# Atom-Photon Interactions 

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## Nomenclature

$\boldsymbol{A} \quad$ Vector potential
$a_{0} \quad$ Bohr radius
$A, A_{b a}$ Einstein $A$ coefficient for spontaneous emission
$\hat{\boldsymbol{A}} \quad$ Field operator
$\hat{a}_{\boldsymbol{k}, \epsilon}, \hat{a}_{\boldsymbol{k}, \epsilon}^{\dagger}$ Creation and annihilation operators
$\alpha \quad$ Absorption coefficient
$\alpha_{\mathrm{fs}} \quad$ Fine-structure constant
$\tilde{\alpha} \quad$ Dynamical polarizability
$\boldsymbol{A}_{\perp} \quad$ Transverse external radiation
$\ldots \quad$ Fluctuation average
$\langle\cdots\rangle$ Statistical average
$\langle\cdots\rangle_{t} \quad$ Statistical average w.r.t. time
$\cdots \quad$ Thermal average
$B_{a b} \quad$ Einstein $B$ coefficient for absorption
$B_{b a} \quad$ Einstein $B$ coefficient for stimulated emission
$\hat{\boldsymbol{B}} \quad$ Magnetic field operator
$\chi \quad$ Susceptibility
CQED Cavity QED
D Dipole moment
$D \quad$ Weighted statistical population
$D_{s} \quad$ Saturated weighted statistical population
$\Delta \quad$ Detuning
$\widetilde{\Delta_{b}}(E)$ Spectral function - Lamb shift
$\mathcal{D}[\sigma]$ Lindblad superoperator
$\tilde{D} \quad$ Complex dipole moment
$\boldsymbol{E} \quad$ Electric field
$\mathcal{E}(r, t)$ Classical electric field
$\mathcal{E}_{0}(L)$ Casimir energy
$\hat{\boldsymbol{E}} \quad$ Electric field operator

EM Electromagnetic
$\epsilon \quad$ Permeability
$\hat{\epsilon} \quad$ Polarization
$\varepsilon_{0} \quad$ Vacuum permitivity
$\tilde{E} \quad$ Complex electric field
$F \quad$ Free energy
$F \quad$ Finesse
$g_{2}(0)-g_{2}(\infty)$ Fano factor
$F(\boldsymbol{k})$ Structure factor
$f_{\mathrm{o}} \quad$ Oscillator strength
$F_{p} \quad$ Purcell factor
FWHM Full-width half-max
$G \quad$ Gibbs potential
$g_{0} \quad$ Atom-photon coupling parameter
$\mathcal{G}^{(1)}\left(r, t^{\prime}, r, t^{\prime}\right)$ Classical correlation function
$g_{1}(\tau)$ Normalized first correlation function
$G^{(1)}\left(r, t^{\prime}, r, t^{\prime}\right)$ Quantum correlation function
$\mathcal{G}^{(2)}\left(r_{1}, t_{1}, r_{2}, t_{2}\right)$ Classical second correlation
$g_{2}(\tau)$ Normalized second correlation function
$G^{(2)}\left(r_{1}, t_{1}, r_{2}, t_{2}\right)$ Quantum second correlation
$\Gamma \quad$ Longitudinal decay rate
$\Gamma \quad$ Non-resonant decay rate
$\Gamma_{0} \quad$ Emission rate of a single atom
$\widetilde{\Gamma_{b}}(E)$ Spectral function - Relaxation
$\gamma_{\perp} \quad$ Transverse decay rate
$\Gamma(s) \quad$ Euler gamma function
$|g\rangle,|e\rangle$ Ground and excited states
$g(\nu)$, DOM Density of modes
$G(\tau) \quad$ Quantum correlations of the vacuum electromagnetic field
$\hat{G}(z)$ Resolvant operator
$\boldsymbol{H} \quad$ Magnetic field
$H \quad$ Enthalpy
$H \quad$ Hamiltonian
$\hbar \quad$ Planck's constant
HBT, HB\&T Hanburry-Brown \& Twiss
$H_{\text {int }} \quad$ Interaction Hamiltonian
$\hat{H} \quad$ Quantum Hamiltonian
$I \quad$ Power intensity
I Radiation power per unit surface
$I(t) \mathrm{d} t$ Detection probability a particle w.r.t. time
$\mathbb{1}$ Identity matrix / operator
$I_{\text {sat }} \quad$ Saturation intensity
$J(t) \quad$ Detector current
$k(t) \quad$ Detector response
$\kappa \quad$ Photon decay rate
$k_{\mathrm{B}} \quad$ Boltzmann's constant
$L \quad$ Orbital angular momentum
$\lambda_{T} \quad$ Thermal wavelength
$L_{\text {cav }}$ Cavity length
$M(\xi, t)$ Generating Function
$\mu_{0} \quad$ Vacuum permeability
$N \quad$ Density
$n \quad$ Refractive index
$\langle n(t)\rangle$ Average number of particles
$\left\langle n^{2}(t)\right\rangle$ Variance of particles
$n_{\mathrm{BE}}(x)$ Bose-Einstein distribution
$\hat{N}_{\boldsymbol{k}, \epsilon} \quad$ Excitation number operator
NMR Nuclear magnetic resonance
$n \quad$ Number of particles
: (...) : Normal ordering

| $\tilde{n}$ | Complex refractive index |
| :---: | :---: |
| $\mathcal{N}(\tau)$ | Memory function |
| $\omega$ | Frequency of light |
| $\Omega_{R}$ | Rabi frequency |
| $\Omega_{v}$ | Rabi frequency of the vacuum |
| $P$ | Momentum |
| $\boldsymbol{P}$ | Polarization density |
| $P$ | Power |
| $p$ | Pressure |
| $\mathcal{P}(\cdot)$ | Cauchy principal value |
| $P(n, t)$ | Detection probability of $n$ particles w.r.t. time |
| $P(\{\alpha\})$ | ) Classical probability distribution |
| $P_{Q}(\{\alpha\})$ | \}) Quantum probability distribution |
| $\mathcal{P}$ | Precession vector |
| $\phi$ | Flux of photons |
| $\phi(\boldsymbol{r}, t)$ | Gauge potential |
| $\varphi$ | Phase difference |
| $\varphi(\nu)$ | Spectral function |
| $P_{\text {rad }}$ | Radiative pressure (power) |
| $\hat{\boldsymbol{p}}_{\text {rad }}$ | Momentum field operator |
| $\theta$ | Stückelberg angle |
| $Q$ | Quality factor |
| $q$ | Charge |
| QED | Quantum electrodynamics |
| $R$ | Reflectivity |
| $R_{\infty}$ | Rydberg constant |
| $R_{a}$ | Collection efficiency |
| $R_{b}(z)$ | Self energy |
| $\hat{\rho}$ | Density matrix |
| $\rho$ | Particle density |

RPA Random phase approximation
$R_{\text {sc }} \quad$ Photon scattering rate
RWA Rotating wave approximation
$\boldsymbol{S} \quad$ Poynting vector
$\boldsymbol{S} \quad$ Spin
$\sigma \quad$ Scattering cross-section
$\sigma_{a b} \quad$ Absorption cross-section
$\sigma_{b a} \quad$ Emission cross-section
$\langle\boldsymbol{\sigma}\rangle \quad$ Bloch vector
$\sigma_{x}, \sigma_{y}, \sigma_{z}$ Pauli matrices
$\hat{\sigma}, \hat{\sigma}^{\dagger} \quad$ Lowering and rasing operators
$\sigma_{R} \quad$ Rayleigh cross-section
$\hat{S}^{ \pm}, \hat{S}_{z}$ Spin operators
$T \quad$ Temperature
$T$ Transmittance
$t \quad$ Time
$T_{1} \quad$ Longitudinal relaxation time
$T_{2} \quad$ Transverse relaxation time
$\tau \quad$ Proper time
$T_{\text {ir }} \quad$ Irreversibility time.
$t_{r} \quad$ Retarded time
$u \quad$ Energy density
$\hat{\boldsymbol{U}} \quad$ Electromagnetic energy operator
$\hat{U}\left(t, t^{\prime}\right)$ Evolution operator
$V \quad$ Volume
$V_{m} \quad$ Mode (modal) volume
$V(\boldsymbol{r}) \quad$ Electrostatic potential
$W \quad$ Energy density
$w_{n} \quad$ Boltzmann factor
$\chi \quad$ Susceptibility
$\xi \quad$ Conjugate variable to number of particles
$\xi(t)$ Time-dependent conjugate variable to number of particles
$Z_{0} \quad$ Classical vacuum impedance
$\frac{h}{e^{2}} \quad$ Quantum vacuum impedance
$\zeta(s)$ Riemann zeta function

## 1 Classical Atom-field Interaction ${ }^{1}$

### 1.1 Interaction between Classical Atoms and Classical Light

This first chapter shall remind us of the known observations.

### 1.1.1 Introduction

Let us talk about the following quantities:

- Refractive index: $n(\omega)$;
- Absorption coefficient: $\alpha(\omega)$;
- Scattering cross-section: $\sigma(\omega)$;
with $\omega$ the frequency of light. What people have noticed is that when one tries to observe these quantities, they behave in a resonance way: they have specific resonances but with finite widths.

These phenomena have led Lorentz (end of 19th century) to the following. Let us start with the following: a) Maxwell equations; b) Matter: set of charges; c) Harmonic oscillations. Although we now know that this is not the correct description, we shall inspect the poet's intentions nonetheless.

### 1.1.2 Isolated Lorentz Atom

Lorentz has started with harmonic oscillators (hence, the name "Lorentzian"). Let us start with an external electric field on an isolated atom:

$$
\begin{equation*}
\boldsymbol{E}=\hat{\epsilon} E \cos (\omega t-\phi), \tag{1.1}
\end{equation*}
$$

where $\hat{\epsilon}$ is the polarization. Introduce charge $q$ and mass $m$ to have the equation of motion

$$
\begin{equation*}
m \frac{\mathrm{~d}^{2} \boldsymbol{r}}{\mathrm{~d} t^{2}}=-m \omega_{0}^{2} r+q E \hat{\epsilon} \cos (\omega t-\phi), \tag{1.2}
\end{equation*}
$$

where $\omega_{0}$ is the characteristic (phenomenological) frequency. Introduce the dipole $\boldsymbol{D}=q \boldsymbol{r}$ to have

$$
\begin{equation*}
\frac{\mathrm{d}^{2} \boldsymbol{D}}{\mathrm{~d} t^{2}}=-m \omega_{0}^{2} \boldsymbol{D}+\frac{q^{2} E}{m} \hat{\epsilon} \cos (\omega t-\phi) . \tag{1.3}
\end{equation*}
$$

We can therefore write the dipole under the slowly varying envelope approximation $\tilde{D}(t)$ as

$$
\begin{equation*}
\boldsymbol{D}(t)=\operatorname{Re}\left[\tilde{D}(t) \hat{\epsilon} \mathrm{e}^{-\mathrm{i} \omega t}\right] . \tag{1.4}
\end{equation*}
$$

Setting $\tilde{E}=E \mathrm{e}^{\mathrm{i} \phi}$ one gets

$$
\begin{equation*}
-\omega^{2} \tilde{D}-2 \mathrm{i} \omega \frac{\mathrm{~d} \tilde{D}}{\mathrm{~d} t}+\frac{\mathrm{d}^{2} \tilde{D}}{\mathrm{~d} t^{2}}=-\omega^{2} \tilde{D}+\frac{q^{2}}{m} \tilde{E} . \tag{1.5}
\end{equation*}
$$

assuming $\tilde{D}(t)$ evolves with a time scale long compared to $\frac{2 \pi}{\omega}$. Hence,

$$
\begin{equation*}
\frac{\mathrm{d} \tilde{D}}{\mathrm{~d} t}=\mathrm{i}\left(\frac{\omega^{2}-\omega_{0}^{2}}{2 \omega}\right) \tilde{D}+\mathrm{i} \frac{q^{2} \tilde{E}}{2 m \omega} . \tag{1.6}
\end{equation*}
$$

[^0]Close to the resonance, $\omega \simeq \omega_{0}$. Hence,

$$
\begin{equation*}
\frac{\omega^{2}-\omega_{0}^{2}}{2 \omega}=\left(\omega-\omega_{0}\right) \frac{\omega+\omega_{0}}{2 \omega} \simeq \omega-\omega_{0} \tag{1.7}
\end{equation*}
$$

We therefore have the stationary solution $\left(\frac{\mathrm{d} \tilde{D}}{\mathrm{~d} t}=0\right)$,

$$
\begin{equation*}
\tilde{D}=\frac{q^{2} \tilde{E}}{2 m \omega\left(\omega_{0}-\omega\right)} . \tag{1.8}
\end{equation*}
$$

### 1.1.3 Energetic Aspect of the Light-atom Interaction

The average power is given by

$$
\begin{equation*}
P=\overline{-\boldsymbol{E} \cdot \frac{\mathrm{d} \boldsymbol{D}}{\mathrm{~d} t}}=\ldots=-\frac{\omega}{2} \operatorname{Im}\left(\tilde{D} \tilde{E}^{*}\right) . \tag{1.9}
\end{equation*}
$$

Define

$$
\begin{equation*}
\tilde{D}=\varepsilon_{0} \tilde{\alpha} \tilde{E} \tag{1.10}
\end{equation*}
$$

where $\tilde{\alpha}$ is the dynamical polarizability

$$
\begin{equation*}
\tilde{\alpha}(\omega)=\alpha+\mathrm{i} \alpha^{\prime} . \tag{1.11}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
P=-\frac{\omega}{2} \alpha^{\prime}|\tilde{E}|^{2} \tag{1.12}
\end{equation*}
$$

Note that what drives the power is the imaginary part $\alpha^{\prime}$. However, in this model, $\tilde{\alpha}$ is strictly real, and therefore,

$$
\begin{equation*}
P=0 \tag{1.13}
\end{equation*}
$$

or, there is no exchange of energy.
Corollary 1.1. This model cannot describe adsorptions or more complex phenomena such as lasers.

### 1.1.4 Phenomenological Approach to Relaxations

We need coupling to the environment (or, relaxation). There are several sources: collision with other atoms, etc. At the end of the day one obtains

$$
\begin{equation*}
\tilde{D}=\frac{q^{2}}{2 m \omega} \cdot \frac{\tilde{E}}{\omega_{0}-\omega-\mathrm{i} \gamma_{d}} \tag{1.14}
\end{equation*}
$$

Using this expression, one can find an expression to $\tilde{\alpha}$ :

$$
\begin{equation*}
\tilde{\alpha}=\frac{q^{2}}{2 m \omega \varepsilon_{0}} \cdot \frac{\omega_{0}-\omega+\mathrm{i} \gamma_{d}}{\left(\omega_{0}-\omega\right)^{2}+\gamma_{d}^{2}} . \tag{1.15}
\end{equation*}
$$

Notice that $\alpha^{\prime} \neq 0$, but it is strictly positive. Hence, we can have absorption but not amplification. In other words,

$$
\begin{equation*}
\tilde{D}=\varepsilon_{0} \tilde{\alpha} \tilde{E} \tag{1.16}
\end{equation*}
$$

### 1.1.5 Connection to Dielectric Media (Atomic Gas)

We want to generalize this discussion to an ensemble of atoms. Let us define polarization density $\boldsymbol{P}=\frac{\text { dipole moment }}{\text { unit volume }}$ and let $N$ be the density. We thus write

$$
\begin{align*}
\boldsymbol{P}=N \boldsymbol{D} & =N \varepsilon_{0} \alpha(\omega) \boldsymbol{E} \\
& \equiv \varepsilon_{0} \chi(\omega) \boldsymbol{E} \tag{1.17}
\end{align*}
$$

with $\chi(\omega)$ the susceptibility given by

$$
\begin{equation*}
\chi(\omega)=N \alpha(\omega)=\frac{N q^{2}}{2 m \omega \varepsilon_{0}} \cdot \frac{\omega_{0}-\omega+\mathrm{i} \gamma_{d}}{\left(\omega_{0}-\omega\right)^{2}+\gamma_{d}^{2}} \tag{1.18}
\end{equation*}
$$

Now we can talk about a refractive index $\tilde{n}$. Let an electric field be

$$
\begin{align*}
\boldsymbol{E}(z)=\boldsymbol{E}_{0} \mathrm{e}^{\mathrm{i} k z} & =\boldsymbol{E}_{0} \mathrm{e}^{\mathrm{i} \tilde{n} k_{0} z} \\
& =\boldsymbol{E}_{0} \mathrm{e}^{\mathrm{i} \operatorname{Re}(\tilde{n}) k_{0} z} \mathrm{e}^{-\operatorname{Im}(\tilde{n}) k_{0} z}, \tag{1.19}
\end{align*}
$$

with $k_{0}$ the wave-number of the incident wave. We now define the refractive index and the absorption coefficient be

$$
\left\{\begin{array}{ll}
n(\omega)=\operatorname{Re}[\tilde{n}(\omega)] & \text { refractive index }  \tag{1.20}\\
a(\omega)=2 k_{0} \operatorname{Im}[\tilde{n}(\omega)] & \text { absorption coefficient }
\end{array} .\right.
$$

The power intensity goes as

$$
\begin{equation*}
\frac{\mathrm{d} I}{\mathrm{~d} z}=-a I \Longrightarrow I(z)=I_{0} \mathrm{e}^{-a z} \tag{1.21}
\end{equation*}
$$

Noe let us reinspect the dipole moment

$$
\begin{align*}
\boldsymbol{D} & =\varepsilon_{0} \boldsymbol{E}+\boldsymbol{P} \\
& =\varepsilon_{0} \boldsymbol{E}+\varepsilon_{0} \chi \boldsymbol{E}  \tag{1.22}\\
& =\varepsilon_{0}(1+\chi) \boldsymbol{E} \\
& \equiv \varepsilon \boldsymbol{E} .
\end{align*}
$$

The refractive index reads

$$
\begin{equation*}
\tilde{n}=\sqrt{\frac{\varepsilon(\omega)}{\varepsilon_{0}}}=\sqrt{1+\chi(\omega)} \simeq 1+\frac{1}{2} \chi(\omega) \tag{1.23}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
n(\omega)=1+\frac{N q^{2}}{2 m \omega \varepsilon_{0}} \frac{\omega_{0}-\omega}{\left(\omega_{0}-\omega\right)^{2}+\gamma_{d}^{2}} \tag{1.24}
\end{equation*}
$$

(See Figure 1.1).

### 1.1.6 Generalization of the Susceptibility

Let us start with a single atom susceptibility,

$$
\begin{equation*}
\chi(\omega)=\frac{N q^{2}}{2 m \varepsilon_{0} \omega} \frac{\omega_{0}-\omega+\mathrm{i} \gamma_{d}}{\left(\omega_{0}-\omega\right)^{2}+\gamma_{d}^{2}} \tag{1.25}
\end{equation*}
$$



Fig. 1.1: The refractive index and the absorption coefficient.
and generalize to the collection of Lorentz atoms

$$
\begin{equation*}
\chi(\omega)=\sum_{j} \frac{N q^{2}}{2 m \varepsilon_{0} \omega_{j}} \cdot f_{\mathrm{o}, j} \cdot \frac{\omega_{j}-\omega+\mathrm{i} \gamma_{d, j}}{\left(\omega_{j}-\omega\right)^{2}+\gamma_{d, j}^{2}} . \tag{1.26}
\end{equation*}
$$

and $f_{\mathrm{o}}$ is the oscillator strength given by quantum mechanical considerations. This is the Thomas-Reiche-Kuhn/Compressibility sum rule.

This expression involves a large set of phenomenological parameters ( $\gamma_{d}, f_{\mathrm{o}}$, etc.).

### 1.1.7 Dipole Radiation

Let us start with an electromagnetic field emitted by an accelerating dipole

$$
\left\{\begin{array}{l}
\boldsymbol{E}(\boldsymbol{r}, t) \simeq \frac{1}{4 \pi \varepsilon_{0} c^{2}}[(\hat{\epsilon} \cdot \hat{r}) \hat{r}-\hat{\epsilon}] \frac{\left|\ddot{\boldsymbol{D}}\left(t_{r}\right)\right|}{r}  \tag{1.27}\\
\boldsymbol{H}(\boldsymbol{r}, t) \simeq \frac{1}{4 \pi c}(\hat{\epsilon} \times \hat{r}) \frac{\left|\ddot{\boldsymbol{D}}\left(t_{r}\right)\right|}{r}
\end{array}\right.
$$

where $t_{r}$ is the retarded time, $t_{r}=t-\frac{r}{c}$. The transport of energy is described by the Poynting vector $\boldsymbol{S}$,

$$
\begin{equation*}
\langle\boldsymbol{S}\rangle=\boldsymbol{E} \times \boldsymbol{H}^{*}+c . c ., \tag{1.28}
\end{equation*}
$$

which leads to

$$
\begin{equation*}
\langle\boldsymbol{S}\rangle=\frac{\hat{r}}{16 \pi^{2} \varepsilon_{0} c^{3}} \frac{|\ddot{\boldsymbol{D}}|^{2}}{r^{2}}\left(1-|\hat{\epsilon} \cdot \hat{r}|^{2}\right) . \tag{1.29}
\end{equation*}
$$

Let us inspect several cases

1. Linear polarization $(\hat{\epsilon}=\hat{z})$ :

$$
\begin{equation*}
1-|\hat{\epsilon} \cdot \hat{r}|^{2}=\sin ^{2} \theta \tag{1.30}
\end{equation*}
$$

This is the doughnut polarization.
2. Circular polarization $\left(\hat{\epsilon}=\mp \frac{\hat{x} \pm \mathrm{i} \hat{y}}{\sqrt{2}}\right)$ :

$$
\begin{equation*}
1-|\hat{\epsilon} \cdot \hat{r}|^{2}=\frac{1}{2}\left(1+\cos ^{2} \theta\right) . \tag{1.31}
\end{equation*}
$$

This is the peanut polarization (see Figure 1.2).


Fig. 1.2: Poynting vector polarizations.

