SEMINAR 3

GEOMETRICAL DESCRIPTION OF VORTICES IN GINZBURG-LANDAU BILLIARDS

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Contents

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1 Introduction

In these notes we discuss the topological nature of some problems in condensed matter physics. This topic has been widely studied in various contexts. In statistical mechanics, the possible stable defects in an ordered system have been classified according to the nature of the order parameter (*e.g.* scalar, vector, matrix) and the space dimensionality of the system using homotopy groups [1]. Then, the discovery of the quantum Hall effects and the role played by stable integers or rational numbers for systems with few or no conserved quantum symmetries have motivated several topological models of quantum condensed matter systems [2, 4]. A combination of these two ideas of defects classification and microscopic quantum models has been used in the description of superfluid ^{3}He [5].

Here, instead of trying an exhaustive review of problems where topological ideas may play a role, we present the basic constituents needed in a geometrical description and calculate the related topological numbers. The ideas and methods developed in mathematics and mathematical physics to solve problems in geometry and topology are pretty sophisticated and sometimes expressed in a way unfamiliar to the physicist. We adopt the language of differential geometry to present this subject, since it is adapted to develop some intuition towards more elaborate concepts like fiber bundles, connexions and topological invariants.

In the last two sections, we shall discuss the problem of superconducting billiards within the Ginzburg-Landau approximation. This problem is interesting for several reasons. First, it is a non trivial example on which topological methods naturally apply to give an elegant solution to the calculation of the ground state energy. It is also a situation for which one can

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address the question of transition between different values of a topological number controlled by the boundary of the system. This question is similar to the transition between quantum Hall plateaus.

Such problems are not only of academic interest. Our motivation has been triggered by a set of new experimental results obtained on small size aluminium disks [6, 7] in a regime where their radius R is comparable with both the coherence length ξ and the London penetration depth λ. The magnetization, as a function of the applied magnetic field, presents a series of jumps with an overall shape reminiscent of type-II superconductors, although a macroscopic sample of aluminium is a genuine type-I superconductor. These notes end with a theoretical analysis of these experimental results.

2 Differentiable manifolds

2.1 Manifolds

A differentiable manifold M of dimension n is a space which looks locally like an open set of $\mathbf{R}^n$. On the vicinity of each point of M one can define a **local coordinate system** where each point is represented by a set of n real numbers $(x^1, ..., x^n)$. There exists geometrical spaces which are not differentiable manifolds like arithmetical ensembles $(\mathbf{Q}, \mathbf{Z}/5\mathbf{Z})$, fractals, objects with branching points (Feynman diagrams). On the other hand vector spaces, spheres, projective spaces, matrix groups $GL(n, \mathbf{R}), SO(2), SO(3)$ (direct isometries), $SU(2), SU(n)$ (complex isometries) are differentiable manifolds. In order to do differential calculus on a manifold, different coordinate systems are required to be **compatible** with each other: the local transformations between one system of coordinates to another have to be smooth and invertible. A quantity attached to a manifold is said to be **geometrical** (or *intrinsic*) if it is independent of the choice of a coordinate system.

Example: The sphere S^2 in Euclidian space $\mathbf{R}^3$ can be endowed with local cartesian coordinates, or with spherical coordinates (latitude and longitude). None of these two systems is global, for instance the spherical coordinates are singular at the poles, although from a geometrical point of view all points on the sphere are equivalent. Therefore the special role played by the poles is an artifact of the spherical coordinate system. On the other hand the tangent plane passing through a point of the sphere is an intrinsic object, although its equation looks very different in cartesian and spherical coordinates.

2.2 *Differential forms and their integration*

2.2.1 Tangent space

Consider a manifold M and p one of its points. Let γ be a curve in M passing through p. In a local coordinate system, the curve γ is given by

$$\begin{aligned} \gamma : [-1,1] &\rightarrow M \\ t &\mapsto (x^1(t), ..., x^n(t)). \end{aligned} \tag{1}$$

For $t = 0$, γ passes through the point p represented by the coordinates $(x^1(0), ..., x^n(0))$. A **tangent vector** to the manifold M at the point p is by definition a tangent vector to a curve passing through p. For instance, the curve γ defines a tangent vector at p $\vec{v}\left(= \frac{\mathrm{d}\gamma}{\mathrm{d}t}\right)_{t=0}$ with components:

$$v^i = \left(\frac{\mathrm{d}x^i}{\mathrm{d}t}\right)_{t=0} . \tag{2}$$

If one uses a different set of local coordinates $(X^1, ..., X^n)$ in the vicinity of p, one finds that the components of $\vec{v}$ are given by

$$V^i = \left(\frac{\mathrm{d}X^i}{\mathrm{d}t}\right)_{t=0} . \tag{3}$$

The **transformation law** from one set of components to the other is

$$V^i = \left(\frac{\partial X^i}{\partial x^j}\right)_p v^j . \tag{4}$$

We use Einstein's convention of summation upon repeated indexes. The transformation law of the components of $\vec{v}$ is inverse of that of the partial derivatives $\frac{\partial}{\partial x^j}$. Hence in "classical" (XIX century) mathematical literature, a tangent vector was said to be *contravariant* and the transformation formula was used as a *definition*: a tangent vector is an object whose components transform according to (4) under a change of the local coordinate system.

The set of all the tangent vectors at p to all curves γ included in M and passing through p is a vector space, called the **the tangent space** of M at p and denoted by T_pM. It has the same dimension as the manifold itself. With the help of (4), one can define a tangent vector, as an invariant object that does not depend on the coordinate system:

$$\vec{v} = v^i \left(\frac{\partial}{\partial x^i}\right)_p = V^i \left(\frac{\partial}{\partial X^i}\right)_p . \tag{5}$$

The vectors

$$e_i = \left(\frac{\partial}{\partial x^i}\right)_p \quad i = 1, ..., n \tag{6}$$

are a basis of the tangent space T_pM. This basis is *local* and depends on the point p. Usually one does not write explicitly the dependence on p. However it is extremely important to keep in mind that two tangent spaces T_pM and $T_{p'}M$ at two different points p and p' are *different* vector spaces. It is not possible *a priori* to compare (*i.e.* add, subtract) two tangent vectors $\vec{v}$ and $\vec{v'}$ at p and p'. A simple way to understand that is to realise that the transformation law (4) is point dependent. Therefore $\vec{v}$ and $\vec{v'}$ could by chance have the same components in a special coordinate system but different components in another system. So it would be a mistake to say that they are "equal".

The *total* **tangent space** TM of M is the collection of all tangent vectors at any point of M. Hence $TM = \bigcup_p T_pM$. There is a natural function π from TM to M called *projection*: to each $\vec{v} \in TM$ is associated $p = \pi(\vec{v})$ which is the point of M at which $\vec{v}$ is tangent to M. It can be shown that TM is a manifold of dimension $2n$: if $\vec{v}$ belongs to TM one needs n components to specify at which point p the vector $\vec{v}$ is tangent and n more components to specify $\vec{v}$ in T_pM.

By definition a **vector-field** is a smooth field of tangent vectors to the manifold M, *i.e.* to each point p is associated a vector $\vec{F}(p)$ tangent to M at p. In a local coordinate system a vector-field is expressed as

$$\vec{F}(x^1, ..., x^n) = F^i(x^1, ..., x^n)\frac{\partial}{\partial x^i}. \tag{7}$$

2.2.2 Forms

A **0-form** is a smooth scalar valued function on the manifold M. A **1-form** is a linear function on vectors. Consider first the case of $\mathbf{R}^3$. A vector $\vec{v} = (x, y, z)$ in $\mathbf{R}^3$ is written in a basis $\vec{v} = x\vec{i} + y\vec{j} + z\vec{k}$. Let ω be a 1-form on $\mathbf{R}^3$. By linearity, $\omega(\vec{v})$ is given by

$$\omega(\vec{v}) = x\omega(\vec{i}) + y\omega(\vec{j}) + z\omega(\vec{k}). \tag{8}$$

Thus the question of calculating $\omega(\vec{v})$ boils down to the question of evaluating it on the vectors of the basis $\vec{i}, \vec{j}, \vec{k}$. In $\mathbf{R}^3$, we define the 1-form $\mathrm{d}x$ by $\mathrm{d}x(\vec{i}) = 1$, $\mathrm{d}x(\vec{j}) = 0$ and $\mathrm{d}x(\vec{k}) = 0$ and equivalently for the 1-forms $\mathrm{d}y$ and $\mathrm{d}z$. We have shown that

$$\omega = \omega(\vec{i})\mathrm{d}x + \omega(\vec{j})\mathrm{d}y + \omega(\vec{k})\mathrm{d}z.$$

The triplet ($\mathrm{d}x$,$\mathrm{d}y$,$\mathrm{d}z$) is a basis for 1- forms called the *dual-basis* of $(\vec{i}, \vec{j}, \vec{k})$.

More generally, one can consider a field of 1-forms on $\mathbf{R}^3$ *i.e.* a 1-form ω with coefficients that vary from point to point. It can be written as

$$\omega = A(x, y, z)\mathrm{d}x + B(x, y, z)\mathrm{d}y + C(x, y, z)\mathrm{d}z.$$

The action of ω on a vector field $\vec{F} = f(x,y,z)\vec{i} + g(x,y,z)\vec{j} + h(x,y,z)\vec{k}$ is

$$\omega(\vec{F})(x,y,z) = Af + Bg + Ch. \tag{9}$$

A **k-form** is a smooth multilinear and antisymmetric function on k-tuples of tangent vectors to M, all of them tangent at the same point. This can be implemented by considering two vectors in the plane, $\vec{v}_1 = x_1\vec{i} + y_1\vec{j}$ and $\vec{v}_2 = x_2\vec{i} + y_2\vec{j}$. Let ϕ be a 2-form, then,

$$\begin{aligned} \phi(\vec{v}_1, \vec{v}_2) &= x_1y_1\phi(\vec{i},\vec{i}) + x_1y_2\phi(\vec{i},\vec{j}) + x_2y_1\phi(\vec{j},\vec{i}) + x_2y_2\phi(\vec{j},\vec{j}) \\ &= (x_1y_2 - x_2y_1)\phi(\vec{i},\vec{j}) \end{aligned} \tag{10}$$

because the antisymmetry condition implies $\phi(\vec{i},\vec{i}) = \phi(\vec{j},\vec{j}) = 0$ and $\phi(\vec{i},\vec{j}) = -\phi(\vec{j},\vec{i})$. The requirement that differential forms be antisymmetric comes from the fact that we need to keep track of orientation.

