Lecture 4

Diffusing photons and superradiance in cold gases

- Model of disorder-Elastic mean free path and group velocity.
- Dicke states- Super- and sub-radiance.
- Scattering properties of Dicke states.
- Multiple scattering and superradiance.
- Comparison with experiments: slow diffusion of light.
- Strong disorder-Effective Hamiltonian.
- Distribution of escape times

Scalar waves in random media

Scalar and monochromatic (k_0) electromagnetic wave: $\psi(r)$ is the electric field, solution of the Helmholtz wave equation:

$$-\nabla^2\psi - k_0^2\mu(r)\psi = k_0^2\psi$$

(compare to Schrödinger equation with disorder)

Disorder potential is continuous : fluctuations of dielectric constant

$$V(r) = -k_0^2\mu(r) = \delta\epsilon/\overline{\epsilon}$$

Gaussian white noise model:
easy to do calculations

$$\langle V(r)\rangle = 0$$
$$\langle V(r)V(r')\rangle = B\delta(r-r')$$

B is related to scattering properties of individual scatterers

Edwards model : N_i identical, localized, randomly distributed scatterers

$$V(r) = \sum_{j=1}^{N_i} v(r - r_j)$$

The potential $v(r)$ is short range compared to k_0^{-1} so that

$$v(r - r_j) = v_0 \delta(r - r_j)$$

Weak potential limit (Born approximation) the scattering cross section of a single scatterer is

$$\sigma = \frac{v_0^2}{4\pi}$$

and $B = n_i v_0^2$ where n_i is the density of scatterers.

Average amplitude of the field

Solution of the wave eq. with a source $j(r)$ is given in terms of the Green's function $G(r,r')$:

$$\psi(r) = \int dr' j(r') G(r,r')$$

Solution of $\left(\nabla^2 + k_0^2 - V(r)\right) G(r,r') = \delta(r-r')$

$G(r,r')$ may be expressed in terms of the free Green's function $G_0(r,r')$ without scattering potential:

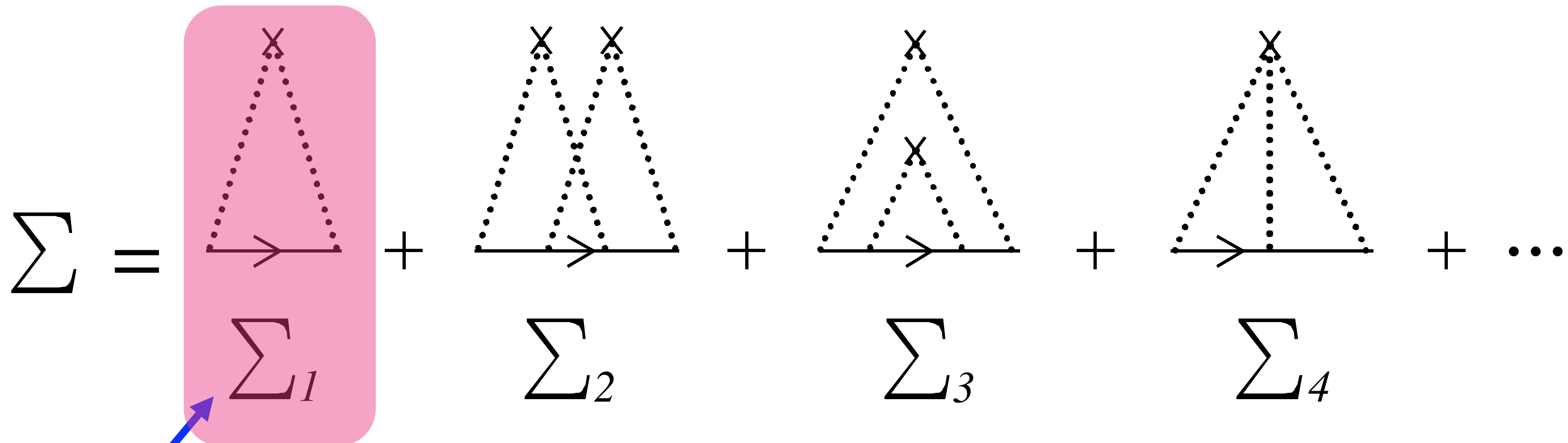
$$G(r,r') = G_0(r,r') - \int dr_1\, G(r,r_1) V(r_1) G_0(r_1,r')$$

Disorder average restores translational invariance and the Fourier transform of the Green's function is $\bar{G}(k)$

