TOPOLOGICAL FEATURES OF THE MAGNETIC RESPONSE IN INHOMOGENEOUS MAGNETIC FIELDS

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Abstract

We present topological features of the magnetic response (orbital and spin) of a two-dimensional non interacting electron gas due to inhomogeneous applied magnetic fields. These issues are analysed from the point of view of the Index theory with a special emphasis on the non perturbative aspects of this response. The limiting case of a Aharonov-Bohm magnetic flux line is studied in details and the results are extended to more general situations.

INTRODUCTION

The aim of this paper is to discuss some features of the magnetic response of a degenerate two-dimensional electron gas in the non interacting limit. Although there is a vast literature devoted to that subject, we would like to present a different point of view which may help to bring new results in some of the left open issues. In those systems, it is quite common to discuss separately the two components (orbital and spin) of the magnetic response. Very often, there is indeed a clear dichotomy between orbital and spin effects which might be due to some physical constraint (e.g. full spin polarization in a strong magnetic field) or to the independence of the two components of the response, for instance for a homogeneous magnetic field where (neglecting the spin-orbit coupling) both the orbital and Zeeman parts in the Hamiltonian do commute. Recently, it was noticed by a number of authors that more complicated situations for instance Dirac fermions in a random field[1, 2] may lead to new and unexpected effects: transition between localized and extended states, multifractal structure, etc. We do not want to discuss here the richness of some specific model but instead to present some features of the total magnetic response in inhomogeneous magnetic fields shared by most of these models in the general case where orbital and spin effects cannot be simply disentangled. These features as we shall see are essentially non perturbative and do require for their study some new tools imported from the Index theory of elliptic operators[3].

The outline of this paper is as follows. In the remaining part of the Introduction, we shall set up a general form for the Hamiltonian we aim to study. Then, we shall discuss its factorizability and define the associated Index. In part 2, we present a detailed study of the magnetic response to a Aharonov-Bohm flux line as a limiting case of inhomogeneous field. This will be the opportunity to discuss the physical meaning of the Index and its relation to the magnetic response. In part 3, some properties of the associated Heat Kernel are outlined. Then in part 4, those results are extended to other systems, and a relation between the spin magnetization and the Index is given.

We start writing a general expression for the magnetic Schrödinger Hamiltonian for a single electron in the two-dimensional plane submitted to a perpendicular magnetic field of strength $B(r)$. We choose the gauge $\mathrm{div}\vec{A} = 0$ such that the vector potential $\vec{A}(x, y)$ obeys the two equations:

$$\begin{cases} \partial_x A_x + \partial_y A_y = 0 \\ \partial_x A_y - \partial_y A_x = B(r) \end{cases} \tag{1}$$

A solution of (1) is $(A_x, A_y) = \frac{\Phi(r)}{2\pi r^2}(-y, x)$, and the function $\Phi(r)$ is related to the magnetic field by $B(r) = \frac{1}{2\pi r}\partial_r\Phi(r)$ such that the Schrödinger Hamiltonian is

$$\frac{2m}{\hbar^2}H = -\frac{1}{r}\partial_r(r\partial_r) + (\frac{i}{r}\partial_\theta + \frac{\phi(r)}{r})^2 \tag{2}$$

where $\phi(r) \equiv \frac{e}{\hbar c}\Phi(r) = \frac{\Phi(r)}{\Phi_0}$. This Hamiltonian can be expressed as well in terms of the two (formally) self-adjoint first order differential operators D and $D^\dagger$ as

$$\frac{2m}{\hbar^2}H = DD^\dagger - \frac{2\pi}{\Phi_0}B(r) \tag{3}$$

where $D = e^{i\theta}(\partial_r + \hat{J})$ and $D^\dagger = e^{-i\theta}(-\partial_r + \hat{J})$ such that $[D, D^\dagger] = 4\pi B(r)$. The operator $\hat{J} = \frac{i}{r}\partial_\theta + \frac{\phi(r)}{r}$ describes the azimuthal current.