Examples:
1. In $\mathbf{R}^3$ the 2-form $\mathrm{d}x \wedge \mathrm{d}y$ is defined by $\mathrm{d}x \wedge \mathrm{d}y(\vec{i},\vec{j}) = 1$, $\mathrm{d}x \wedge \mathrm{d}y(\vec{j},\vec{k}) = 0$ and $\mathrm{d}x \wedge \mathrm{d}y(\vec{k},\vec{i}) = 0$ and equivalently for the 2-forms $\mathrm{d}y \wedge \mathrm{d}z$ and $\mathrm{d}z \wedge \mathrm{d}x$.
2. More generally, in $\mathbf{R}^n$, the dual basis of the canonical basis $(e_1, ..., e_n)$ is denoted by $(\mathrm{d}x^1, ..., \mathrm{d}x^n)$. It satisfies $\mathrm{d}x^i(e_j) = \delta^i_j$, where δ^i_j is the Kronecker delta. One defines the k-form $\mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge ... \wedge \mathrm{d}x^{i_k}$, such that if $(\vec{v}_1, \vec{v}_2, ..., \vec{v}_k)$ is a k-tuple of vectors of $\mathbf{R}^n$, the quantity $\mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge ... \wedge \mathrm{d}x^{i_k}(\vec{v}_1, \vec{v}_2, ..., \vec{v}_k)$ is the $k \times k$ determinant of the components of the vectors $\vec{v}_i$ along the directions defined by $(e_{i_1}, e_{i_2} ... \, e_{i_k})$. The geometric interpretation of this number is known: it is the volume of the projection of the parallelepiped generated by $(\vec{v}_1, \vec{v}_2, ..., \vec{v}_k)$ on the linear space spanned by $(e_{i_1}, e_{i_2}, ..., e_{i_k})$. From example 2, one can prove that the set of k-forms $(\mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge ... \wedge \mathrm{d}x^{i_k})$ with $i_1 < i_2 < ... < i_k$ is a basis of the vector space of the k-forms in $\mathbf{R}^n$. Hence the space of k-forms has dimension $\frac{n!}{k!(n-k)!}$. In particular all n-forms on $\mathbf{R}^n$ are proportional to $(\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge ... \wedge \mathrm{d}x^n)$, which is nothing but the determinant.

On an general manifold M, one constructs k-forms locally for each tangent space T_pM. For instance, a **1-form** ω on M is a smooth linear function on T_pM for every point p on M. Similarly, a k-form w on a manifold M is a smooth collection of k-forms on each tangent space T_pM. For each p one defines a local basis $((\mathrm{d}x^1)_p, ..., (\mathrm{d}x^n)_p)$ dual to the basis (6) $\left(e_1 = \left(\frac{\partial}{\partial x^1}\right)_p, ..., e_n = \left(\frac{\partial}{\partial x^n}\right)_p\right)$ of T_pM. Then any k-form w on M can be written in a local system of coordinates as follows:

$$w = \sum A_{i_1,...,i_k} \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge ... \wedge \mathrm{d}x^{i_k} \tag{11}$$

where the $A_{i_1,...,i_k}(p)$ are real functions. Once again, the p dependence is not explicitly stated.

2.2.3 Wedge-product

A 2-form ψ can be constructed by forming the *wedge-product* $\psi = \omega_1 \wedge \omega_2$ of two smooth 1-forms ω_1 and ω_2 *via:*

$$\psi(\vec{u}, \vec{v}) = \omega_1(\vec{u})\omega_2(\vec{v}) - \omega_1(\vec{v})\omega_2(\vec{u}) \tag{12}$$

for any vectors $\vec{u}$ and $\vec{v}$. More generally, if ϕ is a k-form and ψ a l-form one can construct a $(k+l)$-form $\phi \wedge \psi$ which acts on $(k+l)$-tuples of tangent vectors at a point, by antisymmetrizing correctly the product $\phi\psi$. We shall not write the explicit formula here. It is the same as the one used to construct a fermionic (antisymmetric) wave-function of $(k+l)$ variables starting with two fermionic wave functions of respectively k and l variables. One has in particular

$$\phi \wedge \psi = (-1)^{kl} \psi \wedge \phi. \tag{13}$$

2.2.4 The exterior derivative

The exterior derivative d generalizes the usual operations of vector calculus. The exterior derivative d transforms k-forms into $(k+1)$-forms. Let for instance A be a 0-form (a function), its exterior derivative $\mathrm{d}A$ is the 1-form

$$\mathrm{d}A = \frac{\partial A}{\partial x^1}\mathrm{d}x^1 + ... + \frac{\partial A}{\partial x^n}\mathrm{d}x^n. \tag{14}$$

If $\vec{v}$ is a tangent vector then

$$\mathrm{d}A(\vec{v}) = \frac{\partial A}{\partial x^i} v^i$$

is the rate of variation of the function A in the direction of $\vec{v}$. This quantity is usually denoted by $(\nabla A).\vec{v}$ or $\nabla_{\vec{v}} A$. We have seen that a k-form w can be written in a local basis

$$w = \sum A_{i_1,\ldots,i_k} \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \ldots \wedge \mathrm{d}x^{i_k}$$

where the $A_{i_1,\ldots,i_k}$ are real functions. The exterior derivative operates on w by acting on each of the coefficients $A_{i_1,\ldots,i_k}$ *via* (14). For example, for a 1-form $\phi = A\,\mathrm{d}x + B\,\mathrm{d}y + C\mathrm{d}z$, one obtains

$$\begin{aligned} \mathrm{d}\phi &= \mathrm{d}A \wedge \mathrm{d}x + \mathrm{d}B \wedge \mathrm{d}y + \mathrm{d}C \wedge \mathrm{d}z \\ &= (C_y - B_z)\mathrm{d}y \wedge \mathrm{d}z + (A_z - C_x)\mathrm{d}z \wedge \mathrm{d}x + (B_x - A_y)\mathrm{d}x \wedge \mathrm{d}y. \end{aligned} \tag{15}$$

An important property of the exterior derivative is that it gives 0 when applied twice,

$$d^2\phi = 0 \tag{16}$$

for any k-form ϕ. This follows from the Schwartz identity for partial derivatives:

$$\frac{\partial^2 A}{\partial x_i \partial x_j} = \frac{\partial^2 A}{\partial x_j \partial x_i}.$$

There is a Leibniz rule for the exterior derivative of a k-form ϕ and a l-form ψ

$$\mathrm{d}(\phi \wedge \psi) = (\mathrm{d}\phi) \wedge \psi + (-1)^k \phi \wedge \mathrm{d}\psi. \tag{17}$$

Examples: In the Euclidean space $\mathbf{R}^3$, a vector field $\vec{F}$ is defined at each point (x, y, z) by $\vec{F} = A\vec{i} + B\vec{j} + C\vec{k}$, where A, B, C are smooth functions of the coordinates. And let f be a function. Its gradient is the vector field $\nabla f = f_x \vec{i} + f_y \vec{j} + f_z \vec{k}$. Similarly, the application of the exterior derivative d on the 0-form (function) f gives the 1-form $\mathrm{d}f = f_x \,\mathrm{d}x + f_y \,\mathrm{d}y + f_z \,\mathrm{d}z$. Hence, to the 1-form $\mathrm{d}f$ is associated the vector field ∇f. The action of the rotational on the vector field $\vec{F}$ gives another vector field

$$\nabla \times \vec{F} = (C_y - B_z)\vec{i} + (A_z - C_x)\vec{j} + (B_x - A_y)\vec{k},$$

and the exterior derivative d operates on a 1-form $\phi = A \,\mathrm{d}x + B \,\mathrm{d}y + C \,\mathrm{d}z$ to give the corresponding 2-form

$$\mathrm{d}\phi = (C_y - B_z)\mathrm{d}y \wedge \mathrm{d}z + (A_z - C_x)\mathrm{d}z \wedge \mathrm{d}x + (B_x - A_y)\mathrm{d}x \wedge \mathrm{d}y.$$

Finally, one can check that the divergence of a vector field corresponds to d acting on a 2-form to generate a 3-form. In summary, gradient, curl and divergence result from the application of d to 0-forms, 1-forms and 2-forms respectively. The relations $\nabla \times \nabla A = \vec{0}$ and $\nabla . \nabla \times F = 0$ are simply a consequence of $d^2 = 0$.

2.2.5 Closed and exact forms

If ϕ is a differential form defined on a manifold M with the property $\mathrm{d}\,\phi = 0$, then ϕ is said to be **closed**. If it has the property that $\phi = \mathrm{d}\,\psi$ for some form ψ on each point in M, then ϕ is **exact**. It follows from these definitions and from $d^2 = 0$, that **every exact form is closed.** But the reciprocal is not true with the important exception that on a simply connected domain M, *i.e.* a domain in which every closed curve can be continuously deformed to a point through deformations that remain in M, every closed 1-form is exact. We shall discuss later in more detail the 1-form

$$\omega = \frac{-y}{x^2 + y^2}\mathrm{d}x + \frac{x}{x^2 + y^2}\mathrm{d}y. \tag{18}$$

It is closed ($\mathrm{d}\omega = 0$) but not exact on $M = \mathbf{R}^2 \backslash (0, 0)$.

2.2.6 Integration of forms

A k-dimensional submanifold (or k-chain) of a manifold M of dimension $n \geq k$ is a subset which can be parametrized with only k coordinates. In this terminology, a 0-chain is a point, a 1-chain is a curve and a 2-chain a surface. k-forms are the right objects to be integrated over k-chains. Indeed k-forms were invented by Elie Cartan for this purpose! (Their similarity with determinants is not by chance: when changing variables in an oriented integral one must multiply the integrand by a determinant and k-forms were built to produce such a determinant by a change of the local coordinate system.) We shall simply consider the integral of 1-forms over 1-chains in the Euclidean space $\mathbf{R}^3$. Let $\mathcal{C}$ be an oriented smooth curve (*i.e.* a 1-chain) parameterized by $\vec{r}(t) = (x(t), y(t), z(t))$ for $t \in I = [-1, 1]$ and let $\omega = A\ \mathrm{d}x + B\ \mathrm{d}y + C\ \mathrm{d}z$ be a smooth 1-form on $\mathcal{C}$. Then,

$$\int_{\mathcal{C}} \omega = \int_{-1}^{1} (A(x,y,z)x'(t) + B(x,y,z)y'(t) + C(x,y,z)z'(t))\,\mathrm{d}t. \qquad (19)$$

Let $\vec{F} = A\vec{i} + B\vec{j} + C\vec{k}$ be the vector field which corresponds to the 1-form ω. We obtain the more familiar expression

$$\int_{\mathcal{C}} \omega = \int_{\mathcal{C}} \vec{F}.\vec{\mathrm{d}r} \qquad (20)$$

which represents the work of $\vec{F}$ along the curve $\mathcal{C}$.