The radiated power is therefore,

$$
\begin{equation*}
P_{\mathrm{rad}}=\frac{|\ddot{\boldsymbol{D}}|^{2}}{3 \pi \varepsilon_{0} c^{3}} \tag{1.32}
\end{equation*}
$$

And the power absorbed by the field ${ }^{2}$,

$$
\begin{equation*}
P_{\mathrm{abs}}=\frac{\omega}{2} \operatorname{Im}(\alpha(\omega))|\tilde{E}|^{2}=\frac{\omega}{\varepsilon_{0} c} \operatorname{Im}(\alpha(\omega)) I(\boldsymbol{r}), \tag{1.33}
\end{equation*}
$$

where we have defined the intensity of the field,

$$
\begin{equation*}
I(\boldsymbol{r})=\frac{1}{2} \varepsilon_{0} c|\tilde{E}|^{2} \tag{1.34}
\end{equation*}
$$

### 1.2 Mechanical Properties of Classical Light in the Lorentz Model

We want to know if there is a possibility that an external light will act on the atom by getting it to move. Let us calculate some aspects of the mechanical properties.

1. A dipolar force: potential energy of the field $(\operatorname{Re} \tilde{\alpha}(\omega))$.
2. A radiation pressure: related to cycles of absorption and scattering of radiation $(\operatorname{Im} \tilde{\alpha}(\omega))$.

### 1.2.1 The Dipolar Force

The potential energy is given by

$$
\begin{equation*}
V_{d}=-\boldsymbol{D} \cdot \boldsymbol{E} \tag{1.35}
\end{equation*}
$$

[^1]Now, let us decompose $\boldsymbol{D}$ into two, forward and backward, parts

$$
\begin{equation*}
\boldsymbol{D}=\boldsymbol{D}^{(+)}+\boldsymbol{D}^{(-)}, \quad \boldsymbol{D}^{( \pm)} \propto \mathrm{e}^{ \pm \mathrm{i} \omega t} \tag{1.36}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
V_{d}=-\left(\boldsymbol{D}^{(+)}+\boldsymbol{D}^{(-)}\right) \cdot\left(\boldsymbol{E}^{(+)}+\boldsymbol{E}^{(-)}\right) . \tag{1.37}
\end{equation*}
$$

The next step is the rotating wave approximation (RWA): neglect the $2 \omega$ frequent $\boldsymbol{D}^{(+)} \boldsymbol{E}^{(+)}$and $\boldsymbol{D}^{(-)} \boldsymbol{E}^{(-)}$parts. Hence,

$$
\begin{align*}
V_{d} & =-\left(\boldsymbol{D}^{(+)} \cdot \boldsymbol{E}^{(-)}+\boldsymbol{D}^{(-)} \cdot \boldsymbol{E}^{(+)}\right) \\
& =-\operatorname{Re}(\tilde{\alpha}(\omega))\left|\boldsymbol{E}^{(+)}\right|^{2} \tag{1.38}
\end{align*}
$$

Therefore,

$$
\begin{equation*}
V_{d}=-\frac{1}{\varepsilon_{0} c} \operatorname{Re}(\tilde{\alpha}(\omega)) I(\boldsymbol{r}) \tag{1.39}
\end{equation*}
$$

Remark 1.1. We have obtained the classical vacuum impedance

$$
\begin{equation*}
\frac{1}{\varepsilon_{0} c}=\mu_{0} c=\sqrt{\frac{\mu_{0}}{\varepsilon_{0}}} \equiv Z_{0}=377[\Omega] \tag{1.40}
\end{equation*}
$$

On the other hand, there is another natural-unit impedance,

$$
\begin{equation*}
\frac{h}{e^{2}}=28[\mathrm{k} \Omega] \tag{1.41}
\end{equation*}
$$

which is the quantum impedance. Notice that the classical effects are much smaller than the quantum one by the fine-structure constant,

$$
\begin{equation*}
\frac{Z_{0}}{h / e^{2}}=\alpha_{\mathrm{fs}} \simeq \frac{1}{137} \tag{1.42}
\end{equation*}
$$

Now, write the potential as

$$
\begin{align*}
V_{d} & =-Z_{0} \operatorname{Re}(\tilde{\alpha}(\omega)) I(\boldsymbol{r}) \\
& =-\frac{q^{2}}{2 m \varepsilon_{0} \omega c} \frac{\omega_{0}-\omega}{\left(\omega_{0}-\omega\right)^{2}+\gamma_{d}^{2}} I(\boldsymbol{r}) . \tag{1.43}
\end{align*}
$$

The force is given by

$$
\begin{equation*}
\boldsymbol{F}_{d}=-\nabla V_{d} \propto-\nabla I(\boldsymbol{r}) . \tag{1.44}
\end{equation*}
$$

The sign of $\boldsymbol{F}_{d}$ depends on $\Delta \equiv \omega-\omega_{0}$ (detuning):

- $\Delta>0$ (blue detuning) : $V_{d}>0$;
- $\Delta<0$ (red detuning) : $V_{d}<0$.


### 1.2.2 Radiative Pressure ${ }^{3}$

The radiative pressureis given by (in units of power)

$$
\begin{equation*}
P_{\mathrm{rad}}=\frac{\omega^{4}|\tilde{\alpha}(\omega)|^{2}}{6 \pi \varepsilon_{0}^{2} c^{4}} I(r) \equiv \sigma_{R} I(\boldsymbol{r}) \tag{1.45}
\end{equation*}
$$

Here, $\sigma_{R}$ is the Rayleigh cross-section (scattering). The law of $\sigma_{R} \propto \omega^{4}$ is also obtained by the full quantum derivation (see Chap. 2 of Akkermans and Montambaux [2011]).

Remark 1.2. This is the Optical theorem (see Chap. 2, App. 2 of Akkermans and Montambaux [2011]). It means that the powers and fluxes are basically the same:

$$
\begin{equation*}
P_{\mathrm{rad}}=P_{\mathrm{abs}} \tag{1.46}
\end{equation*}
$$

Rearranging everything one has

$$
\begin{equation*}
\operatorname{Im}(\tilde{\alpha}(\omega))=\frac{1}{4 \pi \varepsilon_{0}} \frac{2 \omega^{3}}{3 c^{3}}|\tilde{\alpha}(\omega)|^{2} \tag{1.47}
\end{equation*}
$$

Note that the radiation pressure is proportional to the imaginary part of $\tilde{\alpha}$. Therefore,

$$
\begin{equation*}
P_{\mathrm{rad}}=Z_{0} \omega\left(\frac{\omega}{\omega_{0}}\right)^{2} \operatorname{Im}(\tilde{\alpha}(\omega)) I(\boldsymbol{r}) . \tag{1.48}
\end{equation*}
$$

and the $\left(\frac{\omega}{\omega_{0}}\right)^{2}$ part is only close to resonance.
Let us ${ }^{4}$ introduce the photon scattering rate,

$$
\begin{equation*}
R_{\mathrm{sc}}=\frac{P_{\mathrm{rad}}}{\hbar \omega} \tag{1.49}
\end{equation*}
$$

This lets us define how the force is related to the radiative pressure:

$$
\begin{align*}
\boldsymbol{F}_{\mathrm{rad}} & =\hbar \boldsymbol{k} R_{\mathrm{sc}} \\
& =\frac{Z_{0}}{c} \frac{\omega^{3}}{\omega_{0}^{2}} \operatorname{Im}(\tilde{\alpha}(\omega)) I(\boldsymbol{r}, \omega) . \tag{1.50}
\end{align*}
$$

Rearranging one gets

$$
\begin{equation*}
\boldsymbol{F}_{\mathrm{rad}}=\hbar \boldsymbol{k} \frac{\gamma_{d}^{3}}{\Delta^{2}+\gamma_{d}^{2}} \cdot \frac{I\left(\omega_{0}\right)}{I_{\mathrm{sat}}} \tag{1.51}
\end{equation*}
$$

Here we have defined the saturation intensity $I_{\mathrm{sat}}=\frac{\hbar \omega \gamma_{d}}{2 \sigma_{0}}$ with the scattering cross-section $\sigma_{0}=\frac{q^{2}}{m \varepsilon_{0} c^{2} \gamma_{d}}$.

[^2]
### 1.2.3 Laser Cooling - Optical Molasses

Take a Lorentz atom with velocity $\boldsymbol{v}$. What would be the effect of an external radiation?

We shall apply 2 monochromatic lasers on the atom from opposite directions,

$$
\begin{equation*}
\boldsymbol{F}_{\mathrm{rad}}=\hbar \boldsymbol{k} \gamma_{d}^{3}\left(\frac{1}{\Delta_{1}^{2}+\gamma_{d}^{2}}-\frac{1}{\Delta_{2}^{2}+\gamma_{d}^{2}}\right) \frac{I}{I_{\mathrm{sat}}} . \tag{1.52}
\end{equation*}
$$

Lasers \#1 and \#2 have the same frequency and intensity, but there is a different detuning due to the Doppler shift,

$$
\begin{equation*}
\Delta_{1}=\Delta-\boldsymbol{k} \cdot \boldsymbol{v}, \quad \Delta_{2}=\Delta+\boldsymbol{k} \cdot \boldsymbol{v} . \tag{1.53}
\end{equation*}
$$

Let us assume small velocity, $v \ll \frac{|\Delta|}{k}$. Expand with Taylor series to the first order,

$$
\begin{equation*}
\boldsymbol{F}_{\mathrm{rad}} \simeq \frac{\hbar k^{2} \gamma_{d}^{3}}{2} \cdot \frac{\Delta}{\left(\Delta^{2}+\gamma_{d}^{2}\right)^{2}} \cdot \frac{I}{I_{\mathrm{sat}}} \boldsymbol{v} \tag{1.54}
\end{equation*}
$$

Hence, $\boldsymbol{F}_{\text {rad }} \propto \boldsymbol{v}$.
Remark 1.3. The Doppler shift is responsible for the relaxation. Thus, the microscopic origin of damping is the Doppler shift (first pointed out by Einstein [1917]).

### 1.3 Blackbody Radiation

### 1.3.1 The Phenomenon

Consider any cavity within matter at equilibrium temperature $T$ full of radiation and radiates. The radiation is universal: it's independent of the material, shape, size of the cavity, etc. The light (radiation) depends on $T$. Note that one cannot focus the light within the cavity.

This universality is expressed by 3 laws:

1. Wien's displacement law:

$$
\begin{equation*}
u(\nu, T)=T^{3} f\left(\frac{\nu}{T}\right) \tag{1.55}
\end{equation*}
$$

where $u$ is the energy density. This is due to purely thermodynamic considerations.
2. As a consequence of the above,

$$
\begin{equation*}
\lambda_{\max } T=A . \tag{1.56}
\end{equation*}
$$

3. Stefan's law: the total energy radiated at fixed $T$ is given by

$$
\begin{equation*}
u(T)=\sigma T^{4} \tag{1.57}
\end{equation*}
$$

Here $A$ and $\sigma$ are universal constants.
For a long time there was a puzzle as there was no theoretical explanation for the blackbody curve (see Fig. 1.3). Planck has found a fit

$$
\begin{equation*}
u(\nu, T)=\frac{\alpha \nu^{3}}{\mathrm{e}^{\beta \nu / T}-1}, \tag{1.58}
\end{equation*}
$$

where $\alpha$ and $\beta$ are fitting constants.


Fig. 1.3: Blackbody radiation curves.

### 1.3.2 Electromagnetic Radiation - Basics

Let us be more qualitative. Take a cavity $L \times L \times L$ with an electromagnetic field $(\boldsymbol{E}, \boldsymbol{H})$. The energy density is given (in the wrong units) by

$$
\begin{equation*}
W=\frac{1}{8 \pi}\left(E^{2}+H^{2}\right) \tag{1.59}
\end{equation*}
$$

and the Poynting vector

$$
\begin{equation*}
\boldsymbol{S}=\frac{c}{4 \pi} \boldsymbol{E} \times \boldsymbol{H} \tag{1.60}
\end{equation*}
$$

The diffraction equation (Helmholtz) for $\boldsymbol{E}$ is given by

$$
\begin{equation*}
\nabla^{2} \boldsymbol{E}+\frac{\partial^{2} \boldsymbol{E}}{\partial t^{2}}=0 \tag{1.61}
\end{equation*}
$$

with the appropriate boundary conditions on the wall. We shall look for planewave solutions,

$$
\begin{equation*}
(\boldsymbol{E}, \boldsymbol{H})=\left(\boldsymbol{E}_{0}, \boldsymbol{H}_{0}\right) \times \mathrm{e}^{\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{x}-\omega t)} \tag{1.62}
\end{equation*}
$$

The dispersion is given by

$$
\begin{equation*}
\omega^{2}=c^{2} k^{2} \tag{1.63}
\end{equation*}
$$

with $\boldsymbol{k}$ quantized,

$$
\begin{equation*}
|\boldsymbol{k}|=\frac{2 \pi}{L}\left(n_{x}, n_{y}, n_{z}\right), n_{i} \in \mathbb{Z} \tag{1.64}
\end{equation*}
$$

Here $\boldsymbol{E}, \boldsymbol{H}$ and $\boldsymbol{k}$ are orthogonal.
In this case,

$$
\begin{equation*}
\boldsymbol{E} \times \boldsymbol{H}=\frac{1}{2}\left(E^{2}+H^{2}\right) \hat{n}, \tag{1.65}
\end{equation*}
$$

with $\boldsymbol{k}=k \hat{n}$. Therefore,

$$
\begin{equation*}
\boldsymbol{S}=c W \hat{n} . \tag{1.66}
\end{equation*}
$$

The total energy is given by

$$
\begin{equation*}
E=\int \mathrm{d} V W \tag{1.67}
\end{equation*}
$$

and the total momentum

$$
\begin{equation*}
\boldsymbol{P}=\frac{1}{c^{2}} \int \mathrm{~d} V \boldsymbol{S}=\frac{E}{c} \hat{n} . \tag{1.68}
\end{equation*}
$$

### 1.3.3 Definition of Blackbody Radiation ${ }^{5}$

The basic idea of the blackbody radiation is that one has a cavity in thermodynamic equilibrium (fixed temperature $T$ ). Inside this cavity, which is almost closed (for pedagogical reasons) there is a radiation. This radiation 1. cannot be focused; 2. is unpolarized; 3. is at thermal equilibrium. The distribution of the energy density of this radiation is given by the Planck distribution.


Fig. 1.4: Blackbody radiation box.
We shall now consider the radiation as a thermodynamic fluid. Although it is a strange description, we shall justify it later. We shall derive the equation of state and later - Planck's law.

### 1.3.4 Basics of Radiation

Let us consider plane wave solutions

$$
\left\{\begin{array}{l}
W=\frac{1}{8 \pi}\left(E^{2}+H^{2}\right)  \tag{1.69}\\
\boldsymbol{S}=\frac{c}{4 \pi} \boldsymbol{E} \times \boldsymbol{H}
\end{array}\right.
$$

where $\boldsymbol{E} \perp \boldsymbol{H}$ and $\boldsymbol{S}=S \hat{n}$. We shall also set

$$
\begin{equation*}
\omega^{2}=c^{2} k^{2} ; \quad \boldsymbol{k}=\frac{2 \pi}{L}\left(n_{x}, n_{y}, n_{z}\right), n_{i} \in \mathbb{Z} \tag{1.70}
\end{equation*}
$$

Last time we also saw that

$$
\begin{equation*}
\boldsymbol{E} \times \boldsymbol{H}=\frac{1}{2}\left(E^{2}+H^{2}\right) \hat{n} \tag{1.71}
\end{equation*}
$$

The total energy is given by

$$
\begin{equation*}
E=\int \mathrm{d} V W \tag{1.72}
\end{equation*}
$$

with the momentum

$$
\begin{equation*}
\boldsymbol{P}=\frac{1}{c^{2}} \int \mathrm{~d} V \boldsymbol{S}=\frac{E}{c} \hat{n} \tag{1.73}
\end{equation*}
$$

### 1.3.5 Poor Man's Derivation

Let us consider a unit of radiation with momentum $\boldsymbol{P}$ hitting a wall. The change of the momentum is given by

$$
\begin{equation*}
\Delta P=2 \frac{E}{c} \tag{1.74}
\end{equation*}
$$

[^3]The radiation power per unit surface is given by

$$
\begin{equation*}
I \equiv \frac{E}{S \cdot t} \Longrightarrow E=I \cdot t \cdot S \tag{1.75}
\end{equation*}
$$

with $t$ a unit of time. Combining both equations one has

$$
\begin{equation*}
\Delta P=\underbrace{F}_{\text {force }} \cdot t=\underbrace{p}_{\text {pressure }} \cdot S \cdot t=p \frac{E}{I}=2 \frac{E}{c} . \tag{1.76}
\end{equation*}
$$

Where for the last two equalities we used both previous equations. Hence,

$$
\begin{equation*}
p=2 \frac{I}{c} . \tag{1.77}
\end{equation*}
$$

Now, consider our unit of radiation hits the wall with angle $\theta$. One thus has

$$
\begin{equation*}
p=2 \frac{I}{c} \cos ^{2} \theta \Longrightarrow p=2 \frac{I}{c} \overline{\cos ^{2} \theta} \tag{1.78}
\end{equation*}
$$

and the average is taken over a closed cavity. Thus,

$$
\begin{equation*}
p=\frac{1}{3} \frac{I}{c}=\frac{1}{3} W=\frac{1}{3} \frac{E}{V} \tag{1.79}
\end{equation*}
$$

where the factor $\frac{1}{3}$ is actually the dimension $\frac{1}{d}$.

### 1.3.6 The General Derivation

The general derivation is given in Eric's Quantim3 course. We use the action formalism for the field

$$
\begin{equation*}
F_{\mu \nu} \rightarrow \text { action } S \tag{1.80}
\end{equation*}
$$

Noether's theorem ensures us that the energy momentum tensor $T_{\mu \nu}$ is traceless

$$
T_{\mu}^{\mu}=\operatorname{Tr}\left(\begin{array}{lll}
\stackrel{E}{V} & &  \tag{1.81}\\
& & \\
& & \\
& & \\
& & \\
&
\end{array}\right)=0
$$

so that Eq. (1.79) is retrieved. For a full derivation see Chap. 34 of Landau and Lifshitz, (Vol. 2) [1987].

### 1.3.7 Temperature

We know the equation for the ideal gas

$$
\begin{equation*}
p V=\frac{2}{3} U(T) \tag{1.82}
\end{equation*}
$$

How do we then introduce the temperature $T$ into Eq. (1.79). The solution is to use the equipartition theorem.

The equipartition theorem tells us that

$$
\begin{equation*}
E=\frac{1}{2} k_{\mathrm{B}} T \times(\# \text { degrees of freedom }) \tag{1.83}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
p=\frac{1}{3} \frac{E}{V} u(T), \tag{1.84}
\end{equation*}
$$

where $u(T)$ is the energy density. We also note that we cannot have $u(T, X)$ due to the Carnot cycle.
"Proof" of $\mathbf{p}=\frac{1}{3} \mathbf{u}(\mathbf{T})$. Let us consider a vector potential of wave in vacuum,

$$
\begin{equation*}
\Delta \boldsymbol{A}=\frac{1}{c^{2}} \frac{\partial^{2} \boldsymbol{A}}{\partial t^{2}} \tag{1.85}
\end{equation*}
$$

Add the Coulomb gauge,

$$
\begin{equation*}
\operatorname{div} \boldsymbol{A}=0 \tag{1.86}
\end{equation*}
$$

to have

$$
\begin{equation*}
\boldsymbol{H}=\operatorname{rot} \boldsymbol{A} ; \quad \boldsymbol{E}=-\frac{\partial \boldsymbol{A}}{\partial t} \tag{1.87}
\end{equation*}
$$

Now, $\boldsymbol{A}(\boldsymbol{x}, t)$ can be Fourier expanded (with periodic boundary conditions) so that

$$
\begin{equation*}
\boldsymbol{A}=\sum_{\boldsymbol{k}} \boldsymbol{a}(\boldsymbol{k}, t) \mathrm{e}^{\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}}+\boldsymbol{a}^{*}(\boldsymbol{k}, t) \mathrm{e}^{-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}} \tag{1.88}
\end{equation*}
$$

where we set the modes $\boldsymbol{a}(\boldsymbol{k}, t)$ to be orthonormal; also, $\boldsymbol{k} \cdot \boldsymbol{a}=0$. Plug this expansion into the wave equation to have

$$
\left\{\begin{array}{l}
\ddot{\boldsymbol{a}}(\boldsymbol{k}, t)+\omega^{2} \boldsymbol{a}(\boldsymbol{k}, t)=0  \tag{1.89}\\
\omega=c|\boldsymbol{k}|
\end{array}\right.
$$

Therefore, one has the energy

$$
\begin{equation*}
E=\int \mathrm{d}^{3} \boldsymbol{x} \frac{1}{8 \pi}\left(E^{2}+H^{2}\right)=\frac{V}{2 \pi} \sum_{\boldsymbol{k}}|\dot{\boldsymbol{a}}(\boldsymbol{k}, t)|^{2}+\omega^{2}|\boldsymbol{a}(\boldsymbol{k}, t)|^{2} \tag{1.90}
\end{equation*}
$$

In other words, the energy is written as a sum of harmonic oscillators.
Hence, the thermodynamic energy is

$$
\begin{equation*}
\langle E\rangle=\sum_{k} 2 \times \frac{1}{2} k_{\mathrm{B}} T \tag{1.91}
\end{equation*}
$$

In other words, the temperature is defined via the radiation modes. Also, the modes are quantized

$$
\begin{equation*}
2 \pi \nu=\omega=c|\boldsymbol{k}|=c \frac{\pi}{L} \sqrt{n_{x}^{2}+n_{y}^{2}+n_{z}^{2}} \tag{1.92}
\end{equation*}
$$

therefore,

$$
\begin{equation*}
\langle E\rangle=\int_{0}^{\infty} \mathrm{d} \nu g(\nu) k_{\mathrm{B}} \tag{1.93}
\end{equation*}
$$

where $g(\nu)$ is the density of modes (DOM). Now, in the continuum limit, $L \rightarrow$ $\infty$, one has

$$
\begin{align*}
\sum_{k}=\sum_{k_{x}=0}^{\infty} \sum_{k_{y}=0}^{\infty} \sum_{k_{z}=0}^{\infty} & =\left(\frac{L}{2 \pi}\right)^{3} \int_{-\infty}^{\infty} \mathrm{d} k_{x} \mathrm{~d} k_{y} \mathrm{~d} k_{z} \\
& =\frac{L^{3}}{8 \pi^{3}} \int \mathrm{~d}^{3} \boldsymbol{k}  \tag{1.94}\\
& =\frac{L^{3}}{8 \pi^{3}} \int 4 \pi k^{2} \mathrm{~d} k \\
& =\frac{L^{3}}{2 \pi^{2}} \int_{0}^{\infty}\left(\frac{\omega}{c}\right)^{2} \frac{\mathrm{~d} \omega}{c}=\frac{V}{2 \pi^{2}} \int_{0}^{\infty}(2 \pi)^{3}\left(\frac{\nu}{c}\right)^{2} \frac{\mathrm{~d} \nu}{c} .
\end{align*}
$$

Hence,

$$
\begin{equation*}
g(\nu)=2 \times \frac{4 \pi V}{c^{3}} \nu^{2} \tag{1.95}
\end{equation*}
$$

where the $2 \times$ factor comes from the polarization.
Finally, one has

$$
\begin{equation*}
E(T)=\int_{0}^{\infty} \mathrm{d} \nu g(\nu) k_{\mathrm{B}} \equiv \int \mathrm{~d} \nu E(\nu, T) . \tag{1.96}
\end{equation*}
$$

And we have defined the energy of the spectral energy by

$$
\begin{equation*}
u(\nu, T) \equiv \frac{E(\nu, T)}{V}=\frac{8 \pi \nu^{2}}{c^{3}} k_{\mathrm{B}} \tag{1.97}
\end{equation*}
$$

This is the Rayleigh-Jeans formula.


