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Chains of random impedances

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Résumé. — On calcule le coefficient de transmission d'une onde électrique de fréquence ω à travers un réseau linéaire de dipôles électriques d'impédances aléatoires en fonction de ω et de N, la taille du réseau. Deux types de désordre sont étudiés : le désordre faible où tous les moments des impédances aléatoires sont définis et le désordre fort où aucun de ces moments n'est défini. Les longueurs caractéristiques associées aux phénomènes présents simultanément, de localisation et de diffusion sont obtenues à partir du coefficient de transmission, ainsi que la fréquence de coupure du filtre passe-bas équivalent. Pour la plupart des situations, la dissipation impose sa longueur caractéristique et sa fréquence au coefficient de transmission. Toutefois une situation particulière est envisagée où le phénomène de localisation pourrait être observé en dépit des effets de dissipation ou de diffusion.

Abstract. — The transmission of an electrical wave of frequency ω through a random ladder network is calculated at low frequency in terms of the scaling variables ω and N (the size of the chain). Two classes of disorder are considered : weak disorder where all the moments exist, and strong disorder where no moment can be defined. The characteristic lengths — localization or diffusion — are obtained from the transmission coefficient and the cut-off frequency for the band low-pass filter. For most situations dissipation imposes its characteristic length and frequency dependence on the transmission coefficient. A special situation is found where the localization phenomenon could be observed above the dissipation or diffusion effects.

The problem of propagation of a wave in a one dimensional random medium has now been studied extensively and is well understood. It is established that all the modes are exponentially localized while the threshold of localization is the zero frequency. The critical exponent ν_0 for the localization length $\xi_0 \cong \omega^{-\nu_0}$ has been found [1, 7] to be 2 for a weakly disordered medium. However the question of the universality of this value with the various kind of disorder usually considered has not been answered so far and constitutes part of the motivation for this study. Alternatively, the diffusion of a particle in a random one-dimensional medium has been also studied extensively by various authors [8, 10]. The space-time scaling relation $\xi_D \approx t^{\nu_D}$ or $\omega^{-\nu_D}$ for asymptotically long time or low frequency has been obtained for various classes of disorder : $\nu_D = 1/2$ for weak disorder and $\nu_D \leqslant 1/2$ for strong disorder. The analysis of both effects — localization and diffusion — has not been attempted and this represents the second part of our study. This mixed situation is present in many linear mechanical systems where dissipation or viscous forces act simultaneously with the reactive or harmonic forces. A unified description of the most general linear systems deals with an electrical network built up with resistances, capacitances and inductances. When only inductances and capacitances are present the ladder network corresponds to a harmonic chain [11-12] while the resistances and capacitances describe the diffusion process [8].

This article is organized in 5 sections. In section 1, a general expression for the transmission coefficient T of a propagating wave of low frequency ω through a chain of random impedances is obtained from the lowest order term of a series expansion in the fluctuations around the average impedances (in the appendix the detailed calculation of this coefficient is reported). The correct averaging is then performed and produces the $\langle \ln T \rangle$ expression. The case of weak disorder is studied in section 2 and the new frequency dependence of a mixed network — reactive and dissipative impedances — is then obtained. For this mixed network the dissipation plays the dominant rôle at low frequency. In section 3 the class of strong disorder is considered for the random impedances. A new expression for the localization length is deduced. A special type of random chain is also studied for which the effects of localization could dominate those of dissipation at low frequency. Space correlations of phases are given in section 4 in terms of a characteristic length while conclusions are drawn in section 5.

1. The low frequency transmission coefficient of a chain of length N.

We consider a sequence of $N + 1$ random impedances Z_n ($n = 0, 1, 2, ..., N$) linked with two semi-infinite ladder networks of self-inductances L alternating with equal capacities C (see Fig. 1). Let V_n be the voltage at terminal n; the current equation at node n for frequency ω is :

$$\frac{V_{n-1} - V_n}{Z_n} - \left(\frac{V_n - V_{n+1}}{Z_{n+1}}\right) = i\omega C V_n . \quad (1)$$

By using the current intensity I_n through the impedance Z_n in the section n, $I_n = (V_{n-1} - V_n)/Z_n$, the current equation (1) can be changed into :

$$I_{n-1} - 2 I_n + I_{n+1} = i\omega C Z_n I_n \quad (2)$$

where the disorder, *via* the impedances Z_n, now becomes diagonal. For weak disorder, there is an average value for Z_n : $\overline{Z_n} = Z$ and we start by solving the equation (2) for the average chain :

$$I_{n-1} - 2 I_n + I_{n+1} = i\omega C \overline{Z} I_n . \quad (3)$$

The solutions are given by the associated characteristic equation

$$X^2 - 2 X\left(1 + \frac{i\omega C\overline{Z}}{2}\right) + 1 = 0 . \quad (4)$$