Exercise: The winding number
1. Show that on the unit circle S^1 parameterized by $r(\theta) = (\cos\theta, \sin\theta)$ in the plane, the 1-form ω given by (18) is such that $\int_{S^1} \omega = 2\pi$.
2. Show that if γ is a path connecting two points P and Q of the plane $\mathbf{R}^2 \backslash (0,0)$ the integral $\int_\gamma \omega$ mesures the difference of the polar angles of Q and P, the center of the polar coordinates being (0,0).
3. Deduce from 2. that if γ is a closed path that encircles n times the point (0,0), $W(\gamma,(0,0)) \equiv \frac{1}{2\pi}\int_\gamma \omega = n$. In particular if γ is a closed path that does not encircle (0,0), this integral is equal to zero.
4. The mapping $W(\gamma,(0,0))$ defined from the space of closed curves in $\mathbf{R}^2 \backslash (0,0)$ to the set of rational integers $\mathbf{Z}$ is called **the winding number**. It allows to classify different type of curves in $\mathbf{R}^2 \backslash (0,0)$. It is a simple example of a **topological invariant** (see Sect. 2.3). Two curves with the same winding number are said to be *homologous*.

2.2.7 Theorem of Stokes

Let M be a compact oriented smooth manifold of dimension n with boundary ∂M (possibly empty) and let ∂M be given the induced orientation [8,9].

For a $(n-1)$-form ϕ, we have

$$\int_M \mathrm{d}\phi = \int_{\partial M} \phi. \tag{21}$$

The integral $\int_C \omega$ of a k-form is said to be path-independent if the value of this integral depends only on the boundary ∂C of the oriented k-chain C. This implies that $\int_C \omega = 0$ for every closed k-chain. This property can be used to state the important result: a form ω defined on a manifold M is exact iff $\int_C \omega$ is path-independent on M. For instance, the 1-form (18) is not exact.

All these properties are generalizations of well-known results in vector calculus. For instance, a vector field $\vec{F}$ is conservative if $\vec{F} = \nabla f$ for some function f (the potential). Let ω be the 1-form associated to $\vec{F}$. A conservative $\vec{F}$ corresponds to $\omega = df$ so that ω is exact. This implies that ω is closed and since $d\omega$ corresponds to $\nabla \times \vec{F}$, this implies $\nabla \times \vec{F} = 0$ as well. Electrostatics results from the fact that in $\mathbf{R}^3$ a closed 1-form is exact, hence the electric field is conservative. Another consequence is the widely used result (*e.g.* in thermodynamics) that a 1-form $\phi = p\mathrm{d}x + q\ \mathrm{d}y$ cannot be exact unless $\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}$.

2.3 *Topological invariants of a manifold*

2.3.1 Motivations

Two manifolds are said to be **homeomorphic** if there is a continuous mapping from each other, with continuous inverse mapping. A **topological invariant** is an intrinsic characteristic of a manifold conserved by homeomorphism. These invariants reveal important features and help to classify different types of manifolds. Topological invariants can be numbers, scalars, polynomials, differential forms or more general algebraic sets such as groups, or algebras. Their importance for condensed matter physics was recognized in a seminal paper of Toulouse and Kleman [1] for the study of defects (vortices, nodal lines, textures, anomalies) and their stability as a function of external parameters. The analysis of [1] depends on general characteristics of the system under study (*e.g.* dimensionality of space, nature and symmetries of the order parameter) and not on the precise form of the equations governing the system. Loosely speaking, different types of defects correspond to different (non homeomorphic) geometrical structures. Therefore topological invariants can help to distinguish between them. Of course this general scheme does not tell how to compute the relevant invariant in a given problem. A nice example is the Aharonov-Bohm effect in an infinite plane where the relevant invariant is the *winding number* (defined in the exercise of Sect. 2.2.6). A thorough study was done in [10]. In the last chapter of these notes we discuss two dimensional superconductors.

There the winding number measures the circulation of the phase of the order parameter around the vortices. In the following paragraphs we merely give a taste of the vast subject of invariants in a manifold.

2.3.2 The Euler-Poincaré characteristic

The Euler-Poincaré characteristic $\chi(M)$ of a manifold M is the oldest and the most celebrated topological invariant. We shall explain how to calculate it for a surface. Any given surface S can be tiled by triangles (of different sizes and shape). Such a partition is called a *triangulation.* For any triangulation of S, denote by V, E and F the number of vertices, edges and faces respectively. Then $\chi(S)$ is defined by

$$\chi(S) = V - E + F. \tag{22}$$

This number does not depend on the chosen triangulation and is a characteristic of the surface S [11].
Exercise: Show for the sphere S^2 that $\chi(S^2) = 2$. Show for the torus T^2 that $\chi(T^2) = 0$.
For a three dimensional manifold one needs to use tetrahedra to perform a "triangulation" and the formula for χ becomes $\chi = V - E + F - T$, where T is the number of tetrahedra. It is possible to generalize this notion to higher dimensional manifolds [9].

2.3.3 De Rham's cohomology

We have seen (Sect. 2.2.5) that any exact form is closed but that the reverse is not true. Forms which are closed but *not* exact reveal important topological features and help to classify different types of manifolds. The kth **cohomology group** of De Rham of the manifold M, $H^k(M)$, is the set of closed k-forms modulo the exact k-forms. It is a finite dimensional vector space, its dimension b_k is called the kth Betti number. This integer is a topological invariant of the manifold M. A beautiful result [9,12] is that the alternating sequence of Betti numbers is the Euler-Poincaré characteristics of the manifold, namely

$$\chi(M) = \sum_{r=0}^{n} (-1)^r b_r. \tag{23}$$

This shows that topological properties, as χ, can sometimes be calculated using analytical tools such as differential forms.

2.4 Riemannian manifolds and absolute differential calculus

2.4.1 Riemannian manifolds

A Riemannian manifold is a differentiable manifold M with a local scalar product defined on each tangent space T_pM. If $\vec{v}$ and $\vec{w}$ are two tangent vectors at the same point p then

$$\langle \vec{v} | \vec{w} \rangle_p = \vec{v}.\vec{w} = g_{ij}(p) v^i w^j. \tag{24}$$

In particular the norm of $\vec{v}$ is given by the square-root of $g_{ij}(p)v^i v^j$.

The quantities $g_{ij}(p)$ are the local components of the metric and are called the **metric tensor**. They allow to compute the length $\mathcal{L}(\gamma)$ of a curve γ parameterized as in (1). We recall that $\frac{\mathrm{d}\gamma}{\mathrm{d}t}$ is the tangent vector to γ at the point $p = \gamma(t)$.

$$\mathcal{L}(\gamma) = \int_{-1}^{1} \mathrm{d}t \left(\langle \frac{\mathrm{d}\vec{\gamma}}{\mathrm{d}t} | \frac{\mathrm{d}\vec{\gamma}}{\mathrm{d}t} \rangle_p \right)^{1/2} = \int_{-1}^{1} \mathrm{d}t \left(g_{ij}(p) \frac{\mathrm{d}x^i}{\mathrm{d}t} \frac{\mathrm{d}x^j}{\mathrm{d}t} \right)^{1/2} = \int_0^{\mathcal{L}(\gamma)} \mathrm{d}s. \tag{25}$$

The infinitesimal arc length is denoted by $\mathrm{d}s$. In classical books the metric is written as:

$$\mathrm{d}s^2 = g_{ij} \mathrm{d}x^i \mathrm{d}x^j. \tag{26}$$

A *geodesic curve* between two points p and p' of M is a path of minimal length. Using variational calculus, one can find the Euler-Lagrange equations for a geodesic curve $(x_1(s), ..., x_n(s))$ parameterized by its arc length s (this is a good exercise!). One obtains a system of differential equations:

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}s^2} + \Gamma^i_{jk} \frac{\mathrm{d}x^i}{\mathrm{d}s} \frac{\mathrm{d}x^j}{\mathrm{d}s} = 0 \tag{27}$$

where the *Christoffel symbols* Γ^i_{jk} are given by

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left(\frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right) \text{ with } g^{il} g_{lj} = \delta^i_j, \tag{28}$$

i.e. the matrix g^{il} is the inverse of the metric tensor.

2.4.2 Covariant derivative. Connexion and curvature forms

We have emphasized that on a general manifold there is no way to compare tangent vectors at different points. However, given a smooth vector field, a rather natural question to ask is "What is the infinitesimal variation of a vector field $\vec{Y}$ at point p if one moves in the direction of a vector $\vec{V}$ tangential at the point p to the manifold?" A possible way out is to immerse the manifold in a larger space where coordinates are defined. But this is not intrinsic: it depends on the surrounding space.

One needs an additional structure, called a **connexion**, that allows to compare tangent vectors at different points and to differentiate vector fields (or more generally tensor fields). This method was invented by Levi-Civita and called *absolute differential calculus.* It is possible to give a general definition of a connexion without refering to the metric, as these concepts are independent. However we shall restrict ourselves to Riemannian manifolds.