The Zeeman term describing the coupling of the electron spin to the magnetic field is given by $H_s = -\frac{1}{2}g\mu_B\sigma_z B(r)$, where $\mu_B = \frac{e\hbar}{2mc}$ is the Bohr magneton, g is the gyromagnetic ratio we shall take equal to 2 and σ_z is the Pauli matrix. The total Pauli Hamiltonian (in appropriate units) is

$$\frac{2m}{\hbar^2}H_P = DD^\dagger - 2\pi B(r)(1 + \sigma_z) \tag{4}$$

It has the well known and interesting property to be exactly factorizable regardless of the shape of the magnetic field profile $B(r)$, i.e. it may be rewritten as the product of two first order differential operators (formally self-adjoint) Q and $Q^\dagger$ such that $H = QQ^\dagger$. This is a special example of a $N = 2$ supersymmetric Hamiltonian[4]. This feature is at the origin of the peculiar (non perturbative) topological properties of those systems. For instance, Aharonov and Casher[5] did show explicitly that if the magnetic field is of finite flux, then the ground state degeneracy is $N - 1$ where N is the closest integer to the total flux (in units of Φ_0). This is an example of the Atiyah Singer Index theorem[6] which, for that case states that

$$\mathrm{Index}Q = \mathrm{dimKer}Q - \mathrm{dimKer}Q^\dagger = N - 1 \tag{5}$$

where $\mathrm{dimKer}Q$ (resp. $\mathrm{dimKer}Q^\dagger$) is the (finite) number of zero modes of Q (resp. $Q^\dagger$) i.e. the number of solutions of the first order differential equation $Q\Psi = 0$ (resp. $Q^\dagger\Psi = 0$). The index is an integer and as such a topological invariant in that sense that it remains unchanged under any classically permissible gauge transformation where for instance we may change randomly the profile of the magnetic field $B(r)$ but keeping unchanged the total magnetic flux. This results from the factorizability of the Pauli Hamiltonian a property which in general is not met by Schrödinger Hamiltonians unless either we impose some special boundary conditions or we consider a uniform magnetic field B where the corresponding Hamiltonian (Landau) is factorizable (see (3)) in terms of the operators D and $D^\dagger$ up to a constant $-\frac{2\pi}{\Phi_0}B$, which sets the ground state energy (i.e. the lowest Landau level). In this latter case, it is again possible (at least formally) to define an Index. It turns out to be infinite (and ill defined) which still corresponds to the infinite degeneracy of the lowest Landau level. Since the extensive (i.e. proportional to the surface) degeneracy of the ground state is an important ingredient for the Hall quantization in those systems, it might be tempting to preserve and extend the Index theorem to finite geometries. But then, it can be shown that any local choice of boundary conditions (e.g. Dirichlet or Neumann) destroys the factorizability property of the Schrödinger Hamiltonian and therefore the condition of applicability of the Index theorem with this consequence that the corresponding ground state is always non degenerate. It was shown recently[7] that the proper degeneracy as given by the Index can be restored using a special kind of non local boundary conditions.

To go further and relate these topological features to the magnetic response, we shall first focus on a specific example namely the case of a Aharonov-Bohm magnetic flux line.

THE MAGNETIC RESPONSE OF AHARONOV-BOHM SYSTEMS

We consider now the limiting case of a localized magnetic field of finite flux which corresponds to a Aharonov-Bohm flux line i.e. to a delta function magnetic field $\vec{B}(r) = \Phi\delta(\vec{r})\hat{e}_z = \frac{hc}{e}\phi\delta(\vec{r})\hat{e}_z$ and to a vector potential $\vec{A}(\vec{r}) = \phi\frac{\hat{e}_z\times\vec{r}}{r^2}$, where $\hat{e}_z$ is the unit vector perpendicular to the plane. The corresponding Schrödinger equation is obtained from Eq.(2) using $\phi(r) = \phi$. The angular momentum is a good quantum number and then the equation is separable. In each sector $m \in \mathbb{Z}$, a single valued solution of the radial equation is