Fig. 1.5: The spectral energy.

### 1.3.8 Planck's Distribution

The usual derivation of Planck's spectrum via the quantization of the electromagnetic wave is plainly wrong. Planck himself was opposed to anything quantum at all. His formula was purely a phenomenological one. Although the quantum derivation is mathematically correct, it is physically wrong as it requires the quantization of the electromagnetic field. The blackbody radiation does not require it.

As we shall see shortly,

$$
\begin{equation*}
p V=\frac{1}{3} E(T), \tag{1.98}
\end{equation*}
$$

is the equation of state of the thermal blackbody radiation. We shall derive Planck's law from the thermodynamics alone.

Let us start with

$$
\begin{equation*}
E(T, V)=u(T) \tag{1.99}
\end{equation*}
$$

Now we shall continue with the Stefan-Boltzmann formula

$$
\begin{cases}\mathrm{d} E=-P \mathrm{~d} V+T \mathrm{~d} S & E(V, S)  \tag{1.100}\\ \mathrm{d} F=-S \mathrm{~d} T-P \mathrm{~d} V & F(T, V)=E-T S\end{cases}
$$

where $F$ is the free energy given by the Legendre transform of $E$. Using the relations

$$
\begin{equation*}
\left(\frac{\partial F}{\partial V}\right)_{T}=-P=\left(\frac{\partial E}{\partial V}\right)_{T}-T\left(\frac{\partial S}{\partial V}\right)_{T} \tag{1.101}
\end{equation*}
$$

and Maxwell

$$
\begin{equation*}
\left(\frac{\partial S}{\partial V}\right)_{T}=\left(\frac{\partial P}{\partial T}\right)_{V} \tag{1.102}
\end{equation*}
$$

one has

$$
\begin{equation*}
\left(\frac{\partial E}{\partial V}\right)_{T}=-P+T\left(\frac{\partial P}{\partial T}\right)_{V} \tag{1.103}
\end{equation*}
$$

Now, since

$$
\begin{equation*}
E=V u(T), \quad P=\frac{1}{3} u(T) \tag{1.104}
\end{equation*}
$$

we have

$$
\begin{equation*}
u(T)=-\frac{1}{3} u(T)+\frac{T}{3} \frac{\mathrm{~d} u(T)}{\mathrm{d} T} \tag{1.105}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
\frac{\mathrm{d} u(T)}{\mathrm{d} T}=4 \frac{u(T)}{T} \tag{1.106}
\end{equation*}
$$

so that

$$
\begin{equation*}
u(T)=\sigma T^{4} \tag{1.107}
\end{equation*}
$$

This is the Stefan-Boltzmann equation (1.57). Note that $\sigma$ is a universal constant; it does not depend on $T, V$, etc. This also implies that

$$
\begin{equation*}
P(T)=\frac{\sigma}{3} T^{4} \tag{1.108}
\end{equation*}
$$

Adiabatic Transformation. In this case, $\mathrm{d} S=0$. Thus,

$$
\begin{equation*}
\mathrm{d} E=-P \mathrm{~d} V \tag{1.109}
\end{equation*}
$$

Now, since

$$
\begin{equation*}
E=3 P V \tag{1.110}
\end{equation*}
$$

we have

$$
\begin{equation*}
\mathrm{d} E=3(P \mathrm{~d} V+V \mathrm{~d} P)=-P \mathrm{~d} V \tag{1.111}
\end{equation*}
$$

Thus,

$$
\begin{equation*}
4 \frac{\mathrm{~d} V}{V}+3 \frac{\mathrm{~d} P}{P}=0 \tag{1.112}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
P^{3} V^{4}=\text { const }, \tag{1.113}
\end{equation*}
$$

or

$$
\begin{equation*}
\left(T^{4}\right)^{3} V^{4}=\text { const } \Longrightarrow T^{3} V=\text { const. } \tag{1.114}
\end{equation*}
$$

### 1.3.9 Entropy

From the Maxwell equation, one has

$$
\begin{equation*}
\left(\frac{\partial S}{\partial V}\right)_{T}=\left(\frac{\partial P}{\partial T}\right)_{V} \tag{1.115}
\end{equation*}
$$

or

$$
\begin{equation*}
\left(\frac{\partial S}{\partial V}\right)_{T}=\frac{4}{3} \sigma T^{3} \tag{1.116}
\end{equation*}
$$

Thus,

$$
\begin{equation*}
S(V, T)=\frac{4}{3} \sigma V T^{3}+f(T) \tag{1.117}
\end{equation*}
$$

where $f \xrightarrow{T \rightarrow 0} 0$. Hence,

$$
\begin{equation*}
s(T)=\frac{S(V, T)}{V}=\frac{4}{3} \sigma T^{3} . \tag{1.118}
\end{equation*}
$$

### 1.3.10 Chemical Potential of Blackbody Radiation

Define the enthalpy by

$$
\begin{equation*}
H=E+P V \tag{1.119}
\end{equation*}
$$

One has then

$$
\begin{align*}
H=V \sigma T^{4}+\frac{1}{3} u V & =V \sigma T^{4}+\frac{1}{3} V \sigma T^{4}  \tag{1.120}\\
& =\frac{4}{3} V \sigma T^{4} .
\end{align*}
$$

Now, define the Gibbs potential by

$$
\begin{equation*}
G=H-T S=\frac{4}{3} V \sigma T^{4}-T \frac{4}{3} \sigma V T^{3}=0 \tag{1.121}
\end{equation*}
$$

but since from the Gibbs-Duhem equation,

$$
\begin{equation*}
G=N \mu=0, \tag{1.122}
\end{equation*}
$$

one derives

$$
\begin{equation*}
\mu=0 \tag{1.123}
\end{equation*}
$$

Corollary 1.2. The chemical potential of a radiation, particle or not, must be 0 . This is independent of its nature.

### 1.3.11 Scaling Forms

In the literature one encounters the Wien formula

$$
\begin{equation*}
u(\nu, T)=\nu^{3} f\left(\# \frac{\nu}{k_{\mathrm{B}} T}\right) . \tag{1.124}
\end{equation*}
$$

It means that $\nu$ and $T$ are not independent. Note that this scaling form does not infer Planck distribution.

The $k_{\mathrm{B}}$ in the last equation ${ }^{6}$ is due to unit consideration (from temperature to energy). The last needed factor \# has the units of action. Planck's constant $\hbar$ will come as a natural candidate.

[^4]Proof (Wien formula). Let us look on radiation in a box of size $L^{3}$. From the definitions, its energy is given by

$$
\begin{equation*}
u(T)=\int_{0}^{\infty} \mathrm{d} \nu u(\nu, T) \tag{1.125}
\end{equation*}
$$

The radiation frequency is given by

$$
\begin{equation*}
\omega=2 \pi \nu=c|\boldsymbol{k}|=\frac{c \pi}{L} \sqrt{n_{x}^{2}+n_{y}^{2}+n_{z}^{2}} \tag{1.126}
\end{equation*}
$$

Hence, we can write a scaling law

$$
\begin{equation*}
\nu V^{\frac{1}{3}}=\frac{c}{2} \sqrt{\cdots} \tag{1.127}
\end{equation*}
$$

Notice that this scaling law is independent of the shape of the box; it may be even a potato

$$
\begin{equation*}
\nu V^{\frac{1}{3}}=\text { const. } \tag{1.128}
\end{equation*}
$$

This generalization to a general shape is given by the Weyl expansion (given in his doctorate Weyl [1908]).

On the other hand, last time we have seen another quantity which is constant in an adiabatic transformation,

$$
\begin{equation*}
T^{3} V=\mathrm{const} \Longleftrightarrow T V^{\frac{1}{3}}=\mathrm{const} . \tag{1.129}
\end{equation*}
$$

Thus, we can immediately write,

$$
\begin{equation*}
\frac{\nu}{T}=\text { const. } \tag{1.130}
\end{equation*}
$$

QED.

### 1.3.12 Stefan \& Boltzmann

This constant we have found does not depend on any thermodynamic quantity (thus, "constant"). We can thus phrase the Stefan-Boltzmann law as

$$
\begin{equation*}
u(T)=\sigma T^{4} \tag{1.131}
\end{equation*}
$$

Rephrasing it a bit, one has

$$
\begin{equation*}
\frac{u(T)}{T^{4}}=\sigma=\frac{u\left(T^{\prime}\right)}{T^{\prime 4}} \tag{1.132}
\end{equation*}
$$

Therefore, the definition of $\sigma$ does not depend on the local definition of temperature.

Integrating one has

$$
\begin{equation*}
\frac{1}{T^{4}} \int \mathrm{~d} \nu u(\nu, T)=\frac{1}{T^{\prime 4}} \int \mathrm{~d} \nu^{\prime} u\left(\nu^{\prime}, T^{\prime}\right) \tag{1.133}
\end{equation*}
$$

or

$$
\begin{equation*}
\frac{1}{T^{4}} u(\nu, T) \Delta \nu=\frac{1}{T^{4}} u\left(\nu^{\prime}, T^{\prime}\right) \Delta \nu^{\prime} \tag{1.134}
\end{equation*}
$$

Hence, one obtains

$$
\begin{equation*}
u(\nu, T) \Delta \nu=\left(\frac{T}{T^{\prime}}\right)^{4} u\left(\nu^{\prime}, T^{\prime}\right) \Delta \nu^{\prime} \tag{1.135}
\end{equation*}
$$

Performing the adiabatic transformation we have found earlier, $\frac{\nu}{T}=$ const, one has

$$
\begin{equation*}
\frac{\nu}{T}=\frac{\nu^{\prime}}{T^{\prime}} \Longrightarrow \Delta \nu=\frac{T}{T^{\prime}} \Delta \nu^{\prime} \tag{1.136}
\end{equation*}
$$

Substituting back one has

$$
\begin{equation*}
u(\nu, T)=\left(\frac{T}{T^{\prime}}\right)^{3} u\left(\frac{T^{\prime}}{T} \nu^{\prime}, T^{\prime}\right) \tag{1.137}
\end{equation*}
$$

for all $T$. In particular, for $T^{\prime}=1$. Thus,

$$
\begin{equation*}
u(\nu, T)=T^{3} \underbrace{u\left(\frac{\nu}{T}, 1\right)}_{f\left(\frac{\nu}{T}\right)}, \tag{1.138}
\end{equation*}
$$

hence,

$$
\begin{equation*}
u(\nu, T)=\nu^{3} f\left(\frac{\nu}{T}\right) \tag{1.139}
\end{equation*}
$$

### 1.4 Einstein Model - ( $A, B$ ) Coefficients

All that we did before is to introduce temperature. We have assumed that the radiation is at thermal equilibrium and developed the Planck distribution.

The model we will present now, given by Einstein [1917], will give us the Planck distribution without quantum mechanics. Additionally, it will give us the same results as the Lorentz model. Finally, it will give us amplification of light - lasers.

### 1.4.1 Planck Distribution - Light Amplification

We will shortly prove the following.

- Assuming that matter (atoms) is at thermal equilibrium, the energy density $u(\nu, T)$ has a Planck distribution. This comes from the FluctuationDissipation theorem.


## Assumptions.

1. Gas of atoms: states are $z_{1}, z_{2}, \ldots, z_{n}$ with energies $\epsilon_{1}, \epsilon_{2}, \ldots, \epsilon_{n}$ with statistical weights $p_{1}, p_{2}, \ldots, p_{n}$.
2. The relative occurrence of a state is given by the Boltzmann factor

$$
\begin{equation*}
w_{n}=p_{n} \mathrm{e}^{-\epsilon_{n} / k_{\mathrm{B}} T} \tag{1.140}
\end{equation*}
$$

3. Consider the simplest case of 2 states $z_{a}$ and $z_{b}$ where $\epsilon_{a}<\epsilon_{b}$. There are 3 possible processes.
(a) Absorption. The probability per unit time (rate) is defined by

$$
\begin{equation*}
\mathrm{d} w_{\uparrow} \equiv u(\nu, T) B_{a b} \mathrm{~d} t \tag{1.141a}
\end{equation*}
$$

where the subs $B_{a b}$ mean $a$ to $b$.
(b) Stimulated Emission.

$$
\begin{equation*}
\mathrm{d} w_{\downarrow} \equiv u(\nu, T) B_{b a} \mathrm{~d} t \tag{1.141b}
\end{equation*}
$$

(c) Spontaneous Emission.

$$
\begin{equation*}
\mathrm{d} w_{s}=A_{b a} \mathrm{~d} t \tag{1.141c}
\end{equation*}
$$

The sub $s$ is for "stationary". Note that

$$
\begin{equation*}
A_{b a} \equiv A=\gamma_{d} \tag{1.142}
\end{equation*}
$$

### 1.4.2 Solving the Model

Let us take $N$ atoms of two states $n_{a}=\frac{N_{a}}{N}$ and $n_{b}=\frac{N_{b}}{N}$ such that $N_{a}+N_{b}=N$ (the number of atoms is fixed). Let us write the two equations of evolution for the ground $a$ and excited $b$ states,

$$
\left\{\begin{array}{l}
\frac{\mathrm{d} n_{a}}{\mathrm{~d} t}=p_{\downarrow} n_{b}-p_{\uparrow} n_{a}  \tag{1.143}\\
\frac{\mathrm{~d} n_{b}}{\mathrm{~d} t}=-p_{\downarrow} n_{b}+p_{\uparrow} n_{a},
\end{array}\right.
$$

where

$$
\left\{\begin{array}{l}
p_{\downarrow}=\frac{\mathrm{d} w_{\downarrow}}{\mathrm{d} t}+\frac{\mathrm{d} w_{s}}{\mathrm{~d} t}=B_{b a} u+A  \tag{1.144}\\
p_{\uparrow}=\frac{\mathrm{d} w_{\uparrow}}{\mathrm{d} t}=u B_{a b} .
\end{array}\right.
$$

For a stationary state,

$$
\begin{equation*}
\frac{\mathrm{d} n_{a}}{\mathrm{~d} t}=\frac{\mathrm{d} n_{b}}{\mathrm{~d} t}=0 \tag{1.145}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
p_{\uparrow} n_{a}=p_{\downarrow} n_{b} \tag{1.146}
\end{equation*}
$$

This is the Detailed balance condition.
Remark 1.4. Detailed balance is a condition for a stationary state and not for thermal equilibrium, as usually erroneously thought.

Let's continue. At thermal equilibrium

$$
\left\{\begin{array}{l}
w_{a}=n_{a}=p_{a} \mathrm{e}^{-\epsilon_{a} / k_{\mathrm{B}} T}  \tag{1.147}\\
w_{b}=n_{b}=p_{b} \mathrm{e}^{-\epsilon_{b} / k_{\mathrm{B}} T} .
\end{array}\right.
$$

Combine both to have

$$
\begin{equation*}
p_{a} \mathrm{e}^{-\epsilon_{a} / k_{\mathrm{B}} T} B_{a b} u=p_{b} \mathrm{e}^{-\epsilon_{b} / k_{\mathrm{B}} T}\left(A+B_{b a} u\right) . \tag{1.148}
\end{equation*}
$$

Now, consider the $T \rightarrow \infty$ limit. Here $u \rightarrow \infty$ too. Thus, Eq. (1.148) becomes

$$
\begin{equation*}
p_{a} B_{a b}=p_{b} B_{b a} . \tag{1.149}
\end{equation*}
$$

Combine both to have

$$
\begin{equation*}
p_{b} B_{b a} \mathrm{e}^{-\epsilon_{a} / k_{\mathrm{B}} T} u=p_{b} \mathrm{e}^{-\epsilon_{b} / k_{\mathrm{B}} T}\left(A+B_{b a} u\right) . \tag{1.150}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
u(\nu, T)=\frac{A / B_{b a}}{\mathrm{e}^{\left(\epsilon_{b}-\epsilon_{a}\right) / k_{\mathrm{B}} T}-1} . \tag{1.151}
\end{equation*}
$$

In other words, $u$ has the Planck's distribution.
Now, combine with Wien's formula

$$
\begin{equation*}
u(\nu, T)=\alpha \nu^{3} f\left(\frac{\nu}{T}\right) \tag{1.152}
\end{equation*}
$$

to immediately obtain

$$
\begin{equation*}
\epsilon_{b}-\epsilon_{a}=h \nu \text {. } \tag{1.153}
\end{equation*}
$$

Corollary 1.3. The factor $h$ with the units of action presents natural quantization to the problem, where we have only considered thermodynamic effects.

Continuing further, one has

$$
\begin{equation*}
\frac{A}{B_{b a}}=\alpha \nu^{3} . \tag{1.154}
\end{equation*}
$$

In the Rayleigh-Jeans limit

$$
\begin{equation*}
u(\nu, T)=\frac{8 \pi \nu^{2}}{c^{3}} k_{\mathrm{B}} T \tag{1.155}
\end{equation*}
$$

for $h \nu \ll k_{\mathrm{B}} T$ one has

$$
\begin{equation*}
u(\nu, T) \sim \frac{A}{B_{b a}} \frac{1}{\frac{h \nu}{k_{\mathrm{B}} T}} . \tag{1.156}
\end{equation*}
$$

Thus,

$$
\begin{equation*}
\frac{A}{B_{b a}} \frac{k_{\mathrm{B}} T}{h \nu}=\frac{8 \pi \nu^{2}}{c^{3}} k_{\mathrm{B}} T \tag{1.157}
\end{equation*}
$$

and one converges to

$$
\begin{equation*}
\frac{A}{B_{b a}}=\frac{8 \pi h \nu^{3}}{c^{3}} \tag{1.158}
\end{equation*}
$$

Remark 1.5. If you kill the relaxation-the spontaneous emission factor $A$-one obtains back the Wien formula $u \sim \nu^{3} \mathrm{e}^{-\nu / T}$.

### 1.4.3 Einstein Equations for Other Types of Radiation

This Einstein model is useful not only for describing the statistics of light, but also some light-matter interactions. We shall soon see how we get lasers from it.

We saw that there is a radiation distribution in the system. But since there are atoms, and the radiation moves the atoms, the velocity of the atoms has also got some distribution. One can ask what is this distribution, and see that it is the same Planck distribution. However, one can also ask the opposite question: given a Planck distribution of the atom velocity, what must be the radiation
energy distribution. The answer is the same. These two questions, given by a remark by Einstein in his 1917 paper have given rise to the idea of laser cooling.

Let us start with the basics. Define the energy density by

$$
\begin{equation*}
W=\frac{1}{2} \varepsilon_{0} E^{2} . \tag{1.159}
\end{equation*}
$$

Now, define the flux of photons (rate) $\phi$ by

$$
\begin{equation*}
\frac{P}{h \nu}=\frac{P}{\hbar \omega}=R_{\mathrm{sc}}=\phi . \tag{1.160}
\end{equation*}
$$

Also

$$
\left\{\begin{array}{l}
P=S c W=\phi h \nu  \tag{1.161}\\
W=\frac{P}{S c}=\frac{h}{\lambda} \frac{\phi}{S}
\end{array}\right.
$$

Instead of $u(\nu, T)$ we consider

$$
\begin{equation*}
\rho(\nu) \equiv W \varphi(\nu) \tag{1.162}
\end{equation*}
$$

where $\varphi(\nu)$ is the spectral function (Fig. 1.6), and it is normalized such that

$$
\begin{equation*}
\int_{0}^{\infty} \varphi(\nu) \mathrm{d} \nu=1 \tag{1.163}
\end{equation*}
$$

Note it is not necessarily thermal.


Fig. 1.6: Spectral function.

