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UNIVERSAL FLUCTUATIONS AND LONG-RANGE CORRELATIONS FOR WAVE PROPAGATION IN RANDOM MEDIA

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Invited paper

The last few years, a large amount of effort has been devoted to the study of coherent effects in disordered systems. They essentially dealt with average values over a whole ensemble of samples and gave rise to the so-called coherent corrections of the weak localization regime.

Recently, the transport properties of small disordered (mesoscopic) samples were considered from the view-point of the exact wave configuration (speckle pattern) of a given realization of the random impurity potential.

A new kind of coherent effects appeared in the statistics of various properties of disordered samples like for example the universal conductance fluctuations in metals which are size-independent. These properties are shown to come from the long-range correlations in the fluctuations due to the underlying wave-field.

1. Introduction

The aim of this article is to give a brief summary of the works recently devoted to the study of fluctuations in the propagation of waves or electrons in random media. It has been known for a long time that a wave propagating in a random medium gives rise to an irregular intensity pattern (a speckle pattern) due to the interference of the scattered waves. A huge literature [1] has been devoted in optics to these patterns in order to obtain the intensity distribution law of the speckle spots as well as their contrast. Nevertheless, it has been recently realized [2], in the study of electronic disordered systems of mesoscopic sizes, that the correlations in speckle patterns are of long-range nature when multiple scattering occurs. It was shown at the same time that relative fluctuations of quantities like the total transmitted intensity or the electronic conductance are much larger than what is predicted by the classical theory [3].

2. Fluctuations in disordered mesoscopic electronic systems and in speckle patterns

Until recently, physicists were very used to the idea that transport coefficients, like the electrical conductance of a piece of metal, are well described by average values. For instance a large amount of effort has been devoted to the study of the conductance of disordered metals in order to compare with the theoretical predictions of the localization problem [5]. It appeared to be affected by quantum effects which are now well understood at least in the weakly scattering regime. Spectacular phase coherent effects like magnetoresistance oscillations with period $\phi_0 = h/2e$ (the Sharvin and Sharvin effect [6]) are quantitatively explained by means of the weak localization theory.

The first deviations from the use of averages to characterize completely disordered systems came from experiments done on sub-micron normal metal samples [7]. These systems of so-called mesoscopic size are good metals, like Au–Pd for instance. This means that their average electrical conductance is very large compared to, say, e^2h, or in other words the elastic mean free path is such that $k_F l \gg 1$. These samples are studied at low temperature (typically between 10 and 100 mK) in order the inelastic scattering length $L_{in} \cong 1\ \mu m$ to be larger than their size. The first experiments [8] on these mesoscopic systems consist in measuring their conductance when varying the applied magnetic field in different geometries. The first one is a ring of diameter roughly equal to $1\ \mu m$ placed in a magnetic field $\boldsymbol{B}$ such that the electrons in the disordered metals are affected only by the vector potential $\boldsymbol{A}$. The Fourier transform of the measured conductance variations is directly related to the density of closed electronic trajectories classified according to their trajectories. Among the various contributions, there is one corresponding to the addition of a flux h/e through the ring. It is exactly the well-known Aharanov–Bohm effect but in a disordered system.

The second geometry was used in the experiment done by Benoit et al. [9]. It gives us the voltage fluctuations of a mesoscopic metallic (Sb–Au) sample as a function of the length between the two-measurement probes. They observed that for lengths L smaller than the inelastic mean free path (or generically the phase coherence length L_ϕ), the total rms voltage fluctuations do not depend on L, while for $L > L_\phi$ they recovered a classical behavior.

From the observed behavior of these systems, we can deduce some very characteristic and unusual features.

i) The noisy-like structure observed on the aperiodic part of the conductance oscillations is time-independent and perfectly reproducible at a given temperature. It varies when changing the microscopic impurity configuration.

ii) There exists a Aharonov–Bohm effect with h/e periodicity. This implies interferences between trajectories of electrons which have been scattered many times, and therefore a long-range phase effect exists at the scale of the length of the system, i.e. much larger than the elastic mean free path l.

iii) The amplitude of the conductance fluctuations remains very large even for systems of mesoscopic sizes, much larger than it is predicted classically ($\delta g/\langle g\rangle \propto L^{-d/2}$).

iv) The size-independent behavior of the rms voltage fluctuations implies that transport properties in these systems depend only on L_ϕ but not on the size L of the system as far as $L < L_\phi$.