$$\Psi_m(kr) = aJ_{|m+\phi|}(kr) + bJ_{-|m+\phi|}(kr) \tag{6}$$

where $E = \frac{\hbar^2k^2}{2m}$ is the energy, a and b are constants and $J_\nu(kr)$ are Bessel functions. To describe an impenetrable flux line, we impose the boundary condition $\Psi_m(0) = 0$. Since $J_{-|\nu|}(kr)\sim_{r\to 0}(kr)^{-|\nu|}$ this amounts to take $b = 0$ in order to have square integrable solutions at the origin. This choice is not as innocuous as it seems and we shall comment on it later. By choosing conveniently the normalization of the wavefunction we obtain $a = 1$. Finally, in order to define completely our Hilbert space, we demand the two operators D and $D^\dagger$ to be self adjoint such that for any states $f(r,\theta)$ and $g(r,\theta)$, $\langle f|Dg\rangle = \langle D^\dagger f|g\rangle$. The part $\frac{\phi}{r}$ does not make any problem and by evaluating the radial integral on a disk of radius R (eventually $R \to \infty$), we obtain

$$R\int_0^{2\pi} d\theta e^{i\theta} f^* g|_{r=R} = 0 \tag{7}$$

which is not fulfilled in the general case (there is an exception using Dirichlet boundary conditions). A way to solve this problem is to translate the angular momentum of the eigenfunctions f on the domain of the operator $D^\dagger$ by half a unit and therefore to consider the set $J_{|m+\frac{1}{2}+\phi|}(kr)e^{i(m+\frac{1}{2})\theta}$. Then, on the domain of the operator D, we consider functions g of angular momentum decreased by half a unit. This amounts to consider a spinor like wavefunction, i.e. an effective Pauli Hamiltonian but where we keep only one of the two spin components. We shall come later to this using another point of view.

To characterize the magnetic response of the system, we calculate the (so called) persistent current I and the magnetization $\vec{M}$. Both are obtained from the local current density

$$\vec{j}(r) = \frac{\hbar}{m}\mathrm{Im}(\Psi^*(\vec{\nabla} - \frac{ie}{\hbar c}\vec{A})\Psi) \tag{8}$$

where at zero temperature, we have to sum over all the occupied states up to the Fermi energy. Due to symmetry, only the azimuthal component j_θ is non zero such that $I = \int_0^\infty dr j_\theta(r)$ and the total magnetic moment is $\vec{M} = \frac{\pi}{c}\int_0^\infty dr r^2 j_\theta(r)\hat{e}_z$. ¿From (8) we obtain

$$j_\theta = \int_0^{k_f} \frac{kdk}{2\pi} \sum_{m=-\infty}^{\infty} \frac{m+\phi+\frac{1}{2}}{r} J^2{}_{|m+\frac{1}{2}+\phi|}(kr) \tag{9}$$

such that

$$I = \frac{k_f{}^2}{8\pi} \sum_{m=-\infty}^{\infty} \mathrm{sgn}(m+\frac{1}{2}+\phi) \tag{10}$$

We first have to give a meaning to the divergent series $\eta \equiv \sum_m \mathrm{sgn}(m+\frac{1}{2}+\phi)$. To that purpose, we first derive a relation between η and the eigenvalues of the azimuthal current operator $\hat{J} = \frac{i}{r}\partial_\theta + \frac{\phi+\frac{1}{2}}{r}$. Consider the projection of $\hat{J}$ on a circle of radius $r = R$ (the exact value of R is irrelevant and taken to be one). The spectrum of the projected operator $\hat{J}_P$ is $\lambda = m + \frac{1}{2} + \phi$. Defining after Atiyah, Patodi and Singer[8] the quantity $\eta(\hat{J}_P) = \sum_{\lambda<0} 1 - \sum_{\lambda\geq 0} 1$, we can rewrite it

$$\begin{aligned}\eta(\hat{J}_P) &= \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{dt}{\sqrt{t}} Tr(\hat{J}_P e^{-t\hat{J}_P^2}) \\ &= \sum_\lambda \frac{1}{\sqrt{\pi}}\mathrm{sgn}(\lambda)\int_0^\infty \frac{dt}{\sqrt{t}}|\lambda|e^{-t|\lambda|^2}\end{aligned}$$

which gives $\eta(\hat{J}_P) = \eta$. It is of interest to rewrite $\eta(\hat{J}_P)$ under the form