Now, combining all of our definitions, one has

$$
\begin{equation*}
-\frac{1}{n_{a}} \frac{\mathrm{~d} n_{a}}{\mathrm{~d} t}=p_{\uparrow}=B_{a b} W \varphi(\nu)=B_{a b} \varphi(\nu) \frac{h}{\lambda} \frac{\phi}{S} . \tag{1.164}
\end{equation*}
$$

Consideration of the units gives us

1. Absorption cross-section,

$$
\begin{equation*}
\sigma_{a b}(\nu)=B_{a b} \varphi(\nu) \frac{h}{\lambda} . \tag{1.165a}
\end{equation*}
$$

2. Emission cross-section

$$
\begin{equation*}
\sigma_{b a}(\nu)=B_{b a} \varphi(\nu) \frac{h}{\lambda} \tag{1.165b}
\end{equation*}
$$

$\mathrm{We}^{7}$ can therefore write the rate equation

$$
\begin{align*}
\frac{\mathrm{d} n_{a}}{\mathrm{~d} t} & =\left(A+B_{b a} \rho\right) n_{b}-B_{a b} \rho n_{a} \\
& =A n_{b}+\rho \underbrace{\left(B_{b a} n_{b}-B_{a b} n_{a}\right)}_{\equiv B_{b a} D} \tag{1.166}
\end{align*}
$$

where $D$ is the weighted statistical population,

$$
\begin{equation*}
D \equiv n_{b}-\frac{B_{a b}}{B_{b a}} n_{a}=n_{b}-\frac{p_{b}}{p_{a}} n_{a} \tag{1.167}
\end{equation*}
$$

Here we use the identities

$$
\begin{equation*}
\frac{B_{a b}}{B_{b a}}=\frac{p_{b}}{p_{a}}, \tag{1.168}
\end{equation*}
$$

such that $n=n_{a}+n_{b}=$ const.
Now, we can write a set of coupled equations,

$$
\left\{\begin{array}{l}
n_{b}\left(1+\frac{B_{a b}}{B_{b a}}\right)=D+n \frac{B_{a b}}{B_{b a}}  \tag{1.169}\\
n_{a}\left(1+\frac{B_{a b}}{B_{b a}}\right)=n-D .
\end{array}\right.
$$

Insert these relations into Einstein equations for $\frac{\mathrm{d} n_{a}}{\mathrm{~d} t}, \frac{\mathrm{~d} n_{b}}{\mathrm{~d} t}$ to obtain

$$
\begin{equation*}
\frac{\mathrm{d} D}{\mathrm{~d} t}=-\left[A+\left(B_{a b}+B_{b a}\right) \varphi(\nu) W\right] D-A \frac{B_{a b}}{B_{b a}} n \tag{1.170}
\end{equation*}
$$

Note that what secures the saturation of $D$ is the $A$ factor. Hence, one can write the definition for the saturation,

$$
\begin{equation*}
\frac{\mathrm{d} D}{\mathrm{~d} t}=-\frac{1}{T_{\mathrm{ir}}}\left(D-D_{s}\right) \tag{1.171}
\end{equation*}
$$

This defines $T_{\mathrm{ir}}$ and $D_{s}$. Hence, one has

$$
\begin{equation*}
D(t)=D_{s}+\left(D(0)-D_{s}\right) \mathrm{e}^{-\frac{t}{T_{\mathrm{ir}}}} \tag{1.172}
\end{equation*}
$$

with

$$
\begin{equation*}
D(0)=-\frac{B_{a b}}{B_{b a}} n \tag{1.173}
\end{equation*}
$$

(see Figure 1.7). At large $t$

$$
\begin{equation*}
D(t) \rightarrow D_{s}(W, \nu)=-n \cdot \frac{B_{a b}}{B_{b a}} \cdot \frac{A}{A+\left(B_{a b}+B_{b a}\right) \varphi(\nu) W} \tag{1.174}
\end{equation*}
$$

[^5]

Fig. 1.7: Weighted statistical population.

### 1.4.4 Saturation at large applied light sources

Let us now plot $D_{s}$ versus $W$. We see that for large values of $W>W_{\text {sat }},\left|D_{s}\right|$ saturates. Define $W_{\text {sat }}$ by

$$
\begin{equation*}
D_{s}\left(W_{\mathrm{sat}}\right)=\frac{1}{2} D(0)=-\frac{1}{2} n \frac{p_{b}}{p_{a}} . \tag{1.175}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
W_{\text {sat }}=\frac{A}{\left(B_{a b}+B_{b a}\right) \varphi(\nu)} \Longrightarrow \underbrace{B_{a b} \varphi(\nu) W_{\text {sat }}}_{\text {absorption }}+\underbrace{B_{b a} \varphi(\nu) W_{\text {sat }}}_{\text {emission }}=\underbrace{A}_{\substack{\text { spon. } \\ \text { emission }}} . \tag{1.176}
\end{equation*}
$$

Finally, we have explicit expression as

$$
\left\{\begin{array}{l}
D_{s}(W)=D_{s}(0) \frac{1}{1+\frac{W}{W_{\mathrm{sat}}}}  \tag{1.177}\\
\frac{1}{T_{\mathrm{ir}}}=A\left(1+\frac{W}{W_{\mathrm{sat}}}\right) .
\end{array}\right.
$$

These equations we have, are general and classical.


Fig. 1.8: Saturated weighted statistical population.

## 2 Interaction of Classical Radiation with Quantum Matter ${ }^{8}$

Or, the semi-classical theory of atom-radiation interaction.

### 2.1 Interaction Hamiltonian - Gauge Invariance of Quantum Mechanics

Or, how to obtain the Maxwell equations from Quantum Mechanics.

### 2.1.1 The Hamiltonian

We shall start with the Hamiltonian and develop the theory accordingly. Let us start from the Schrödinger equation

$$
\begin{equation*}
\mathrm{i} \hbar \frac{\partial}{\partial t} \psi(\boldsymbol{r}, t)=\left(\frac{\boldsymbol{p}^{2}}{2 m}+V(\boldsymbol{r}, t)\right) \psi(\boldsymbol{r}, t), \tag{2.1}
\end{equation*}
$$

with $\boldsymbol{p}=\frac{\hbar}{\mathrm{i}} \boldsymbol{\nabla}$. We have a total invariance to the phase of the wave-function

$$
\begin{equation*}
\psi(\boldsymbol{r}, t) \rightarrow \psi^{\prime}(\boldsymbol{r}, t) \equiv \mathrm{e}^{\mathrm{i} \omega} \psi(\boldsymbol{r}, t) . \tag{2.2}
\end{equation*}
$$

This is the global phase transformation. In other words, for any operator $\hat{A}$

$$
\begin{equation*}
\left\langle\psi^{\prime}\right| \hat{A}\left|\psi^{\prime}\right\rangle=\langle\psi| \hat{A}|\psi\rangle . \tag{2.3}
\end{equation*}
$$

A local gauge invariance reads

$$
\begin{equation*}
\psi(\boldsymbol{r}, t) \xrightarrow{\text { G. T. }} \psi^{\prime}(\boldsymbol{r}, t) \equiv \mathrm{e}^{\mathrm{i} \frac{q}{\hbar} \omega(\boldsymbol{r}, t)} \psi(\boldsymbol{r}, t) . \tag{2.4}
\end{equation*}
$$

Put back into the Schrödinger equation to have

$$
\begin{align*}
\mathrm{i} \hbar \frac{\partial}{\partial t} \psi(\boldsymbol{r}, t) & =\mathrm{i} \hbar\left(-\mathrm{i} \frac{q}{\hbar} \frac{\partial \omega}{\partial t} \mathrm{e}^{-\mathrm{i} \frac{q \omega}{\hbar}} \psi^{\prime}+\mathrm{e}^{-\mathrm{i} \frac{q \omega}{\hbar}} \frac{\partial \psi^{\prime}}{\partial t}\right) \\
& =\mathrm{e}^{-\mathrm{i} \frac{q \omega}{\hbar}}\left(\mathrm{i} \hbar \frac{\partial}{\partial t}+q \frac{\partial \omega}{\partial t}\right) \psi^{\prime} . \tag{2.5}
\end{align*}
$$

On the right-hand side one has

$$
\begin{align*}
\frac{\hbar}{\mathrm{i}} \boldsymbol{\nabla} \psi & =\frac{\hbar}{\mathrm{i}}\left(\mathrm{e}^{-\mathrm{i} \frac{q \omega}{\hbar}} \boldsymbol{\nabla} \psi^{\prime}-\mathrm{i} \frac{q}{\hbar}(\boldsymbol{\nabla} \omega) \psi^{\prime} \mathrm{e}^{-\mathrm{i} \frac{q \omega}{\hbar}}\right) \\
& =\mathrm{e}^{-\mathrm{i} \frac{q \omega}{\hbar}}(\boldsymbol{p}-q \boldsymbol{\nabla} \omega) \psi^{\prime} . \tag{2.6}
\end{align*}
$$

Therefore,

$$
\begin{equation*}
\mathrm{i} \hbar \frac{\partial}{\partial t} \psi^{\prime}(\boldsymbol{r}, t)=\left(\frac{1}{2 m}(\boldsymbol{p}-q \boldsymbol{\nabla} \omega)^{2}+V(\boldsymbol{r}, t)-q \frac{\partial \omega}{\partial t}\right) \psi^{\prime}(\boldsymbol{r}, t) . \tag{2.7}
\end{equation*}
$$

Hence, the Schrödinger equation is not locally invariant to $U(1)$ transformation.

[^6]
### 2.1.2 Covariant Derivative

In order to obtain back the symmetry, we must introduce an additional auxiliary field to the system. Let us define the covariant derivative by

$$
\begin{equation*}
D_{t}, \boldsymbol{D}=\left(D_{x}, D_{y}, D_{z}\right) \tag{2.8}
\end{equation*}
$$

such that

$$
\left\{\begin{array}{l}
D_{t} \psi \xrightarrow{\text { G. T. }} D_{t}^{\prime} \psi^{\prime}=\mathrm{e}^{\mathrm{i} \frac{q \omega}{\hbar}} D_{t} \psi  \tag{2.9}\\
\boldsymbol{D} \psi \xrightarrow{\text { G. T. }} \boldsymbol{D}^{\prime} \psi^{\prime}=\mathrm{e}^{\mathrm{i} \frac{q \omega}{\hbar}} \boldsymbol{D} \psi
\end{array}\right.
$$

If we could find such a transformation, we are done.
Let us multiply the Schrödinger equation by $\mathrm{e}^{\mathrm{i} \frac{q \omega}{\hbar}}$

$$
\begin{equation*}
\mathrm{e}^{\mathrm{i} \frac{q \omega}{\hbar}} \partial_{t} \psi=\left(\partial_{t}-\frac{\mathrm{i} q}{\hbar}\left(\partial_{t} \omega\right)\right) \psi^{\prime} \tag{2.10}
\end{equation*}
$$

Introduce a function $\phi(\boldsymbol{r}, t)$ such that

$$
\begin{equation*}
\frac{\mathrm{i} q}{\hbar} \phi \psi^{\prime}=\phi \mathrm{e}^{\mathrm{i} \frac{q \omega}{\hbar}} \psi \frac{\mathrm{i} q}{\hbar} \tag{2.11}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
\left(\partial_{t}+\mathrm{i} \frac{q}{\hbar}\left(\phi-\partial_{t} \omega\right)\right) \psi^{\prime}=\mathrm{e}^{\mathrm{i} \frac{q \omega}{\hbar}}\left(\partial_{t}+\mathrm{i} \frac{q}{\hbar} \phi\right) \psi \tag{2.12}
\end{equation*}
$$

Hence, we have defined the covariant derivative

$$
\begin{equation*}
D_{t} \equiv\left(\partial_{t}+\mathrm{i} \frac{q}{\hbar} \phi(\boldsymbol{r}, t)\right) \tag{2.13}
\end{equation*}
$$

It is a covariant derivative provided that

$$
\begin{equation*}
\phi(\boldsymbol{r}, t) \xrightarrow{\text { G. T. }} \phi^{\prime}(\boldsymbol{r}, t)=\phi(\boldsymbol{r}, t)-\partial_{t} \omega \text {. } \tag{2.14}
\end{equation*}
$$

Similarly, we will calculate the spatial parts

$$
\begin{equation*}
\left(\boldsymbol{\nabla}-\mathrm{i} \frac{q}{\hbar} \nabla \omega\right) \psi^{\prime}=\mathrm{e}^{\mathrm{i} \frac{q \omega}{\hbar}} \boldsymbol{\nabla} \psi \tag{2.15}
\end{equation*}
$$

Introduce a vector potential $\boldsymbol{A}(\boldsymbol{r}, t)$ to obtain

$$
\begin{equation*}
\boldsymbol{D}=\boldsymbol{\nabla}-\mathrm{i} \frac{q}{\hbar} \boldsymbol{A}(\boldsymbol{r}, t) \tag{2.16}
\end{equation*}
$$

and $\boldsymbol{A}(\boldsymbol{r}, t)$ transforms as

$$
\begin{equation*}
\boldsymbol{A}(\boldsymbol{r}, t) \xrightarrow{\text { G. т. }} \boldsymbol{A}^{\prime}(\boldsymbol{r}, t)=\boldsymbol{A}(\boldsymbol{r}, t)+\boldsymbol{\nabla} \omega \text {. } \tag{2.17}
\end{equation*}
$$

Combine both to have

$$
\begin{equation*}
\mathrm{i} \hbar D_{t} \psi=\frac{1}{2 m}\left(\frac{\hbar}{\mathrm{i}} \boldsymbol{D}\right)^{2} \psi \xrightarrow{\text { G. T. }} \mathrm{i} \hbar D_{t}^{\prime} \psi^{\prime}=\frac{1}{2 m}\left(\frac{\hbar}{\mathrm{i}} \boldsymbol{D}^{\prime}\right)^{2} \psi^{\prime} \tag{2.18}
\end{equation*}
$$

The price we had to pay to restore $U(1)$ symmetry is the two gauge fields, $\phi$ and $\boldsymbol{A}$.

### 2.1.3 Invariant Quantities

Now, is it possible to combine both transformations to be independent (invariant) of $\omega$. Since

$$
\left\{\begin{array}{l}
\phi^{\prime}=\phi-\partial_{t} \omega  \tag{2.19}\\
A_{i}^{\prime}=A_{i}+\partial_{i} \omega
\end{array}\right.
$$

one has the coupled equations

$$
\left\{\begin{array}{l}
\partial_{t} \phi^{\prime}=\partial_{t} \phi-\partial_{t}^{2} \omega  \tag{2.20}\\
\partial_{i} \phi^{\prime}=\partial_{i} \phi-\partial_{i} \partial_{t} \omega \\
\partial_{t} A_{i}^{\prime}=\partial_{t} A_{i}+\partial_{t} \partial_{i} \omega \\
\partial_{j} A_{i}^{\prime}=\partial_{i} A_{i}+\partial_{j} \partial_{i} \omega
\end{array}\right.
$$

Combine (b) and (c) to have the invariant

$$
\begin{equation*}
\partial_{i} \phi^{\prime}+\partial_{t} A_{i}^{\prime}=\partial_{i} \phi+\partial_{t} A_{i} \tag{2.21}
\end{equation*}
$$

which has 3 independent quantities. Similarly, from (d) one has

$$
\begin{equation*}
\partial_{j} A_{i}^{\prime}-\partial_{j} A_{i}=\partial_{j} \partial_{i} \omega=\partial_{i} \partial_{j} \omega=\partial_{i} A_{j}^{\prime}-\partial_{i} A_{j} \tag{2.22}
\end{equation*}
$$

which also has 3 independent quantities.
Therefore, there exist

$$
\begin{equation*}
\boldsymbol{E} \equiv-\boldsymbol{\nabla} \phi-\partial_{t} \boldsymbol{A} \tag{2.23}
\end{equation*}
$$

and

$$
\begin{equation*}
B \equiv \nabla \times A \tag{2.24}
\end{equation*}
$$

Remark 2.1. Similarly, we could write the other two non-homogeneous Maxwell equation, but we need to have sources. It will just be more cumbersome, but not harder.
Remark 2.2. We could also add spins. This would give us the gyro-magnetic factor +2 for Pauli equations for a spin $\frac{1}{2}$ particle.

### 2.1.4 Invariant Physical Quantities

There are several physical quantities which remain invariant under the gauge transformation.

Density

$$
\begin{equation*}
\rho(\boldsymbol{r}, t)=|\psi(\boldsymbol{r}, t)|^{2}=\rho^{\prime}(\boldsymbol{r}, t) . \tag{2.25}
\end{equation*}
$$

Position

$$
\begin{equation*}
\left\langle\psi^{\prime}\right| \boldsymbol{R}\left|\psi^{\prime}\right\rangle=\langle\psi| \boldsymbol{R}|\psi\rangle . \tag{2.26}
\end{equation*}
$$

Momentum (p) - not a physical quantity:

$$
\begin{align*}
\left\langle\psi^{\prime}\right| \boldsymbol{p}\left|\psi^{\prime}\right\rangle & =\int \mathrm{d}^{3} \boldsymbol{r} \mathrm{e}^{-\mathrm{i} \frac{q \omega}{\hbar}} \psi^{*}\left(\frac{\hbar}{\mathrm{i}} \boldsymbol{\nabla}\left(\mathrm{e}^{\mathrm{i} \frac{q \omega}{\hbar}} \psi\right)\right) \\
& =\langle\psi| \boldsymbol{p}|\psi\rangle+q\langle\psi| \boldsymbol{\nabla} \omega|\psi\rangle \tag{2.27}
\end{align*}
$$

Velocity

$$
\begin{align*}
\boldsymbol{v}=\frac{1}{m}(\boldsymbol{p}-q \boldsymbol{A}) \xrightarrow{\text { G. T. }} \boldsymbol{v}^{\prime} & =\frac{1}{m}\left(\boldsymbol{p}^{\prime}-q \boldsymbol{A}^{\prime}\right) \\
& =\frac{1}{m}\left(\boldsymbol{p}+q \boldsymbol{\nabla} \omega-q \boldsymbol{A}^{\prime}\right) \\
& =\frac{1}{m}(\boldsymbol{p}-q \boldsymbol{A}) . \tag{2.28}
\end{align*}
$$

Remark 2.3. The quantization of the electric charge + magnetic monopole comes from the gauge invariance. We shall not prove it here; the avid reader is referred to Sakurai and Napolitano, (p. 130) [2013].

### 2.2 Dipolar Interaction

### 2.2.1 Coulomb Gauge

We shall start with the Coulomb gauge

$$
\begin{equation*}
\operatorname{div} \boldsymbol{A}=0 \tag{2.29}
\end{equation*}
$$

The Hamiltonian reads

$$
\begin{equation*}
\hat{H}_{\hat{\boldsymbol{p}} \cdot \boldsymbol{A}_{\perp}} \equiv \frac{1}{2 m}\left(\hat{\boldsymbol{p}}-q \boldsymbol{A}_{\perp}\right)^{2}+V(\boldsymbol{r}) \tag{2.30}
\end{equation*}
$$

where $\boldsymbol{A}_{\perp}$ is the transverse external radiation, and $V(\boldsymbol{r})$ is the electrostatic potential. In this gauge, the electric and magnetic fields read

$$
\left\{\begin{array}{l}
\boldsymbol{E}=-\frac{\partial \boldsymbol{A}_{\perp}}{\partial t}  \tag{2.31}\\
\boldsymbol{B}=\operatorname{rot} \boldsymbol{A}_{\perp}
\end{array}\right.
$$

Define the atomic Hamiltonian as

$$
\begin{equation*}
\hat{H}_{\mathrm{at}}=\frac{1}{2 m} \hat{\boldsymbol{p}}^{2}+V(\boldsymbol{r}) \tag{2.32}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
\hat{H}=\hat{H}_{\mathrm{at}}-\frac{q}{m} \boldsymbol{p} \cdot \boldsymbol{A}_{\perp}+\frac{q^{2}}{2 m} \boldsymbol{A}_{\perp}^{2} \tag{2.33}
\end{equation*}
$$

We used the fact that $\boldsymbol{A}_{\perp}$ and $\boldsymbol{p}$ are commuting

$$
\begin{equation*}
\left[\hat{p}_{j}, \hat{A}_{\perp j}\right]=\frac{\hbar}{\mathrm{i}} \partial_{j} A_{\perp j} \Longrightarrow \boldsymbol{p} \cdot \boldsymbol{A}_{\perp}-\boldsymbol{A}_{\perp} \cdot \boldsymbol{p}=\frac{\hbar}{\mathrm{i}} \operatorname{div} \boldsymbol{A}_{\perp}=0 . \tag{2.34}
\end{equation*}
$$

### 2.2.2 Dipolar Approximation ${ }^{9}$

We shall identify the field $\boldsymbol{A}_{\perp}$ with a laser field with wavelength $\lambda$. Now, if $\lambda \gg\langle\hat{\boldsymbol{r}}\rangle_{\hat{H}_{\mathrm{at}}}$, one can replace $\boldsymbol{A}_{\perp}(\boldsymbol{r}, t)$ with $\boldsymbol{A}_{\perp}(t) \simeq \boldsymbol{A}_{\perp}(\boldsymbol{r}=\mathbf{0}, t)$, where $\mathbf{0}$ is the position of the atom. In other words, we neglect the dependence on the position. This approximation works well with microwave radiation, for example.

Next, we define the equivalence between $\boldsymbol{p} \cdot \boldsymbol{A}$ and $\boldsymbol{d} \cdot \boldsymbol{E}$ Hamiltonians. We shall look for a unitary transformation $\hat{T}(t)$ that gets rid of the time-dependent only $\boldsymbol{A}_{\perp}(t)$. Explicitly,

$$
\begin{equation*}
\hat{T}(t) \hat{\boldsymbol{p}} \hat{T}^{\dagger}(t)=\hat{\boldsymbol{p}}+q \boldsymbol{A}_{\perp}(t) \tag{2.35}
\end{equation*}
$$

[^7]It is easy to find that

$$
\begin{equation*}
\hat{T}(t)=\mathrm{e}^{-\mathrm{i} \frac{q}{\hbar} \hat{r} \cdot \boldsymbol{A}_{\perp}(t)} \tag{2.36}
\end{equation*}
$$

Now we calculate the transformed Hamiltonian. The state transforms as

$$
\begin{equation*}
|\tilde{\psi}(t)\rangle=\hat{T}(t)|\psi(t)\rangle \tag{2.37}
\end{equation*}
$$

Also,

$$
\begin{equation*}
\mathrm{i} \hbar \partial_{t}|\psi\rangle=H(t)|\psi(t)\rangle \Longrightarrow \mathrm{i} \hbar \partial_{t}|\tilde{\psi}\rangle=\tilde{H}(t)|\tilde{\psi}(t)\rangle \tag{2.38}
\end{equation*}
$$

The Hamiltonian transforms as

$$
\begin{equation*}
\tilde{H}(t)=\hat{T}(t) H_{\boldsymbol{p} \cdot \boldsymbol{A}}(t) \hat{T}^{\dagger}(t)+\mathrm{i} \hbar \frac{\partial \hat{T}(t)}{\partial t} \hat{T}^{\dagger}(t) \tag{2.39}
\end{equation*}
$$

Note the second term: it always comes with time-dependent operators such as $\hat{T}(t)$. Hence,

$$
\begin{equation*}
\tilde{H}(t)=\underbrace{\left(\frac{1}{2 m} \hat{\boldsymbol{p}}^{2}+V(\boldsymbol{r})\right)}_{\hat{H}_{\mathrm{at}}}+\mathrm{i} \hbar\left(-\frac{\mathrm{i}}{\hbar} q \boldsymbol{r} \cdot \frac{\mathrm{~d} \boldsymbol{A}_{\perp}(t)}{\mathrm{d} t}\right) \tag{2.40}
\end{equation*}
$$

We therefore have

$$
\begin{equation*}
\tilde{H}(t)=\hat{H}_{\mathrm{at}}+q \boldsymbol{r} \cdot \frac{\mathrm{~d} \boldsymbol{A}_{\perp}(t)}{\mathrm{d} t} \tag{2.41}
\end{equation*}
$$

or

$$
\begin{equation*}
\tilde{H}_{\boldsymbol{d} \cdot \boldsymbol{E}}(t)=\hat{H}_{\mathrm{at}}+\boldsymbol{d} \cdot \boldsymbol{E}(t) \tag{2.42}
\end{equation*}
$$

For additional information, read Chapter 4 of Cohen-Tannoudji, Dupont-Roc, and Grynberg [1989].