$\bar{G}(k)$ is expressed in terms of the *self-energy* $\Sigma(k)$

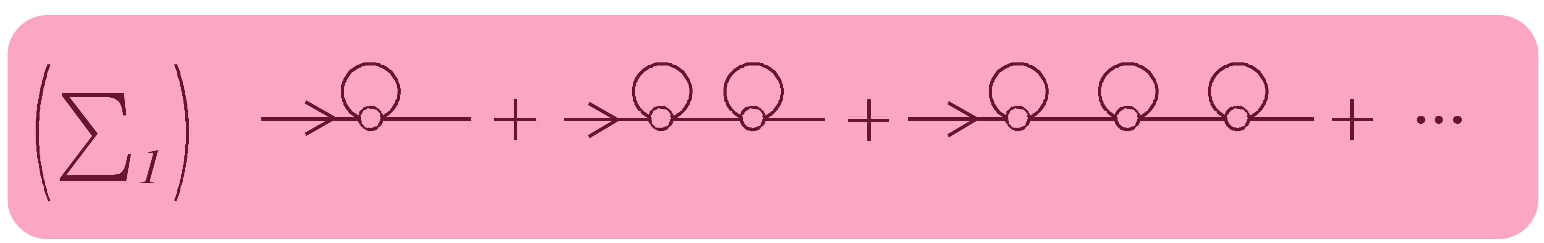
$$\boxed{\bar{G}(k) = G_0(k)\left[1 + \Sigma(k)\bar{G}(k)\right]}$$

the self-energy is given by the sum of irreducible scattering events

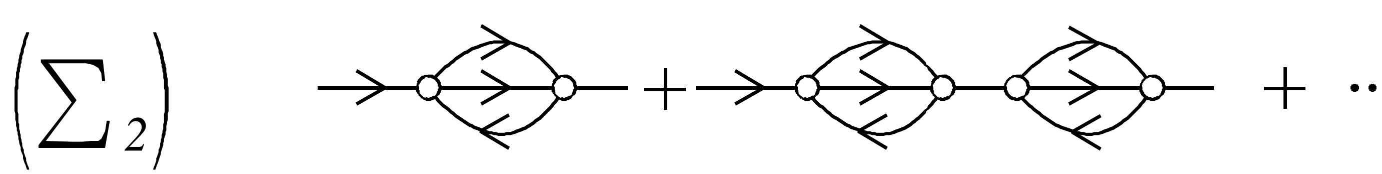


The main contribution to $\Sigma(k)$ neglects interference effects between scatterers,

In real space:



interferences :



The self-energy Σ is a complex valued function. Its imaginary part defines the elastic mean free path l,

$$\boxed{\frac{k_0}{l} = -\operatorname{Im}\Sigma_1(k) = n_i\sigma}$$

the first neglected term provides a correction

$$\operatorname{Im}\Sigma_2(k) = \frac{\pi}{2k_0 l}\operatorname{Im}\Sigma_1(k)$$

identify the small parameter $\frac{1}{k_0 l} \ll 1$: *weak disorder* $\boxed{\Leftrightarrow k_0 l \gg 1}$

Average Green function:

$$\boxed{\bar{G}(r,r') = G_0(r,r')e^{-R/2l} = -\frac{1}{4\pi}\frac{e^{ik_0 R}}{R}e^{-R/2l}}$$

without disorder

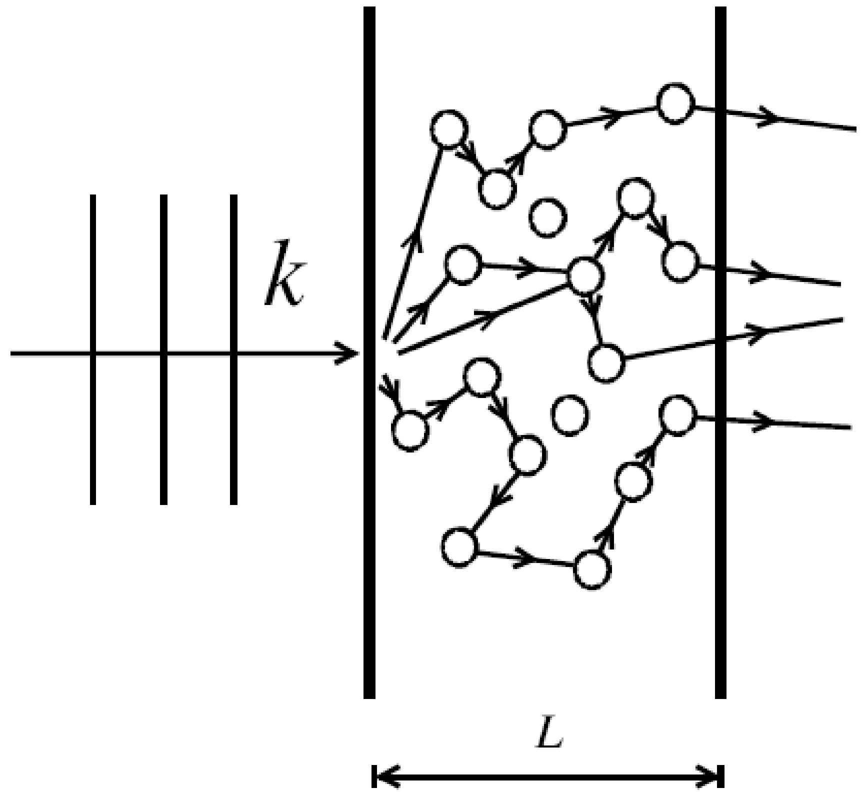
Σ_1 is proportional to the average polarizability of the scattering medium, so that its real part gives the average index of refraction $\eta = ck/\omega$

$$\eta = \left(1 - (\frac{c}{\omega})^2 \mathrm{Re}\Sigma_1\right)^{1/2}$$

The group velocity $v_g = \dfrac{d\omega}{dk}$ of the wave inside the medium is,

$$\frac{c}{v_g} = \eta + \omega \frac{d\eta}{d\omega} = \frac{1}{\eta}\left(1 - (\frac{c^2}{2\omega})\frac{d}{d\omega}\mathrm{Re}\Sigma_1\right)$$