$$\eta(\hat{J}_P) = -\frac{1}{2\sqrt{\pi}}\partial_\phi \int_0^\infty \frac{dt}{t^{\frac{3}{2}}} Z(t,\phi) \tag{11}$$

where $Z(t,\phi) = \sum_m e^{-t(m+\frac{1}{2}+\phi)^2} = Tr(e^{-t\hat{J}_P^2})$ is the partition function (or the Heat Kernel) at temperature $\frac{1}{t}$ of an electron constrained to move on a one-dimensional ring pierced by a Aharonov-Bohm flux. Before calculating η explicitly we notice that it was already considered in different contexts. To study the statistical properties of anyons, it was calculated[9, 10] using a Feynman-Kac integral. In the context of the Index theory of elliptic operators on manifolds with boundary, it was calculated[11] using a zeta function regularization i.e. by writing $\eta = \lim_{s\to 0}\sum_{\lambda\neq 0}\mathrm{sgn}(\lambda)|\lambda|^{-s}$. Here we shall evaluate it using the Poisson summation formula for the partition function $Z(t,\phi)$

$$\begin{aligned}Z(t,\phi) &= \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} dx e^{2i\pi nx} e^{-t(x+\frac{1}{2}+\phi)^2} \\ &= \sqrt{\frac{\pi}{t}} + 2\sqrt{\pi}\sum_{n\neq 0}\frac{1}{\sqrt{t}} e^{\frac{-\pi^2 n^2}{t}} e^{2i\pi n(\phi+\frac{1}{2})}\end{aligned}$$

Inserting this expression in (11), we obtain

$$\eta(\hat{J}_P) = -\frac{1}{2\pi^2}\partial_\phi \sum_{n=1}^{\infty} \frac{\sin^2 \pi n(\phi + \frac{1}{2})}{n^2}$$
$$= 2\{\phi + \frac{1}{2}\} - 1$$

where $\{...\}$ represents the fractional part. The persistent current is then given by

$$I = \frac{E_f}{2}\eta(\hat{J}_P) \tag{12}$$

The amplitude of the total current is proportional to the Fermi energy E_f and is therefore very large. This might be surprising since it is sometimes claimed that in the limit of an infinite system the normal persistent currents should vanish. This is indeed true for a one-dimensional ring of radius $R \to \infty$ but is incorrect in general. This point was discussed[12] and we shall come back to it later when calculating the magnetic moment. We would like now to discuss the topological features of the current and relate it to the Index associated with the Aharonov-Bohm problem. We saw previously that for a factorizable Hamiltonian the Index which counts the zero modes is defined by (5). Unfortunately, the Aharonov-Bohm Hamiltonian is at first sight non factorizable due to the additional factor $-2\pi B(r)$. However, by demanding self-adjointness of D and $D^\dagger$, we ended up with a (factorizable) Pauli Hamiltonian. This result was obtained in a different (and perhaps more physical) way, noticing that the energy spectrum for the conveniently regularized problem[9], is a non analytic function of the reduced flux ϕ. This has its origin in the behaviour of the wavefunction for the angular momentum $m = 0$. There, the unperturbed Hilbert space contains functions which do not vanish at the origin, while for $\phi \neq 0$, they do vanish like $r^{|\phi|}$. This gives rise to singularities in perturbation theory which can be dealt with by adding a repulsive contact term in the Hamiltonian[9]. To calculate the Index, we notice that since there is only one spin component (depending on the sign of the magnetic field), only one of the two operators D and $D^\dagger$ may have zero modes. The solutions of $Df = 0$ are (for large r) of the form $f \propto r^{m-\phi-\frac{1}{2}}$ and their square integrability at infinity requires $m < \phi - \frac{1}{2}$. To obtain well behaved solutions near the origin, we consider instead of the singular flux line a finite cylinder of radius ϵ in a uniform magnetic field B such that the magnetic flux is $\Phi = B\pi\epsilon^2$. The zero modes inside this disk are solutions of $(\partial_r + \frac{i}{r}\partial_\theta + \frac{B}{2}r)f = 0$ which in the sector m are of the form $f(r) \propto r^m e^{-\frac{Br^2}{4}}$. It is straightforward to check that these solutions do match for $r = \epsilon$ where both inside and outside logarithmic derivatives are given by $\partial_r f|_{r=\epsilon} = \frac{1}{\epsilon}(m - \phi - \frac{1}{2})f(\epsilon)$. Near $r = 0$, the zero modes behave like r^m so that square integrability requires $m \geq 0$. Finally, the number of zero modes i.e. the Index is given by Index $= [\phi + \frac{1}{2}]$, where $[...]$ represents the integer part. This corresponds to the degeneracy of the Aharonov-Bohm Hamiltonian (with its boundary condition).