### 2.3 Effect of a Classical EM Field on a Quantum Atom

We shall talk about (a) Rabi oscillations, (b) Feynman \& Vennon description, (c) Bloch sphere, (d) Lindblad equation, (e) Ramsey interference.

### 2.3.1 Approximations and Assumptions

- Assume a monochromatic field $\omega$ with $\lambda \gg\langle\hat{\boldsymbol{r}}\rangle$,

$$
\begin{equation*}
\boldsymbol{E}(t)=\boldsymbol{E}_{0} \cos (\omega t-\boldsymbol{k} \cdot \boldsymbol{r}) \tag{2.43}
\end{equation*}
$$

- A two-level atom: well justified. The levels are degenerate (never justified). Here the detuning is given by

$$
\begin{equation*}
\Delta=\omega-\omega_{0} \tag{2.44}
\end{equation*}
$$

The Hamiltonian is

$$
\begin{equation*}
H_{\mathrm{at}}=\hbar \omega_{0}|e\rangle\langle e| \tag{2.45}
\end{equation*}
$$

Or,

$$
\begin{equation*}
H_{\mathrm{at}}|g\rangle=0 ; \quad H_{\mathrm{at}}|e\rangle=\hbar \omega_{0}|e\rangle \tag{2.46}
\end{equation*}
$$



Fig. 2.1: Two level atom description.

- The atom-field Hamiltonian is given by

$$
\begin{equation*}
H_{\mathrm{AF}}=-\hat{\boldsymbol{d}} \cdot \boldsymbol{E} . \tag{2.47}
\end{equation*}
$$

Theorem 2.1. The atomic eigenstates of $H_{a t}$ always have a well-defined parity.

- We can thus conclude

$$
\begin{equation*}
\underset{\substack{\text { even } \\ \text { odd }}}{ }(\boldsymbol{r}) \rightarrow \hat{\boldsymbol{d}} \psi_{\text {edd }}^{\text {even }}(\boldsymbol{r}) . \tag{2.48}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
\langle i| \boldsymbol{d}|i\rangle=0 \Longleftrightarrow \int \mathrm{~d}^{3} \boldsymbol{r} f(\boldsymbol{r})=0 \tag{2.49}
\end{equation*}
$$

- Since

$$
\begin{equation*}
\mathbb{1}=|e\rangle\langle e|+|g\rangle\langle g|, \tag{2.50}
\end{equation*}
$$

one has

$$
\begin{equation*}
\hat{\boldsymbol{d}}=\mathbb{1} \hat{\boldsymbol{d}} \mathbb{1}=\ldots=\langle g| \hat{\boldsymbol{d}}|e\rangle\left(\hat{\sigma}+\hat{\sigma}^{\dagger}\right), \tag{2.51}
\end{equation*}
$$

where

$$
\begin{equation*}
\hat{\sigma} \equiv|g\rangle\langle e|, \quad \hat{\sigma}^{\dagger} \equiv|e\rangle\langle g| . \tag{2.52}
\end{equation*}
$$

Remark 2.4. Since

$$
\begin{equation*}
\langle g| \hat{\boldsymbol{d}}|e\rangle=\langle e| \hat{\boldsymbol{d}}|g\rangle^{*} \tag{2.53}
\end{equation*}
$$

one can make a choice of phase such that

$$
\begin{equation*}
\langle g| \hat{\boldsymbol{d}}|e\rangle=\langle e| \hat{\boldsymbol{d}}|g\rangle \in \mathbb{R}_{+} \tag{2.54}
\end{equation*}
$$

are real $>0$.

### 2.3.2 The Hamiltonian

Combining everything we have just defined, one writes the Hamiltonian

$$
\begin{equation*}
\hat{H}_{\mathrm{AF}}=-\langle g| \hat{\boldsymbol{d}} \cdot \hat{\boldsymbol{\epsilon}}|e\rangle \frac{E_{0}}{2}\left(\mathrm{e}^{\mathrm{i} \omega t}+\mathrm{e}^{-\mathrm{i} \omega t}\right)\left(\hat{\sigma}+\hat{\sigma}^{\dagger}\right) \tag{2.55}
\end{equation*}
$$

where

$$
\begin{equation*}
\Omega_{R} \equiv \frac{1}{\hbar}\langle g| \hat{\boldsymbol{d}} \cdot \hat{\boldsymbol{\epsilon}}|e\rangle \frac{E_{0}}{2} \tag{2.56}
\end{equation*}
$$

is the Rabi frequency.

Now, let us solve it. Let

$$
\begin{equation*}
|\psi(t)\rangle=c_{g}(t)|g\rangle+c_{e}|e\rangle \tag{2.57}
\end{equation*}
$$

Put the Hamiltonian to have

$$
\begin{equation*}
\mathrm{i} \hbar \frac{\mathrm{~d}}{\mathrm{~d} t}|\psi(t)\rangle=\left(\hat{H}_{\mathrm{at}}+\hat{H}_{\mathrm{AF}}\right)|\psi(t)\rangle \tag{2.58}
\end{equation*}
$$

- For $\hat{H}_{\mathrm{AF}}=0$ one has

$$
\begin{equation*}
c_{e}(t)=\mathrm{e}^{-\mathrm{i} \omega_{0} t} \tag{2.59}
\end{equation*}
$$

- For $\hat{H}_{\mathrm{AF}} \neq 0$ one has the coupled equations

$$
\left\{\begin{array}{l}
\dot{c}_{g}=\mathrm{i} \frac{\Omega_{R}}{2} c_{e} \mathrm{e}^{\mathrm{i} \omega t}\left(1+\mathrm{e}^{-2 \mathrm{i} \omega t}\right)  \tag{2.60}\\
\dot{c}_{e}=-\mathrm{i} \omega_{0} c_{e}+\mathrm{i} \frac{\Omega_{R}}{2} c_{g} \mathrm{e}^{-\mathrm{i} \omega t}\left(1+\mathrm{e}^{+2 \mathrm{i} \omega t}\right)
\end{array}\right.
$$

The last term in the first equation corresponds to going to the excited state while emitting a photon; the last term in the second equation corresponds to absorbing a photon and going to the ground state. We would like to get rid of them.


Fig. 2.2: Rotating wave $2 \omega$ terms. (a) The $\mathrm{e}^{-2 \mathrm{i} \omega t}$ term; (b) the $\mathrm{e}^{+2 \mathrm{i} \omega t}$ term.

### 2.3.3 Rotating Wave Approximation

We would like to solve the just-presented paradox by omitting these two problematic terms. This is not an innocuous approximation. What we finally obtain

$$
\begin{equation*}
\hat{H}_{\mathrm{AF}}^{\mathrm{RWA}}=\frac{\hbar \Omega_{R}}{2}\left(\mathrm{e}^{\mathrm{i} \omega t} \hat{\sigma}+\mathrm{e}^{-\mathrm{i} \omega t} \hat{\sigma}^{\dagger}\right) \tag{2.61}
\end{equation*}
$$

Now, quantum mechanics is the art of unitary transformations. We shall make the following:

$$
\begin{equation*}
\alpha(t)=c_{g}(t), \quad \beta(t)=c_{e}(t) \mathrm{e}^{\mathrm{i} \omega t} \tag{2.62}
\end{equation*}
$$

such that $|\alpha|^{2}+|\beta|^{2}=1$. The Schrödinger equation becomes

$$
\left\{\begin{array}{l}
\partial_{t} \alpha=\mathrm{i} \frac{\Omega_{R}}{2} \beta  \tag{2.63}\\
\partial_{t} \beta=\mathrm{i} \frac{\Omega_{R}}{2} \alpha+\mathrm{i} \Delta \beta
\end{array}\right.
$$

Let us suppose we do not make the rotating wave approximation (RWA). The initial $\hat{H}_{\mathrm{AF}}$ Hamiltonian reads

$$
\begin{equation*}
\partial_{t} \beta-\mathrm{i} \Delta \beta=\mathrm{i} \frac{\Omega_{R}}{2} \alpha(t)\left(1+\mathrm{e}^{+2 \mathrm{i} \omega t}\right) \tag{2.64}
\end{equation*}
$$

The other equation looks even worse. Let

$$
\begin{equation*}
\beta(t)=C(t) \mathrm{e}^{\mathrm{i} \Delta t} \tag{2.65}
\end{equation*}
$$

so that

$$
\begin{equation*}
\partial_{t} C(t)=\mathrm{i} \frac{\Omega_{R}}{2} \alpha(t)\left(1+\mathrm{e}^{\mathrm{i}(2 \omega-\Delta) t}\right) \tag{2.66}
\end{equation*}
$$

For $t \simeq 0$ one has

$$
\begin{align*}
C(t) & \simeq \mathrm{i} \frac{\Omega_{R}}{2} \alpha(0) \int_{0}^{t} \mathrm{~d} t^{\prime}\left(\mathrm{e}^{-\mathrm{i} \Delta t^{\prime}}+\mathrm{e}^{\mathrm{i}(2 \omega-\Delta) t^{\prime}}\right) \\
& =\mathrm{i} \frac{\Omega_{R}}{2} \alpha(0)\left(\frac{\mathrm{e}^{-\mathrm{i} \Delta t}-1}{-\mathrm{i} \Delta}+\frac{\mathrm{e}^{\mathrm{i}(2 \omega-\Delta) t}-1}{\mathrm{i}(2 \omega-\Delta)}\right) \tag{2.67}
\end{align*}
$$

The second term comes from not neglecting RWA. However, the ratio of the amplitudes $\propto \frac{\Delta}{2 \omega} \sim 10^{-4}-10^{-6}$. Hence, we can safely neglect them and use RWA.

Returning to the RWA. Let us solve the equation for $\beta$,

$$
\begin{equation*}
\frac{\mathrm{d}^{2} \beta}{\mathrm{~d} t^{2}}-\mathrm{i} \Delta \frac{\mathrm{~d} \beta}{\mathrm{~d} t}+\frac{\Omega_{R}^{2}}{4} \beta=0 \tag{2.68}
\end{equation*}
$$

Now, let

$$
\left\{\begin{array}{l}
X_{ \pm}=\mathrm{i} \frac{\Delta}{2} \pm \mathrm{i} \frac{\Omega}{2}  \tag{2.69}\\
\Omega=\sqrt{\Omega_{R}^{2}+\Delta^{2}}
\end{array}\right.
$$

and obtain

$$
\begin{equation*}
\beta(t)=\mathrm{e}^{\mathrm{i} \frac{\Delta t}{2}}\left(A \mathrm{e}^{\mathrm{i} \frac{\Omega t}{2}}+B \mathrm{e}^{-\mathrm{i} \frac{\Omega t}{2}}\right) . \tag{2.70}
\end{equation*}
$$

Introducing the initial conditions,

$$
\left\{\begin{array}{l}
\alpha(0)=1  \tag{2.71}\\
\beta(0)=0=A+B
\end{array}\right.
$$

one has

$$
\begin{equation*}
\beta(t)=\mathrm{i} \frac{\Omega_{R}}{\Omega} \mathrm{e}^{\mathrm{i} \frac{\Delta t}{2}} \sin \left(\frac{\Omega t}{2}\right) \tag{2.72}
\end{equation*}
$$

The probability to get from the ground to excited state reads

$$
\begin{equation*}
P_{g \rightarrow e}(t)=|\beta(t)|^{2}=\frac{\Omega_{R}^{2}}{\Omega_{R}^{2}+\Delta^{2}} \cdot \frac{1-\cos \Omega t}{2} \tag{2.73}
\end{equation*}
$$

See Fig. 2.3.

### 2.3.4 Comparison to Einstein Model

Here $\Delta=0$.

- For $t=0 \rightarrow \frac{\pi}{\Omega_{R}}$ atom $|g\rangle \rightarrow|e\rangle$ absorbing $\hbar \omega$.
- For $t=\frac{\pi}{\Omega_{R}} \rightarrow \frac{2 \pi}{\Omega_{R}}$ atom $|e\rangle \rightarrow|g\rangle$ emitting $\hbar \omega$.


Fig. 2.3: Rabi oscillations. In solid blue line $P_{g \rightarrow e}(t)=|\beta(t)|^{2}$; in dashed red line $P_{e \rightarrow g}(t)=|\alpha(t)|^{2}$. (a) Zero detuning, $\Delta=0, \Omega=\Omega_{R}$. (b) Nonzero detuning, $\Delta>0$. Note that the oscillation frequency in (b) is higher than in (a).

One difference is that there is no spontaneous emission in RWA. For $\Omega_{R} \rightarrow 0$ the atom stays in the excited state forever.

There is another, more subtle, difference. In the Einstein model,

$$
\begin{equation*}
-\frac{1}{n_{a}} \frac{\mathrm{~d} n_{a}}{\mathrm{~d} t}=\text { const. } \tag{2.74}
\end{equation*}
$$

However, in RWA

$$
\begin{align*}
-\frac{1}{n_{a}} \frac{\mathrm{~d} n_{a}}{\mathrm{~d} t} & =-\frac{1}{|\alpha(t)|^{2}} \frac{\mathrm{~d}|\alpha(t)|^{2}}{\mathrm{~d} t} \\
& =-\frac{\mathrm{d}}{\mathrm{~d} t} \ln \left(1-\left(\frac{\Omega_{R}}{\Omega}\right)^{2} \sin ^{2} \frac{\Omega t}{2}\right) \\
& \stackrel{=\Omega_{R}}{=}-\Omega_{R} \tan \left(\frac{\Omega_{R} t}{2}\right) . \tag{2.75}
\end{align*}
$$

This is strongly dependent on time. Hence, unlike Einstein model, this model is quantum. The Einstein model, is actually, the Fermi golden rule description of this problem (large $t$ limit), where we sum over all the quantum degrees of freedom. Hence the time independence.

### 2.4 Dressed State Picture ${ }^{10}$

We shall now rephrase all what we have said, but emphasize the coherences. These are the non-diagonal parts of the density matrix. They are purely of quantum nature. Note that we shall use the atomic language (NMR, and such).

[^8]
### 2.4.1 The Dressed State Picture

Let us write the Schrödinger equation of two, excited and ground, states,

$$
\partial_{t}\binom{\tilde{c}_{e}}{c_{g}}=-\mathrm{i}\left(\begin{array}{cc}
-\Delta & \frac{\Omega_{R}}{2}  \tag{2.76}\\
\frac{\Omega_{R}}{2} & 0
\end{array}\right)\binom{\tilde{c}_{e}}{c_{g}} \equiv-\frac{1}{\hbar} \tilde{H}\binom{\tilde{c}_{e}}{c_{g}} .
$$

Here $\tilde{c}_{e}=\mathrm{e}^{\mathrm{i} \omega t} c_{e}$ is the rotated state. Diagonalize $\tilde{H}$ to obtain

$$
\begin{equation*}
E_{ \pm}=\frac{-\hbar \Delta}{2} \pm \frac{\hbar \Omega}{2} \tag{2.77}
\end{equation*}
$$

with $\Omega \equiv \sqrt{\Delta^{2}+\Omega_{R}^{2}}$. The associated eigenstates read
where

$$
\begin{equation*}
\tan 2 \theta=-\frac{\Omega_{R}}{\Delta} . \tag{2.79}
\end{equation*}
$$

This angle $\theta$ is called the Stückelberg angle.
For $\Omega_{R}$ there is a level repulsion that removes degeneracy. This is the AC Stark shift. See Figure 2.4.


Fig. 2.4: The dressed state picture.

### 2.4.2 Rabi Oscillations in the Dressed State Picture

Let us consider two states with $\Delta=0$,

Suppose that we start with an atom in the ground state $|g\rangle$. The time evolution is given by

Note that in the derivation of the third row we have neglected the overall phase of the wavefunction.

At time $\Omega_{R} t=\pi$ one has

$$
\begin{equation*}
\left|\psi\left(t=\frac{\pi}{\Omega_{R}}\right)\right\rangle=|+\rangle+\mathrm{e}^{\mathrm{i} \pi}|-\rangle=|+\rangle-|-\rangle=|e\rangle . \tag{2.82}
\end{equation*}
$$

and at time $\Omega_{R} t=2 \pi$ one has

$$
\begin{equation*}
\left|\psi\left(t=\frac{2 \pi}{\Omega_{R}}\right)\right\rangle=|+\rangle+\mathrm{e}^{2 \mathrm{i} \pi}|-\rangle=|g\rangle . \tag{2.83}
\end{equation*}
$$

### 2.4.3 The Bloch Sphere

This description emphasized the role of coherences. Start with the basis $|e\rangle,|g\rangle$. Define the lowering operator by

$$
\left\{\begin{align*}
\hat{\sigma}|e\rangle & =|g\rangle  \tag{2.84}\\
\hat{\sigma}|g\rangle & =0
\end{align*}\right.
$$

Explicitly,

$$
\hat{\sigma}=\left(\begin{array}{ll}
0 & 1  \tag{2.85}\\
0 & 0
\end{array}\right) .
$$

We can now define the Pauli matrices

$$
\begin{align*}
\hat{\sigma}+\hat{\sigma}^{\dagger} & =\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right)=\sigma_{x} \\
-\mathrm{i}\left(\hat{\sigma}-\hat{\sigma}^{\dagger}\right) & =\left(\begin{array}{cc}
0 & -\mathrm{i} \\
\mathrm{i} & 0
\end{array}\right)=\sigma_{y}  \tag{2.86}\\
|e\rangle\langle e|-|g\rangle\langle g| & =\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right)=\sigma_{z} .
\end{align*}
$$

Also,

$$
\begin{equation*}
\sigma_{z}=\hat{\sigma}^{\dagger} \hat{\sigma}-\hat{\sigma} \hat{\sigma}^{\dagger}=\left[\hat{\sigma}^{\dagger}, \hat{\sigma}\right] . \tag{2.87}
\end{equation*}
$$

The Pauli matrices obey the algebra

$$
\left\{\begin{array}{l}
{\left[\sigma_{\alpha}, \sigma_{\beta}\right]=2 \mathrm{i} \varepsilon_{\alpha \beta \gamma} \sigma_{\gamma}}  \tag{2.88}\\
\left\{\sigma_{\alpha}, \sigma_{\beta}\right\}=2 \delta_{\alpha \beta} .
\end{array}\right.
$$

The density matrix $\hat{\rho}$ is defined by

$$
\hat{\rho}=|\psi(t)\rangle\langle\psi(t)|=\binom{\tilde{c}_{e}}{c_{g}}\left(\begin{array}{cc}
\tilde{c}_{e} & c_{g}
\end{array}\right)=\left(\begin{array}{ll}
\left|c_{e}\right|^{2} & \tilde{c}_{e}^{*} c_{g}  \tag{2.89}\\
\tilde{c}_{e} c_{g}^{*} & \left|c_{g}\right|^{2}
\end{array}\right) .
$$

The diagonal quantities $\left|c_{e}\right|^{2}$ and $\left|c_{g}\right|^{2}$ are the populations; they are classical and sum to 1 . The non-diagonal quantities $\tilde{c}_{e} c_{g}^{*}$ and $\tilde{c}_{e}^{*} c_{g}$ are called the coherences; they are purely quantum.

We can relate the four matrix elements of the density matrix to the expectation values of the operators we have defined previously

$$
\begin{align*}
\langle\hat{\sigma}\rangle & =\operatorname{Tr}(|g\rangle\langle e| \hat{\rho})=\tilde{\rho}_{e g}  \tag{2.90a}\\
\left\langle\hat{\sigma}^{\dagger}\right\rangle & =\operatorname{Tr}(|e\rangle\langle g| \hat{\rho})=\tilde{\rho}_{g e}  \tag{2.90b}\\
\left\langle\hat{\sigma}^{\dagger} \hat{\sigma}\right\rangle & =\operatorname{Tr}(|e\rangle\langle e| \hat{\rho})=\rho_{e e}  \tag{2.90c}\\
\left\langle\hat{\sigma} \hat{\sigma}^{\dagger}\right\rangle & =\operatorname{Tr}(|g\rangle\langle g| \hat{\rho})=\rho_{g g} \tag{2.90~d}
\end{align*}
$$

where we used the notations

$$
\begin{equation*}
\tilde{\rho}_{\alpha \beta}=\tilde{c}_{\alpha} \tilde{c}_{\beta}^{*}, \quad \tilde{c}_{g}=c_{g}, \quad \tilde{c}_{e}=\mathrm{e}^{\mathrm{i} \omega t} c_{e} \tag{2.91}
\end{equation*}
$$

### 2.4.4 Time Evolution of the Density Matrix

Let us look on the time evolution of $\tilde{\rho}_{\alpha \beta}$ :

$$
\begin{align*}
\partial_{t} \tilde{\rho}_{e e}=\partial_{t}\left(\tilde{c}_{e} \tilde{c}_{e}^{*}\right) & =\tilde{c}_{e}^{*} \partial_{t} \tilde{c}_{e}+\text { c.c. } \\
& =\mathrm{i} \Delta \tilde{c}_{e} \tilde{c}_{e}^{*}-\mathrm{i} \frac{\Omega_{R}}{2} c_{g} \tilde{c}_{e}^{*}+\text { c.c. }  \tag{2.92}\\
& =\mathrm{i} \frac{\Omega_{R}}{2}\left(\tilde{\rho}_{e g}-\tilde{\rho}_{g e}\right)
\end{align*}
$$

where in the second line we used the Schrödinger equation. Similarly,

$$
\begin{align*}
& \partial_{t} \tilde{\rho}_{g g}=-\partial_{t} \tilde{\rho}_{e e}  \tag{2.93a}\\
& \partial_{t} \tilde{\rho}_{g e}=-\mathrm{i} \Delta \tilde{\rho}_{g e}-\mathrm{i} \frac{\Omega_{R}}{2}\left(\tilde{\rho}_{e e}-\tilde{\rho}_{g g}\right)=\partial_{t} \tilde{\rho}_{e g} \tag{2.93b}
\end{align*}
$$

The time evolution of $\hat{\rho}$ is also given by the Schrödinger-von Neumann equation,

$$
\begin{equation*}
\partial_{t} \tilde{\rho}=-\frac{\mathrm{i}}{\hbar}[\tilde{H}, \tilde{\rho}] . \tag{2.94}
\end{equation*}
$$

### 2.4.5 Physical Interpretation of the Density Matrix Solutions

Let us inspect $\hat{\rho}$. There are $4 \times 2=8$ real parameters, but we have several constraints:

- The populations are reals (-2 parameters);
- $\rho_{e e}+\rho_{g g}=1$ (-1 parameter);
- $\tilde{\rho}_{e g}=\tilde{\rho}_{g e}^{*}$ ( -2 parameters).

