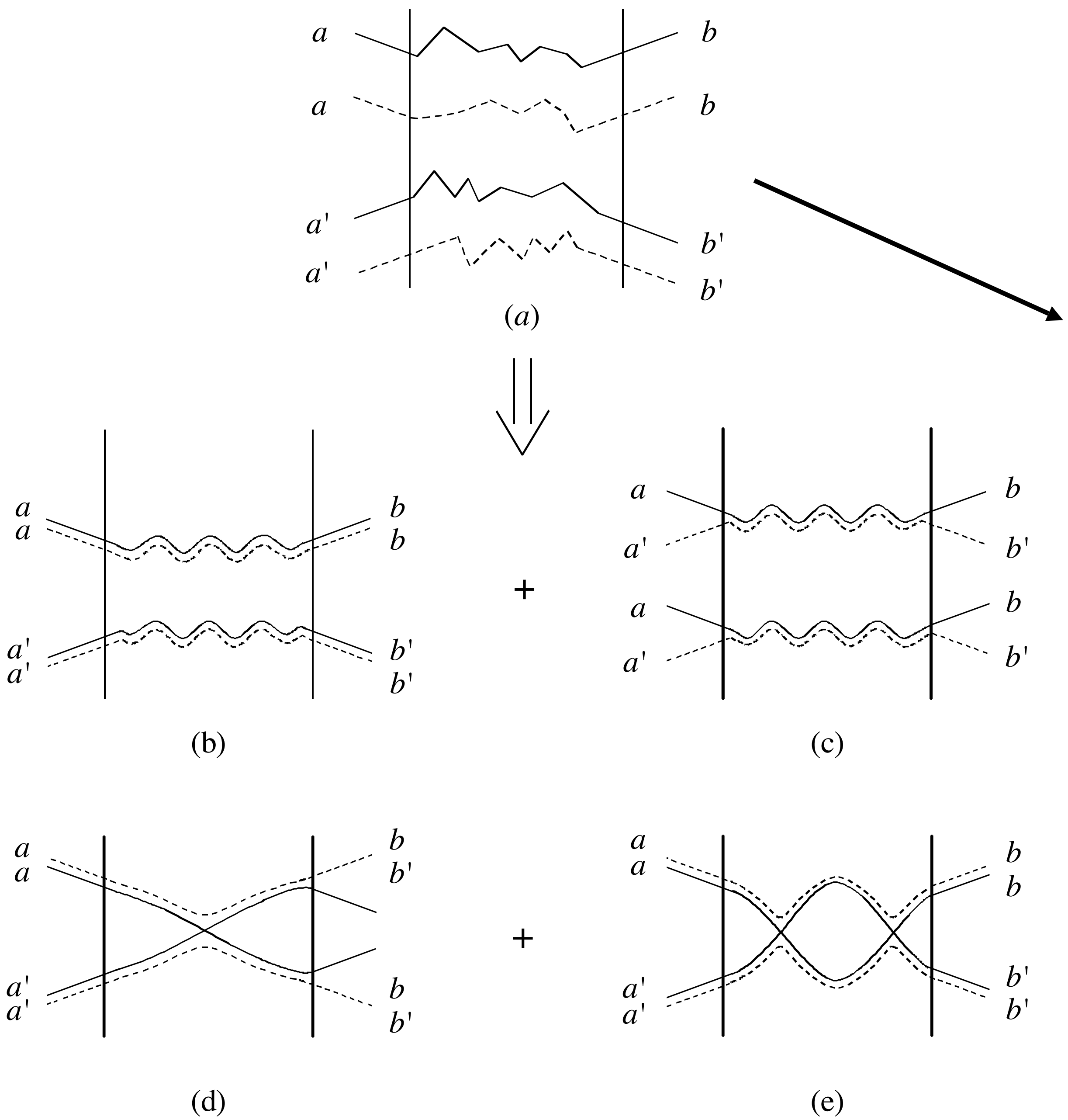
$\rightarrow$ $\phi_0/2$ periodicity of the Sharvin effect