A Riemannian manifold can canonically be endowed with a connexion provided by the Christoffel symbols Γ^i_{jk} (28). Consider a vector field $\vec{Y}$ on the manifold. One defines its covariant derivative $\nabla_{\vec{V}}\vec{Y}$ along the direction of the vector $\vec{V}$ tangential at the point p to the manifold by

$$\nabla_{\vec{V}}\vec{Y} = \left(\frac{\partial Y^i}{\partial x^j}V^j + \Gamma^i_{jk}V^jY^k\right)e_i \tag{29}$$

where $e_i = \frac{\partial}{\partial x^i}$ is the i^{th} vector of the basis (6) of the tangent space T_pM at the point p. The quantity $\nabla_{\vec{V}}\vec{Y}$ is a vector, tangent to the manifold M at the point p. It represents the total rate of variation of $\vec{Y}$ along the direction of $\vec{V}$. The first term in (29) is nothing but the *convective* derivative of $\vec{Y}$ along the direction of $\vec{V}$. This term is familiar in hydrodynamics (*e.g.* in Euler and Navier-Stokes equations). The second term represents the derivative of the vectors e_i of the local basis of T_pM along the direction of $\vec{V}$. Such a term is familiar from mechanics when one uses non-cartesian coordinates (*e.g.* polar or spherical), it reflects that the coordinate system is local and point dependent. One can rewrite (29) as follows:

$$\begin{aligned} \nabla_{\vec{V}}\vec{Y} &= \nabla_{\vec{V}}Y^ie_i = \mathrm{d}Y^i(\vec{V})e_i + Y^k\nabla_{\vec{V}}e_k \\ \text{with } \nabla_{\vec{V}}e_k &= V^j\nabla_{e_j}e_k \\ \text{and } \nabla_{e_j}e_k &= \Gamma^i_{jk}e_i. \end{aligned} \tag{30}$$

It is analogous to the well-known expression in mechanics of the derivative of a vector in the rotating frame

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(v^ie_i) = \frac{\partial\vec{v}}{\partial t} + \vec{\Omega}\times\vec{v}.$$

***Parallel Transport:** The formula (29) for the covariant derivative $\nabla_{\vec{V}}\vec{Y}$ allows to compute the variation of a vector field along a given direction. Inversely, from a tangent vector $\vec{Y}_0$ at a point p_0 and a curve γ going from p_0 to p_1, one can construct from (29) a vector field $\vec{Y}$ along the curve γ such that $\vec{Y}(P_0) = \vec{Y}_0$. Such an operation is called *the parallel transport of*

$\vec{Y}_0$ along γ. The vector field $\vec{Y}$ is determined by

$$0 = \nabla_{\vec{V}} \vec{Y} \text{ with } \vec{V} = \frac{\mathrm{d}\gamma}{\mathrm{d}t}.$$

It is therefore obtained by solving the system of differential equations:

$$0 = \frac{\partial Y^i}{\partial x^j} \frac{\mathrm{d}x^j}{\mathrm{d}t} + \Gamma^i_{jk} \frac{\mathrm{d}x^j}{\mathrm{d}t} Y^k = \frac{\mathrm{d}Y^i}{\mathrm{d}t} + \Gamma^i_{jk} \frac{\mathrm{d}x^j}{\mathrm{d}t} Y^k \tag{31}$$

with initial condition $\vec{Y}(0) = \vec{Y}_0$. If one considers the special case $\vec{Y} = \vec{V} = \frac{\mathrm{d}\gamma}{\mathrm{d}s}$ where s is the arc-length of the curve γ, equation (31) becomes identical to the formula defining a geodesic (27). The equation of a geodesic is thus $\nabla_{\vec{V}} \vec{V} = 0$ with $\vec{V} = \frac{\mathrm{d}\gamma}{\mathrm{d}s}$. In other words a geodesic is a curve which is parallel transported along itself, as this is well-known for straight lines in Euclidean spaces.

Remark: We did not give any explanation for the appearance of the Christoffel symbols in the formula (30) for the covariant derivative of the vectors of the basis. This is in fact the heart of Levi-Civita's construction. We encourage the reader to develop an intuition on this formula through his/her own readings [11, 13, 14]. A nice (and historical) approach is provided by surface theory in $\mathbf{R}^3$. A tangent vector $\vec{V}$ at a point p of the surface is transported to an infinitesimally close point p' by first bringing it parallel to itself from p to p'. As $\vec{V}$ has no reason to be tangent to the surface at p', one projects $\vec{V}$ on the tangent plane at p' and obtains a new vector $\vec{V}'$ tangent at p'. The variation of $\vec{V}$ through this operation provides the formula for the covariant derivative (this is explained in [11]). Another approach is *via* "Tensor Calculus": the first term in the r.h.s. of (29) is not a tensor, so one has to add to it something to have a tensorial derivative. The Christoffel symbols are not tensors themselves but adding them to the first term turns the sum into a tensor. This approach is well explained in [15], or in books about General Relativity [16, 17].

We now give a more compact expression for the covariant derivative. One first notices that $\nabla_{\vec{V}} \vec{Y}$ is linear in $\vec{V}$, *i.e.* $\nabla \vec{Y}$ can thus be viewed as a linear operator which acting on $\vec{V}$ produces another tangent vector at p. Equivalently, $\nabla \vec{Y}$ is tangent-vector valued 1-form and can be written as

$$\nabla \vec{Y} = \left(\mathrm{d}Y^i + \Gamma^i_{jk} \mathrm{d}x^j \; Y^k\right) e_i.$$

From this expression we see that ∇ itself acts as a linear operator on the vector $\vec{Y}$. Recalling that a matrix L acts on $\vec{Y}$ as $L\vec{Y} = L^i_k Y^k e_i$, we write:

$$\nabla = \mathrm{d} + \omega \tag{32}$$

where ω is a matrix whose coefficients are 1-forms. Precisely ω has matrix elements

$$\omega^i_k = \Gamma^i_{jk} \mathrm{d}x^j. \tag{33}$$

The matrix-valued 1-form ω is called the **connexion** 1-form and ∇ the **covariant derivative.** The **curvature** Ω of a Riemannian Manifold is defined by the matrix valued 2-form

$$\Omega = \mathrm{d}\omega + \omega \wedge \omega \tag{34}$$

i.e. for two vectors $\vec{X}$,$\vec{Y}$ on the manifold, $\Omega(\vec{X}, \vec{Y})$ is a matrix. Notice that in general the connexion 1-form can be written as

$$\omega = A_j \mathrm{d}x^j \tag{35}$$

where the A_j's are matrices. Then, $\omega \wedge \omega = \sum_{i<j} \mathrm{d}x^i \wedge \mathrm{d}x^j$ is non zero when the matrices do not commute.

Exercise: Find in a book the formula of the Riemann curvature tensor (see *e.g.* Spivak [13], Vol II, Chap. 4, p. 189) in terms of the Christoffel symbols. Using (33), verify that Ω defined in (34) is indeed the Riemann tensor.

2.5 The Laplacian

We saw that the spaces of k-forms and $(n-k)$-forms over a manifold of dimension n have the same dimension. It is possible to define a duality transformation $*$, called *the Hodge star,* between these two spaces. In the Euclidean space $\mathbf{R}^3$ the Hodge duality is given by:

$$\begin{aligned} *1 &= \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\ *\mathrm{d}x = \mathrm{d}y \wedge \mathrm{d}z, \ *\mathrm{d}y &= \mathrm{d}z \wedge \mathrm{d}x, \ *\mathrm{d}z = \mathrm{d}x \wedge \mathrm{d}y, \\ *(\mathrm{d}x \wedge \mathrm{d}y) = \mathrm{d}z, \ *(\mathrm{d}y \wedge \mathrm{d}z) &= \mathrm{d}x, \ *(\mathrm{d}z \wedge \mathrm{d}x) = \mathrm{d}y \\ *(\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z) &= 1. \end{aligned} \tag{36}$$

More generally the duality $*$ operation on a manifold *depends on the local metric*, and is given by:

$$*(\mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge ... \wedge \mathrm{d}x^{i_k}) = |g|^{1/2} \mathrm{d}x^{i_{k+1}} \wedge ... \wedge \mathrm{d}x^{i_n} \tag{37}$$

where $(i_1, i_2, ..., i_k, i_{k+1}, ..., i_n)$ is a permutation of $(1, 2, ..., n)$ with positive signature and $|g|$ is the determinant of the metric (g_{ij}) (*i.e.* $|g|^{1/2}$ is the volume element). If ϕ is a k-form, one can check (exercise!) that $** \phi = (-1)^{k(n-k)} \phi$.

The Hodge duality allows to define a scalar product in the space of k-forms by

$$(\phi_k, \psi_k) = \int_M \phi_k \wedge * \psi_k \tag{38}$$

where the integral is over the manifold.

Exercise: Show that the scalar product of two 1-forms $\phi = A_1 \, \mathrm{d}x + B_1 \, \mathrm{d}y + C_1 \, \mathrm{d}z$ and $\psi = A_2 \, \mathrm{d}x + B_2 \, \mathrm{d}y + C_2 \, \mathrm{d}z$ in $\mathbf{R}^3$ is

$$(\phi, \psi) = \int_{R^3} (A_1 A_2 + B_1 B_2 + C_1 C_2) \mathrm{d}x \mathrm{d}y \mathrm{d}z.$$

The exterior derivative d maps k-forms onto $(k+1)$-forms. One defines the adjoint exterior derivative δ which maps k-forms onto $(k-1)$-forms *via* $(\phi_k, d\psi_{k-1}) = (\delta\phi_k, \psi_{k-1})$.

Exercise:
1. Show by integrating by part [18] that in a n-dimensional vector space

$$\delta = (-1)^{nk+n+1} * d * .$$

2. Deduce that $\delta^2 = 0$. A *co-exact* k-form ω satisfies the global condition $\omega = \delta\phi$ where ϕ is a $(k+1)$-form. A *co-closed* k-form ω satisfies $\delta\omega = 0$.

The **Laplacian** is an operation which takes k-forms onto k-forms and generalizes the usual Laplacian on functions. It is defined by

$$\triangle = (d + \delta)^2 = d\delta + \delta d. \tag{39}$$

Exercise: In $\mathbf{R}^3$ show that if f is a function, one has $\triangle f = -\left(\partial_x{}^2 f + \partial_y{}^2 f + \partial_z{}^2 f \right)$.

A k-form ω_k such that $\triangle\omega_k = 0$ is called **harmonic.** For a smooth enough k-form, a necessary and sufficient condition for harmonicity is to be closed and co-closed:

$$\triangle\omega_k = 0 \Longleftrightarrow d\omega_k = 0 \text{ and } \quad \delta\omega_k = 0.$$

Hodge's theorem: On a compact manifold without boundary any k-form ω_k can always be decomposed into the sum of an exact form $d\alpha_{k-1}$, a co-exact form $\delta\beta_{k+1}$ and a harmonic form γ_k [19, 20]:

$$\omega_k = d\alpha_{k-1} + \delta\beta_{k+1} + \gamma_k. \tag{40}$$

This very important result is well-known in 3d vector analysis of electromagnetism of continuous media as the Helmholtz decomposition according to which for a closed and compact manifold or for a controlled growth of the fields at infinity, any smooth vector field $\vec{B}$ can be decomposed as $\vec{B} = \nabla\phi + \nabla \times \vec{M} + \vec{H}$, where $\vec{H}$ has both vanishing curl and divergence.