By defining :

$$\cos \phi = 1 + \frac{i\omega C\overline{Z}}{2} . \quad (5)$$

The solutions have the canonical form $X = e^{\pm i\phi}$ where ϕ can be a complex number. This step is useful to define a basis for the disordered chain for which the solutions of (2) are sought in the following form :

$$I_n = A_n e^{in\phi} + B_n e^{-in\phi} \quad (6)$$

where A_n and B_n can be complex numbers obeying the continuity equation :

$$A_n e^{in\phi} + B_n e^{-in\phi} = A_{n-1} e^{in\phi} + B_{n-1} e^{-in\phi} . \quad (7)$$

The coefficients of adjacent intensities are connected by the transfer matrix :

$$\begin{bmatrix} A_n \\ B_n \end{bmatrix} = \theta_n \begin{bmatrix} A_{n-1} \\ B_{n-1} \end{bmatrix} \quad (8a)$$

$$\theta_n = \begin{bmatrix} 1 + \xi_n & \xi_n e^{-2in\phi} \\ -\xi_n e^{2in\phi} & 1 - \xi_n \end{bmatrix} \quad (8b)$$

where

$$\xi_n = \frac{\omega C g_n \overline{Z}}{2 \sin \phi} \quad (8c)$$

$$g_n = \frac{Z_n}{\overline{Z}} - 1 . \quad (8d)$$

Here ξ_n or g_n describe the fluctuations of the random impedances around the average value $\overline{Z}$. For the random ladder network of figure 1, the coefficients A_N and B_N of the current at the section N are obtained from A_0 and B_0 by the product of the N transfer matrices θ_n :

$$\begin{bmatrix} A_N \\ B_N \end{bmatrix} = \left[\prod_{n=1}^{N} \theta_n\right] \begin{bmatrix} A_0 \\ B_0 \end{bmatrix} . \quad (9)$$

With the aim of determining the frequency dependence of the characteristic length of this problem we now expand the product to first order in the fluctuations ξ_n (equivalent to the Born approximation in scattering theory) :

$$\prod_{n=1}^{N} \theta_n \cong \begin{bmatrix} 1 + \sum_{n=1}^{N} \xi_n & \sum_{n=1}^{N} \xi_n e^{-2in\phi} \\ -\sum_{n=1}^{N} \xi_n e^{2in\phi} & 1 - \sum_{n=1}^{N} \xi_n \end{bmatrix} \quad (10)$$

which gives the relations :

$$\begin{cases} A_N = A_0\left(1 + \sum_{n=1}^{N} \xi_n\right) + B_0 \sum_{n=1}^{N} \xi_n e^{-2in\phi} \\ B_N = - A_0 \sum_{n=1}^{N} \xi_n e^{2in\phi} + B_0\left(1 - \sum_{n=1}^{N} \xi_n\right) . \end{cases} \quad (11)$$

Both regular chains at the ends of the random network support incident, reflected and transmitted waves of

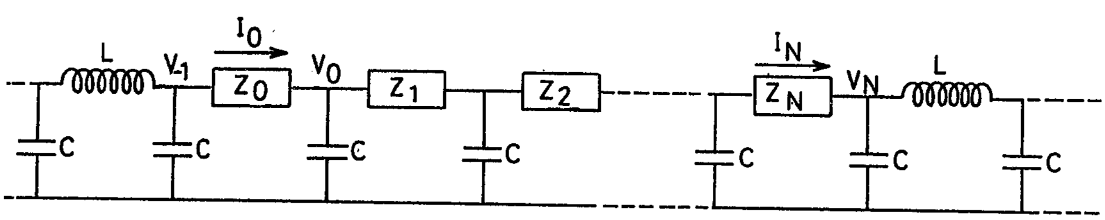

Fig. 1. — Chain of random impedances. The random $Z_n C$ chain is connected with two regular semi-infinite LC networks.

dispersion relation

$$k = \omega \sqrt{LC} \tag{12}$$

where k is the wave vector. This dispersion relation is a special case of relation (5) where $\overline{Z} = i\omega L$ and $\phi = k \ll 1$ in the approximation of long wave length. The transmission coefficient through the random network is obtained by imposing the following expressions for the intensities :

$$I_n = \begin{cases} e^{ikn} + r\, e^{-ikn} & n < 0 \\ A_n\, e^{in\phi} + B_n\, e^{-in\phi} & 0 \leqslant n \leqslant N \\ t\, e^{ikn} & n > N . \end{cases} \tag{13}$$