Summary: multiple scattering



Characteristic lengths:

Wavelength: λ_0

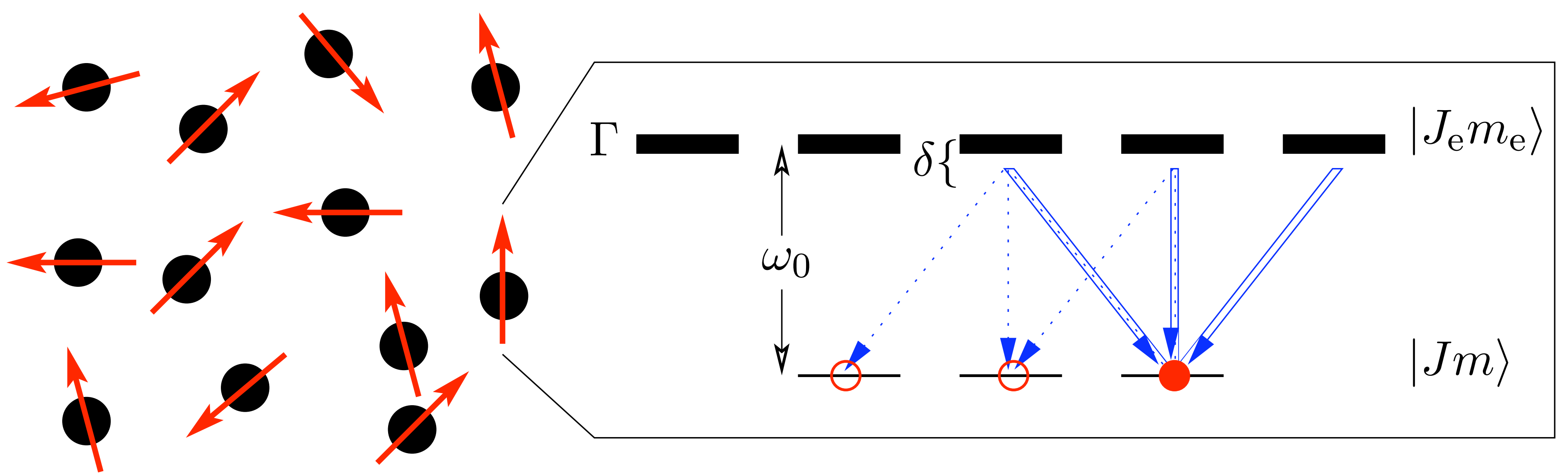
Elastic mean free path: $l = \dfrac{1}{n_i \sigma} \gg n_i^{-1/d}$

density of scatterers

scattering cross section

Weak disorder $\lambda_0 \ll l \Leftrightarrow$ independent scattering events

Multiple and resonant scattering of photons by a cold atomic gas.



Scattering cross section and elastic mean free path

The scattering cross section σ is related to the elastic mean free path l by $l = 1/n_i\sigma$

$$\frac{\lambda}{l} = n_i a_{JJ_e} \frac{3\lambda^3}{2\pi} \frac{1}{1+(2\delta/\Gamma)^2}$$

with $a_{JJ_e} = \frac{1}{3}\frac{2J_e+1}{2J+1}$

Resonant scattering is much more efficient than Rayleigh scattering

We have used a model of disorder where scatterers are independent : Edwards model or white noise

In atomic gases, there are cooperative effects *(superradiance, subradiance)* that lead to an interacting potential between pairs of atoms.

Dicke states:

$$|g\rangle = |J_g = 0, m_g = 0\rangle$$
$$|e\rangle = |J_e = 1, m_e\rangle \; \textit{natural width } \Gamma$$

Pair of two-level atoms in their ground state + absorption of a photon.
Unperturbed 0-photon states :

$$\textit{Singlet Dicke state}: |00\rangle = \frac{1}{\sqrt{2}}\left[|e_1 g_2\rangle - |g_1 e_2\rangle\right]$$

Triplet Dicke states :

$$|11\rangle = |e_1 e_2\rangle, \; |10\rangle = \frac{1}{\sqrt{2}}\left[|e_1 g_2\rangle + |g_1 e_2\rangle\right], \; |1-1\rangle = |g_1 g_2\rangle$$