There is another way to obtain the Index which may shed some more light on its physical interpretation. To that purpose, we consider the scattering description of the Aharonov-Bohm effect[13]. Berry et al.[14] proposed in that context to study the phase $\chi(r)$ of the scattered part of the wavefunction and in particular the dislocations of the wavefronts defined as points where the modulus of the wavefunction vanishes[15]. The circulation $\frac{1}{2\pi}\oint_c d\chi = \frac{1}{2\pi}\oint_c \vec{\nabla}\chi \cdot d\vec{r}$ of this phase over a close contour encircling the dislocation (i.e. the flux line) is an integer equals to $[\phi + \frac{1}{2}]$ i.e. to the Index. This result is not fortuitous and was clearly recognised in the mathematical literature[16] where the equivalence between the winding number around a singular point as introduced by

Poincare[17], the degree of a continuous map from the circle to the punctured complex plane and the Index of a conveniently defined elliptic operator was discussed. Each of those different points of view depends on the way we look at this problem. This way is geometric when considering instead of the initial map, say f, the mapping $\frac{f}{|f|}$ from the circle to the circle and count algebraically the number of intersections of the path with an arbitrary ray emanating from the origin. It is combinatorial when approximating the initial mapping by a piece-wise linear path and use combinatorial methods. It is differential in the way Berry et al.[14] considered it and analytic when studied from the point of view of the Index of an elliptic operator. It is the equivalence of those descriptions which is part of the richness of this problem. Although the differential point of view may appear at first sight more physical, the approach using the Index of an operator is more systematic which is useful in such non perturbative issues. Finally, comparing the different points of view, we arrive to the result

$$\text{Index} = [\phi + \frac{1}{2}] = \frac{1}{2\pi} \oint_c \vec{\nabla}\chi \cdot d\vec{r} \tag{13}$$

Here, the first equality is obtained by calculating the zero modes of D, and the second comes from the definition of the azimuthal current density which tells us that the integral over a closed contour can be evaluated either over a circle of large radius thus using the scattering form of the wavefunction[14] or on a circle of radius $r \to 0$ thus retaining in the series of Bessel functions (9) only the lowest indices which corresponds to a calculation of the zero modes. The Index as a function of the flux ϕ shows plateaus and jumps at half integers. As explained by Berry et al.[14], these jumps correspond to a long range reorganisation of the wavefronts in contrast to the behaviour at integers which describes local changes of the wavefronts around the flux line. It might be interesting at this stage to compare the previous results with those obtained in the context of superfluids or superconductors[18]. For superfluids, the gradient of the phase of the macroscopic Onsager-Feynman wavefunction measures the superfluid velocity and then the total current. Being a gradient, the latter describes an irrotational flow such that the circulation of the velocity on a closed curve is quantized as observed experimentally by Vinen[19]. As an outcome, the force exerted on a body immersed in the fluid vanishes (d'Alembert paradox). For a superconductor, the bulk current density vanishes (Meissner effect), such that from the circulation on a closed curve we obtain the quantization of the magnetic flux. But this is different from the Index theorem which states that the circulation on a closed curve of the gradient of the phase which must be an integer, depends on the flux as given by (13).

Considering the sum of $\eta(\hat{J}_P)$ and of the Index, we obtain the relation

$$\text{Index} + \eta(\hat{J}_P) = \phi \tag{14}$$

which in other words gives a sum rule between the radial and azimuthal integrals of the current density j_θ. This relation is the expression of a general result[8] which generalises to non compact spaces the Atiyah Singer Index theorem.