Matching of these solutions at both ends $n = 0$ and $n = N$ provides the four relations :

$$\begin{cases} I_{-2} - 2\, I_{-1} + I_0 = -\,\omega^2\, LCI_{-1} \\ I_{-1} - 2\, I_0 + I_1 = i\omega CZ_0\, I_0 \\ I_{N+1} - 2\, I_N + I_{N-1} = i\omega CZ_N\, I_N \\ I_{N+2} - 2\, I_{N+1} + I_N = -\,\omega^2\, LCI_{N+1} . \end{cases} \tag{14}$$

By using (12) and (13) in (14) a linear system of four unknown variables r, t, A_0 et B_0 is obtained, a system which can be solved directly. The energy flow through the network is defined [11] by the local relation :

$$J_n = \mathrm{Re}\,\{ I_n^*\, V_n \} . \tag{15}$$

The transmission coefficient T, ratio of the transmitted energy flow to the incident energy flow, is equal to $| t |^2$.

In the appendix, the expression $\ln T = \ln | t |^2$ is derived up to lowest order in ξ_n or g_n (Born approximation for $| t |^2$). Taking into account the randomness of the impedances Z_n, we define the average $\langle \ln T \rangle$ over the distribution function of Z_n. It has been shown by various authors [1, 13] that the correct proceeding is to average the transmission coefficient over a disordered chain since it is only $\langle \ln T \rangle$ which obeys the central limit theorem for large N. This property is due to the multiplicative character of the transmission coefficient through different media which becomes extensive in the $\ln T$ formulation. This averaging procedure provides a great simplification when successive random impedances are assumed uncorrelated. We finally obtain the following expression :

$$\langle \ln T \rangle \cong \ln \left| \frac{4\, \phi k}{(\phi + k)^2} \right|^2 - 2\, N\, \mathrm{Im}\, \phi - \frac{N \sigma^2(\omega)\, | \phi |^2}{4} \tag{16}$$

where ϕ is given by the low frequency approximation of (5) :

$$\phi \cong i^{3/2} (\omega C \overline{Z})^{1/2} . \tag{17}$$

The first two terms are not related to the disorder. Furthermore the first term does not depend on the size N of the chain, it originates [11] from the mismatch of the impedances at the junctions between the semi-infinite LC network and the chain of average impedance $\overline{Z}$. For large and disordered chains it is negligible. The second term is proportional to the imaginary part of ϕ therefore to the dissipation in the circuit. The last term makes use of $\sigma^2(\omega)$, the variance of the $| g_n |$. It is this term which is perturbative and derived to lowest order in $| g_n |^2$. However we know from the Furstenberg's [14] convergence theorem that at any order in g_n a variation in N of $\langle \ln T \rangle$ is expected but with more complicated frequency dependence than in the non perturbative regime. Only at low frequency a very simple power law dependence is obtained for $\langle \ln T \rangle$.

2. Weak disorder.

In this section we consider disorder situations where the various related moments of the distribution function of Z_n are well-defined.

2.1 PURE REACTIVE RANDOM NETWORK. — The random impedances Z_n are of self-inductance nature L_n :

$$Z_n = i\omega L_n . \tag{18}$$

A section of the ladder network is made up of capacitance C and inductance L_n. This configuration can be mapped onto the harmonic chain of atoms where L_n and C stand for the masses and the inverse harmonic restoring force respectively.

It can also be put in correspondance with the Schrödinger equation of the random Kronig-Penney model [1]. By tuning the $\overline{L}$ (the average inductance) to L in order to eliminate the reflexions at the junctions, one finds from (17) that $\phi = +\,k$, $\mathrm{Im}\, \phi = 0$ and $\sigma_L^2 = \langle (L_n - \overline{L})^2 / \overline{L}^2 \rangle$ are independent of the frequency. Then,

$$\langle \ln T \rangle = -\frac{N k^2\, \sigma_L^2}{4} . \tag{19}$$

This result has previously been derived for the disordered harmonic chain [2, 3, 5, 15]. With the dispersion relation (12), $\langle \ln T \rangle$ becomes proportional to $-\,N\omega^2$. This result can be understood in a simple way : The $\omega = 0$ mode is extended owing to global translation invariance. It is the mobility edge of localization at 1 dimension. For small but non-zero frequencies, the modes become localized. The localization length $\xi_0(\omega)$ is usually defined by the standard relation (in units of the length of one section) :