Therefore, we have altogether $8-5=3$ independent quantities. This motivates using a 3 -vector in $\mathbb{R}^{3}$ to represent the atomic state.

We shall use the following 3 basis vectors

$$
\begin{align*}
\left\langle\sigma_{x}\right\rangle & =\langle\hat{\sigma}\rangle+\left\langle\hat{\sigma}^{\dagger}\right\rangle & & =\tilde{\rho}_{e g}+\tilde{\rho}_{g e}  \tag{2.95a}\\
\left\langle\sigma_{y}\right\rangle & =-\mathrm{i}\left(\langle\hat{\sigma}\rangle-\left\langle\hat{\sigma}^{\dagger}\right\rangle\right) & & =\mathrm{i}\left(\tilde{\rho}_{e g}-\tilde{\rho}_{g e}\right)  \tag{2.95b}\\
\left\langle\sigma_{z}\right\rangle & =|e\rangle\langle e|-|g\rangle\langle g| & & =\rho_{e e}-\rho_{g g} . \tag{2.95c}
\end{align*}
$$

Rearranging one has the following time evolution,

$$
\begin{cases}\partial_{t}\left\langle\sigma_{x}\right\rangle & =\Delta\left\langle\sigma_{y}\right\rangle  \tag{2.96}\\ \partial_{t}\left\langle\sigma_{y}\right\rangle & =\Delta\left\langle\sigma_{x}\right\rangle-\Omega_{R}\left\langle\sigma_{z}\right\rangle \\ \partial_{t}\left\langle\sigma_{z}\right\rangle & =\Omega_{R}\left\langle\sigma_{y}\right\rangle\end{cases}
$$

These equations, also called the Optical Bloch Equations are completely equivalent to the Schrödinger equation and to the dressed state picture we have shown earlier.


Fig. 2.5: The Bloch sphere.

### 2.4.6 Bloch Vector

Let us define the vector

$$
\begin{equation*}
\langle\boldsymbol{\sigma}\rangle \equiv\left\langle\sigma_{x}\right\rangle \hat{e}_{x}+\left\langle\sigma_{y}\right\rangle \hat{e}_{y}+\left\langle\sigma_{z}\right\rangle \hat{e}_{z} \tag{2.97}
\end{equation*}
$$

Its evolution is given, as we saw, by

$$
\begin{equation*}
\partial_{t}\langle\boldsymbol{\sigma}\rangle=-\Delta \hat{e}_{z} \times\langle\boldsymbol{\sigma}\rangle+\Omega_{R} \hat{e}_{x} \times\langle\boldsymbol{\sigma}\rangle \tag{2.98}
\end{equation*}
$$

Define a precession vector by

$$
\begin{equation*}
\mathcal{P}=\Omega_{R} \hat{e}_{x}-\Delta \hat{e}_{z} \tag{2.99}
\end{equation*}
$$

to have the concise

$$
\begin{equation*}
\partial_{t}\langle\boldsymbol{\sigma}\rangle=\boldsymbol{P} \times\langle\boldsymbol{\sigma}\rangle \tag{2.100}
\end{equation*}
$$

This is analogous to the Larmor precession of a magnetic moment in a magnetic field

$$
\begin{equation*}
\boldsymbol{\tau}=\partial_{t} \boldsymbol{L}=\boldsymbol{\Omega} \times \boldsymbol{L} \tag{2.101}
\end{equation*}
$$

This is the Feynman, Vernon, and Hellwarth [1957] representation.
Remark 2.5. This entire description is for a spin- $\frac{1}{2}$ particle. Hence, we have two states, ground and excited.
Remark 2.6. The length of the Bloch vector is constant (for a pure state),

$$
\begin{align*}
|\langle\boldsymbol{\sigma}\rangle|^{2} & =\left\langle\sigma_{x}\right\rangle^{2}+\left\langle\sigma_{y}\right\rangle^{2}+\left\langle\sigma_{z}\right\rangle^{2} \\
& =\left(\tilde{\rho}_{e g}+\tilde{\rho}_{g e}\right)^{2}-\left(\tilde{\rho}_{e g}-\tilde{\rho}_{g e}\right)^{2}+\left(\rho_{e e}-\rho_{g g}\right)^{2} \\
& =4 \tilde{\rho}_{e g} \tilde{\rho}_{g e}+\rho_{e e}-\rho_{g g}-2 \rho_{e e} \rho_{g g} \\
& =\left(\rho_{e e}+\rho_{g g}\right)^{2}=1 . \tag{2.102}
\end{align*}
$$

The last equality is because $\tilde{\rho}_{e g} \tilde{\rho}_{g e}=\rho_{e e} \rho_{g g}$ in a pure state $(\operatorname{det} \hat{\rho}=0)$.
Corollary 2.1. Dynamics of the Bloch vector are depicted on a sphere of a fixed radius 1 .


Fig. 2.6: Rabi oscillations in the language of the Bloch sphere. Here we used $\Delta=0$ and $\mathcal{P}=\Omega_{R} \hat{e}_{x}$

### 2.4.7 Transverse Components $(x, y)$ of the Bloch Vector

Let us recall the dipole moment

$$
\begin{equation*}
\hat{\boldsymbol{d}}=\langle g| \hat{\boldsymbol{d}}|e\rangle\left(\hat{\sigma}+\hat{\sigma}^{\dagger}\right) . \tag{2.103}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
\langle\hat{\boldsymbol{d}}\rangle=\langle g| \hat{\boldsymbol{d}}|e\rangle\left\langle\sigma_{x}\right\rangle . \tag{2.104}
\end{equation*}
$$

### 2.4.8 Discussion ${ }^{11}$

We thus have 3 equivalent descriptions:

1. Schrödinger equation.
2. Density matrix $\tilde{\rho}=\left(\begin{array}{c}\rho_{e e} \\ \tilde{\rho}_{e g} \\ \tilde{\rho}_{e g} \\ \rho_{g g}\end{array}\right)$. Here the density matrix transforms as $\partial_{t} \tilde{\rho}=-\frac{\mathrm{i}}{\hbar}\left[\tilde{H}_{\mathrm{A}}+\tilde{H}_{\mathrm{AF}}, \tilde{\rho}\right]$ with $\tilde{\rho}_{e e}=\rho_{e e}$ and $\tilde{\rho}_{e g}=\rho_{e g} \mathrm{e}^{\mathrm{i} \omega t}$.
3. Bloch equations. The vector $\langle\boldsymbol{\sigma}\rangle=\left(\left\langle\sigma_{x}\right\rangle,\left\langle\sigma_{y}\right\rangle,\left\langle\sigma_{z}\right\rangle\right)$ with the definitions as in Eqs. (2.95). The evolution is given by $\partial_{t}\langle\boldsymbol{\sigma}\rangle=\boldsymbol{\mathcal { P }} \times\langle\boldsymbol{\sigma}\rangle$, and a pure state is given by $|\langle\boldsymbol{\sigma}\rangle|^{2}=1$.

### 2.4.9 Atomic Time Bookkeeping - Ramsey Interference

Remark 2.7. Ramsey has received a Nobel prize for it.
We saw that the Rabi oscillations gives is the energy level difference $\omega_{0}$. However, it is difficult to measure it accurately (at least in Rabi's time). Now, how can we do it? The idea is to use an interferometry similarly to the Young experiment, but in time rather than in space.

Consider the following picture (Fig. 2.7). Let $\tau$ be the time an atom spends inside the cavity, with the distance between cavities given by $L=v T$.

[^9]

Fig. 2.7: Ramsey Experiment. The atoms flow from right to left. There is no evolution outside of the cavity.

## Assumptions:

- Assume to be close to resonance, $\Delta \ll \Omega_{R}$ (or, $\Omega=\sqrt{\Omega_{R}^{2}+\Delta^{2}} \simeq \Omega_{R}$
- The time inside the cavity $\tau$ is at resonance: $\Omega_{R} \tau=\frac{\pi}{2}$ (a $\frac{\pi}{2}$-pulse).

Hence, in the dressed-state picture, starting with

$$
\begin{equation*}
\left|\psi_{0}(0)\right\rangle=|g\rangle=|+\rangle+|-\rangle, \tag{2.105}
\end{equation*}
$$

we have a phase accumulation inside the first cavity of

$$
\begin{equation*}
\left|\psi_{1}(\tau)\right\rangle=|+\rangle+\mathrm{e}^{\mathrm{i} \Omega_{R} \tau}|-\rangle=(1+\mathrm{i})|g\rangle+(1-\mathrm{i})|e\rangle . \tag{2.106}
\end{equation*}
$$

Between the cavities, there is no field $(\Omega=0)$. Thus, the atoms obtain an additional phase of

$$
\begin{equation*}
\left|\psi_{2}(\tau)\right\rangle=|+\rangle+\mathrm{e}^{\mathrm{i} \frac{\pi}{2}+\mathrm{i} \Delta \cdot T}|-\rangle \tag{2.107}
\end{equation*}
$$

Finally, after the second cavity one has

Now, we have several possibilities.

- If $\Delta \cdot T=2 \pi n$ for $n \in \mathbb{Z}$ then

$$
\begin{equation*}
\left|\psi_{f}(\tau)\right\rangle=|+\rangle-|-\rangle=|e\rangle ; \tag{2.109}
\end{equation*}
$$

- If $\Delta \cdot T=\pi+2 \pi n$ for $n \in \mathbb{Z}$ then

$$
\begin{equation*}
\left|\psi_{f}(\tau)\right\rangle=|+\rangle+|-\rangle=|g\rangle \tag{2.110}
\end{equation*}
$$

Generally,

$$
\begin{equation*}
P_{e}(T)=\cos ^{2}\left(\frac{\Delta T}{2}\right)=\frac{1+\cos \Delta T}{2} \tag{2.111}
\end{equation*}
$$

with $\Delta T=\pi \Longrightarrow \Delta=\frac{\pi}{T}$.

### 2.5 Justifying the Classical Chapter

In this section we shall compare the Lorenz atom and the Einstein model with the current semiclassical model.


Fig. 2.8: Ramsey Oscillations.

### 2.5.1 Optical Bloch Equations

Let us recall the evolution equations for the density matrix

$$
\left\{\begin{array}{l}
\partial_{t} \tilde{\rho}_{e e}=+\mathrm{i} \frac{\Omega_{R}}{2}\left(\tilde{\rho}_{e g}-\tilde{\rho}_{g e}\right)  \tag{2.112}\\
\partial_{t} \tilde{\rho}_{g g}=-\mathrm{i} \frac{\Omega_{R}}{2}\left(\tilde{\rho}_{e g}-\tilde{\rho}_{g e}\right) \\
\partial_{t} \tilde{\rho}_{g e}=-\mathrm{i} \Delta \tilde{\rho}_{g e}-\mathrm{i} \frac{\Omega_{R}}{2}\left(\rho_{e e}-\rho_{g g}\right) \\
\partial_{t} \tilde{\rho}_{e g}=+\mathrm{i} \Delta \tilde{\rho}_{g e}+\mathrm{i} \frac{\Omega_{R}}{2}\left(\rho_{e e}-\rho_{g g}\right) .
\end{array}\right.
$$

Now, in order to compare to the Einstein (or Lorenz), we have to add relaxation. It can only be done in a phenomenological way. For $\Delta=\Omega_{R}=0$ one has

$$
\left\{\begin{array}{l}
\partial_{t} \rho_{e e}=-\Gamma \rho_{e e}  \tag{2.113}\\
\partial_{t} \rho_{g g}=+\Gamma \rho_{e e}
\end{array}\right.
$$

Now, we have an additional relaxation of the coherent terms, which does not exist classically:

$$
\left\{\begin{array}{l}
\partial_{t} \tilde{\rho}_{g e}=-\gamma_{\perp} \tilde{\rho}_{g e}  \tag{2.114}\\
\partial_{t} \tilde{\rho}_{e g}=-\gamma_{\perp} \tilde{\rho}_{g e}
\end{array}\right.
$$

We shall denote

- $\Gamma$ : longitudinal decay rate; $\Gamma^{-1}=T_{1}$ (NMR language);
- $\gamma_{\perp}$ : transverse decay rate; $\gamma_{\perp}^{-1}=T_{2}$ (NMR language).

We shall usually assume that the rate of decoherence much larger than the longitudinal decay rate: $\gamma_{\perp} \gg \Gamma$ or $T_{2} \ll T_{1}$. Finally, we write the complete optical Bloch equations

$$
\left\{\begin{array}{l}
\partial_{t} \tilde{\rho}_{e e}=+\mathrm{i} \frac{\Omega_{R}}{2}\left(\tilde{\rho}_{e g}-\tilde{\rho}_{g e}\right)-\Gamma \rho_{e e}  \tag{2.115}\\
\partial_{t} \tilde{\rho}_{g g}=-\mathrm{i} \frac{\Omega_{R}}{2}\left(\tilde{\rho}_{e g}-\tilde{\rho}_{g e}\right)+\Gamma \rho_{e e} \\
\partial_{t} \tilde{\rho}_{g e}=-\left(\gamma_{\perp}+\mathrm{i} \Delta\right) \tilde{\rho}_{g e}-\mathrm{i} \frac{\Omega_{R}}{2}\left(\rho_{e e}-\rho_{g g}\right) \\
\partial_{t} \tilde{\rho}_{e g}=-\left(\gamma_{\perp}-\mathrm{i} \Delta\right) \tilde{\rho}_{g e}+\mathrm{i} \frac{\Omega_{R}}{2}\left(\rho_{e e}-\rho_{g g}\right)
\end{array}\right.
$$

If one writes the optical Bloch equations in the language of the Bloch vectors, one has

$$
\left\{\begin{array}{l}
\partial_{t}\left\langle\sigma_{x}\right\rangle=\Delta\left\langle\sigma_{y}\right\rangle-\gamma_{\perp}\left\langle\sigma_{x}\right\rangle  \tag{2.116}\\
\partial_{t}\left\langle\sigma_{y}\right\rangle=-\Delta\left\langle\sigma_{x}\right\rangle-\Omega_{R}\left\langle\sigma_{y}\right\rangle-\gamma_{\perp}\left\langle\sigma_{y}\right\rangle \\
\partial_{t}\left\langle\sigma_{z}\right\rangle=\Omega_{R}\left\langle\sigma_{y}\right\rangle-\Gamma\left(\left\langle\sigma_{z}\right\rangle+1\right) .
\end{array}\right.
$$

It means that $\gamma_{\perp}$ kills the coherences, whereas $\Gamma$ brings the Bloch vector to -1 , i.e., to the ground state.

### 2.5.2 Lindblad Operator Form

If we try to write the equations in the operator form, they cannot look like in the usual Liouville equation. One needs some other ingredient. Explicitly,

$$
\begin{equation*}
\partial_{t} \tilde{\rho}=-\frac{\mathrm{i}}{\hbar}\left[\tilde{H}_{\mathrm{A}}+\tilde{H}_{\mathrm{AF}}, \tilde{\rho}\right]+\Gamma \mathcal{D}[\sigma] \tilde{\rho}+\gamma_{c} \mathcal{D}\left[\sigma_{z}\right] \tilde{\rho} \tag{2.117}
\end{equation*}
$$

with

$$
\begin{equation*}
\gamma_{c}=\gamma_{\perp}-\frac{\Gamma}{2} \tag{2.118}
\end{equation*}
$$

The term $\mathcal{D}[\sigma]$ called Lindblad superoperator, and given by

$$
\begin{equation*}
\mathcal{D}[c] \rho=c \rho c^{\dagger}-\frac{1}{2}\left(c^{\dagger} c \rho+\rho c^{\dagger} c\right) \tag{2.119}
\end{equation*}
$$

This is the canonical form for independent Markov processes, and the subject of current research interest.

### 2.5.3 Example - Natural Damping

Consider an atom interacting only with an electromagnetic field $\omega_{0}$. The atom levels have some width given by the relaxation rate (we shall discuss everything in the next chapter). Here $\gamma_{c}=0$. Hence, the Lindblad equation reads

$$
\begin{equation*}
\partial_{t} \tilde{\rho}=-\frac{\mathrm{i}}{\hbar}\left[\tilde{H}_{\mathrm{A}}+\tilde{H}_{\mathrm{AF}}, \tilde{\rho}\right]+\Gamma \mathcal{D}[\sigma] \tilde{\rho} . \tag{2.120}
\end{equation*}
$$

Now, define

$$
\begin{align*}
\tilde{H}_{\mathrm{eff}} & =\tilde{H}_{\mathrm{A}}+\tilde{H}_{\mathrm{AF}}-\mathrm{i} \frac{\hbar \Gamma}{2} \sigma^{\dagger} \sigma \\
& \equiv\left(\hbar \omega_{0}-\mathrm{i} \frac{\hbar \Gamma}{2}\right) \underbrace{\sigma^{\dagger} \sigma}_{|e\rangle\langle e|}+\tilde{H}_{\mathrm{AF}}, \tag{2.121}
\end{align*}
$$

Notice that $\tilde{H}_{\text {eff }}$ is not Hermitian. Note that the energy level separation has an imaginary part, which can be interpreted as an energy level width.

Therefore, we can write

$$
\begin{equation*}
\partial_{t} \tilde{\rho}=-\frac{\mathrm{i}}{\hbar}\left(\tilde{H}_{\mathrm{eff}} \tilde{\rho}-\tilde{\rho} \tilde{H}_{\mathrm{eff}}^{\dagger}\right)+\underbrace{\Gamma \sigma \tilde{\rho} \sigma^{\dagger}}_{\Gamma \rho_{e e}|g\rangle\langle g|} . \tag{2.122}
\end{equation*}
$$

This form will be extremely useful near the end of the course, where we would learn superradiance.

### 2.5.4 Quantum Harmonic Oscillators

Next, we would like to compare this semiclassical approach to the Lorentz atom and Einstein model.

The Lorentz model is a model of harmonic oscillators. Let us remind ourselves some theory of quantum harmonic oscillators. The Hamiltonian is given by

$$
\begin{equation*}
H=\frac{p^{2}}{2 m}+\frac{1}{2} m \omega_{0}^{2} x^{2}=\hbar \omega_{0}\left(a^{\dagger} a+\frac{1}{2}\right), \tag{2.123}
\end{equation*}
$$

where

$$
\begin{equation*}
a=\frac{1}{\sqrt{2}}\left(\frac{x}{x_{0}}+\mathrm{i} \frac{x_{0} p}{\hbar}\right), \quad x_{0} \equiv \sqrt{\frac{\hbar}{m \omega_{0}}}, \tag{2.124}
\end{equation*}
$$

and

$$
\begin{equation*}
x=\frac{x_{0}}{\sqrt{2}}\left(a+a^{\dagger}\right), \quad p=\frac{\hbar}{\mathrm{i} \sqrt{2} x_{0}}\left(a+a^{\dagger}\right) . \tag{2.125}
\end{equation*}
$$

One has the canonical permutation relations

$$
\begin{equation*}
[x, p]=\mathrm{i} \hbar \Longleftrightarrow\left[a, a^{\dagger}\right]=1 \tag{2.126}
\end{equation*}
$$

The eigenstates of the Hamiltonian are given by

$$
\begin{equation*}
E_{n}=\hbar \omega_{0}\left(n+\frac{1}{2}\right) \tag{2.127}
\end{equation*}
$$

with

$$
\left\{\begin{array}{l}
a|n\rangle=\sqrt{n}|n-1\rangle  \tag{2.128}\\
a^{\dagger}|n\rangle=\sqrt{n+1}|n+1\rangle .
\end{array}\right.
$$

The eigenstates of $a$ are called coherent quantum states

$$
\begin{equation*}
a|\alpha\rangle=\alpha|\alpha\rangle \tag{2.129}
\end{equation*}
$$

with

$$
\begin{equation*}
|\alpha\rangle=\sum_{n=0}^{\infty} \frac{\alpha^{n}}{\sqrt{n!}} \mathrm{e}^{-\frac{1}{2}|\alpha|^{2}}|n\rangle \tag{2.130}
\end{equation*}
$$

### 2.5.5 Damped Quantum Harmonic Oscillators ${ }^{12}$

The Lindblad form is given by

$$
\begin{equation*}
\partial_{t} \rho=-\frac{\mathrm{i}}{\hbar}[H, \rho]+\gamma \mathcal{D}[a] \rho, \tag{2.131}
\end{equation*}
$$

with $H$ the Hamiltonian of the harmonic oscillator, and the dissipation

$$
\begin{equation*}
\mathcal{D}[a] \rho=a \rho a^{\dagger}-\frac{1}{2}\left(a^{\dagger} a \rho+\rho a^{\dagger} a\right) \tag{2.132}
\end{equation*}
$$

For any operator $\hat{A}$ we can write its dynamics

$$
\partial_{t}\langle\hat{A}\rangle=\operatorname{Tr}\left(\hat{A} \partial_{t} \rho\right)
$$

[^10]\[

$$
\begin{equation*}
=-\frac{\mathrm{i}}{2 m \hbar}\left\langle\left[\hat{A}, \rho^{2}\right]\right\rangle-\frac{\mathrm{i} m \omega_{0}^{2}}{\hbar}\left\langle\left[\hat{A}, x^{2}\right]\right\rangle+\frac{\gamma}{2}\left\langle a^{\dagger}[\hat{A}, a]+\left[a^{\dagger}, \hat{A}\right] a\right\rangle \tag{2.133}
\end{equation*}
$$

\]

Now, recall that

$$
\begin{align*}
{[x, f(p)] } & =\mathrm{i} \hbar \partial_{p} f(p) \\
{[x, a] } & =-\frac{x_{0}}{\sqrt{2}}=-\left[x, a^{\dagger}\right] \tag{2.134}
\end{align*}
$$

Thus, for $\hat{A}=\hat{x}$,

$$
\begin{equation*}
\left\langle a^{\dagger}[\hat{A}, a]+\left[a^{\dagger}, \hat{A}\right] a\right\rangle=-\frac{x_{0}}{\sqrt{2}}\left\langle a+a^{\dagger}\right\rangle=-\langle x\rangle . \tag{2.135}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
\partial_{t}\langle x\rangle=\frac{\langle p\rangle}{m}-\frac{\gamma}{2}\langle x\rangle \text {. } \tag{2.136}
\end{equation*}
$$

Similarly, due to the fact that

$$
\begin{align*}
{[p, f(x)] } & =-\mathrm{i} \hbar \partial_{x} f(x) \\
{[p, a]=\left[p, a^{\dagger}\right] } & =\frac{\hbar}{\sqrt{2} \mathrm{i} x_{0}} \tag{2.137}
\end{align*}
$$

Hence, for $\hat{A}=\hat{p}$ one has

$$
\begin{equation*}
\partial_{t}\langle p\rangle=-m \omega_{0}^{2}\langle x\rangle-\frac{\gamma}{2}\langle p\rangle . \tag{2.138}
\end{equation*}
$$

Combine both to obtain

$$
\begin{equation*}
\partial_{t}^{2}\langle x\rangle+\gamma \partial_{t}\langle x\rangle+\left(\omega_{0}^{2}+\frac{1}{4} \gamma^{2}\right)\langle x\rangle=0 \tag{2.139}
\end{equation*}
$$

This is exactly the same as the classical damped harmonic oscillator, up to the $\frac{1}{4} \gamma^{2}$ factor.