Remark: if ω_k is closed then $d\delta\beta_{k+1} = 0$, which implies, as d and δ are adjoint, that $\delta\beta_{k+1} = 0$. Hence $\omega_k = d\alpha_{k-1} + \gamma_k$. Therefore ω_k and γ_k belong to the same cohomology class. Denoting by $\text{Harm}^k(M)$ the space of harmonic k-forms on M, the Hodge theorem establishes an isomorphism between the two spaces $H^k(M)$ and $\text{Harm}^k(M)$ and according to (23), the Euler-Poincaré characteristic is given by

$$\chi(M) = \sum_{k=0}^{n} (-1)^k \text{dim Harm}^k(M). \tag{41}$$

This relation shows that properties of the Laplacian probe the topology of the manifold M [20].

2.6 Bibliography

We do not intend to be exhaustive, we just list books we used and give some (subjective) comments. A classical and outstanding introduction to Geometry in general is [21]. A "modern-classic" more inclined toward topology and analysis is [22]. A good way to start with differential geometry is to study curves and surfaces. A very pleasant book on these topics, full of results, figures and historical comments is [23]. Some books begin with curves and surfaces and then introduce the general concept of n dimensional manifolds. This can be a very helpful point of view since it is not too formal [11, 15, 24]. More recent works are [25] with emphasis on applications, [19] more inclined towards algebra and [20] devoted to analysis on manifolds and well adapted to theoretical physicists. An exhaustive treatment is to be found in [8, 9, 13] and the following volumes. We owe a lot to the review paper [26] written for high energy physics and Yang-Mills theories. For a pedagogic introduction to differential forms, see [27].

3 Fiber bundles and their topology

3.1 Introduction

The concepts of connexion and curvature that we have defined for Riemannian manifolds can be extended to a more general structure called a **fiber bundle**. A fiber bundle is a manifold $\mathbf{X}$ that locally looks like the product of two simpler manifolds. For instance the torus $T^2 = S^1 \times S^1$ is globally the product of two 1-dimensional circles S^1. A cylinder is also a global product $S^1 \times [-1, 1]$. A Möbius strip is locally the product of an arc of S^1 by $[-1, 1]$ but not globally. More precisely, a fiber bundle is a triplet $(\mathbf{X}, M, \pi)$ where $\mathbf{X}$ is a manifold (*the total space*), M (*the base space*) a submanifold of $\mathbf{X}$ and π (*the projection*) a smooth function from $\mathbf{X}$ to M:

$$\pi : \mathbf{X} \rightarrow M$$

$$x \mapsto \pi(x) = p. \tag{42}$$

The inverse image $\pi^{-1}(p)$ of any point $p \in M$ is called the *fiber* above p and is isomorphic to a given manifold F. Intuitively, a fiber bundle is a collection of identical (isomorphic) manifolds F which depend on a parameter p belonging to the base manifold M. When the fiber F is a vector-space $\mathbf{R}^n$ or $\mathbf{C}^n$ the bundle is called a *vector-bundle* (see *e.g.* [28]).

A **section** s of a fiber bundle is a smooth function which associates to each point $p \in M$ a element of the fiber above p. Therefore one has $\pi \circ s = Id_M$ where Id_M is the Identity function in M.

Examples:
1. The product $M_1 \times M_2$ of two manifolds M_1 and M_2 is a fiber bundle called the trivial bundle. One can take either M_1 or M_2 to be the base and the other one to be the fiber.
2. The total tangent space of M is a vector bundle also called the *tangent bundle.* The base space is the manifold M itself and the fiber above the point p is T_pM which is homeomorphic to $\mathbf{R}^n$. The projection is exactly the one defined in Section 2.2.1. And what is a section of the tangent bundle? A vector-field!
3. More generally the space of k-forms is also a vector bundle over the manifold M.

The fiber bundles described here are assumed to be *locally trivial:* for any point p of the base M there is a neighborhood U_p such that the restriction of the bundle over U_p, *i.e.* $\pi^{-1}(U_p)$, is homeomorphic to $U_p \times F$.

3.2 Local symmetries. Connexion and curvature

In physical applications one considers vector bundles where the base space is the physical space and the fiber is a representation of a continuous symmetry group, *i.e.* a Lie group $\mathbf{G}$ with Lie algebra $\mathcal{G}$. At each point, the order parameter is an element of the fiber over that point and a local symmetry group acts upon it. The order parameter is precisely a section of a vector bundle. It is natural to ask about the variation of the order parameter when one moves from one point of the base to another. However, although all fibers are homeomorphic to the same vector space, there is no intrinsic way to identify two fibers over two different points p and p' of the base. In other words, given a section s, the two vectors $s(p)$ and $s(p')$ can not be compared *a priori.* This is exactly the same problem that we encountered for vectors fields, which are sections of the tangent bundle. The extra-structure needed to compute the variation of a section s along a given direction tangent to the base is called a *fiber bundle connexion* and is again denoted by ∇. The expression of ∇ is obtained by reinterpreting the formula (32):

$$\nabla = d + \omega. \tag{43}$$

Here d is the usual operation of differentiation of functions and ω is a matrix-valued 1-form (33) on the base space M that can be written

$$\omega = A_i dx^i$$

where $(x^1, ..., x^n)$ is a local system of coordinates on the base M. We impose that ω represents at each point an infinitesimal transformation of the group $\mathbf{G}$ *i.e.* the matrices A_i belong to the Lie algebra $\mathcal{G}$. The *curvature* Ω of the connexion is also given by the same formula as above (34):

$$\Omega = \mathrm{d}\omega + \omega \wedge \omega.$$

Important examples:
1. For the tangent bundle, we take the symmetry Lie group to be the group of invertible matrices $GL(n, \mathbf{R})$. The associated Lie algebra is just the group of all $n \times n$ matrices. Connexions on a Riemannian manifold are a special case of fiber bundle connexions.
2. We study now a 2-dimensional physical system $M = \mathbf{R}^2$ where at each point an order parameter (or wavefunction) is defined and is a complex number $\psi = |\psi| e^{i\chi}$. Hence the fiber is $F = \mathbf{C}$. The Lie group for Maxwell electromagnetism is $U(1)$ and the Lie algebra is $i\mathbf{R}$. Then, the connexion 1-form (or gauge potential) can be written $\omega = -iA$ where $A = A_x\ \mathrm{d}x + A_y\ \mathrm{d}y$, A_x and A_y being real functions (1×1 matrices). Since the Lie group is abelian, the curvature reduces to $\Omega = \mathrm{d}\omega = -iB$ where $B = (\partial_x A_y - \partial_y A_x)\ \mathrm{d}x \wedge\ \mathrm{d}y$ is nothing but the magnetic field. The covariant derivative given by (43) is then $\nabla = d - iA$. This is indeed the differential operator that appears when one studies the Schrödinger equation in a magnetic field, or the Ginzburg-Landau model for a superconductor.
3. In the Gross-Pitaevskii description for rotating superfluid ^{4}He the connexion is the Coriolis term $\vec{\Omega} \times \vec{r}$, $\vec{\Omega}$ being here the angular velocity.
4. The Yang-Mills connexion is obtained by the same construction as before with a non-abelian Lie group (typically $SU(2)$). The reader is referred to the original paper of Yang and Mills [29]. The theory of fiber bundles is the right setting for gauge symmetries in high energy physics [30, 31].

3.3 Chern classes

The cylinder and the Moebius strip have both S^1 as base space and $[-1, 1]$ as fiber, but they are not homeomorphic. One would like to classify different types of fiber bundles with given base and fiber. **Characteristic classes** are cohomology classes of the base space that are topological invariants of the fiber bundle [32].

We shall study the **Chern classes** for fiber bundles with Lie group $\mathbf{G} = GL(n, \mathbf{C})$, the group of invertible complex $n \times n$ matrices. The Chern-Weil theorem gives an explicit construction of these classes starting from a connexion 1-form ω and the associated curvature Ω. This theorem also proves that the invariants obtained *do not depend* on the chosen connexion (see [33] for a most enlightning exposition).

We recall that the curvature Ω is a 2-form whose coefficients are $n \times n$ matrices with complex coefficients. A polynomial $P(\Omega)$ is said *invariant* if for any matrix g in $GL(n, \mathbf{C})$,

$$P(\Omega) = P(g^{-1}\Omega g).$$

For example, $\det(1 + \frac{i}{2\pi}\Omega)$ and $\mathrm{tr}(e^{\frac{i\Omega}{2\pi}})$ are invariant polynomials.

Theorem (Chern-Weil): *If P is an invariant polynomial and Ω curvature 2-form, then*

(i) $P(\Omega)$ is a closed differential form ($dP = 0$).

(ii) If ω' is another connexion on the same fiber bundle and Ω' the associated curvature, then there exists a form Q such that $P(\Omega') - P(\Omega) = dQ$.

Property (i) shows that $P(\Omega)$ defines a cohomology class of the base space M and (ii) proves that this class does not depend on the chosen connexion Ω. $P(\Omega)$ is called a *characteristic class.* In particular, integrals of $P(\Omega)$ over cycles (i.e. submanifolds without boundaries) will provide topological invariants [33, 34].

This construction can be generalized to other Lie groups and the polynomial P has a different expression according to the associated the Lie algebra. Here, we shall consider only the *Chern classes* defined from

$$P(\Omega) = \det\left(1 + \frac{i}{2\pi}\Omega\right) = 1 + c_1(\Omega) + c_2(\Omega) + ... \tag{44}$$

where $c_i(\Omega)$ is a scalar-valued $2i$-form called the i^{th} Chern class. From the expansion of the determinant we obtain the expressions

$$\begin{aligned} c_0 &= 1 \\ c_1 &= \frac{i}{2\pi}\mathrm{tr}(\Omega) \\ c_2 &= \frac{1}{8\pi^2}(\mathrm{tr}\Omega \wedge \Omega - \mathrm{tr}\Omega \wedge \mathrm{tr}\Omega) \end{aligned} \tag{45}$$

and $c_i = 0$ when $2i$ is greater than the dimension of M. Since $dP = 0$, each Chern form is closed as well: $dc_i(\Omega) = 0$. The i^{th} Chern class belongs to the cohomology group $H^{2i}M$. A remarkable fact is that the Chern forms define *integer* cohomology classes: the integral of $c_i(\Omega)$ over any $2i$-cycle C *i.e.* any oriented submanifold C of M, of dimension $2i$ and without boundary is an integer, depending upon C but independent from the connexion 1-form ω over the fiber bundle:

$$\int_C c_i(\Omega) = n \in \mathbf{Z}. \tag{46}$$

This integer n is a topological invariant of M called a *Chern number.*

Remark: if $P(\Omega)$ is homogeneous of degree r (see *e.g.* (45)) an explicit formula can be given for the form Q that appears in the Chern-Weil theorem:

$$Q = r\int_0^1 \mathrm{d}tP(\omega' - \omega, \Omega_t, ..., \Omega_t) \tag{47}$$

where Ω_t is the curvature associated to the interpolating connexion $\omega_t = t\omega' + (1-t)\omega$.