$$\langle \ln T(\omega) \rangle = -\frac{N}{\xi_0(\omega)} . \tag{20}$$

Here we obtain

$$\xi_0(\omega) = \frac{4}{\omega^2\, LC \sigma^2} . \tag{21}$$

The linear dependence of $\langle \ln T \rangle$ in N comes from the uncorrelated distribution of random impedances to lowest order in the fluctuation expansion or, at any order, from the Furstenberg's theorem. The ω^2 variation of $\xi_0^{-1}(\omega)$ is simply due to the weight of a local impedance fluctuation in ω^2 as in the harmonic disordered chain where any mass fluctuations contribute as ω^2 to the force equations. Another way to present the result [19] is to consider the network as a low-pass filter for which the transmission coefficient vanishes above some critical frequency ω_c. We know from the general theorem of localization in one dimension that at any finite frequency, the coefficient of transmission vanishes for the infinite network. For a finite network of size N, a low-pass band filter subsists up to a critical frequency $\omega_c(N)$. This cut-off frequency is simply evaluated by writing :

$$\xi_0(\omega_c) \gtrsim N \tag{22}$$

which produces here :

$$\omega_c \sim N^{-1/2} . \tag{23}$$

This scaling relation is characteristic of the low frequency regime of standard localization.

2.2 PURE DISSIPATIVE RANDOM NETWORK. — All the random impedances are resistances $Z_n = R_n$. From (17),

$$\phi \cong i^{3/2}(\omega \bar{R} C)^{1/2} \tag{24}$$

which produces damped waves, as expected, in the average medium.

Since

$$\operatorname{Im} \phi \cong \left(\frac{\omega \bar{R} C}{2}\right)^{1/2} \tag{25}$$

the dissipative term contributes to $\ln T$ by an attenuation in $-2N(\omega RC/2)^{1/2}$. The fluctuations in R_n contribute through $|\phi|^2$ by a term proportional to ω while $\sigma^2(\omega) = \left\langle \left(\frac{R_n}{\bar{R}} - 1\right)^2 \right\rangle$ is independent of frequency. Finally, by adding the two terms we find :

$$\langle \ln T \rangle \cong -N(2\,\omega \bar{R} C)^{1/2} - N \frac{\omega \bar{R} C \sigma^2}{4} . \tag{26}$$

For low frequency the dominant term is the first one coming from the average dissipative medium (weak disorder does not affect strongly the scaling relation of the latter). It is well known that this purely dissipative case corresponds to the problem of diffusion of a particle in a random medium. The diffusion length $\xi_D(\omega)$ is obtained from (25) by rewritting :

$$\langle \ln T \rangle \cong -\frac{N}{\xi_D(\omega)} \tag{27}$$

which gives :

$$\xi_D(\omega) \cong (2\,\bar{R} C \omega)^{-1/2} . \tag{28}$$

This is the standard space-time relation for diffusion where ξ is proportional to $t^{1/2}$. The cut-off frequency corresponding to this case is now $\omega_c \sim N^{-2}$: the pass band width is strongly reduced compared to the localization situation (A).

2.3 THE MIXED REACTIVE-DISSIPATIVE NETWORK. — Since weak disorder does not affect the scaling relation of the dissipative case, we consider for simplicity only the fluctuations in the impedances :

$$Z_n = R + i\omega L_n . \tag{29}$$

The dominant term for $\omega \to 0$ is indeed the resistivity part of ϕ so that the dissipative term is unchanged from (26). In contrast the term of disorder is changed by the frequency dependence of the variance $\sigma^2(\omega)$:

$$\sigma^2(\omega) = \langle |g_n(\omega)|^2 \rangle = \omega^2 \left(\frac{\bar{L}}{R}\right)^2 \sigma_L^2 \tag{30}$$

where σ_L^2 is the variance of the random inductances. With ϕ given by (24) we see :

$$\langle \ln T \rangle \cong -N(2\,\omega RC)^{1/2} - \frac{N\omega^3}{4} \frac{\bar{L}^2 C}{R} \sigma_L^2 . \tag{31}$$

The main conclusion of this general case is that the disordered term is not the dominant one for the transmission coefficient. In the presence of dissipation in the one dimensional network, the localization effect, responsible for the term in $N\omega^3$, is negligible compared with dissipation at low frequency. In terms of characteristic lengths, the diffusion length $\xi_D \sim \omega^{-1/2}$ is much shorter than the localization length $\xi_0 \sim \omega^{-3}$. This situation must be enhanced even further in higher dimensions since the localization length is even longer (for example $\xi_0 \sim e^{1/\omega^2}$ in two dimensions [16]).

3. Strong disorder.

One-dimensional systems are very sensitive to the occurrence of a cut, even when the probability of this event is vanishingly small. This situation has been considered in detail by Alexander *et al.* [8] for the case of diffusion in a random medium corresponding to the purely dissipative network. In this section we are going to generalize this result to the case of localization (reactive network) and mixed chains.