$P_{int}(r, r', t)$ is obtained from the *covariant* diffusion equation

effective charge 2e

$$\left(\frac{1}{\tau_\phi} + \frac{\partial}{\partial t} - D\left[\nabla_{r'} + i\frac{2e}{\hbar}\mathbf{A}(r')\right]^2\right) P_{int}(r, r', t) = \delta(r - r')\delta(t)$$

Fluctuations and correlations



Slab geometry

transmission coefficient

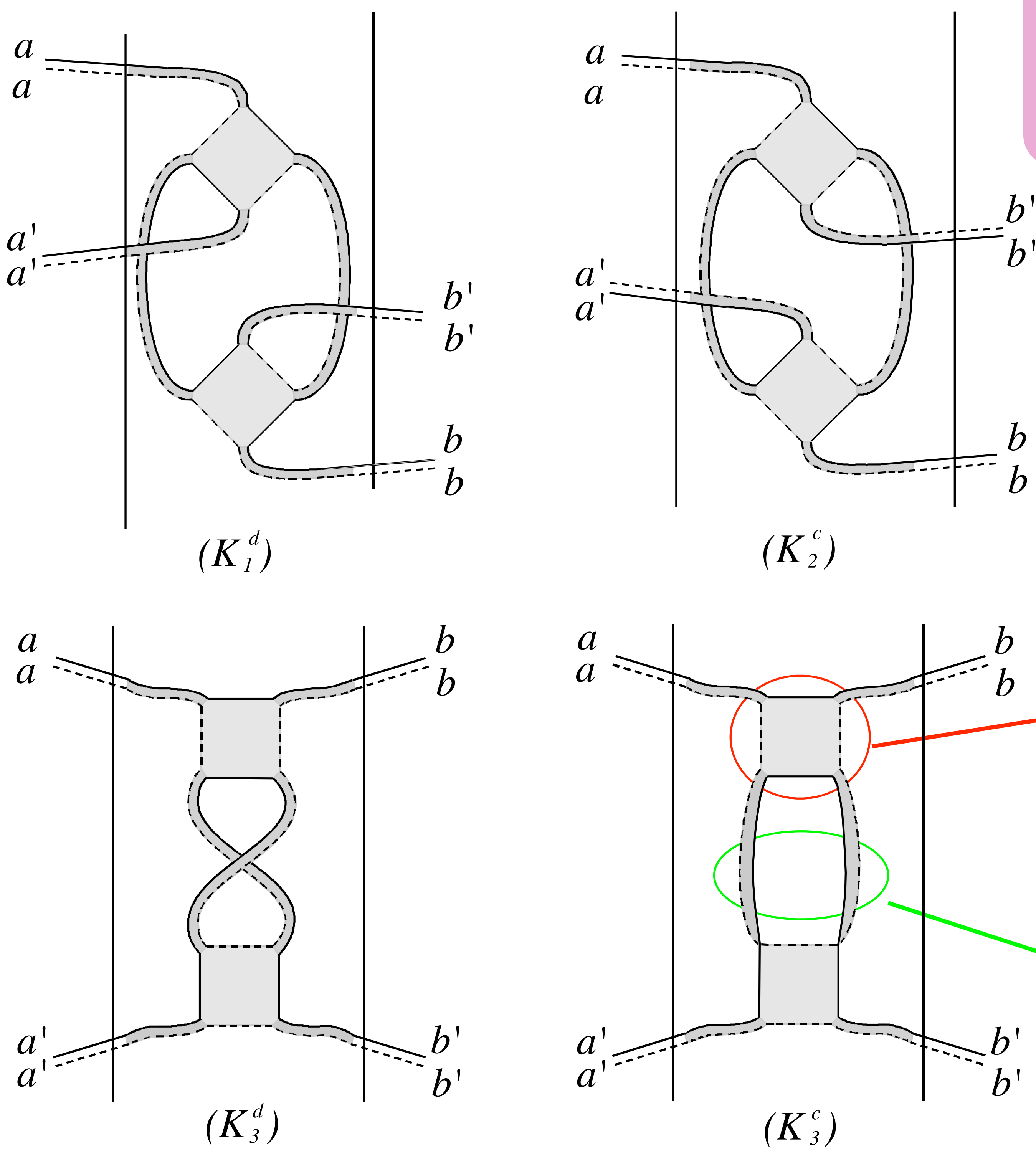
$$\boxed{T_{ab} = \left| t_{ab} \right|^2}$$

correlations involve the product of 4 complex amplitudes with or without quantum crossings