It might be also of interest to rewrite (14) using for $\frac{I}{E_f}$ its expression in terms of scattering phase shifts[12], $\frac{I}{E_f} = \frac{1}{\pi}\partial_\phi \sum_m \delta_m(\phi)$ where the phase shifts for a spin one half are given [20] by $\delta_m(\phi) = \frac{\pi}{2}(|m + \phi + \frac{1}{2}| - |m|)$. Then,

$$\text{Index} + \frac{1}{\pi}\partial_\phi \sum_m \delta_m(\phi) = \phi \tag{15}$$

Under this form, it is easy to see that the phase shift term represents the boundary contribution to the Index theorem for non compact manifolds.

The relation between the persistent currents and the scattering phase shifts suggests an interesting analogy between this problem and the screening of an electric charge as given by the Friedel sum rule[21]. In the latter case, inserting an external charge Z in a metal the electrical neutrality is expressed self-consistently by the sum rule $\pi Z = \delta(E_f)$ where $\delta(E_f)$ is the total scattering phase shift calculated at the Fermi energy describing the scattering of electrons by the external charge. For the Aharonov-Bohm case, we can interpret the persistent currents in a similar way saying that the magnetic flux line is screened by current loops of electrons sitting at infinity. Using (15), the current can be expressed in terms of a topological invariant, namely the Index. The Friedel sum rule may be understood similarly. The total phase shift which in principle depends on the microscopic details of the potential created by the external charge is in fact a function of Z only, irrespective of the way this charge is distributed.

Finally, we need to evaluate the total magnetic moment. Substituting (9) into the Biot-Savart law gives

$$M = \mu_B \int_0^R r^2 dr \int_0^{k_f} kdk \sum_{m=-\infty}^{\infty} \frac{m+\phi+\frac{1}{2}}{r} J^2{}_{|m+\frac{1}{2}+\phi|}(kr). \tag{16}$$

Using the Euler-Maclaurin summation formula[22] we obtain an asymptotic expansion in terms of the large parameter $k_f R$ such that the magnetic moment can be written as a series $\sum_{m=-\infty}^{\infty} F(m+\phi+\frac{1}{2})$, where the function F is

$$F(x) = x\mu_B \int_0^R rdr \int_0^{k_f} kdk J^2{}_{|x|}(kr).$$

A naive application of the Euler-Maclaurin summation formula would give zero due to the vanishing of the function $F(x)$ at plus and minus infinity, for all finite k_f and R. However this function is singular at $x = 0$, where its second derivative is discontinuous with a jump equal to $-2\mu_B$ (this result has small corrections of the order of $\mu_B/k_f R$). Then, the usual derivation of the Euler-Maclaurin summation formula[23] should be revised, taking into account the singularity at $m+\phi+\frac{1}{2} = 0$. Finding an integer m_1 such that $m_1+\phi+\frac{1}{2} < 0 < m_1+\phi+\frac{3}{2}$, it turns out that the important parameter (which determines the position of the singularity) in the derivation of the modified Euler-Maclaurin summation formula is $m_1+\phi+\frac{3}{2}$, which is equal to the fractional part of the total flux ϕ. One can prove then that the correction itself is proportional to the jump of the second derivative times the Bernoulli polynomial of third order, evaluated at $\{\phi\}$. More precisely we obtain

$$M \sim \frac{\mu_B}{3} B_3(\{\phi\}) + O(\mu_B/k_f R), \tag{17}$$

where $B_3(x) = x^3 - 3x^2/2 + x/2$ is a Bernoulli polynomial. The magnetization of the system, which is according to the definition the magnetic moment per unit area, vanishes in the thermodynamic limit. Indeed, (17) corresponds to the finite magnetic moment of an infinite system. Therefore, even though the current is large, the magnetic response is experimentally inaccessible.