Examples:
1. Consider the $U(1)$ abelian bundle described by the curvature $\Omega = -iB$ where B is the magnetic field. On a two dimensional base space, we have from (44)

$$P(\Omega) = 1 + \frac{B}{2\pi}$$

and the only non-zero Chern class is $c_1(B) = \frac{B}{2\pi}$. Then, if M is a closed surface without boundary

$$\frac{1}{2\pi}\int_M B = n \in \mathbf{Z} \tag{48}$$

corresponds to a well-known flux quantization condition [36].
2. More generally, on a manifold M, all the k-fold exterior products $\Omega, \Omega\wedge\Omega, \Omega\wedge\Omega\wedge\Omega$ etc. ($k \leq \frac{1}{2}\mathrm{dim}M$) lead to topological invariants. If $\mathrm{dim}M = 4$, only the two first are relevant

$$\begin{aligned} \int_M \Omega \propto \oint \vec{B}.\mathrm{d}\vec{S} &= n \in \mathbf{Z} \\ \int_M \Omega\wedge\Omega \propto \int \vec{E}.\vec{B}\mathrm{d}^4x &= m \in \mathbf{Z}. \end{aligned} \tag{49}$$

To end this section, we discuss the issue of what makes certain physical quantities topological numbers and some others not. To that purpose, we consider two examples taken from geometry and electromagnetism.

Example 1: Consider the 2-sphere S^2 of radius R. Its curvature is $K = \frac{1}{R^2}$. Then the area of the sphere $\int_{S^2} \mathrm{d}S = 4\pi R^2$ is a metric invariant. But

$\int_{S^2} \Omega = \int_{S^2} K \mathrm{d}S = 4\pi$ is a topological invariant and is conserved by smooth deformations of the sphere that change the metric.

Example 2: For electromagnetism, the electric charge inside a closed surface S is given by

$$\frac{1}{4\pi}\int_S \vec{E}.\vec{\mathrm{d}S} = \frac{1}{4\pi}\int \nabla.\vec{E}\mathrm{d}V.$$

This statement is equivalent to $\nabla.\vec{E} = \rho$ (charge density). The conservation of the electric charge in time is $\frac{\mathrm{d}}{\mathrm{d}t}\oint_S \vec{E}.\vec{\mathrm{d}S} = 0$. It is equivalent to the Maxwell-Faraday equation. These two relations do not refer to topological invariance although it is sometimes asserted that a topological invariant is obtained by integrating a total divergence. The electric charge is a counterexample of this statement.

3.4 *Manifolds with a boundary: Chern-Simons classes*

3.4.1 The Gauss-Bonnet theorem

We recall some facts concerning a surface S embedded in $\mathbf{R}^3$ [23]. To investigate geometrical properties, Euler had the idea to study curves drawn on S. More precisely, he considered *normal sections* at a point p, *i.e.* plane curves obtained by intersecting the surface S with its normal plane at a point p. Euler proved that such all normal sections have a curvature κ at p limited by two extremal values κ_1 and κ_2 called *principal curvatures.* Moreover, if the principal curvatures are not equal, there is one curve C_1 having curvature κ_1 and another C_2, perpendicular to the first one, having curvature κ_2. If a normal section makes an angle ϕ with C_1, its curvature is given by $\kappa = \kappa_1 \cos^2\phi + \kappa_2 \sin^2\phi$.

Exercise: Prove this theorem of Euler (hint: write the local equation of the surface as $z = z(x, y)$ and Taylor-expand to the second order [13, 23]).

The product of the two principal curvatures κ_1 and κ_2 at a point p is denoted by $K = \kappa_1\kappa_2$. The scalar K is the **Gaussian Curvature** of the surface S at the point p. The Gaussian curvature is conserved under deformations that preserve the metric (*e.g.* bending a surface). It is a metric invariant, that only depends on the intrinsic geometry of the surface and not on the surrounding space, although the above definition involves this embedding space. Gauss proved this fact by discovering an explicit formula for K in terms of the local metric g_{ij} and its derivatives. This is the *Theorema Egregium* (1827), a most "remarkable theorem" [11, 15, 23].

Suggestion: Do not try to prove the Theorema Egregium.

Exercise: Check that the Gaussian curvature of a plane is 0. Prove that the Gaussian curvature of the sphere S^2 of radius R is $K = 1/R^2$. Since S^2 and the plane have different Gaussian curvatures it is not possible to make an isometric mapping, even local, between them. There is no faithful plane map of the Earth that would respect the distances.

The **Gauss-Bonnet theorem** [11] for a closed surface S establishes a relation between the curvature, which is a metric invariant and the Euler-Poincaré characteristic of S:

$$\frac{1}{2\pi}\int\int_S K\mathrm{d}S = \chi(S). \tag{50}$$

The quantity $\frac{1}{2\pi}\int\int_S K\mathrm{d}S$ where $\mathrm{d}S$ is the surface element, is sometimes called *curvatura integra.* The Gauss-Bonnet theorem establishes that the curvatura integra is a topological invariant of closed surfaces. It has numerous and profound implications. For example [15], as the reader can check, it implies that no metric of negative curvature can be defined on sphere, or that no metric with strictly positive or strictly negative curvature can be defined on a torus (for a torus, $\chi = 0$).

One should notice the similarity between the theorem of Gauss-Bonnet (50) and the Chern numbers (46) defined as topological invariants obtained by integrating a Chern class. The curvature K of a surface can be defined as a characteristic class (called the Euler class). So in fact these two expressions are not only similar: they have the same origin. Chern developed the theory of characteristic classes while studying higher dimensional generalizations of the Gauss-Bonnet theorem [35].

3.4.2 Surfaces with boundary

We now consider that the surface S has a boundary and is *oriented.* An orientation means that an outward unitary normal vector $\vec{N}$ can be defined in a coherent manner throughout the surface (for instance this is not possible for a Moebius strip). Let $\gamma(s)$ be a space curve parameterized by its arc-length s, with $\mathrm{d}s^2 = \mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2$. Suppose that γ is drawn on the surface S. To each point $p \in S$ on the curve, one associates a local, orthonormal and direct frame $(\vec{t}, \vec{u}, \vec{N})$ where $\vec{t} = \frac{\mathrm{d}\gamma}{\mathrm{d}s}$ is a unitary tangent vector to the curve γ, $\vec{u}$ is perpendicular to $\vec{t}$ and belongs to the tangent plane of S at p and $\vec{N}$ is the normal vector at p. One has in particular

$$\vec{u} = \vec{N} \times \vec{t}.$$

The curvature vector $\vec{k}$ of the curve γ is defined by

$$\vec{k} = \frac{\mathrm{d}\vec{t}}{\mathrm{d}s}\cdot \tag{51}$$

As $\vec{t}$ is unitary, the curvature vector $\vec{k}$ has components on $\vec{u}$ and $\vec{N}$. Its projection on $\vec{N}$ is called the *normal curvature* of γ, whereas the projection on $\vec{u}$ defines the *geodesic curvature* k_g:

$$k_g = \vec{k}.\vec{u} = \left(\vec{t} \times \frac{d\vec{t}}{ds}\right) . \vec{N}. \qquad (52)$$

The geodesic curvature k_g vanishes for a geodesic curve of the surface S [15, 23]. We can now give the generalization of the Gauss-Bonnet theorem if the the boundary ∂S is not empty. As the curvatura integra is not an integer in general, a boundary contribution must be added [11]:

$$\frac{1}{2\pi}\int\int_S K dS + \frac{1}{2\pi}\oint_{\partial S} k_g dl = \chi(S) \qquad (53)$$

where K and k_g are respectively the curvature of the manifold S and the geodesic curvature of its boundary.

Example: A spherical cap in the north hemisphere, containing the north pole and limited by circle of latitude θ_0, is topologically equivalent to a disk *i.e.* to a triangle. Hence its Euler-Poincaré characteristic is equal to 1. The geodesic curvature of the circle of latitude θ_0 is $k_g = \frac{1}{R}\text{cotan}\theta_0$. The Gauss-Bonnet theorem (53) gives indeed

$$\frac{\text{Area of the cap}}{2\pi R^2} + \frac{1}{2\pi} 2\pi R k_g \sin\theta_0 = 1. \qquad (54)$$

For $\theta_0 = \frac{\pi}{2}$, *i.e.* at the equator (half-sphere case), the boundary is a geodesic and $k_g = 0$ so that only the first term in (53) contributes.

3.4.3 Secondary characteristic classes

We showed that the Gauss-Bonnet theorem need to be generalized to incorporate the contribution of a boundary. Is it possible to modify the characteristic classes theory accordingly? The answer is again contained in the Chern-Weil theorem. Since $P(\Omega') - P(\Omega) = dQ$, the integral of dQ vanishes if the manifold has no boundary. But if there is a boundary ∂M, then by Stokes' theorem

$$\int_M dQ = \int_{\partial M} Q \qquad (55)$$

needs not be zero and corresponds precisely the contribution of a "geodesic curvature". Hence (46) rewrites [34]:

$$\int_M c_i(\Omega) - \int_{\partial M} Q(\omega, \omega_0) = n \in \mathbf{Z} \qquad (56)$$

where ω is the connexion 1-form associated to the curvature Ω and ω_0 is a well chosen connexion that compensates boundary effects. The forms Q associated to each Chern forms c_i are called **Chern-Simons classes** or *secondary* characteristic classes. We now present some explicit calculations. The characteristic class associated to $\Omega = \mathrm{d}\omega + \omega \wedge \omega$ for the Yang-Mills case is $\mathrm{tr}(\Omega \wedge \Omega)$. Consider another connexion ω' and the interpolation between ω and ω' defined by

$$\begin{aligned} \omega_t &= t\omega' + (1-t)\omega \\ \Omega_t &= \mathrm{d}\omega_t + \omega_t \wedge \omega_t \end{aligned} \tag{57}$$

for $t \in [0, 1]$. Define $\alpha = \omega - \omega'$ so that by (47), Q is given by

$$Q(\omega, \omega') = 2 \int_0^1 \mathrm{tr}(\alpha \wedge \Omega_t)\mathrm{d}t. \tag{58}$$