Correlation function of the transmission coefficient :

$$\boxed{C_{aba'b'} = \frac{\overline{\delta T_{ab} \delta T_{a'b'}}}{\overline{T}_{ab} \overline{T}_{a'b'}}}$$

2-quantum crossings correlations



(K_1^d) (K_2^c) (K_3^d) (K_3^c)

$$C^{(3)}_{aba'b'} = \frac{12}{g^2}\frac{D^2}{L^4}\int_0^L\int_0^L dz dz' P_{int}(z,z')^2$$

$$= \frac{2}{15}\frac{1}{g^2}$$

Crossings: coherent effects, spatially localized in a volume $\lambda^{d-1} l$

Long range diffusons (classical)

Propagation of coherent effect over long distances

Universal conductance fluctuations

Landauer description : $G = \frac{e^2}{h} \sum_{ab} T_{ab}$

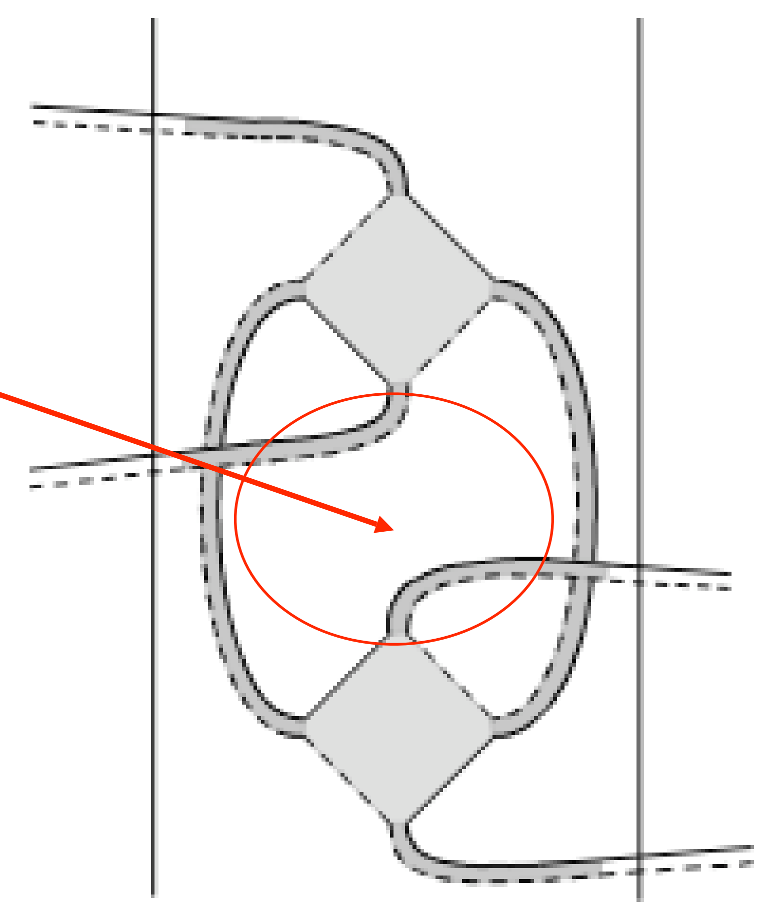
0 crossing: $\overline{G}^2 = G_{cl}^2 = \left(e^2/h\right)^2 \, g^2$

1 crossing: vanishes due to the summation over the channels.