Let us now describe the counterpart of this kind of experiments in the domain of electromagnetic waves propagating in solid or liquid suspensions. Two main systems are usually considered. The first one is an aqueous suspension of polystyrene beads with high volume fraction ($\cong 10\%$). The diameter of the beads used is roughly $0.5\ \mu$m and the transport elastic mean free path is $l^* \cong 20\ \mu$m. The wavelength $\lambda \approx 0.5\ \mu$m is such that those systems are equivalent of good metals in the sense that the ratio $l/\lambda \cong 10^2$ is much larger than one (weak localization regime). But it is necessary to study these suspensions at short time scales (typically $\Delta t \cong 100\ \mu$s) in order to avoid self-averaging effects due to the thermal Brownian motion of the scatterers [11]. Another system I shall discuss more in detail is the solid suspension of TiO_2 particles embedded in polystyrene considered by Genack and Drake [12]. It has roughly the same characteristics of the liquid solution. The experiment consists in measuring on a given speckle spot of the total transmitted pattern the intensity correlation function when varying the wavelength of the incident laser light. The averaging procedure is realized by summing over many speckle spots. They showed that this correlation function is exponentially decreasing with a characteristic frequency $\Delta\omega = D/L^2$, where $D = \frac{1}{3}vl$ is the diffusion constant of the light in the system, while L is its width. This correlation frequency is much smaller than the expected value $1/\tau$ of speckle patterns obtained from single-scattering systems.

The net conclusion of all those experimental results would be that the standard theory of fluctuations in the electronic conductance as well as for speckle patterns seems to be unable to describe them. Before going further let us recall the main features of this classical theory.

3. "Classical" theory of fluctuations for optical speckle patterns and for electronic conductance

We consider the situation of a slab of thickness L and surface W containing a suspension of elastic scatterers. A plane wave of wavelength λ is incident on its surface and we are interested in the light intensity I_θ emerging in a given angle θ. It is convenient to introduce the number N of channels given as usual by $N = W/\lambda^2$ or in other words the total number of speckle spots in the pattern. The emerging intensity I_θ is the sum of many sources. The relative phase of those sources can be assumed to be random due to the multiple scattering inside the medium. Therefore, we can suppose that the amplitudes contributing to I_θ are independent random variables leading to

$$\langle I_\theta^2 \rangle = 2\langle I_\theta \rangle^2 . \tag{1}$$

This gives rise to the well-known Rayleigh distribution for the intensity. From eq. (1), we can deduce that the speckle contrast between the different spots given by the relative fluctuation of I_θ is unity.

We are now interested in the fluctuations of the total intensity defined by

$$I = \sum_{i=1}^{N} I_{\theta i} \tag{2}$$

which is the quantity experimentally measured.

Let us now make the fundamental assumption that all the channels are uncorrelated, which is very reasonable for single scattering situations. Therefore we obtain for the variance of I

$$\mathrm{Var}\, I = N\langle I_\theta \rangle^2 . \tag{3}$$

The relative fluctuations of the total intensity or equivalently those of the transmission coefficient T_b for an incident plane wave in the channel b are then given by the well-known result:

$$\frac{\delta I}{\langle I \rangle} = \frac{\delta T_b}{\langle T_b \rangle} = \frac{1}{\sqrt{N}} , \tag{4}$$

where $\delta I \equiv (\mathrm{Var}\, I)^{1/2}$. An equivalent result can be obtained for the relative fluctuations of the electronic conductance, $\delta g / \langle g \rangle$, under the same assumption of uncorrelated channels if one remembers that the dimensionless conductance g (in units of e^2/h) can be expressed as a function of the transmission coefficient according to the Landauer formula

$$g = \sum_{ab} T_{ab} . \tag{5}$$

It differs from the previous situation because we have now to sum over all the incident channels. The same kind of calculation leads to

$$\frac{\delta g}{\langle g \rangle} = \frac{1}{N} \tag{6}$$

and

$$\delta g = \frac{l}{L} \, . \tag{7}$$

These expressions (4), (5) and (6) are in contradiction with experimental results presented above and lead to the conclusion that the assumption of independent channels must be seriously revisited.

4. Universal fluctuations and long-range correlations

The theoretical challenge raised by the discrepancy between experiments and the theory of classical fluctuations results in a large literature. I would like to sketch briefly what are the three main approaches within which, at least to my taste, most of this literature can be classified.

The first one is built on the Landauer formulation of the transport in disordered systems. More precisely earlier works from Azbel [13] and Büttiker et al. [14] tried to generalize the one channel Landauer formula of the conductance and proposed at that time that magnetoresistance oscillations with period h/e corresponding to the usual Aharonov–Bohm effect could be observable in small rings. Later, a numerical simulation of the magnetoresistance of small wires and rings done by Stone [16] reproduced the experimental result including the aperiodic fluctuations and the Aharonov–Bohm effect. These simulations did use the Azbel and Büttiker et al. multichannel Landauer formula, which appeared to contain all the relevant ingredients to go beyond the classical theory.