Since $\Omega_t = \Omega - t\mathrm{d}\alpha + t^2 \alpha \wedge \alpha - t\alpha \wedge \omega - t\omega \wedge \alpha$, then

$$Q(\omega, \omega') = 2\mathrm{tr}\left(\alpha \wedge \Omega - \frac{1}{2}\alpha \wedge \mathrm{d}\alpha + \frac{1}{3}\alpha \wedge \alpha \wedge \alpha - \frac{1}{2}\alpha \wedge \alpha \wedge \omega - \frac{1}{2}\alpha \wedge \omega \wedge \alpha\right).$$

The cyclicity of the trace together with the property (13) of the wedge-product provides

$$Q(\omega, \omega') = \mathrm{tr}\left(2\alpha \wedge \Omega - \alpha \wedge \mathrm{d}\alpha - 2\alpha \wedge \omega \wedge \alpha + \frac{2}{3}\alpha \wedge \alpha \wedge \alpha\right).$$

Taking $\omega' = 0$ we recover a famous expression for the Chern-Simons connexion [44] namely

$$Q(\omega) = \mathrm{tr}\left(\omega \wedge \mathrm{d}\omega + \frac{2}{3}\omega \wedge \omega \wedge \omega\right). \tag{59}$$

When applied to the abelian connexion U(1) on a domain M of the 2d plane, we obtain $Q(A, A') = A - A'$ and the curvature is related to the magnetic field $\Omega = -iB$. The vector potential A' to be equal to $\nabla\chi$, where χ is the phase of the order parameter $\psi = |\psi|\mathrm{e}^{i\chi}$. Then, according to (56) we have

$$\int_M B + \oint_{\partial M} (\nabla\chi - A) = n \in \mathbf{Z} \tag{60}$$

which corresponds to the well-known fluxoid expression. By analogy with the Gauss-Bonnet theorem for a surface with boundary (53), we can say that B is a curvature and that $(\nabla\chi - A)$, which we shall interpret in Section 5, plays the role of a geodesic curvature.

3.5 The Weitzenböck formula

The definition of the Laplacian on forms (39) is not always adapted to describe problems we aim to solve. For a quantum particle moving in a magnetic field described by a $U(1)$-connexion or for the related Ginzburg-Landau equation, the Hamiltonian H (or the free energy) is given in terms of the covariant derivative $\nabla = d - iA$ and of its adjoint ∇^* with respect to the scalar product on 1-forms defined in (38), *e.g.* $H = \frac{1}{2}\nabla^*\nabla$ in units where both the mass of the particle and $\hbar$ are set to one. Thus the question arises to relate the covariant derivative to the Laplacian.

We first denote by D the operator $D = d + \delta$ so that the Laplacian $\Delta = d\delta + \delta d = D^2$. On a flat space and in the dual basis $\epsilon_i = dx^i$, we have $Df = \sum_i \epsilon_i \partial_i f$ so that $D^2 f = -\sum_i \partial_i^2 f$ coincides with the usual Laplacian on functions. But more generally for a Riemannian manifold a curvature term, the local basis vectors ϵ_i are also dynamical variables so we write $Df = \sum_i \epsilon_i \nabla_i f$, where ∇_i is the covariant derivative along the i-direction, then,

$$D^2 f = -\sum_i \nabla_i^2 f + \sum_{j<i} \epsilon_j \epsilon_i (\nabla_j \nabla_i - \nabla_i \nabla_j) f.$$

The second term in the rhs corresponds precisely to the curvature K so that we have the general expression

$$D^2 f = \nabla^* \nabla f + K f \tag{61}$$

known as the Weitzenböck formula [20, 37].

Coming back to the example of a quantum particle in a magnetic field, we can identify the curvature K with the magnetic field. For the case of a constant magnetic field, the K-term in (61) is just a constant so that the Hamiltonian coincides with the Laplacian. Therefore, on a compact manifold without boundary, the Schrödinger equation is geometrical and admits topological invariants, namely the total quantized magnetic flux. This is not true anymore in the presence of boundaries. For a non-uniform magnetic field, the curvature K becomes a local function of the coordinates and the problem defined by the Hamiltonian $H = \frac{1}{2}\nabla^*\nabla$ has no geometrical features. To recover a geometrical formulation, Aharonov and Casher [38] proved that a Zeeman term $(\frac{1}{2}\vec{\sigma}.\vec{B})$, playing the role of K in (61), must be added. Similar non-geometrical features appear for the Pauli or Dirac equations. These equations admit zero-modes solutions of the type $Df = 0$ whose number n is an Atiyah-Singer Index. But this number *does not* correspond to a Chern number of the type (48) since the magnetic field is not a dynamical variable of the problem but only a parameter and its magnetic flux is not quantized. To obtain an example of a $U(1)$-connexion with topological invariants, we consider now the case of a two-dimensional superconductor described by the Ginzburg-Landau equations, where both the order parameter and the magnetic field are dynamical variables.

4 The dual point of Ginzburg-Landau equations for an infinite system

The existence and stability of vortices in superfluid or superconducting systems have been mainly studied for the case of infinite systems or in a limit where boundary effects do not play an essential role. Among the large variety of methods available to study vortices in superfluids or superconductors, we choose to work in the framework of the Ginzburg-Landau expression for the free energy. We consider the case of a finite 2d superconducting bounded domain (a billiard) and study the existence and stability of vortices. The superconducting state is characterized by a complex order parameter. For infinite systems, nonlinear functionals given by Ginzburg-Landau, Gross-Pitaevskii or Higgs expressions admit vortex like solutions. These solutions are characterized by topological numbers *e.g.* the number n of vortices. How can these results be extended to finite size systems? Is there a mechanism by which boundary conditions may allow to select a state with a given number of vortices?

4.1 The Ginzburg-Landau equations

The Ginzburg-Landau equations describe a superconducting billiard if both the order parameter and the vector potential have a slow spatial variation. The expression of the Ginzburg-Landau energy density a is

$$a = a_0 + a_2|\psi|^2 + a_4|\psi|^4 + a_1\left|\left(\vec{\nabla} - i\frac{2e}{\hbar c}\vec{A}\right)\psi\right|^2 + \frac{B^2}{8\pi} \qquad (62)$$

where $\psi = |\psi|e^{i\chi}$ is the complex-valued order parameter, B is the magnetic field and the a_i's are real parameters. Defining [39] $\xi^2 = \frac{a_1}{|a_2|}$, $\lambda^2 = \frac{1}{4\pi}\sqrt{2}\left(\frac{\hbar c}{2e}\right)^2 \frac{a_4}{a_1|a_2|}$, the dimensionless free energy $\mathcal{F}$ is

$$\mathcal{F} = \int_\Omega \frac{1}{2}|B|^2 + \kappa^2|1 - |\psi|^2|^2 + |(\vec{\nabla} - i\vec{A})\psi|^2 \qquad (63)$$

where ψ is measured in units of $\psi_0 = \sqrt{\frac{|a_2|}{2a_4}}$, B in units of $\frac{\phi_0}{4\pi\lambda^2}$, and the lengths in units of $\lambda\sqrt{2}$. The numerical factor $\sqrt{2}$ is for further convenience. The ratio $\kappa = \frac{\lambda}{\xi}$ is the only free parameter in (63) and it determines, in the limit of an infinite system, whether the sample is a type-I or type-II superconductor [39]. The integral is over the two-dimensional domain Ω of the superconducting sample. The Ginzburg-Landau equations for the order parameter ψ and for the magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$ are obtained from a variation of $\mathcal{F}$. They are nonlinear second order differential equations. Their solutions are not known except for some particular cases.

4.2 The Bogomol'nyi identities

For the special value $\kappa = \frac{1}{\sqrt{2}}$, the equations for ψ and $\vec{A}$ can be reduced to first order differential equations. This special point was first used by Sarma [41] in his discussion of type-I *vs.* type-II superconductors and then identified by Bogomol'nyi [40] in the more general context of stability and integrability of classical solutions of some quantum field theories. This special point is also called a duality point. We first review some properties of the Ginzburg-Landau free energy at the duality point. We use the following identity true for two dimensional systems

$$|(\vec{\nabla} - i\vec{A})\psi|^2 = |\mathcal{D}\psi|^2 + \vec{\nabla} \times \vec{\jmath} + B|\psi|^2 \tag{64}$$

where $\vec{\jmath} = \mathrm{Im}(\psi^* \vec{\nabla}\psi) - |\psi|^2 \vec{A}$ is the current density and the operator $\mathcal{D}$ is defined as $\mathcal{D} = \partial_x + i\partial_y - i(A_x + iA_y)$. This relation is a relative of the Weitzenböck formula (61). At the duality point $\kappa = \frac{1}{\sqrt{2}}$ the expression (63) for $\mathcal{F}$ can be rewritten using (64) as

$$\mathcal{F} = \int_\Omega \left(\frac{1}{2}|B - 1 + |\psi|^2|^2 + |\mathcal{D}\psi|^2 \right) + \oint_{\partial\Omega} (\vec{\jmath} + \vec{A}).\vec{dl} \tag{65}$$

where the last integral over the boundary $\partial\Omega$ of the system results from Stokes theorem. For an infinite system, we impose [40] the usual conditions for a superconductor, namely $|\psi| \to 1$ and $\vec{\jmath} \to 0$ at infinity. The boundary term in (65) then becomes

$$\oint_{\partial\Omega} (\vec{\jmath} + \vec{A}).\vec{dl} = \oint_{\partial\Omega} \left(\frac{\vec{\jmath}}{|\psi|^2} + \vec{A} \right).\vec{dl}. \tag{66}$$

This last integral is the fluxoid. It is quantized and is equal to

$$\oint_{\partial\Omega} \vec{\nabla}\chi.\vec{dl} = 2\pi n.$$

The integer n is the winding number of the order parameter ψ and as such is a topological characteristic of the system. Using (66), we see that n is also the total magnetic flux through the system:

$$\int_\Omega B = n. \tag{67}$$

As we interpreted B as a first Chern class, this relation is similar to (46). The extremal values of $\mathcal{F}$, are obtained when the two Bogomol'nyi [40] equations are satisfied

$$\begin{aligned} \mathcal{D}\psi &= 0 \\ B &= 1 - |\psi|^2. \end{aligned} \tag{68}$$

In this case, the free energy $\mathcal{F}$ is given by

$$\frac{1}{2\pi}\mathcal{F} = n. \tag{69}$$

Therefore the free energy itself is a topological invariant. The two Bogomoln'yi equations can be decoupled and $|\psi|$ is a solution of the second order nonlinear equation

$$\nabla^2 ln|\psi|^2 = 2(|\psi|^2 - 1) \tag{70}$$

which is related to the Liouville equation.