2 crossings: correction $\overline{\delta G^2} \propto \overline{G}^2 / g^2 = (e^2/h)^2$ universal

(very different from the classical self-averaging limit $\overline{\delta G^2} \sim L^{d-4}$)

Depends on the distribution of closed loops



$$\frac{\overline{\delta G^2}}{G_{cl}^2} \sim \frac{1}{g^2} \int_0^{\tau_D} Z(t) \frac{t \, dt}{\tau_D^2}$$

Quantum conductance fluctuations

Classical self-averaging limit : $\frac{\delta G}{\overline{G}} = \frac{1}{N} = \left(\frac{L_\varphi}{L}\right)^{d/2}$

where $\delta G = \sqrt{\overline{G^2} - \overline{G}^2}$ and $\overline{G} = \sigma L^{d-2}$

$\overline{\cdots}$ is the average over disorder.

$$\delta G^2 \propto L^{d-4}$$

In contrast, a mesocopic quantum system is such that :

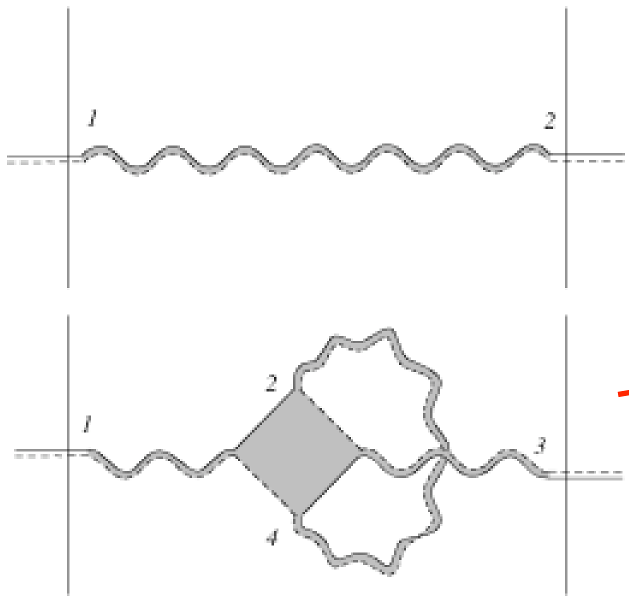
$$\delta G \simeq \frac{e^2}{h}$$

Fluctuations are quantum, large and independent of the source of disorder : they are called universal.

In the mesoscopic limit, the electrical conductance is not self-averaging.

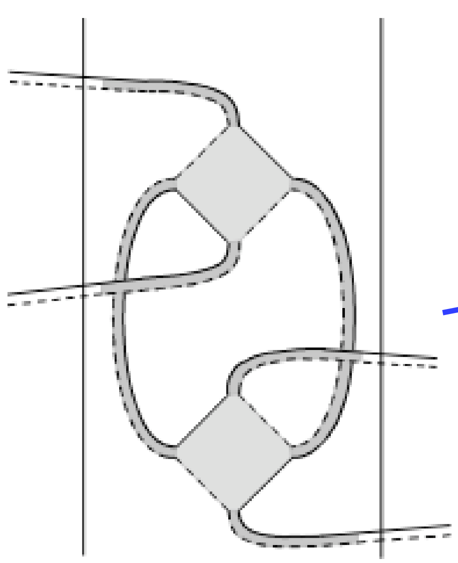
Summarize :

Weak localization corrections to the electrical conductance



$$\frac{\Delta G}{G_{cl}} \sim -\frac{1}{g}\int_0^{\tau_D} Z(t)\frac{dt}{\tau_D}$$