Second order in perturbation theory in the coupling to photons

$$\varepsilon V_e(r) = -\varepsilon \frac{\hbar\Gamma}{2}\frac{\cos k_0 r}{k_0 r}$$

$$\Gamma^{(\varepsilon)} = \Gamma\left(1+\varepsilon\frac{\sin k_0 r}{k_0 r}\right)$$

Superradiance

$$V_e(r) \propto -\frac{1}{r}$$

(attractive at short distance)

Superradiant state $\varepsilon = +1$

$$\frac{1}{\sqrt{2}}\left[\left|e_1 g_2\right\rangle + \left|g_1 e_2\right\rangle\right]$$

$$\Gamma^{(+1)} = 2\Gamma \ \ (\text{for } r = 0)$$

Characteristics of superradiance

Subradiant state $\varepsilon = -1$

$$\frac{1}{\sqrt{2}}\left[\left|e_1 g_2\right\rangle - \left|g_1 e_2\right\rangle\right]$$

$$\Gamma^{(-1)} = 0$$

Photon is trapped by the two atoms

Scattering properties of Dicke states

Scattering amplitudes of a photon by pairs of atoms in *superradiant* T^+ or *subradiant* T^- states are:

$$T^+ = Ae^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{R}} \cos\left(\frac{\mathbf{k}\cdot\mathbf{r}}{2}\right) \cos\left(\frac{\mathbf{k}'\cdot\mathbf{r}}{2}\right) G^+$$

$$|\mathbf{k}| = |\mathbf{k}'| = k_0$$

$$T^- = Ae^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{R}} \sin\left(\frac{\mathbf{k}\cdot\mathbf{r}}{2}\right) \sin\left(\frac{\mathbf{k}'\cdot\mathbf{r}}{2}\right) G^-$$

$$G^\pm = \frac{1}{\hbar\left(\delta + i\frac{\Gamma}{2} \pm \frac{\Gamma}{2}\frac{e^{ik_0 r}}{k_0 r}\right)}$$

$$\mathbf{R} = \frac{\mathbf{r_1} + \mathbf{r_2}}{2} \qquad \mathbf{r} = \mathbf{r_1} - \mathbf{r_2}$$

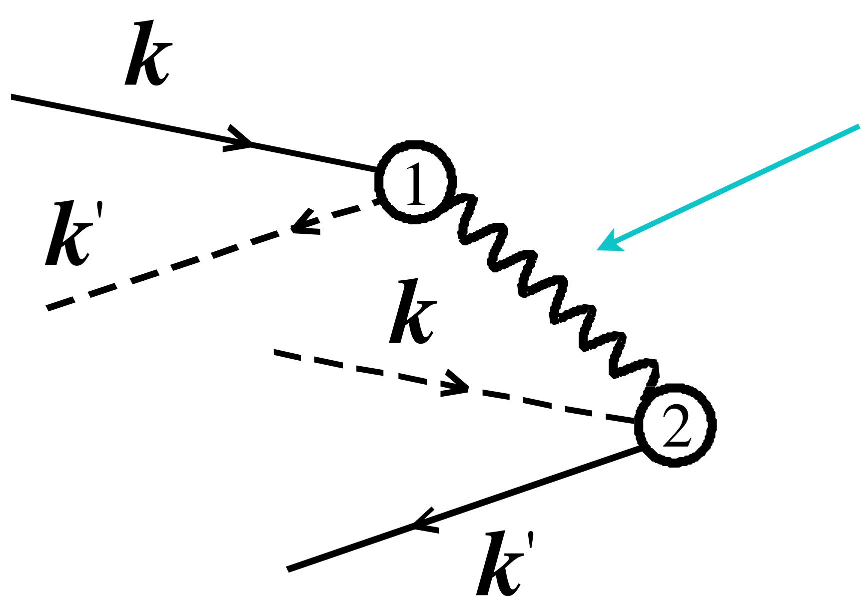
$$\delta = \omega - \omega_0 \quad \text{(detuning)}$$

Photon frequency   two-level spacing

At short distance $k_0 r \ll 1$ between the two atoms, the subradiant term T^- is negligible compared to the superradiant term T^+

Superradiance and Cooperon

The scattering diagram of a photon on a superradiant state is analogous to a Cooperon



infinite number of exchanges of a virtual photon

$$|I|^2 = 2\left|\frac{t^2G_0}{1-t^2G_0^2}\right|^2 [1+\cos(\mathbf{k}+\mathbf{k}')\cdot(\mathbf{r_1}-\mathbf{r_2})]$$

Multiple scattering and superradiance

Multiple scattering of a photon by atoms in superradiant states, i.e. coupled by the attractive potential $V_e(r) \propto -1/r$

Use Edwards model to calculate the self-energy $\Sigma_e^{(1)}$ in the weak disorder limit $k_0 l \gg 1$

$$\Sigma_e^{(1)} = \frac{6\pi n_i}{k_0 r_m} \int_0^{r_m} \frac{dr}{\frac{\delta}{\Gamma} + \frac{1}{2k_0 r} + i}$$

n_i atomic density

r_m maximum separation between the two atoms.