Let us consider first the purely reactive network with large fluctuations of L_n described by the probability density function $p(L_n)$:

$$p(L_n) = (1 - \alpha)\, L_0^{1-\alpha}\, L_n^{-(2-\alpha)} \tag{32}$$

for $L_n \in [L_0, +\infty[$ and $\alpha \in]0, 1[$ (L_0 : minimum value of L_n). All the moments of this distribution diverge and no effective medium can be defined for this case of strong disorder. Therefore the previous method cannot be used and some modifications must enter in the previous expression for the transmission coefficient (16). The basic idea is to truncate the distribution function (32) up to a special value $\tilde{L}_n$ defined as the most probable value of the largest fluctuation of L_n on a chain of size N. Let us call $\eta(\tilde{L}_n)$ the probability that the inductance L_n is less than $\tilde{L}_n$:

$$\eta(\tilde{L}_n) = 1 - \left(\frac{L_0}{\tilde{L}_n}\right)^{1-\alpha} \qquad (33)$$

The probability of finding a chain of N random inductances with only one fluctuation larger than $\tilde{L}_n$ is equal to $\eta^N(1-\eta)$. The most probable value of the large fluctuation $\tilde{L}_n$ is obtained by maximizing $\eta^N(1-\eta)$ which gives $N = \eta(\tilde{L}_n)\,[1-\eta(\tilde{L}_n)]^{-1}$. For large size one finds :

$$\tilde{L}_n = L_0\, N^{\frac{1}{1-\alpha}}. \qquad (34)$$

The truncated distribution for finite size is then :

$$p'(L_n) \cong \begin{cases} (1-\alpha)\, L_0^{1-\alpha}\, L_n^{-(2-\alpha)} & L_n \leqslant \tilde{L}_n \\ 0 & \text{otherwise}. \end{cases} \qquad (35)$$

This is for the same conditions as in (32). The moments are now well-defined.

$$\bar{L} \cong \frac{1-\alpha}{\alpha} L_0\, N^{\frac{\alpha}{1-\alpha}}$$
$$\sigma_L^2 = \frac{\langle (L_n - \bar{L})^2 \rangle}{\bar{L}^2} = \frac{\alpha^2}{1-\alpha^2} N. \qquad (36)$$

The expression (19) must be changed both for k and σ_L since

$$k^2 = \omega^2\, \bar{L} C \cong \frac{1-\alpha}{\alpha}\, \omega^2\, L_0\, C N^{\frac{\alpha}{1-\alpha}} \qquad (37)$$

which gives

$$\langle \ln T \rangle \cong - \frac{\alpha}{4(1+\alpha)}\, \omega^2\, L_0\, C N^{\frac{2-\alpha}{1-\alpha}}. \qquad (38)$$

The expression (38) defines a new scaling relation through the factor $N^{[(2-\alpha)/(1-\alpha)]}\,\omega^2$ which provides a new localization length ξ_0 as well as cut-off frequency for the pass band :

$$\xi_0 \approx \omega^{-2\left(\frac{1-\alpha}{2-\alpha}\right)}$$
$$\omega_c \approx N^{-\frac{1}{2}\cdot\frac{2-\alpha}{1-\alpha}} \qquad (39)$$

As α is defined in]0, 1[, the exponent ν of $\omega^{-\nu}$ is also reduced to the interval]0, 1[instead of 2 for a weak disorder. This leads to an important shortening of the localization length. This is expected since strong disorder enhances considerably localization phenomena.

Consider now the mixed network with constant resistance but random impedances. Suppose that these impedances have the probability distribution given by (32). The second term of the expression (31) of $\langle \ln T \rangle$ must be changed by taking into account the N-dependence of $\bar{L}$ and σ_L (36) :

$$\langle \ln T \rangle \cong - N(2\,\omega RC)^{1/2} - \frac{L_0^2\, C}{4\,R}\,\frac{1-\alpha}{1+\alpha}\,\omega^3\, N^{\frac{2}{1-\alpha}}. \qquad (40)$$

The scaling between N and ω is different in the two terms. While the dissipative term gives the $N\omega^{1/2}$ dependence, the localization one now exhibits a new dependence $\omega^3\, N^{(2/1-\alpha)}$. The limit of very low frequency now shows two regimes. Let us call ω_{CD} the cut-off frequency for the dissipative term ($\omega_{CD} \sim N^{-2}$) and ω_{CO} the same cut-off for the second term. One finds :

$$\omega_{CO} \cong N^{-\frac{2}{3(1-\alpha)}} \qquad (41)$$

which becomes less than or equal to ω_{CD} for $2/3 < \alpha < 1$. For these values of α and for $\omega \gtrsim \omega_{CO}$ the transmission coefficient is controlled by the localization phenomena instead of the dissipation. It is, to our knowledge, the only situation where localization can be observed at low frequency in the presence of dissipation.