Conductance fluctuations

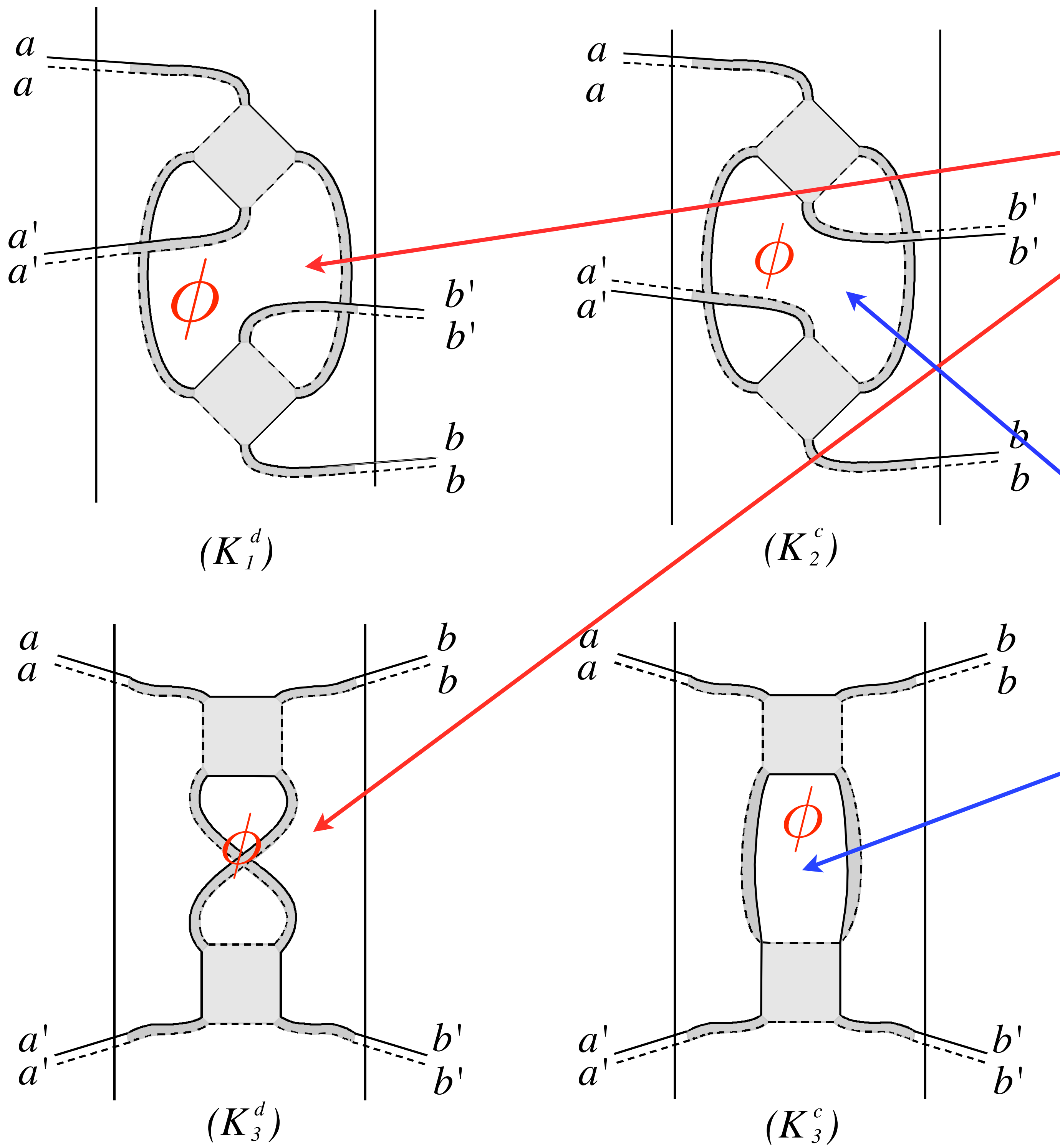


$$\frac{\overline{\delta G^2}}{G_{cl}^2} \sim \frac{1}{g^2}\int_0^{\tau_D} Z(t)\,\frac{t\,dt}{\tau_D^2}$$

$$Z(t)=\int dr\,P_{cl}(r,r,t)=\left(\frac{\tau_D}{4\pi t}\right)^{d/2}$$

Dephasing and decoherence

Universal conductance fluctuations



(K_1^d) (K_2^c) (K_3^d) (K_3^c)