A second approach developed simultaneously by Altschuller [17] and Lee and Stone [18] consists in doing a perturbation expansion in the small parameter $(k_F l)^{-1}$ as is usually done in the weak localization theory. They calculated the correlation function of the conductivity σ to zeroth order in $(k_F l)^{-1}$ with the result

$$\delta\sigma^2 \equiv \langle \sigma^2 - \langle \sigma \rangle^2 \rangle \cong \left(\frac{e^2}{\hbar}\right)^2 L^{4-d} \, , \tag{8}$$

where L is the thickness of the sample supposed to be infinite in the other directions. From the relation $g = \sigma L^{d-2}$ which is assumed to remain valid for the random variables themselves, we obtain

$$\frac{\delta g}{\langle g \rangle} \cong \frac{1}{L^{d-2}} \tag{9}$$

for the relative fluctuations of the dimensionless conductance. Since $\langle g \rangle \sim L^{d-2}$, eq. (9) implies that the variance of the conductance δg is of order unity, i.e. does not depend on the size L of the sample neither on the disorder characterized by the elastic mean free path l.

Finally, the third approach based on some properties of the eigenvalues of large random matrices leads Imry [19] to the conclusion that the number of independent channels is not N as it was assumed in the classical theory but $N_{\text{eff}} = Nl/L$. Inserting this result which takes into account the correlations between the channels in eq. (8) of the classical theory, one obtains

$$\frac{\delta g}{\langle g \rangle} = \frac{1}{N_{\text{eff}}} = \frac{L}{Nl} \,. \tag{10}$$

From Ohm's law $g = \sigma L^{d-2}$, we can calculate $\langle g \rangle = lN/L$ so that $\delta g \cong 1$, which coincides with the result of eq. (11) obtained within perturbation theory. The result given above for N_{eff} is strongly related with the experimental result obtained in optics by Genack and Drake [12]. Starting from their results it is indeed possible to obtain the above expression for N_{eff}. They got the result that the characteristic frequency of the intensity correlation function is $\Delta\omega = D/L^2$. The transmission coefficient or the total intensity for an incident plane wave is determined by the number $\mathcal{N}$ of levels in the bandwidth $\Delta\omega$ so that

$$\mathcal{N} = \nu V \, \Delta\omega \,, \tag{11}$$

where $\nu = 8\pi/v\lambda^2$ is the number of energy states per unit energy and unit volume.

Then, we obtain

$$\mathcal{N} = \frac{8\pi}{3} \frac{l}{\lambda^2} \frac{W}{L} = \frac{8\pi}{3} N_{\text{eff}} \,. \tag{12}$$

This result shows unambiguously that the classical assumption of uncorrelated channels fails badly when it is applied to systems undergoing multiple scattering. Let us also mention that this third approach based on properties of random matrices was also used to recover the above result and to calculate the correlation functions between the channels in reflexion and in transmission

[20]. Actually a slightly different approach [21] using a "maximum entropy" criterion was proposed in order to obtain the equation of motion of the probability density of the transfer matrices. Without going into details the interested reader can find in the proposed references, I prefer to give the final results which look particularly clear. The correlation function between channels in transmission defined by $C^{\mathrm{T}}_{aba'b'} \equiv \langle T_{ab} T_{a'b'} \rangle - \langle T_{ab} \rangle \langle T_{a'b'} \rangle$ is given by

$$C^{\mathrm{T}}_{aba'b'} = \langle T_{ab} \rangle \langle T_{a'b'} \rangle [\delta_{aa'} \delta_{bb'} + \tfrac{2}{3} \langle T \rangle^{-1} (\delta_{aa'} + \delta_{bb'}) + \tfrac{2}{15} \langle T \rangle^{-2}] , \tag{13}$$

where T_{ab} is defined as discussed above as the transmission coefficient of the intensity for an incident plane wave in the channel a emerging along the channel b. Moreover, we have $\langle T_{ab} \rangle = l/NL$, and $\langle T \rangle = Nl/L$ is proportional to the conductance. Finally, let us precise that eq. (13) has been obtained to leading order in $N \gg 1$ and $L/l \gg 1$. In reflexion, with obvious notations and in the same limit one obtains

$$\begin{aligned} C^{\mathrm{R}}_{aba'b'} = \langle R_{ab} \rangle \langle R_{a'b'} \rangle [&(1 + \delta_{ab})(\delta_{aa'} \delta_{bb'} + \delta_{ab'} \delta_{a'b}) \\ &+ \langle R \rangle^{-1} (\delta_{ab} \delta_{a'b'} \delta_{aa'} - \delta_{aa'} - \delta_{bb'} - \delta_{ab'} - \delta_{ba'}) + \tfrac{32}{15} \langle R \rangle^{-2}] . \end{aligned} \tag{14}$$