Finally, for completeness, let us deduce the diffusion length for the case of random resistances where the resistances follow a law of distribution analogous to (32). By a similar method one can obtain the moments of the truncated distribution :

$$\langle R \rangle = \frac{1-\alpha}{\alpha}\, R_0\, N^{\frac{\alpha}{1-\alpha}}$$
$$\sigma_R^2 = \frac{\alpha^2}{1-\alpha^2} N. \qquad (42)$$

Expression (26) now becomes

$$\langle \ln T \rangle \cong - \left(\frac{2\,CR_0(1-\alpha)}{\alpha}\right)^{1/2} \omega^{1/2}\, N^{\frac{2-\alpha}{2(1-\alpha)}} - \frac{C\alpha R_0}{4(1+\alpha)}\,\omega N^{\frac{2-\alpha}{1-\alpha}}. \qquad (43)$$

The scaling variable is now $\omega^{1/2}\, N^{[(2-\alpha)/2(1-\alpha)]}$ for both terms, which produce the « diffusion » length :

$$\xi_D(\omega) \sim \omega^{-\frac{1-\alpha}{2-\alpha}} \qquad (44)$$

and the cut-off frequency $\omega_c \sim N^{-\frac{2-\alpha}{1-\alpha}}$ in agreement with reference [8].

4. Phase correlation function.

We would now like to consider the effect of disorder on the phases of A_n and B_n. In the case of the random impedances network we can generalize previous approaches to this problem [17, 18] due to the fact that the disorder introduces only one single characteristic length describing both amplitude and phase-correlation decay.

We define the phase-correlation function $q(p)$ as the ratio of the phase at some point p to the initial phase ϕ_0 so that

$$q(p) \equiv \frac{\phi_p}{\phi_0} \tag{45}$$

where we defined the phases ϕ_n by :

$$\begin{cases} A_n = | A_n | \, \mathrm{e}^{-i\phi_n} \\ B_n = | B_n | \, \mathrm{e}^{-i\phi'_n} . \end{cases} \tag{46}$$

We can easily show that phase-correlation functions have the same scaling form for both cases :

$$q(p) = \frac{\phi_p}{\phi_0} \quad \text{and} \quad q'(p) = \frac{\phi'_p}{\phi'_0},$$

corresponding to A_p and B_p respectively. Following Azbel, we consider the only relevant extensive quantity which obeys a central limit theorem for $N \to \infty$, to describe phase correlations, $\langle \ln q \rangle$.

Starting from (11) and using the notation introduced in the appendix, we get the following form :

$$\frac{\phi_p}{\phi_0} \cong 1 + \sum_{n=1}^{p} \mathrm{Im}\,(\xi_n), \tag{47}$$

with a first order expansion of tg $(\phi_L - \phi_0)$. Then we have

$$\left\langle \ln \frac{\phi_p}{\phi_0} \right\rangle \cong -\frac{1}{2} \sum_{n=1}^{p} \langle [\mathrm{Im}\,(\xi_n)]^2 \rangle, \tag{48}$$

where the linear terms in ξ_n of zero mean-value are eliminated.

This expression for $\langle \ln q(p) \rangle$ allows us to obtain the scaling form of the phase-correlation function in the three previous cases :

i) *purely reactive network* : $\mathrm{Im}\,(\xi_n) = \dfrac{\omega\sqrt{LC}}{2} g_n$

$$\langle \ln q(p) \rangle \cong -\frac{LC\omega^2 N \sigma_L^2}{8},$$

ii) *purely dissipative random network* :

$$\mathrm{Im}\,(\xi_n) = \left(\frac{\omega RC}{2}\right)^{1/2} g_n$$

$$\langle \ln q(p) \rangle \cong -\frac{RC\, \sigma_R^2 N\omega}{4},$$

iii) *mixed reactive-dissipative network* :

$$\mathrm{Im}\,(\xi_n) = L\left(\frac{C}{2R}\right)^{1/2} \omega^{3/2} g_n$$

$$\langle \ln q(p) \rangle \cong -\frac{L^2 C}{4R} \sigma_L^2 N\omega^3 .$$

Then the influence of disorder on the phase correlation of the amplitudes A_n and B_n has the same scaling form as $\langle \ln T \rangle$. This was expected since in 1D, the disorder can be described by a single characteristic length.