It should be noticed that the set of equations (68, 70) has been obtained without any assumption on the nature of the magnetic field and appears in various other situations, *e.g.* Higgs [43], Yang-Mills [42] and Chern-Simons [44] field theories. It was proven that these equations admit families of vortex solutions [43]. For infinite systems, it can be shown that each vortex carries one flux quantum and that the winding number n is equal to the number of vortices in the system. However for an infinite system there is no mechanism to select the value of n. It will be precisely the role of the boundary of a finite system to introduce such a selection mechanism and to determine n, according to the applied magnetic field.

5 The superconducting billiard

From now on, we shall study finite size systems in an external magnetic field. The question then arises to know if they can sustain stable vortex solutions and what their behaviour is, as a function of the applied field. An interesting approach has been developed by Bethuel and coworkers [47]. They considered the case of a billiard Ω without applied magnetic field but with the boundary condition for the order parameter $\psi|_\Omega = g(\theta)$ with $g(\theta) = e^{i\phi(\theta)}$ and a prescribed winding number $\frac{1}{2i\pi}\int_0^{2\pi}\frac{\partial g}{\partial\theta}d\theta = n$. In the London limit, *i.e.* $\kappa \to \infty$, $|\psi|$ is 1 almost everywhere but because of the degree n on the boundary, $|\psi|$ must vanish n times in the bulk therefore leading to vortices. An extension of this approach to the case where a magnetic field is applied on the system has been proposed in [48] where it is shown by a variational argument that vortex solutions have a lower energy when the magnetic field is increased. By the same method, it is also possible to discuss the type of vortices and their distribution as a function of the geometry of the billiard. Numerical simulations [45] of the Ginzburg-Landau equations for a long parallelepiped in a uniform magnetic field show that the physical picture derived for $\kappa = \frac{1}{\sqrt{2}}$, namely the existence of stationary vortex solutions whose number depends on the magnetic field, remains valid for quite a large range of values of κ, and the corresponding change of free energy is small [46]. We shall therefore study the case $\kappa = \frac{1}{\sqrt{2}}$, *i.e.* the duality point and extend the previous approach to a system with finite size where boundary effects are important.

5.1 The zero current line

In a finite system, there are in general non-zero edge currents and the order parameter is not equal to 1 on the boundary. Hence, the identification of the boundary integral in (65) with the fluxoid (66) is not possible anymore, and the free energy can not be minimized just by imposing Bogomol'nyi equations (68). However, the currents on the boundary of the system screen the external magnetic field and therefore produce a magnetic moment (a circulation) opposite to the direction of the field, whereas vortices in the bulk of the system produce a magnetic moment along the direction of the applied field. Hence currents in the bulk circulate in a direction opposite to those at the boundary. If one assumes cylindrical symmetry, $\vec{\jmath}$ has only an azimuthal component, with opposite signs in the bulk and on the edge of the system (the radial component is zero since $\vec{\jmath}$ is divergence free). Thus, there exists a circle Γ on which $\vec{\jmath}$ vanishes. This allows us to separate the domain Ω into two concentric subdomains $\Omega = \Omega_1 \cup \Omega_2$ such that the boundary $\partial\Omega_1$ is the curve Γ. On $\partial\Omega_1$, the current density $\vec{\jmath}$ is zero, therefore

$$\oint_{\partial\Omega_1} \vec{\jmath}.\vec{dl} = \oint_{\partial\Omega_1} \frac{\vec{\jmath}}{|\psi|^2}.\vec{dl} = 0. \tag{71}$$

Thus one deduces as above that Bogomol'nyi and Liouville equations are valid in the finite domain Ω_1 as in the case of the infinite plane. The magnetic flux $\Phi(\Omega_1)$ is calculated using the fluxoid and (71) so that

$$\Phi(\Omega_1) = n - \oint_{\partial\Omega_1} \frac{\vec{\jmath}.\vec{dl}}{|\psi|^2} = n.$$

As before n is the winding number, *i.e.* $\oint_{\partial\Omega_1} \vec{\nabla}\chi.\vec{dl} = 2\pi n$, as well as the number of vortices [49] in Ω_1. The free energy in Ω_1 is

$$\mathcal{F}(\Omega_1) = 2\pi n. \tag{72}$$

The contribution of Ω_2 to the free energy is given by (2) and can be expressed using the phase and the modulus of the order parameter ψ

$$\mathcal{F}(\Omega_2) = \int_{\Omega_2} (\nabla|\psi|)^2 + |\psi|^2|\vec{\nabla}\chi - \vec{A}|^2 + \frac{B^2}{2} + \frac{(1-|\psi|^2)^2}{2}. \tag{73}$$

The boundary conditions for both the magnetic field $B(R)$ and the vector potential $A(R)$ adapted to a flat disk geometry are provided by the condition $\phi = \phi_e$ where ϕ_e is the flux of the applied field B_e namely, $\phi_e = \frac{B_e R^2}{\phi_0}$ and $\phi_0 = \frac{hc}{2e}$. It implies that at the boundary $B(R)$ is larger than the applied field B_e due to the distorsion of the flux lines near the edge of the system.

5.2 *A selection mechanism and topological phase transitions*

The analysis presented in [49] leads for the total Gibbs potential of the billiard to the expression

$$\begin{aligned}\frac{1}{2\pi}\mathcal{G}(n,\phi_e) &= \frac{1}{2\pi}\mathcal{F}(n,\phi_e)-\frac{2\lambda^2}{R^2}{\phi_e}^2\\ &= n+\frac{\lambda\sqrt{2}}{R}(n-\phi_e)^2-\frac{1}{2}\left(\frac{\lambda\sqrt{2}}{R}\right)^3(n-\phi_e)^4-\frac{2\lambda^2}{R^2}{\phi_e}^2 \quad (74)\end{aligned}$$

This relation consists in a set of quartic functions indexed by the integer n. The minimum of the Gibbs potential is the envelop curve defined by the equation $\frac{\partial\mathcal{G}}{\partial n}|_{\phi_e}=0$, *i.e.* the system chooses its winding number n in order to minimize $\mathcal{G}$. This provides a relation between the number n of vortices in the system and the applied magnetic field ϕ_e. In the limit of a large enough $\frac{R}{\lambda}$, the quartic term is negligible and the Gibbs potential reduces to a set of parabolas. The winding number n is then given by

$$n=\left[\phi_e-\frac{R}{2\sqrt{2}\lambda}+\frac{1}{2}\right]. \tag{75}$$

The magnetization of the system $M=-\frac{\partial\mathcal{G}}{\partial\phi_e}$ is given by

$$-M=\frac{2\sqrt{2}\lambda}{R}(\phi_e-n)-\frac{4\lambda^2}{R^2}\phi_e. \tag{76}$$

For ϕ_e smaller that $\frac{R}{2\sqrt{2}\lambda}$, we have $n=0$ and $(-M)$ increases linearly with the external flux. This corresponds to the London regime before the first vortex enters the system. The field H_1 at which the first vortex enters the system corresponds to $\mathcal{G}(n=0)=\mathcal{G}(n=1)$, *i.e.* to $H_1=\frac{\phi_0}{2\pi\sqrt{2}R\lambda}+\frac{\phi_0}{2\pi R^2}$. The subsequent vortices enter one by one for each crossing $\mathcal{G}(n+1)=\mathcal{G}(n)$; this happens periodically in the applied field, with a period equal to $\Delta H=\frac{\phi_0}{\pi R^2}$. This gives rise to a discontinuity of the magnetization $\Delta M=\frac{2\sqrt{2}\lambda}{R}$.

5.3 *A geometrical expression of the Gibbs potential for finite systems*

For infinite system, with the boundary conditions $|\psi|\to 1$ and $\vec{\jmath}\to 0$ at infinity, we showed that the free energy at the dual point $\kappa=\frac{1}{\sqrt{2}}$ is a topological invariant proportional (69) to the fluxoid n, which represents also the number of vortices in the system:

$$\frac{1}{2\pi}\mathcal{F}=\int B=n. \tag{77}$$

This relation is a property of the dual point at which the Ginzburg-Landau functional has a geometrical interpretation and admits the quantized magnetic flux as a topological invariant. For a finite system the fluxoid quantification can be expressed as

$$\frac{1}{2\pi}\int_{\Omega} \vec{B}.\mathrm{d}\vec{S} + \oint_{\partial\Omega} \frac{\vec{\jmath}}{|\psi|^2}.\mathrm{d}\vec{l} = n. \tag{78}$$

This relation is analogous to the Gauss-Bonnet theorem for a surface with a boundary (53) or more generally to a topological invariant obtained by summing a Chern class in the bulk and a Chern-Simons class on the boundary (56). Recalling (60), we see that the magnetic field B plays the role of a curvature K, and the quantity $\frac{\vec{\jmath}}{|\psi|^2} = \nabla\chi - A$ is similar to a *geodesic curvature* k_{g}. For a system with cylindrical symmetry, one can show [49] that $\frac{\vec{\jmath}}{|\psi|^2} = n - \phi_{\mathrm{e}}$. Therefore equation (74) can be rewritten as

$$\frac{1}{2\pi}\mathcal{F} = \int_{\Omega} B + \int_{\partial\Omega} \eta\left(\frac{\vec{\jmath}}{|\psi|^2}\right) \equiv \int K + \oint \eta(k_{\mathrm{g}}). \tag{79}$$

Comparing (79) with (77) we conclude that the boundary correction is a functional of the geodesic curvature. In the preceeding section, we obtained an explicit formula for the function η as an even fourth order polynomial in the geodesic curvature.

This geometric interpretation makes us believe that an expression such as (79) is fairly general. It could be well suited, as an Ansatz, to describe finite systems which are known to have a topological description in the infinite limit; for example, a suitable generalization of (79) to $SU(2)$ symmetry could describe superfluid ^{3}Helium in a bounded domain.

We have presented a geometrical formulation for the Ginzburg-Landau problem which we now briefly summarize.

i) For a certain theory, like Ginzburg-Landau or other functionals of this type, it may appear stable singular solutions (*e.g.* vortices) whose nature is determined by the related homotopy groups of the Toulouse-Kleman [1] approach. To these solutions are associated topological numbers in the Bogomol'nyi limit.

ii) The existence of topological numbers signals the occurence of a geometrical description of the problem.

What is it good for? First, we notice that topological quantities describe global features of the problem *i.e.* behaviour in the large by opposition to the local behaviour obtained from solutions of differential equations. Then, by identifying physical quantities in terms of global topological invariants, we do not need to solve the local equations to obtain the behaviour of the system. As a result, we may say that an important goal of a geometrical description is to obtain physical quantities in terms of global topological expressions. When it is possible, it is very much rewarding.

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