HEAT KERNEL AND PARTITION FUNCTION FOR THE AHARONOV-BOHM PROBLEM

¿From (11), we obtained an expression for the persistent current I in terms of an integral over the partition function $Z(t, \phi)$ of an electron moving on a one-dimensional ring pierced by a Aharonov-Bohm flux at an effective temperature $\frac{1}{t}$. On the other hand, since the persistent current is a thermodynamic quantity, it is possible in principle to express it in terms of the Heat Kernel (or the partition function) $P(\beta,\phi) = Tr(e^{-\beta H(\phi)})$ where $H(\phi)$ is the Hamiltonian defined by (2) for a constant ϕ and where β is the inverse temperature. To obtain a relation between these two quantities, we start writing the thermodynamic grand potential $\Omega(\beta,\phi) = \int_0^\infty dE f(E,\mu,\beta) N(E)$ where f is the Fermi-Dirac distribution function, μ the chemical potential and $N(E)$ the integrated density of states (the counting function). The current is then given by $I = -\frac{\partial \Omega}{\partial \phi}$ in the appropriate units. We then define after Sondheimer and Wilson[24] the function $z(E,\phi)$ by

$$\frac{P(\beta,\phi)}{\beta^2} = \int_0^\infty dE e^{-\beta E} z(E,\phi) \tag{18}$$

The grand potential as a function of $z(E,\phi)$ rewrites $\Omega(\beta,\phi) = \int_0^\infty dE \frac{\partial f}{\partial E} z(E,\phi)$. In the asymptotic limit $\beta\mu \gg 1$, i.e. for very low temperatures we obtain $I = \frac{\partial z(\mu,\phi)}{\partial \phi}$ from which we deduce

$$\frac{1}{\beta^2}\frac{\partial P(\beta,\phi)}{\partial \phi} = \int_0^\infty dE e^{-\beta E} I(E,\phi) \tag{19}$$

By inserting (11) for I, we obtain

$$\frac{\partial P(\beta,\phi)}{\partial \phi} = -\frac{1}{4\sqrt{\pi}} \partial_\phi \int_0^\infty \frac{dt}{t^{\frac{3}{2}}} Z(t,\phi) \tag{20}$$

which relates the partition function $P(\beta,\phi)$ of the two-dimensional problem to the partition function $Z(t,\phi)$ of an effective one-dimensional Aharonov-Bohm problem.

BEYOND THE AHARONOV-BOHM CASE

We would like now to discuss the extension of the previous results to more general magnetic field configurations. We noticed that the Pauli Hamiltonian (4) is always factorizable irrespective of the field configuration $B(r)$. Using this property, the ground state degeneracy is given by the Index. In addition to that result we obtained for the case of a flux line a relation (14) between the Index and $\eta(\hat{J}_P)$. This relation holds for any factorizable Hamiltonian. Is it possible to interpret it in terms of the magnetic response of the system? To that purpose we consider once again the simpler case of a uniform magnetic field. There, the orbital and spin parts of the magnetization can be separated. In the radial gauge $\vec{A} = \frac{B}{2}(-y,x)$, the Pauli Hamiltonian is $H = \frac{1}{2m}(\vec{p} - \frac{e}{c}\vec{A})^2 - \mu_B B\sigma_z \equiv H_0 - \mu_B B \sigma_z$. This can be written in a matricial form

$$H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}$$

where $H_\pm = H_0 \pm \mu_B B$. Defining the supercharge $Q = \frac{1}{\sqrt{4m}}\vec{\Pi}.\vec{\sigma} = \frac{1}{\sqrt{4m}}(\Pi_x\sigma_x + \Pi_y\sigma_y)$ where $\Pi_x = p_x - \frac{e}{c}A_x$ and $\Pi_y = p_y - \frac{e}{c}A_y$, we can write the Hamiltonian under the supersymmetric form $H = 2Q^2$. The associated creation and annihilation operators A and $A^\dagger$ are given by $A = \frac{1}{\sqrt{2m}}(\Pi_x - i\Pi_y)$ and $A^\dagger = \frac{1}{\sqrt{2m}}(\Pi_x + i\Pi_y)$, such that $H_+ = AA^\dagger$ and $H_- = A^\dagger A$. According to (5), the Index is then given[25] by Index $=$ $\dim\mathrm{Ker}A - \dim\mathrm{Ker}A^\dagger$. To go further and connect it with the thermodynamics of the system, we notice that $\mathrm{Ker}A = \mathrm{Ker}A^\dagger A = \mathrm{Ker}H_-$ and $\mathrm{Ker}A^\dagger = \mathrm{Ker}AA^\dagger = \mathrm{Ker}H_+$.