### 2.5.6 Equivalence with the Lorentz Atom

In this case, we need to add the atom-field interaction

$$
\begin{equation*}
H_{\mathrm{AF}}=-\boldsymbol{d} \cdot \boldsymbol{E}=e \hat{x} E_{0} \mathrm{e}^{-\mathrm{i} \omega t}+\text { c.c. } \tag{2.140}
\end{equation*}
$$

Such that

$$
\begin{equation*}
\left[p, H_{\mathrm{AF}}\right]=-\mathrm{i} \hbar e E_{0} \mathrm{e}^{-\mathrm{i} \omega t}+\mathrm{c} . \mathrm{c} . \tag{2.141}
\end{equation*}
$$

We now need to adjust the equation of evolution of the momentum to

$$
\begin{equation*}
\partial_{t}\langle p\rangle=-m \omega_{0}^{2}\langle x\rangle-\frac{\gamma}{2}\langle p\rangle-\left(e E_{0} \mathrm{e}^{-\mathrm{i} \omega t}+\text { c.c. }\right), \tag{2.142}
\end{equation*}
$$

thus,

$$
\begin{equation*}
\partial_{t}^{2}\langle x\rangle+\gamma \partial_{t}\langle x\rangle+\left(\omega_{0}^{2}+\frac{1}{4} \gamma^{2}\right)\langle x\rangle=-e E_{0} \mathrm{e}^{-\mathrm{i} \omega t}+\text { c.c. } \tag{2.143}
\end{equation*}
$$

Again, this is the classical equation.

### 2.5.7 Connection with the 2-level Atom

We shall work in the weak driving limit ( $E_{0} \rightarrow 0$ and $\Omega_{R} \rightarrow 0$ ). In this assumption, only the first 2 states $|0\rangle$ and $|1\rangle$ will be populated. We shall make the following distinction

$$
\begin{align*}
|1\rangle & \leftrightarrow|e\rangle \\
|0\rangle & \leftrightarrow|g\rangle  \tag{2.144}\\
a & \leftrightarrow \sigma=|g\rangle\langle e| .
\end{align*}
$$

With this dictionary in place, one can deduce the evolution equations

$$
\begin{equation*}
\partial_{t} \rho=-\mathrm{i} \omega_{0}\left[\sigma^{\dagger} \sigma, \rho\right]-\frac{\mathrm{i}}{\hbar}\left[H_{\mathrm{AF}}, \rho\right]+\gamma \mathcal{D}[\sigma] \rho \tag{2.145}
\end{equation*}
$$

This is exactly the same as the optical Bloch equation we saw previously.
Remark 2.8. We ought to introduce the notion of oscillator strength $f_{\mathrm{o}}$ by changing $\frac{e}{m} \rightarrow \frac{e}{m} f_{\mathrm{o}}$. The dipole moment reads

$$
\begin{equation*}
\hat{d}_{z}=\langle g| \hat{d}_{z}|e\rangle\left(\hat{\sigma}+\hat{\sigma}^{\dagger}\right) . \tag{2.146}
\end{equation*}
$$

The usual dipole reads

$$
\begin{equation*}
e \hat{x}=\frac{e x_{0}}{\sqrt{2}}\left(a+a^{\dagger}\right)=\sqrt{\frac{e^{2} \hbar}{2 m \omega_{0}}}\left(a+a^{\dagger}\right), \tag{2.147}
\end{equation*}
$$

but now we upgrade it to

$$
\begin{equation*}
\left.e \hat{x} \sqrt{\frac{e^{2} \hbar \cdot f_{\mathrm{o}}}{2 m \omega_{0}}}\left(a+a^{\dagger}\right) \Longrightarrow\left|\langle g| \hat{d}_{z}\right| e\right\rangle\left.\right|^{2}=\frac{e^{2} \hbar f_{\mathrm{o}}}{2 m \omega_{0}} \text {. } \tag{2.148}
\end{equation*}
$$

### 2.5.8 Relation between the Optical Bloch Equations and the Einstein Model

Recall that we had the dynamics of two quantities: the populations and the coherences,

$$
\left\{\begin{array}{l}
\partial_{t} \rho_{e e}=\mathrm{i} \frac{\Omega_{R}}{2}\left(\tilde{\rho}_{e g}-\tilde{\rho}_{g e}\right)-\Gamma \rho_{e e}  \tag{2.149}\\
\partial_{t} \tilde{\rho}_{e g}=-\left(\gamma_{\perp}-\mathrm{i} \Delta\right) \tilde{\rho}_{e g}+\mathrm{i} \frac{\Omega_{R}}{2}\left(\rho_{e e}-\rho_{g g}\right) .
\end{array}\right.
$$

We have also assumed that $\gamma_{\perp} \gg \Gamma$ (the classical limit), meaning we kill the coherences much faster than the populations. Explicitly,

$$
\begin{equation*}
\partial_{t} \tilde{\rho}_{e g}=\partial_{t} \tilde{\rho}_{g e}=0 . \tag{2.150}
\end{equation*}
$$

Therefore,

$$
\begin{align*}
& \left(\gamma_{\perp}-\mathrm{i} \Delta\right) \tilde{\rho}_{e g}=+\mathrm{i} \frac{\Omega_{R}}{2}\left(\rho_{e e}-\rho_{g g}\right)  \tag{2.151a}\\
& \left(\gamma_{\perp}+\mathrm{i} \Delta\right) \tilde{\rho}_{g e}=-\mathrm{i} \frac{\Omega_{R}}{2}\left(\rho_{e e}-\rho_{g g}\right) \tag{2.151b}
\end{align*}
$$

Massaging a bit Eqs. (2.151a) and (2.151b) one obtains

$$
\begin{equation*}
\gamma_{\perp}\left(1+\frac{\Delta^{2}}{\gamma_{\perp}^{2}}\right)\left(\tilde{\rho}_{e g}-\tilde{\rho}_{g e}\right)=\mathrm{i} \Omega_{R}\left(\rho_{e e}-\rho_{g g}\right) \tag{2.152}
\end{equation*}
$$

Insert into the classical limit condition (2.150) to have

$$
\begin{equation*}
\partial_{t} \rho_{e e}=-\Gamma \rho_{e e}-\frac{\Omega_{R}^{2}}{2 \gamma_{\perp}\left(1+\frac{\Delta^{2}}{\gamma_{\perp}^{2}}\right)}\left(\rho_{e e}-\rho_{g g}\right) \tag{2.153}
\end{equation*}
$$

This is the Einstein equation with $\rho_{e e} \leftrightarrow \frac{N_{2}}{N}$ and $\Gamma \leftrightarrow A$. This proves that Einstein equation is classical, since we have killed all the coherences.

## 3 Quantization of the Electromagnetic Field ${ }^{13}$

### 3.1 Introduction to Quantization

There are several ways to quantize the electromagnetic field. One of the ways, presented in the Quantim3 course, is to list all the symmetries of the Lorentz group, and derive the canonical equations. This will not be the approach in this chapter to avoid boredom for those who have already seen it, and relativistic effects.

### 3.1.1 A Reminder

The vector potential $\boldsymbol{A}(\boldsymbol{r}, t)$ obeys the wave equation

$$
\begin{equation*}
\left(\Delta-\frac{1}{c^{2}} \partial_{t}^{2}\right) \boldsymbol{A}=0 \tag{3.1}
\end{equation*}
$$

A few observations:

- Plane wave solutions: $\mathrm{e}^{\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r}-\omega t)}$.
- The dispersion is linear: $\omega=c|\boldsymbol{k}|$.
- Polarization is given by $\hat{\epsilon}$. It can be linear or circular, thus complex.
- Gauge condition: Coulomb gauge

$$
\begin{equation*}
\operatorname{div} \boldsymbol{A}=0 \Longleftrightarrow \hat{\epsilon} \cdot \boldsymbol{k}=0 . \tag{3.2}
\end{equation*}
$$

These are transverse waves. Note that $\hat{\epsilon}$ depends on $\boldsymbol{k}$.
Thus, we will talk about transverse wave solutions.
Let us inspect a discrete distribution of modes in a volume $V$,

$$
\begin{equation*}
\boldsymbol{A}(\boldsymbol{r}, t)=A_{\boldsymbol{k}} \hat{\epsilon}_{\boldsymbol{k}} \frac{\mathrm{e}^{\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r}-\omega t)}}{\sqrt{V}}+A_{\boldsymbol{k}}^{*} \hat{\epsilon}_{\boldsymbol{k}}^{*} \frac{\mathrm{e}^{-\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r}-\omega t)}}{\sqrt{V}} \tag{3.3}
\end{equation*}
$$

with

$$
\begin{equation*}
\hat{\epsilon}_{\boldsymbol{k}} \hat{\epsilon}_{\boldsymbol{k}}^{*}=1 \tag{3.4}
\end{equation*}
$$

In the limit $V \rightarrow \infty$ one replaces a sum with an integral,

$$
\begin{equation*}
\frac{1}{V} \sum_{\boldsymbol{k}} f(\boldsymbol{k}) \rightarrow \int \frac{\mathrm{d}^{3} \boldsymbol{k}}{{(2 \pi)^{3}}^{3}} f(k) . \tag{3.5}
\end{equation*}
$$

Define the electric and magnetic fields by

$$
\begin{align*}
& \boldsymbol{E}=-\partial_{t} \boldsymbol{A} \\
& \boldsymbol{B}=\boldsymbol{\nabla} \times \boldsymbol{A} . \tag{3.6}
\end{align*}
$$

Hence,

$$
\begin{align*}
& \boldsymbol{E}=\mathrm{i} \omega A_{\boldsymbol{k}} \hat{\epsilon}_{\boldsymbol{k}} \frac{\mathrm{e}^{\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r}-\omega t)}}{\sqrt{V}}-\mathrm{i} \omega A_{\boldsymbol{k}}^{*} \hat{\epsilon}_{\boldsymbol{k}} \frac{\mathrm{e}^{-\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r}-\omega t)}}{\sqrt{V}} \\
& \boldsymbol{B}=\mathrm{i} A_{\boldsymbol{k}} \boldsymbol{k} \times \hat{\epsilon}_{\boldsymbol{k}} \frac{\mathrm{e}^{\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r}-\omega t)}}{\sqrt{V}}-\mathrm{i} A_{\boldsymbol{k}}^{*} \boldsymbol{k} \times \hat{\epsilon}_{\boldsymbol{k}}^{*} \frac{\mathrm{e}^{-\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r}-\omega t)}}{\sqrt{V}} . \tag{3.7}
\end{align*}
$$

[^11]Now, define the energy density of the electromagnetic field by

$$
\begin{equation*}
u(\boldsymbol{r}, t)=\frac{\varepsilon_{0}}{2}\left(E^{2}(\boldsymbol{r}, t)+c^{2} B^{2}(\boldsymbol{r}, t)\right) . \tag{3.8}
\end{equation*}
$$

Using the identity

$$
\begin{equation*}
\left(\hat{\epsilon}_{\boldsymbol{k}} \times \boldsymbol{k}\right) \cdot\left(\hat{\epsilon}_{\boldsymbol{k}}^{*} \times \boldsymbol{k}\right)=k^{2}=\frac{\omega^{2}}{c^{2}} \tag{3.9}
\end{equation*}
$$

one obtains

$$
\begin{equation*}
\overline{u(\boldsymbol{r})}=\frac{1}{T} \int_{0}^{T} \mathrm{~d} t u(\boldsymbol{r}, t)=2 \varepsilon_{0} \omega^{2} \frac{\left|A_{\boldsymbol{k}}\right|^{2}}{V} . \tag{3.10}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
U=2 \varepsilon_{0} \omega^{2}\left|A_{\boldsymbol{k}}\right|^{2} \tag{3.11}
\end{equation*}
$$

### 3.1.2 Intermezzo on the Derivation of a Quantization Relation

We would like to know, if indeed as expected,

$$
\begin{equation*}
U=V \bar{u} \stackrel{?}{=} \hbar \omega N \tag{3.12}
\end{equation*}
$$

with $N$ an integer number is a quantum phenomenon. We shall show that it is not quantum.

Recall from Chapter 1.3 that from thermodynamic considerations we have had

$$
\begin{equation*}
E=V u(T)=V \int \mathrm{~d} \nu u(\nu, T) \tag{3.13}
\end{equation*}
$$

We have also deduced the entropy relation

$$
\begin{equation*}
S=V \int \mathrm{~d} \nu s(\nu, T) \tag{3.14}
\end{equation*}
$$

The entropy density had the following relation

$$
\begin{equation*}
s(T)=\frac{4}{3} \frac{u(T)}{T} \Longrightarrow s(\nu, T)=\frac{4}{3} \frac{u(\nu, T)}{T} \tag{3.15}
\end{equation*}
$$

We have also seen the first law of thermodynamics

$$
\begin{equation*}
\mathrm{d} E=T \mathrm{~d} S-p \mathrm{~d} V \Longrightarrow \mathrm{~d} S=\frac{1}{T} \mathrm{~d} E+\frac{p}{T} \mathrm{~d} V \tag{3.16}
\end{equation*}
$$

Hence,

$$
\begin{align*}
\frac{1}{T}=\left(\frac{\partial S}{\partial E}\right)_{V} & =\left(\frac{\partial S}{\partial T}\right)_{V}\left(\frac{\partial T}{\partial E}\right)_{V}  \tag{3.17}\\
& =\frac{\mathrm{d} s(\nu, T)}{\mathrm{d} T} \frac{\mathrm{~d} T}{\mathrm{~d} u(\nu, T)}=\frac{\mathrm{d} s(\nu, T)}{\mathrm{d} u(\nu, T)}
\end{align*}
$$

Such that

$$
\begin{equation*}
\frac{1}{T} \propto \frac{\mathrm{~d} s}{\mathrm{~d} u} \tag{3.18}
\end{equation*}
$$

We have also derived Wien's law

$$
\begin{equation*}
u(\nu, T)=\alpha \nu^{3} \mathrm{e}^{-\beta \nu / T} \tag{3.19}
\end{equation*}
$$

with

$$
\begin{equation*}
\alpha=\frac{8 \pi h}{c^{3}}, \quad \beta=\frac{h}{k_{\mathrm{B}}}, \tag{3.20}
\end{equation*}
$$

under the (so called strongly quantum) limit

$$
\begin{equation*}
h \nu \gg k_{\mathrm{B}} T . \tag{3.21}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
\frac{1}{T}=-\frac{k_{\mathrm{B}}}{h \nu} \ln \left(\frac{u(\nu, T)}{\alpha \nu^{3}}\right)=\frac{\mathrm{d} s}{\mathrm{~d} u}, \tag{3.22}
\end{equation*}
$$

hence,

$$
\begin{equation*}
s(u, \nu)=-\frac{k_{\mathrm{B}}}{h \nu} \cdot u \cdot\left(\ln \frac{u}{\alpha \nu^{3}}-1\right), \tag{3.23}
\end{equation*}
$$

so that

$$
\begin{equation*}
S(E, \nu, V)=V \cdot s(u, \nu) \tag{3.24}
\end{equation*}
$$

Now, make a transformation at a fixed $(E, \nu)$ changing $V$ to have

$$
\begin{equation*}
S(E, \nu, V)-S\left(E, \nu, V_{0}\right)=\frac{k_{\mathrm{B}} E}{h \nu} \ln \left(\frac{V}{V_{0}}\right) . \tag{3.25}
\end{equation*}
$$

We have arrived to the ideal gas law

$$
\begin{equation*}
S(E, V)=N k_{\mathrm{B}} \ln \left(\frac{V}{N \lambda^{3}(T)}\right)+\text { const. } \tag{3.26}
\end{equation*}
$$

so that

$$
\begin{equation*}
S(E, V)-S\left(E, V_{0}\right)=N k_{\mathrm{B}} \ln \left(\frac{V}{V_{0}}\right) \tag{3.27}
\end{equation*}
$$

Einstein's reasoning [Einstein, 1905] was, verbatim, that if the equations look exactly as in the ideal gas law, then the radiation behaves as particles too. Therefore, one can identify $N$ with the number of radiation particles (today: "photons"). Hence,

$$
\begin{equation*}
\frac{k_{\mathrm{B}} E}{h \nu}=N k_{B} \Longrightarrow E=h \nu N \tag{3.28}
\end{equation*}
$$

Therefore, the fact that radiation has a molecular aspect can be derived from purely classical thermodynamic considerations. In terms of energy, it can be written as

$$
\begin{equation*}
U=\hbar \omega n_{\boldsymbol{k}} \tag{3.29}
\end{equation*}
$$

From equations (3.11) and (3.29) we have

$$
\begin{equation*}
\left|A_{\boldsymbol{k}}\right|^{2}=\frac{\hbar}{2 \varepsilon_{0} \omega} n_{\boldsymbol{k}} . \tag{3.30}
\end{equation*}
$$

We will come back to this equation once in a while.

### 3.1.3 Excitation of Matter (Absorption and Emission of Radiation) ${ }^{14}$

In order to calculate the rates classically, we shall use the Fermi golden rule. For the Hamiltonian given by

$$
\begin{equation*}
H^{\prime}=H_{a}=\frac{q}{m} \boldsymbol{p} \cdot \boldsymbol{A}_{\perp}(\hat{\boldsymbol{r}}=\mathbf{0}), \tag{3.31}
\end{equation*}
$$

with

$$
\begin{equation*}
\boldsymbol{A}_{\perp}(t)=\boldsymbol{A}_{\perp} \mathrm{e}^{\mathrm{i} \omega t} . \tag{3.32}
\end{equation*}
$$

Hence, the Fermi golden rule states

$$
\begin{equation*}
\left.\Gamma_{a \rightarrow b}=\frac{2 \pi}{\hbar}\left|\langle b| \frac{q}{m} \hat{\boldsymbol{p}} \cdot \boldsymbol{A}_{\perp}(\mathbf{0})\right| a\right\rangle\left.\right|^{2} \delta\left(E_{b}-E_{a}-\hbar \omega\right), \tag{3.33}
\end{equation*}
$$

where $\delta\left(E_{b}-E_{a}-\hbar \omega\right)$ accounts for the conservation of energy.
Now, rearranging the items a bit one has

$$
\begin{align*}
\Gamma_{a \rightarrow b} & \left.=\frac{2 \pi}{\hbar} \frac{q^{2}}{m^{2}}\left|\boldsymbol{A}_{\perp}(\mathbf{0})\right|^{2}\left|\langle b| \hat{\epsilon}_{\boldsymbol{k}} \cdot \hat{\boldsymbol{p}}\right| a\right\rangle\left.\right|^{2} \delta\left(E_{b}-E_{a}-\hbar \omega\right) \\
& \left.=\frac{2 \pi}{\hbar} \frac{\hbar q^{2}}{2 \varepsilon_{0} \omega m^{2}} n_{\boldsymbol{k}} \cdot\left|\langle b| \hat{\epsilon}_{\boldsymbol{k}} \cdot \hat{\boldsymbol{p}}\right| a\right\rangle\left.\right|^{2} \delta\left(E_{b}-E_{a}-\hbar \omega\right), \tag{3.34}
\end{align*}
$$

where in the last equality we used Eq. (3.30). Note the slight abuse of notations: $\hat{\boldsymbol{p}}$ refers to operator, whereas $\hat{\epsilon}_{\boldsymbol{k}}$ to a unit vector.

### 3.2 Quantum Radiation

In the following section we shall derive and present the usual notation of quantum radiation. We shall introduce the notion of a photon as well as several of its unique properties.

### 3.2.1 Fock (Hilbert) Space

Define the Fock space for the radiation,

$$
\begin{equation*}
\underbrace{\text { Hilbert }}_{\left(e^{-}, \text {matter }\right)} \otimes \underbrace{\text { Fock }}_{(\text {radiation })} . \tag{3.35}
\end{equation*}
$$

The initial state is given by

$$
\begin{equation*}
\left|\psi_{a}\right\rangle=\left|a, \ldots, n_{\boldsymbol{k}}, \ldots\right\rangle=|a\rangle \otimes\left|\ldots, n_{\boldsymbol{k}}, \ldots\right\rangle . \tag{3.36}
\end{equation*}
$$

We now want to calculate the probability to find after a time $t$ the state

$$
\begin{equation*}
\left|\psi_{b}\right\rangle=\left|b, \ldots, n_{\boldsymbol{k}}-1, \ldots\right\rangle, \tag{3.37}
\end{equation*}
$$

i.e., after an absorption of a single radiation quantum.