5. Conclusions.

Let us summarize the main results of this article. The electrical one-dimensional network is made up of a finite number N of random reactive and dissipative impedances. The propagation of an electrical wave is studied by means of the transmission coefficient T through the network. Simple results for the scaling variables ω and N can be obtained only at low frequency. In this regime an effective medium is built up by average impedances while the fluctuations around the mean-values are treated by a perturbation expansion to lowest order in the fluctuations g_n. The correct averaging on $\langle \ln T \rangle$ is performed and produces the scaling dependence as well as the characteristic lengths (localization and diffusion) and the cut-off frequency of the low-pass band filter. For weak disorder (all the moments of the distribution function exist) we find for the general chain a strong dominance of the dissipation over localization, reflected by the fact that the shortest characteristic length is that of diffusion. Strong disorder is considered in terms of distribution laws for which no moments exist. The extreme sensitivity of the one-dimensional systems to this type of disorder leads to a marked, change in the scaling relations, in particular a new frequency dependence of the localization length is derived. A special situation is envisaged where there is no randomness in the dissipation while the reactive impedances are strongly fluctuating. For this situation only, there is a domain of frequency where the localization becomes dominant over the dissipation.

The domain of validity of the low frequency expansion is limited by values of $\langle \ln T \rangle \gtrsim -1$ or, equivalently, by the cut-off frequency ω_c of the low-pass band filter. At higher frequency we have estimated the following terms in the expansion which show a similar scaling dependence. This indicates a range of validity probably broader than might be expected from the lowest term of the perturbation expansion. However, at higher frequencies, we know from Azbel's recent numerical work [1] that « passing modes » emerge from a background of very small transmission coefficients for discrete values of ω in the case of a random, purely reactive chain. We believe that any dissipation term will affect these « passing » modes and smooth considerably the frequency dependence of the transmission coefficient. This smoothing conjugated with strong damping of the dissipation would probably prevent the observation of the localization phenomena through the transmission peaks even at high frequency, except for the special situations described in section 3.

Many physical systems can be put into correspondence with the electrical network described here.

Vibrations of random masses corresponds to the purely reactive case while the electrical resistance could describe the viscosity of phonons coming from the anharmonic *normal* collisions or, more generally, inelastic scattering. Mechanics or linearized hydrodynamics have electrical network analogues. The recent proposal [19] to observe localization of shallow water waves by a random bottom belongs to the class of random impedance problem in one or two dimensional networks. Since the frequency dependence of the localization length is even more divergent, $\xi \sim \exp(1/\omega^2)$, [16] in two dimensions the viscous length will be shorter, preventing the observation of the localization phenomena at low frequency.

Appendix.

DERIVATION OF $\langle \ln T \rangle$. — Starting from (14) we obtain two different systems coupling respectively (A_0, B_0, r) and (A_0, B_0, t) :

$$\begin{cases} A_0 + B_0 = 1 + r \\ A_0\,e^{-i\phi} + B_0\,e^{i\phi} - r(e^{ik} - i\omega C\bar{Z}g_1) = e^{-ik} - i\omega C\bar{Z}g_1 \end{cases} \tag{A.1}$$

and

$$\begin{cases} t\,e^{ik(N+1)} + aA_0 - bB_0 = 0 \\ t\,e^{ikN} + cA_0 - dB_0 = 0 \end{cases} \tag{A.2}$$

where a, b, c, d, are defined by :

$$\begin{aligned} a &\equiv e^{-i(N+1)\phi}\,S_1 - e^{i(N+1)\phi}(1 + S_2) \\ b &\equiv e^{i(N+1)\phi}\,S_3 + e^{-i(N+1)\phi}(1 - S_2) \\ c &\equiv e^{-iN\phi}\,S_1 - e^{iN\phi}(1 + S_2) \\ d &\equiv e^{iN\phi}\,S_3 + e^{-iN\phi}(1 - S_2) \end{aligned}$$

and

$$S_1 \equiv \sum_{p=1}^{N} \xi_p\,e^{2ip\phi} \qquad S_2 \equiv \sum_{p=1}^{N} \xi_p \qquad S_3 \equiv \sum_{p=1}^{N} \xi_p\,e^{-2ip\phi}.$$

Eliminating r and t in (A.1) and (A.2) one obtains :

$$\begin{cases} A_0(e^{-i\phi} - e^{ik} + iC\omega\bar{Z}g_1) + B_0(e^{i\phi} - e^{ik} + iC\omega\bar{Z}g_1) = e^{-ik} - e^{ik} \\ A_0(a - c\,e^{ik}) + B_0(d\,e^{ik} - b) = 0\,. \end{cases} \tag{A.3}$$

Let D be the determinant of this system, then the resolution of (A.3) gives

$$A_0 = \frac{2\,i \sin k}{D}(b - d\,e^{ik}) \quad \text{and} \quad B_0 = \frac{2\,i \sin k}{D}(a - c\,e^{ik})$$

and we find the following expression for t :

$$t\,e^{ikN} = \frac{4 \sin k . \sin \phi}{D}(1 - S_2^2 + S_1\,S_3)\,. \tag{A.4}$$