Eq. (13) for $C^{\mathrm{T}}_{aba'b'}$ coincides with eq. (3) of ref. [22] in the limit $W \ll L$. The calculations of ref. [22] being obtained via a perturbation expansion according to what we called the second approach are another confirmation of the equivalence of the various approaches. Let us now discuss the main features of eqs. (13) and (14). The first term in the expression of $C^{\mathrm{T}}_{aba'b'}$ which is the dominant one is nothing but the familiar local intensity fluctuations of speckle patterns giving rise to eq. (1). The second term describes long-range spatial correlations between the channels. It gives the dominant contribution to the quantity Var T_b which describes fluctuations of the total emergent intensity for an incident plane wave. We obtain

$$\mathrm{Var}\, T_b = \frac{2}{3} \frac{l}{NL} . \tag{15}$$

This term was also predicted by Stephen and Cwillich [23] by means of weak scattering perturbation expansion. Finally, the third term appears to give the dominant contribution when we are calculating the variance, Var T, of the total transmission coefficient summed over all the incident channels or in other words the variance of the conductance. It gives the already well-known result

Var $T \cong c^{te}$. The constant which is not universal at all appears in this case to be equal to $\frac{2}{15}$.

The different contributions appearing in eq. (4) for $C^{R}_{aba'b'}$ cannot be separated so easily. The first term gives, like for C^{T}, the usual speckle contribution. Let us remark that it reveals the familiar enhanced backscattering phenomenon [24] when calculating averages. The second and third terms describe correlations between the reflected channels. It is worthwhile to remark that Var R, which is the counterpart of Var T or Var g, is determined by all the terms appearing in eq. (4) and not only by the last one as before. They all contribute to give Var $R = \frac{2}{15}$ = Var T which ensures the conservation of energy. There are corrections to Var R proportional to $(l/L)^2$ and not to l/L as found by Lee [25] who supposed that the channels in reflexion were not correlated which strongly contradicts eq. (4). For all results, see table I.

Table I
Summary of the various regimes obtained respectively from the classical theory of fluctuations developed in section 3 and from long-range correlations as explained in section 4. We recall that $\delta X^2 \equiv \langle X^2 - \langle X \rangle^2 \rangle$. The model-dependent numerical value $\frac{2}{15}$ is obtained from calculations of ref. [20].

	Classical theory	Fluctuations in multiple scattering theory
$\langle T_{ab} \rangle$	$\frac{l}{NL}$	$\frac{l}{NL}$
$\langle T^2_{ab} \rangle$	$2\langle T_{ab} \rangle^2$	$2\langle T_{ab} \rangle^2$
$\langle T_b \rangle$	$\frac{l}{L}$	$\frac{l}{L}$
δT^2_b	$\frac{l^2}{NL^2}$	$\frac{l}{NL}$
$\langle g \rangle$	$\frac{Nl}{L}$	$\frac{Nl}{L}$
$\delta g^2 = \delta T^2$	$\left(\frac{l}{L}\right)^2$	$\frac{2}{15}$
$\langle R_{ab} \rangle$	$\frac{1}{N}\left(1 - \frac{l}{L}\right)$	$\frac{1}{N}\left(1 - \frac{l}{L}\right)$
$\langle R^2_{ab} \rangle$	$2\langle R_{ab} \rangle^2$	$2\langle R_{ab} \rangle^2$
δR^2_b	$\frac{1}{N}\left(1 - \frac{l}{L}\right)^2$	$\frac{l}{NL}$
δR^2	$\left(1 - \frac{l}{L}\right)^2$	$\frac{2}{15} + \mathcal{O}\left(\frac{l}{L}\right)^2$

5. Conclusion

I have presented in this article a brief and superficial review on this exploding field of physics now sometimes called "mesophysics". New kinds of theoretical approaches, actually already in gestation since a long time, were developed somewhat independently. They are explaining quite well the experiments but the proof of their equivalence needs still a lot of effort.

I would like also to emphasize that optical experiments seem to be more appropriate to test theoretical predictions since the fluctuations should be larger due to the fact that we are not integrating over incident channels. Moreover, the transmission or reflexion coefficients are directly related to the theory which is not the case for the two-points-conductance expression in electronic systems as was discussed in section 2.

To conclude on some open problems, I must mention a question related to the approach of the localization transition, i.e. for $\langle g \rangle \sim e^2/h$. It seems at first sight that no modification has to be introduced by universal fluctuations theory since it gives only numbers independent of any of the characteristics of the system. Therefore no new scaling parameter is introduced. Nevertheless, it has been recently predicted [27] that the distribution function of conductance fluctuations should have a non-universal log-normal tail which can be neglected in the regime $\langle g \rangle \gg e^2/h$ but becomes predominant near the localization transition. There the failure of the one-parameter scaling theory should appear. All these questions deserve further studies for the future.

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