$$\frac{k_0}{l} = -\mathrm{Im}\Sigma_e^{(1)}$$

Elastic mean free path

$$\frac{c}{v_g} = \frac{1}{\eta}\left(1 - \frac{c^2}{2\omega}\frac{d}{d\omega}\mathrm{Re}\Sigma_e^{(1)}\right)$$

Group velocity

Index of refraction: $\eta = \left(1 - \left(\frac{c}{\omega}\right)^2 \mathrm{Re}\Sigma_e^{(1)}\right)^{1/2}$

Absence of divergence of the group velocity

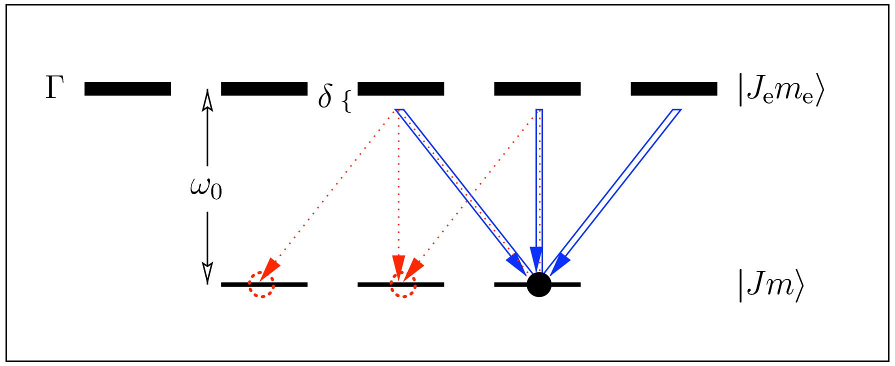
- The group velocity at resonance is

$$\frac{c}{v_g(0)} = 1 + \frac{4\pi n_i}{k_0^3}\frac{\omega_0}{\Gamma}(k_0 r_m)^2$$