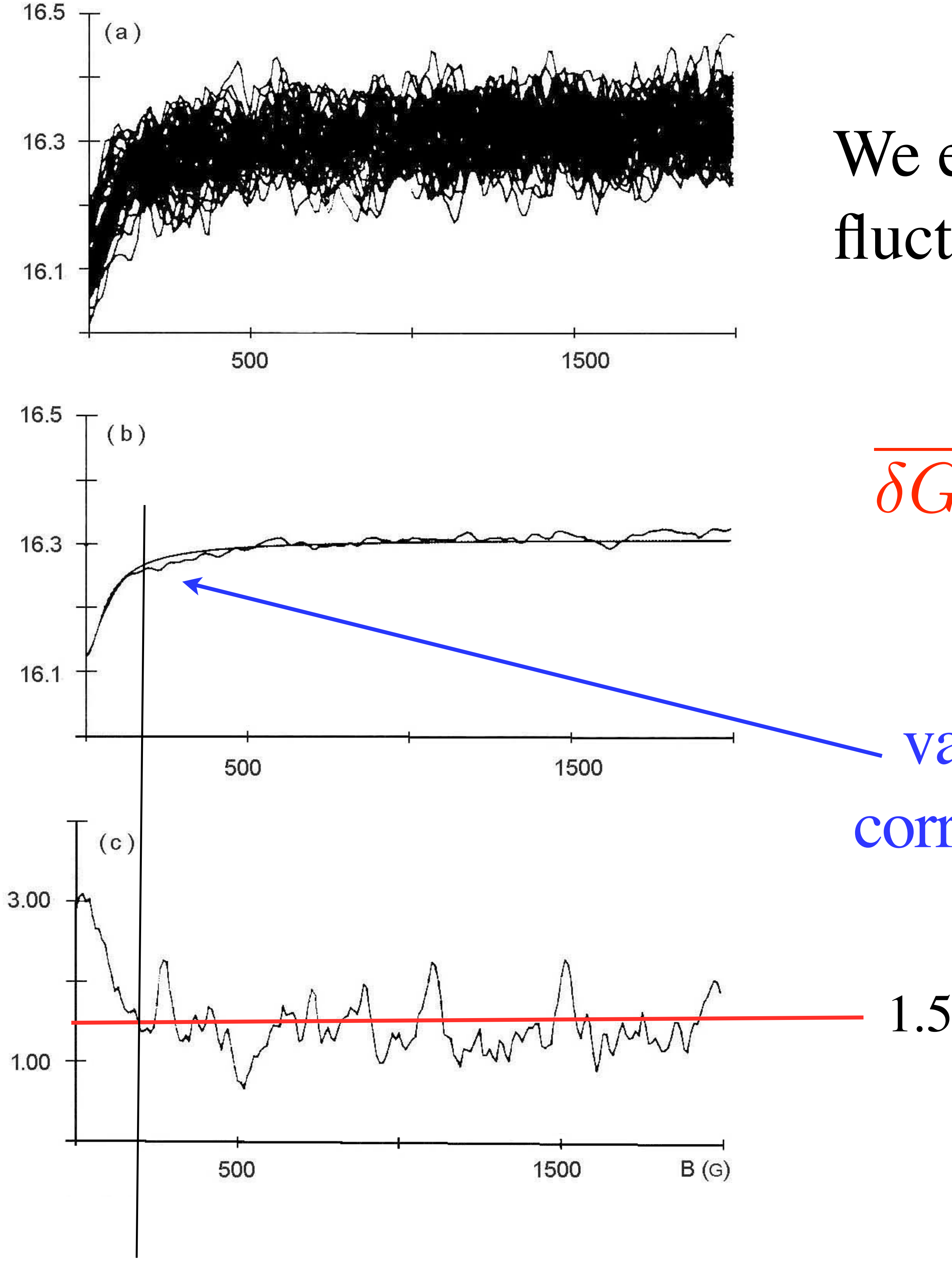
2 Diffusons

2 Cooperons

sensitive to an applied Aharonov-Bohm magnetic flux

ϕ

46 *Si*-doped *GaAs* samples at 45 mK



We expect the conductance fluctuations to be reduced by a factor 2

$$\overline{\delta G^2} \xrightarrow{\phi} \frac{\overline{\delta G^2}}{2}$$

vanishing of the weak localization correction for the same magnetic field

In the presence of incoherent processes $L > L_\phi$:

$$\overline{\delta G^2} \to 0$$