This result gives the expression of the transfer matrix to the lowest order in ξ, and, the quadratic terms, S_2^2 and $S_1\,S_3$ in (A.4) are of the second order in the fluctuation ξ. Actually, we should have to take into account in the expansion (10) of the transfer matrix, the terms $\xi_n\,\xi_p$ $(n \neq p)$. But for a complete disorder without site correlation where ξ_n and ξ_p are independent random variables, the contribution of these terms vanishes after averaging. This is a well-known result for the incoherent scattering in a random medium with the Born approximation where the two sites scattering is destroyed by the averaging of the interference effects and yields finally a cross-section proportional to the impurity concentration.

Substituting the expression of D into (A.4) one obtains :

$$t\,e^{ikN} = \frac{4\,k.\phi(1 - S_2^2 + S_1\,S_3)}{e^{-iN\phi}[(\phi + k)^2(1 - S_2)] + (k^2 - \phi^2)(S_1\,e^{-iN\phi} + S_3\,e^{iN\phi})} \tag{A.5}$$

where we have done an expansion in $e^{\pm i\phi}$ and $e^{\pm ik}$ and the following non restrictive hypothesis :

$$\mathrm{Im}\,\phi \geqslant 0 .$$

Taking first the squared modulus of (A.5) and afterwards its logarithm :

$$\ln T = \ln \left| \frac{4\,\phi k}{(\phi + k)^2} \right|^2 - 2\,N\,\mathrm{Im}\,\phi + \ln | 1 - S_2^2 + S_1\,S_3 |^2 - $$
$$- \ln \left| 1 - S_2 + \frac{k^2 - \phi^2}{(k + \phi)^2} (S_1 + e^{2iN\phi}\,S_3) \right|^2 \tag{A.6}$$

the term $\ln | 1 - S_2^2 + S_1\,S_3 |^2$ gives :

$$- 2\,\mathrm{Re}\,(S_2^2) + 2\,\mathrm{Re}\,(S_1\,S_3)$$

which is zero when averaged. The last term in (A.6) gives the contribution :

$$- | S_2 |^2 + 2\,\mathrm{Re} \left[\lambda^* \left(\sum_{n=1}^{N} | \xi_n |^2\, e^{-2in\phi^*} + e^{-2iN\phi^*} \sum_{n=1}^{N} | \xi_n |^2\, e^{2in\phi^*} \right) \right]$$

where we have defined $\lambda \equiv \dfrac{k^2 - \phi^2}{(k + \phi)^2}$ and we have eliminated the linear terms in ξ_n, which have zero mean-values and, we have taken into account the fact that the ξ_n are non-correlated variables.

In the presence of dissipative terms, one can make the assumption $| \phi | \gg | k |$ in the zero-frequency limit and $\langle \ln T \rangle$ reduces to :

$$\langle \ln T \rangle = \ln \left| \frac{4\,\phi k}{(\phi + k)^2} \right|^2 - 2\,N\,\mathrm{Im}\,\phi - \langle | S_2 |^2 \rangle \tag{A.7}$$

where

$$\langle | S_2 |^2 \rangle = \left\langle \sum_{n=1}^{N} | \xi_n |^2 \right\rangle$$
$$\langle | S_2 |^2 \rangle = \frac{| \phi |^2}{4} \sum_{n=1}^{N} \langle | g_n |^2 \rangle$$
$$\langle | S_2 |^2 \rangle = \frac{| \phi |^2}{4} N\sigma^2(\omega) \tag{A.8}$$

where the notation $\sigma^2(\omega)$ was introduced to reflect the fact that g_n could be ω-dependent.

Substituting (A8) in the original formula (A7) leads to :

$$\langle \ln T \rangle = \ln \left| \frac{4\,\phi k}{(\phi + k)^2} \right|^2 - 2\,L\,\mathrm{Im}\,\phi - \frac{N\,| \phi |^2}{4}\,\sigma^2(\omega) \tag{A.9}$$

as was given in the text.

If there is no dissipation (purely reactive network), then $\phi = k$ and the original formula (A5) becomes :

$$t = \frac{1 - S_2^2 + S_1\,S_3}{1 - S_2}$$

and

$$\langle \ln T \rangle = - \langle | S_2 |^2 \rangle$$

because of the cancellation of $\langle - 2\,\mathrm{Re}\,(S_2^2) + 2\,\mathrm{Re}\,(S_1\,S_3) \rangle$ and then

$$\langle \ln T \rangle = - \frac{Nk^2\,\sigma^2}{4} \tag{A.10}$$

which is a result given directly by (A9) with $\phi = k$.

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