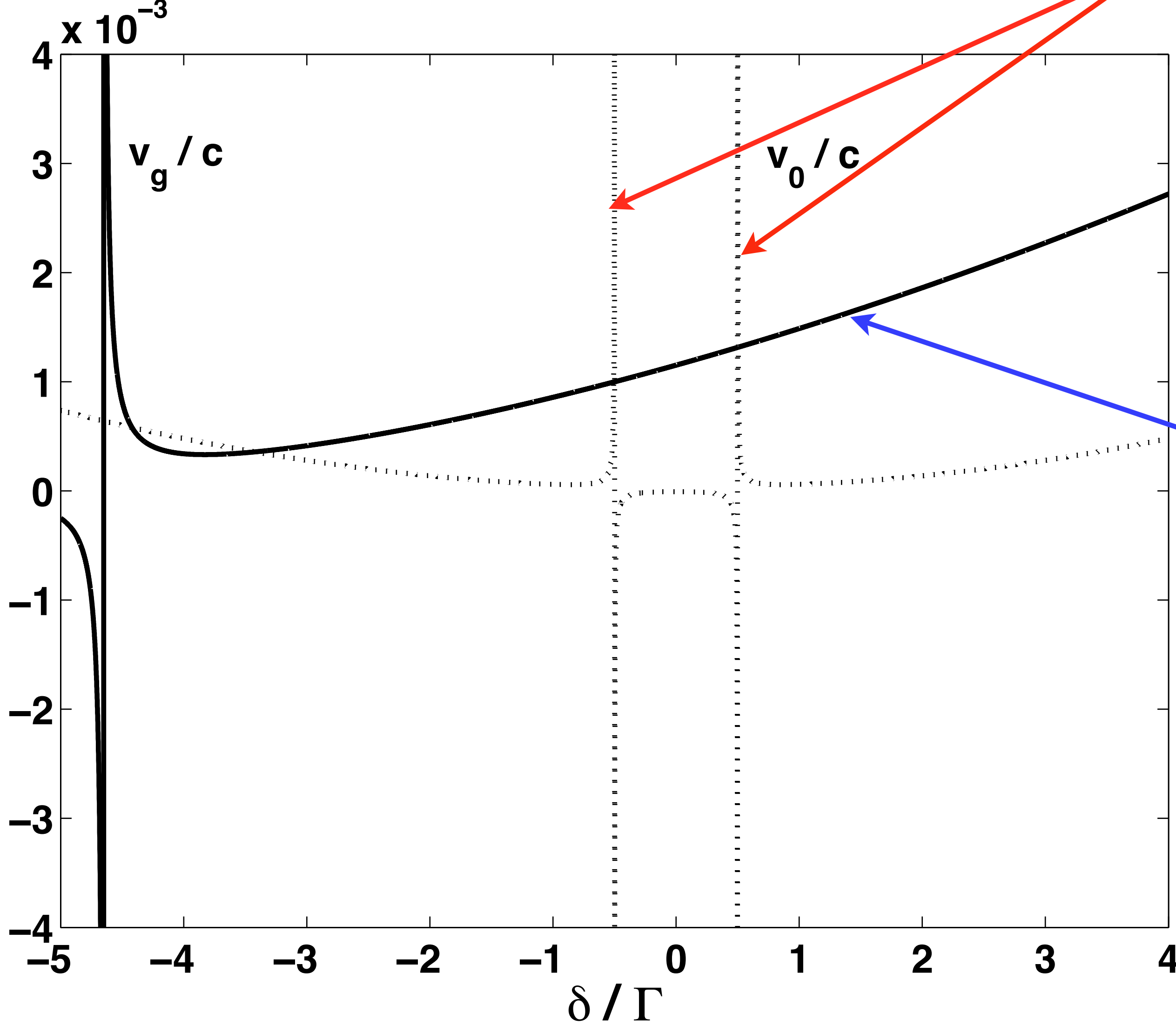
$\simeq 0.26$

$$\frac{v_g(0)}{c} \simeq 3.84 \times 10^{-5}$$

$\simeq 10^5$ for ^{85}Rb



Divergence of the group velocity for scattering by independent atoms



Group velocity for superradiant states

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Slow Diffusion of Light in a Cold Atomic Cloud

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We study the diffusive propagation of multiply scattered light in an optically thick cloud of cold rubidium atoms illuminated by a quasiresonant laser beam. In the vicinity of a sharp atomic resonance, the energy transport velocity of the scattered light is almost 5 orders of magnitude smaller than the vacuum speed of light, reducing strongly the diffusion constant. We verify the theoretical prediction of a frequency-independent transport time around the resonance. We also observe the effect of the residual velocity of the atoms at long times.

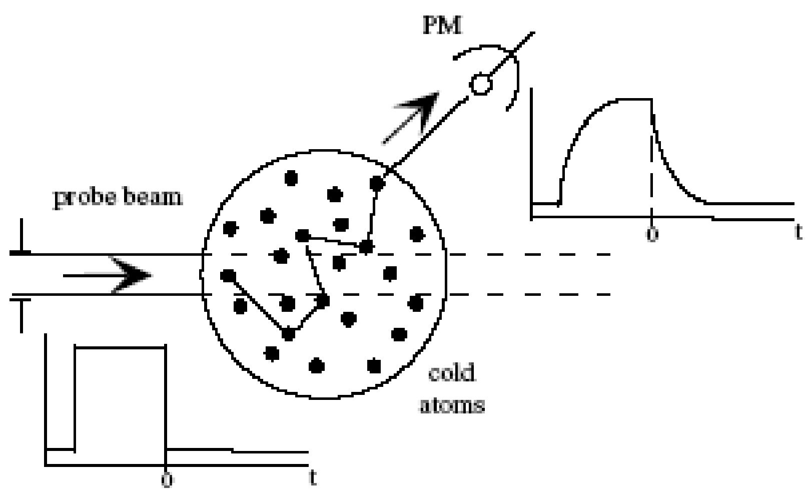


FIG. 1. Radiation trapping (RT) experimental scheme. A pulsed probe beam is sent through the center of a laser-cooled atomic cloud. The transmitted diffuse light is collected as a function of time in a solid angle close to the forward direction.