Since the spectra of H_+ and H_- are identical except perhaps for the zero modes, it is possible to rewrite the Index using the regularization

$$\text{Index} = \lim_{t\to 0} \text{Tr}(e^{-tH_+} - e^{-tH_-}) = \lim_{t\to 0} \text{Tr}(\sigma_z e^{-tH}) \tag{21}$$

The spectrum of the Pauli Hamiltonian is given as for the Landau case by a set of infinitely degenerate Landau levels. The supersymmetric pairing of all the excited states means that they do not participate to the spin magnetization since there are no states with respectively spin up and spin down which are unpaired on an excited Landau level. Then, a finite spin magnetization stems only from possible unpaired states in the lowest Landau level. For a non interacting electron gas, the Fermi energy E_f is determined by fixing the total number N of electrons through $N = \text{Tr}(\Theta(E_f - H))$. The spin magnetization can be written $M_S = \mu_B(N_+ - N_-)$ where N_+ and N_- are the number of electrons with spin up and spin down respectively. Using a regularization equivalent to (21) we obtain the result

$$M_S = \mu_B \text{Tr}(\sigma_z \Theta(E_f - H)) = \mu_B \text{Index} \tag{22}$$

In order to calculate explicitly the Index, we write the traces in (21)

$$\text{Index} = \lim_{t\to 0} \int d^2x \{\langle x|e^{-tH_+}|x\rangle - \langle x|e^{-tH_-}|x\rangle\} \tag{23}$$

The Heat Kernels for respectively spin up and spin down which do appear in (23) are given for the case of a uniform magnetic field by

$$\int d^2x \langle x|e^{-tH_\pm}|x\rangle = \frac{1}{2}\frac{\Phi}{\Phi_0} e^{\pm\frac{\hbar\omega_c t}{2}} \frac{1}{\sinh(\frac{\hbar\omega_c t}{2})} \tag{24}$$

where $\omega_c = \frac{eB}{mc}$ is the cyclotron frequency. The Index is effectively t-independent and given by the total magnetic flux (in units of the flux quantum Φ_0) and is therefore infinite.

The outcome of this is the relation between the Index and the spin magnetization given by (22). As anticipated from the role of the Zeeman term for the Pauli Hamiltonian to be factorizable, the Index appears as a measure of the spin magnetization. Nevertheless, this result is very peculiar and applies only for the case of a uniform magnetic field, where the orbital and Zeeman parts in the Hamiltonian do commute, such that the respective magnetic responses are independent. In the general case of a inhomogeneous magnetic field, it is not anymore possible to disentangle those two parts. Another peculiarity of the uniform field case is that the Index is infinite and formally ill-defined. For a magnetic field of finite flux, it is not anymore the case and the total magnetic response is still given by (14) as a consequence of the Atiyah, Patodi, Singer[8] theorem.

CONCLUSION

We have presented a study of the magnetic response of a degenerate non interacting electron gas in a inhomogeneous magnetic field from the point of view of the Index Theory. For the limiting case of a Aharonov-Bohm flux line, it allowed us to find a relation between the persistent current and a topological invariant, namely the Index of an operator and to calculate it. We obtained a relation between this Index and the winding number defined geometrically from the behaviour of the wavefunction. To provide another physical interpretation of the Index, we showed that for the case of a uniform magnetic field, it is proportional to the spin magnetization and we presented a general regularization scheme to calculate it from the partition function (or the Heat Kernel). This connection to the magnetic response can be generalised to other systems as an application of the Atiyah, Patodi, Singer Theorem. The extension of this point of view to study transport coefficients in the Quantum Hall regime is appealing. It allows[26] to relate the Hall conductance to a suitably defined Index for systems with a boundary. This generalises other approaches using topological numbers[27, 28]. Among other problems where the point of view presented here may be relevant, is the puzzle of mesoscopic persistent currents in small (but multichannel) rings. There, as discussed by Leggett[29], the existence of nodal lines where the wavefunction vanishes would involve more sophisticated topological tools in order to make some progress.

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