Weak disorder: $k_0 l \simeq 2 \times 10^3 \gg 1$

$$\frac{v_g(0)}{c} \simeq 3.1 \times 10^{-5}$$

$$D \simeq 0.66 m^2/s$$

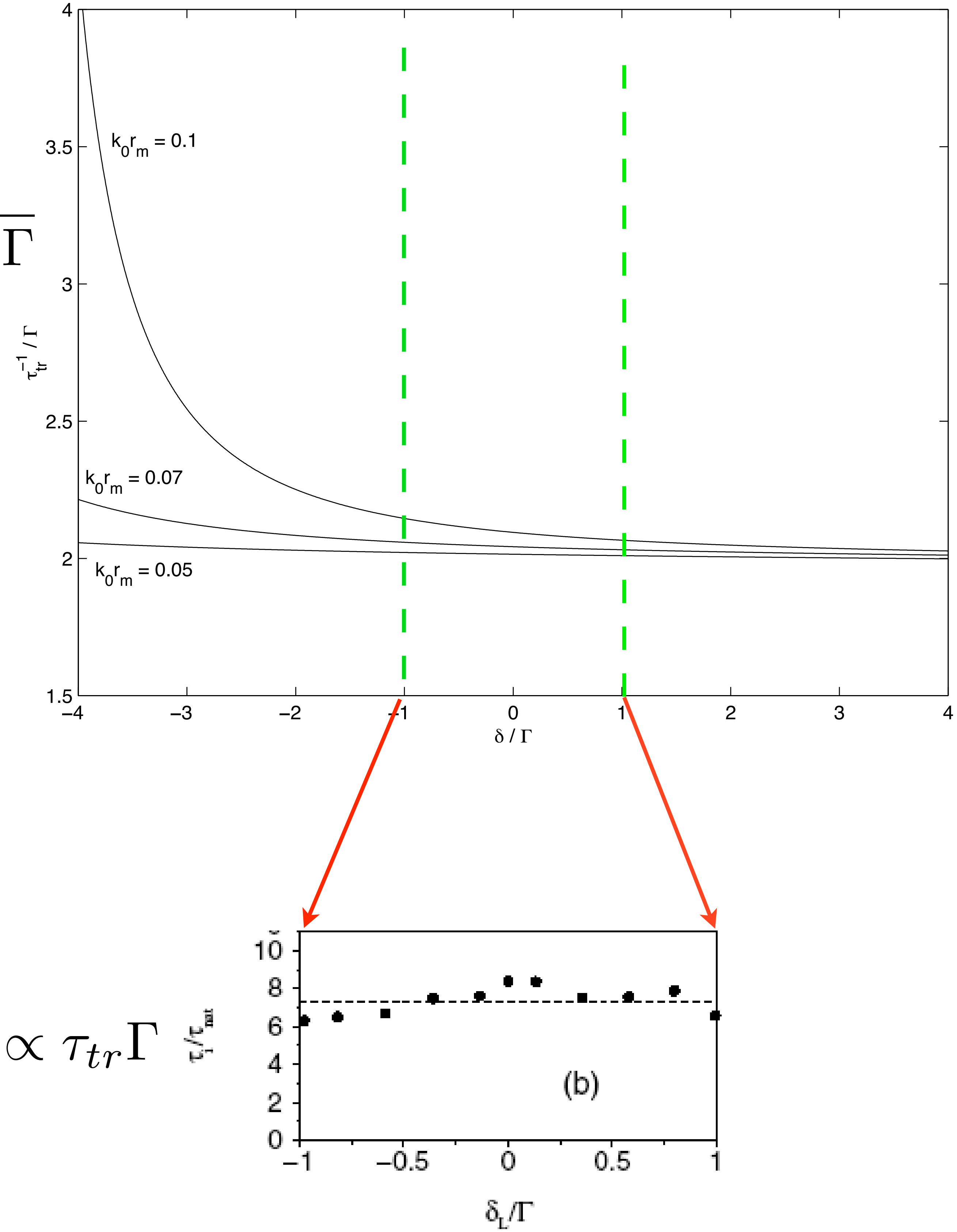
$$\longrightarrow \quad k_0 r_m \simeq 0.51$$

Transport time is defined by:

$$\tau_{tr}(\delta) = \frac{l}{v_g} = 3\frac{D}{v_g^2}$$

Transport time τ_{tr} depends weakly on the frequency ω

$$\frac{1}{\tau_{tr}\Gamma}$$



$$\propto \tau_{tr}\Gamma$$

Strong disorder

Onset of a photon localization transition: for a critical amount of disorder (density of atoms), there is a phase transition from delocalized to localized photon states.

Ioffe -Regel criterion: $\lambda \simeq l$ where $l = 1/n_i\sigma$

For resonant atom-photon scattering: $\boxed{\dfrac{\lambda}{l} = n_i a_{JJ_e} \dfrac{3\lambda^3}{2\pi} \dfrac{1}{1+(2\delta/\Gamma)^2}}$

At resonance, $\delta = 0$ so that, $\dfrac{\lambda}{l} \simeq n_i \lambda^3$

Cold vapor of ^{85}Rb : $\dfrac{\lambda}{l} \simeq 10^{-4}$ (far from localization)

while $n_i \simeq 10^{14}\,\mathrm{cm}^{-3}$ and $\dfrac{\lambda}{l} \simeq 3.4$

Cooperative effects and dipole-dipole resonant interation

Dipolar atoms are not independent scatterers.
There are long range dipole-dipole interactions.

Moreover, when two resonant scatterers are close enough, collective states appear (Dicke states) that change substantially the nature of the localization transition.

Characterize the onset of localization transition by mean of the **distribution P(t) of escape times.**

Effective Hamiltonian

Atoms = collection of resonant two-level systems:

Ground state: $|g_i\rangle$ Excited state: $|e_i\rangle$

$$H_{int} = -i\hbar\frac{\Gamma}{2}\sum_{i=1}^{N_i}|e_i\rangle\langle e_i| - \hbar\frac{\Gamma}{2}\sum_{i\neq j}\frac{e^{ik|\mathbf{r}_i-\mathbf{r}_j|}}{k|\mathbf{r}_i-\mathbf{r}_j|}d_i^+ d_j^-$$

with $d_i^+ = d|e_i\rangle\langle g_i|$ and $d_i^- = (d_i^+)^\dagger$

Diagonal elements: spontaneous emission of isolated atoms

Off-diagonal terms: modification of the spontaneous emission due to collective effects and dipole-dipole resonant interaction.

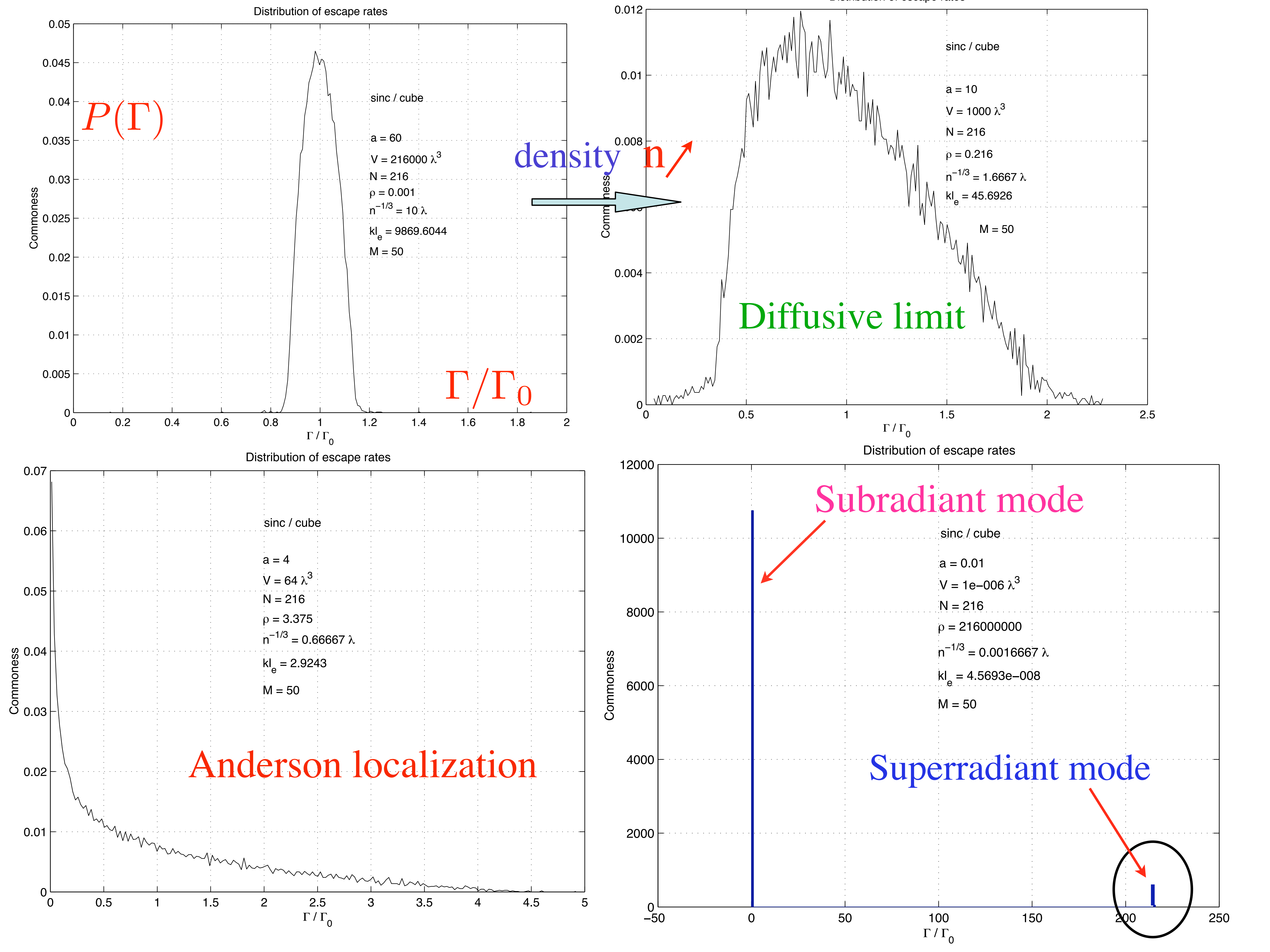
Distribution of escape times

Probability $\pi(t)$ that a photodetector placed outside the atomic cloud will detect a photon at time t .

$$\pi(t) = \Gamma \sum_{i,j} \left\langle \frac{\sin(k\, r_{ij})}{k\, r_{ij}} d_i^+(t) d_j^-(t) \right\rangle$$

Probability to detect a photon between times 0 and t :

$$P(t) = \int_0^t \pi(t')\, dt' \xrightarrow[\text{Laplace transform}]{} P(\Gamma)$$



Distribution of escape rates
$P(\Gamma)$
sinc / cube
a = 60
V = 216000 λ^3
N = 216
ρ = 0.001
n^−1/3 = 10 λ
kl_e = 9869.6044
M = 50
Γ/Γ_0
Commoness
Γ / Γ_0
density n
sinc / cube
a = 10
V = 1000 λ^3
N = 216
ρ = 0.216
n^−1/3 = 1.6667 λ
kl_e = 45.6926
M = 50
Diffusive limit
Distribution of escape rates
sinc / cube
a = 4
V = 64 λ^3
N = 216
ρ = 3.375
n^−1/3 = 0.66667 λ
kl_e = 2.9243
M = 50
Anderson localization
Distribution of escape rates
Subradiant mode
sinc / cube
a = 0.01
V = 1e−006 λ^3
N = 216
ρ = 216000000
n^−1/3 = 0.0016667 λ
kl_e = 4.5693e−008
M = 50
Superradiant mode