Define an operator $\hat{W}$ by

$$
\begin{equation*}
\left.\Gamma_{a \rightarrow b}=\frac{2 \pi}{\hbar}\left|\left\langle b, \ldots, n_{\boldsymbol{k}}-1, \ldots\right| \hat{W}\right| a, \ldots, n_{\boldsymbol{k}}, \ldots\right\rangle\left.\right|^{2} . \tag{3.38}
\end{equation*}
$$

[^12]Necessarily, $\hat{W}$ is the product of the two operators

$$
\begin{equation*}
-q \sqrt{\frac{\hbar}{2 \varepsilon_{0} \omega m^{2}}}\left(\hat{\epsilon}_{\boldsymbol{k}} \cdot \hat{\boldsymbol{p}}\right) \otimes \hat{a}_{\boldsymbol{k}, \hat{\epsilon}_{\boldsymbol{k}}} \tag{3.39}
\end{equation*}
$$

It is easy to check that the operator $\hat{a}_{\boldsymbol{k}, \hat{\epsilon}_{\boldsymbol{k}}}$ is dependent on $n_{\boldsymbol{k}}$ by

$$
\begin{equation*}
\left.\left|\left\langle\ldots, n_{\boldsymbol{k}}-1, \ldots\right| \hat{a}_{\boldsymbol{k}, \hat{\epsilon}_{\boldsymbol{k}}}\right| \ldots, n_{\boldsymbol{k}}, \ldots\right\rangle\left.\right|^{2}=n_{\boldsymbol{k}}>0 \tag{3.40}
\end{equation*}
$$

or

$$
\begin{equation*}
\left\langle\ldots, n_{\boldsymbol{k}}-1, \ldots\right| \hat{a}_{\boldsymbol{k}, \hat{\epsilon}_{\boldsymbol{k}}}\left|\ldots, n_{\boldsymbol{k}}, \ldots\right\rangle=\sqrt{n_{\boldsymbol{k}}}>0 \tag{3.41}
\end{equation*}
$$

Thus, we have the following identity

$$
\hat{a}_{\boldsymbol{k}, \hat{\epsilon}_{\boldsymbol{k}}}|n\rangle_{\boldsymbol{k}, \hat{\epsilon}_{\boldsymbol{k}}}= \begin{cases}\sqrt{n}|n-1\rangle_{\boldsymbol{k}} & n>0  \tag{3.42}\\ 0 & n=0\end{cases}
$$

Finally, we have all the ingredients for our Fock space.

1. States $|n\rangle$ constitute a basis of the Fock space associated to the mode $(\boldsymbol{k}, \hat{\epsilon})$.
2. The algebra of the operators $\left(\hat{a}_{\boldsymbol{k}, \epsilon}, \hat{a}_{\boldsymbol{k}, \epsilon}^{\dagger}\right)$ is the one of a Harmonic oscillator,

$$
\begin{equation*}
\left[\hat{a}_{\boldsymbol{k}, \epsilon}, \hat{a}_{\boldsymbol{k}, \epsilon}^{\dagger}\right]=1 \tag{3.43}
\end{equation*}
$$

3. Between distinct spaces (modes),

$$
\begin{align*}
{\left[\hat{a}_{\boldsymbol{k}, \epsilon}, \hat{a}_{\boldsymbol{k}^{\prime}, \epsilon^{\prime}}^{\dagger}\right] } & =\delta_{\boldsymbol{k}, \boldsymbol{k}^{\prime}} \delta_{\epsilon, \epsilon^{\prime}}  \tag{3.44a}\\
{[\hat{a}, \hat{a}]=\left[\hat{a}^{\dagger}, \hat{a}^{\dagger}\right] } & =0 . \tag{3.44b}
\end{align*}
$$

### 3.2.2 Field Operators

We can now write the field operators of the electromagnetic field

$$
\begin{align*}
& H^{\prime}=-\frac{q}{m} \sum_{\boldsymbol{k}, \hat{\epsilon}} \sqrt{\frac{\hbar}{2 \varepsilon_{0} \omega}}[\overbrace{\left(\hat{\epsilon}_{\boldsymbol{k}} \cdot \hat{\boldsymbol{p}}\right) \hat{a}_{\boldsymbol{k}, \epsilon} \mathrm{e}^{-\mathrm{i} \omega t}}^{\text {absorption }} \\
&+\underbrace{\left(\hat{\epsilon}_{\boldsymbol{k}}^{*} \cdot \hat{\boldsymbol{p}}\right) \hat{a}_{\boldsymbol{k}, \epsilon}^{\dagger} \mathrm{e}^{+\mathrm{i} \omega t}}_{\text {emission }}] \tag{3.45}
\end{align*}
$$

$$
\equiv-\frac{q}{m} \hat{\boldsymbol{p}} \cdot \boldsymbol{A}_{\perp}(\mathbf{0}) .
$$

Hence,

$$
\begin{equation*}
\hat{\boldsymbol{A}}(\boldsymbol{r}, t)=\sum_{\boldsymbol{k}, \hat{\epsilon}} \sqrt{\frac{\hbar}{2 \varepsilon_{0} \omega}}\left(\hat{a}_{\boldsymbol{k}, \epsilon} \hat{\epsilon}_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r}-\omega t)}+\hat{a}_{\boldsymbol{k}, \epsilon}^{\dagger} \hat{\epsilon}_{\boldsymbol{k}}^{*} \mathrm{e}^{-\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r}-\omega t)}\right) \tag{3.46}
\end{equation*}
$$

Remark 3.1. This operator $\hat{\boldsymbol{A}}$ lives only in the Fock space. There is no matter for the Hilbert space.
Remark 3.2. The $\boldsymbol{r}$ in $\hat{\boldsymbol{A}}(\boldsymbol{r}, t)$ is not an operator; it is not a dynamical quantity, but simply a position. It does not live in a Fock space.

### 3.2.3 More Field Operators

Define two other field operators for the electric and magnetic fields as usual,

$$
\begin{equation*}
\hat{\boldsymbol{E}}=-\partial_{t} \hat{\boldsymbol{A}} ; \quad \hat{\boldsymbol{B}}=\nabla \times \hat{\boldsymbol{A}} \tag{3.47}
\end{equation*}
$$

In the same manner, define the energy operator,

$$
\begin{equation*}
\hat{U}=\frac{\varepsilon_{0}}{2} \int_{V} \mathrm{~d}^{3} \boldsymbol{r}\left(\hat{\boldsymbol{E}}^{2}+c^{2} \hat{\boldsymbol{B}}^{2}\right) . \tag{3.48}
\end{equation*}
$$

Note that one must check that these definition make sense when you plug in the states $\left|\psi_{a}\right\rangle$ and $\left|\psi_{b}\right\rangle$, but we shall leave that question for later.

After a bit of tedious algebra one obtains

$$
\begin{equation*}
\hat{U}=\sum_{\boldsymbol{k}, \hat{\epsilon}} \frac{\hbar \omega}{2}\left(\hat{a}_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}}^{\dagger}+\hat{a}_{\boldsymbol{k}}^{\dagger} \hat{a}_{\boldsymbol{k}}\right) \tag{3.49}
\end{equation*}
$$

### 3.2.4 Number of Photons Operator

Define a Hermitian operator

$$
\begin{equation*}
\hat{N}_{\boldsymbol{k}, \hat{\epsilon}} \equiv \hat{a}_{\boldsymbol{k}, \hat{\epsilon}}^{\dagger} \hat{a}_{\boldsymbol{k}, \hat{\epsilon}}, \tag{3.50}
\end{equation*}
$$

such that

$$
\begin{equation*}
\hat{N}_{\boldsymbol{k}, \hat{\epsilon}}|n\rangle_{\boldsymbol{k}, \hat{\epsilon}}=n|n\rangle_{\boldsymbol{k}, \hat{\epsilon}} . \tag{3.51}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
\hat{a}_{\boldsymbol{k}, \hat{\epsilon}} \hat{a}_{\boldsymbol{k}, \hat{\epsilon}}^{\dagger}=\hat{N}_{\boldsymbol{k}, \hat{\epsilon}}+\mathbb{1} . \tag{3.52}
\end{equation*}
$$

Thus,

$$
\begin{equation*}
\hat{U}=\sum_{\boldsymbol{k}, \hat{\epsilon}} \frac{\hbar \omega}{2}\left(\hat{N}_{\boldsymbol{k}, \hat{\epsilon}}+\frac{1}{2} \mathbb{1}\right) . \tag{3.53}
\end{equation*}
$$

Remark 3.3. Recall again that the entire argument was classical due to Einstein. Usually, one gives the quantized operators and states that it is quantum mechanics, but this is simply wrong, as we saw.
Remark 3.4. The factor $\frac{1}{2}$ in $\hat{U}$ is artificial. It has nothing to do with vacuum expectation value or some quantized magic as is often states. As we have seen, we can get rid of it without much effort. That is called normal ordering. However, it is relevant to the relativistic nature of the electromagnetic wave, but we shall not deal with it here.

Remark 3.5. This case is similar to the case of fluids, where one can do only with thermodynamics and does not need statistical mechanics. The power of statistical mechanics is only needed when one needs to probe the discrete nature of the fluid (condensates, etc.). Here we shall later use this description in order to probe the discrete nature of the electromagnetic radiation. So far, it is not needed.

### 3.2.5 Momentum of Radiation

Recall that classically the Poynting vector is given by

$$
\begin{equation*}
\boldsymbol{S}=\varepsilon_{0} c^{2} \boldsymbol{E} \times \boldsymbol{B} \tag{3.54}
\end{equation*}
$$

so that the linear momentum of radiation reads

$$
\begin{equation*}
\boldsymbol{p}_{\mathrm{rad}}=\varepsilon_{0} \int_{V} \mathrm{~d}^{3} \boldsymbol{r} \boldsymbol{E}(\boldsymbol{r}, t) \times \boldsymbol{B}(\boldsymbol{r}, t) . \tag{3.55}
\end{equation*}
$$

Let us do some dimension analysis

$$
\begin{align*}
{\left[\boldsymbol{p}_{\mathrm{rad}}\right] } & =\left[\varepsilon_{0}\right][V][E][B] \\
& =\left[\varepsilon_{0}\right]\left[\frac{E^{2}}{c}\right][V]  \tag{3.56}\\
& =\left[\frac{J}{V}\right]\left[\frac{1}{c}\right][V]=\left[\frac{J}{c}\right]
\end{align*}
$$

After a bit of algebra one has

$$
\begin{equation*}
\hat{\boldsymbol{p}}_{\mathrm{rad}}=\sum_{\boldsymbol{k}, \hat{\epsilon}} \hbar \boldsymbol{k} \hat{N}_{\boldsymbol{k}, \hat{\epsilon}} \tag{3.57}
\end{equation*}
$$

Note that it has nothing to do with the de Broglie wavelength. We have thus derived the $\hbar \boldsymbol{k}=$ momentum correspondence without any ad hoc definitions.

### 3.2.6 Photons

Let us make some observations and definitions.
Photons: Excitations of the quantum electromagnetic field;
Energy: $\hbar \omega$;
Momentum: $\hbar \boldsymbol{k}$;
Special Relativity: Energy and momentum are components of a 4 -vector, whose length is a Lorentz invariant

$$
\begin{align*}
\left(m c^{2}\right)^{2} & =E^{2}-p^{2} c^{2} \\
& =(\hbar \omega)^{2}-(\hbar k)^{2} c^{2} \stackrel{c k=\omega}{=} 0 \tag{3.58}
\end{align*}
$$

Velocity: $\boldsymbol{v}=(\boldsymbol{p} / E)^{-1}=c$.
Proper time interval: At $\mathrm{d} \tau$ one has

$$
\begin{equation*}
(\mathrm{d} \tau)^{2}-\left(\frac{\mathrm{d} \boldsymbol{r}}{c}\right)^{2}=(\mathrm{d} \tau)^{2} \stackrel{\text { photon }}{=} 0 \tag{3.59}
\end{equation*}
$$

since $\mathrm{d} \tau$ is a Lorentz invariant. Note that the proper time of the photon does not change; the photon does not evolve.

Remark 3.6. The photon is a purely quantum particle; it does not have a classical limit. One cannot write a Schrödinger equation for the photon, since it doesn't have mass. The photon does not have position or time operators; it cannot be thus measured.

### 3.2.7 Physical Properties of Quantum Radiation

We have defined a Fock space - how many excitations are there in a mode $|n\rangle_{\boldsymbol{k}}$. Since

$$
\begin{equation*}
\left[\hat{N}_{\boldsymbol{k}, \epsilon}, \hat{a}_{\boldsymbol{k}^{\prime}, \epsilon^{\prime}}\right]=-\delta_{\boldsymbol{k}, \boldsymbol{k}^{\prime}} \delta_{\epsilon, \epsilon^{\prime}} \hat{a}_{\boldsymbol{k}}, \tag{3.60}
\end{equation*}
$$

then $\hat{\boldsymbol{E}}$ and $\hat{\boldsymbol{B}}$ cannot be diagonalized in $\{|n\rangle\}$ basis. Moreover, since

$$
\begin{equation*}
{ }_{\boldsymbol{k}}\langle n| \hat{a}_{\boldsymbol{k}}|n\rangle_{\boldsymbol{k}}={ }_{\boldsymbol{k}}\langle n| \hat{a}_{\boldsymbol{k}}^{\dagger}|n\rangle_{\boldsymbol{k}} \tag{3.61}
\end{equation*}
$$

then

$$
\begin{equation*}
{ }_{\boldsymbol{k}}\langle n| \hat{E}(\boldsymbol{r}, t)|n\rangle_{\boldsymbol{k}}=0 \tag{3.62}
\end{equation*}
$$

for all $|n\rangle$ and $\boldsymbol{k}$.
However, one might say that the real quantity needed is the fluctuations around the mean $\left\langle E^{2}\right\rangle$. Let us calculate it. Since

$$
\begin{equation*}
\langle 0| \hat{a}_{\boldsymbol{k}, \epsilon} \hat{a}_{\boldsymbol{k}^{\prime}, \epsilon^{\prime}}^{\dagger}|0\rangle \sim \delta_{\boldsymbol{k}, \boldsymbol{k}^{\prime}} \delta_{\epsilon, \epsilon^{\prime}}, \tag{3.63}
\end{equation*}
$$

one has

$$
\begin{equation*}
\langle 0| \hat{E}^{2}(\boldsymbol{r}, t)|0\rangle=\sum_{\boldsymbol{k}, \hat{\epsilon}} \hbar \omega \rightarrow \infty . \tag{3.64}
\end{equation*}
$$

In other words, the fluctuations diverge. We have still learned nothing.

### 3.3 Spin of the Photon ${ }^{15}$

The main purpose of our endeavor in this chapter was to define matter (atom) + radiation within a quantum framework:

$$
\begin{equation*}
\underbrace{\text { Hilbert }}_{\left(e^{-}, \text {matter }\right)} \otimes \underbrace{\text { Fock }}_{(\text {radiation })} \tag{3.65}
\end{equation*}
$$

This way we can write Hamiltonians etc. We saw that it is possible to write excitations of quantum radiation; these excitations we call photon. To complete the discussion we need to introduce spin to the network.

### 3.3.1 Angular Momenta of Radiation

Recall the expression for the classical total angular momentum of the radiation,

$$
\begin{equation*}
\boldsymbol{J}_{\mathrm{rad}}=\varepsilon_{0} \int \mathrm{~d}^{3} \boldsymbol{r} \boldsymbol{r} \times(\boldsymbol{E}(\boldsymbol{r}, t) \times \boldsymbol{B}(\boldsymbol{r}, t)) \tag{3.66}
\end{equation*}
$$

We shall also employ the Coulomb gauge

$$
\begin{equation*}
\boldsymbol{B}=\boldsymbol{\nabla} \times \boldsymbol{A}, \quad \boldsymbol{\nabla} \cdot \boldsymbol{E}=0 \tag{3.67}
\end{equation*}
$$

In order to continue, we shall write the components of $\boldsymbol{J}_{\text {rad }}$. Start with $\boldsymbol{E} \times \boldsymbol{B}$ to have

$$
\begin{align*}
(\boldsymbol{E} \times \boldsymbol{B})_{k} & =\varepsilon_{k l m} E_{l} B_{m} \\
& =\underbrace{\varepsilon_{k l m} \varepsilon_{m i j}}_{\delta_{k i} \delta_{l j}-\delta_{k j} \delta_{l i}} E_{l} \nabla_{i} B_{j} . \tag{3.68}
\end{align*}
$$

[^13]Hence, using the condition $\boldsymbol{\nabla} \cdot \boldsymbol{E}=0$,

$$
\begin{equation*}
(\boldsymbol{E} \times \boldsymbol{B})_{k}=E_{l} \nabla_{k} A_{l}-\nabla_{l}\left(E_{l} A_{k}\right) . \tag{3.69}
\end{equation*}
$$

This implies

$$
\begin{equation*}
(\boldsymbol{r} \times(\boldsymbol{E} \times \boldsymbol{B}))_{i}=E_{l}(\boldsymbol{r} \times \boldsymbol{\nabla})_{i} A_{l}-\nabla_{l}\left(E_{4} \varepsilon_{i j k} r_{j} A_{k}\right)+\varepsilon_{i l k} E_{l} A_{k} \tag{3.70}
\end{equation*}
$$

The second term $\nabla_{l}\left(E_{l} \varepsilon_{i j k} r_{j} A_{k}\right)$ is a divergence and vanishes after integration at infinity. But, if there are boundaries, we may have topological effects. These come from this term only.

We thus can write $\boldsymbol{J}_{\text {rad }}$ as

$$
\begin{equation*}
\boldsymbol{J}_{\mathrm{rad}}=\boldsymbol{L}+\boldsymbol{S} \tag{3.71}
\end{equation*}
$$

with

$$
\begin{align*}
\boldsymbol{L} & =\varepsilon_{0} \int \mathrm{~d}^{3} \boldsymbol{r} \sum_{j=1}^{3} E_{j} \boldsymbol{r} \times \boldsymbol{\nabla} A_{j}  \tag{3.72a}\\
\boldsymbol{S} & =\varepsilon_{0} \int \mathrm{~d}^{3} \boldsymbol{r} \boldsymbol{E} \times \boldsymbol{A} \tag{3.72b}
\end{align*}
$$

One of this quantities, $\boldsymbol{L}$, is not specific for the electromagnetic field because

- First, it is dependent on $\boldsymbol{r}$.
- Second, it can be generalized to a scalar field $\varphi: \boldsymbol{L} \propto \int \mathrm{d}^{3} \boldsymbol{r}\left(-\partial_{t} \varphi\right)(\boldsymbol{r} \times \boldsymbol{\nabla} \varphi)$.

This $\boldsymbol{L}$ is called the orbital angular momentum of the radiation. The second quantity $\boldsymbol{S}$ is called the spin, and it is intrinsic to the electromagnetic field.

### 3.3.2 Calculation of Spin

Let us write the quantum counterpart to the spin we have just found,

$$
\begin{equation*}
\hat{\boldsymbol{S}}=-\mathrm{i} \hbar \sum_{\boldsymbol{k}}\left(\sum_{T_{1}=1}^{2} \hat{\epsilon}_{\boldsymbol{k}, T_{1}} \hat{a}_{\boldsymbol{k}, T_{1}}\right) \times\left(\sum_{T_{2}=1}^{2} \hat{\epsilon}_{\boldsymbol{k}, T_{2}} \hat{a}_{\boldsymbol{k}, T_{2}}\right) \tag{3.73}
\end{equation*}
$$

with $T_{1,2}$ the linear polarizations and

$$
\begin{equation*}
\hat{\epsilon}_{\boldsymbol{k}, 1} \times \hat{\epsilon}_{\boldsymbol{k}, 2}=\frac{\boldsymbol{k}}{|\boldsymbol{k}|} . \tag{3.74}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
\hat{\boldsymbol{S}}=-\mathrm{i} \hbar \sum_{\boldsymbol{k}} \frac{\boldsymbol{k}}{|\boldsymbol{k}|}\left(\hat{a}_{\boldsymbol{k}, 1}^{\dagger} \hat{a}_{\boldsymbol{k}, 2}-\hat{a}_{\boldsymbol{k}, 2}^{\dagger} \hat{a}_{\boldsymbol{k}, 1}\right) . \tag{3.75}
\end{equation*}
$$

Now, the polarization basis we have used was the linear polarization basis. However, it is not very useful to our case. We would make the transformation to a circular polarization basis. Define

$$
\begin{equation*}
\hat{\epsilon}_{ \pm}=\mp \frac{1}{\sqrt{2}}\left(\hat{\epsilon}_{\boldsymbol{k}, 1} \pm \mathrm{i} \hat{\epsilon}_{\boldsymbol{k}, 2}\right) \tag{3.76}
\end{equation*}
$$



Fig. 3.1: Linear polarization directions.
so that

$$
\left\{\begin{array}{l}
\hat{\epsilon}_{\lambda}^{*} \cdot \hat{\epsilon}_{\lambda}=\delta_{\lambda \lambda^{\prime}}  \tag{3.77}\\
\hat{\epsilon}_{\lambda}^{*} \times \hat{\epsilon}_{\lambda}=\mathrm{i} \lambda \frac{k}{|\boldsymbol{k}|} \delta_{\lambda \lambda^{\prime}}
\end{array} \quad \text { for } \lambda= \pm 1 .\right.
$$

We shall also redefine our creation and annihilation operators

$$
\left\{\begin{array}{l}
\hat{a}_{ \pm}=\mp \frac{1}{\sqrt{2}}\left(\hat{a}_{1} \mp \mathrm{i} \hat{a}_{2}\right)  \tag{3.78}\\
\hat{a}_{ \pm}^{\dagger}= \pm \frac{1}{\sqrt{2}}\left(\hat{a}_{1}^{\dagger} \pm \mathrm{i} \hat{a}_{2}^{\dagger}\right),
\end{array}\right.
$$

with the commutation relations

$$
\begin{equation*}
\left[\hat{a}_{\boldsymbol{k}, \lambda}, \hat{a}_{\boldsymbol{k}^{\prime}, \lambda}^{\dagger}\right]=\delta_{\boldsymbol{k}, \boldsymbol{k}^{\prime}} \delta_{\lambda, \lambda^{\prime}} \tag{3.79}
\end{equation*}
$$

Therefore, we have what we wanted,

$$
\begin{equation*}
\hat{\boldsymbol{S}}=\hbar \sum_{\boldsymbol{k}, \lambda} \lambda \hat{k} \underbrace{\hat{a}_{\boldsymbol{k}, \lambda}^{\dagger} \hat{a}_{\boldsymbol{k}, \lambda}}_{\hat{N}_{\boldsymbol{k}, \lambda}}, \tag{3.80}
\end{equation*}
$$

meaning the linear dependence on $\hat{N}_{\boldsymbol{k}, \lambda}$.

### 3.3.3 Observations

Since

$$
\begin{equation*}
|1\rangle_{\boldsymbol{k}, \pm}=\hat{a}_{\boldsymbol{k}, \pm}^{\dagger}|0\rangle_{\boldsymbol{k}, \pm}, \tag{3.81}
\end{equation*}
$$

the polarization is the eigenvalue of the spin operator on the eigenvector

$$
\begin{equation*}
\frac{\boldsymbol{k}}{|\boldsymbol{k}|} \cdot \hat{\boldsymbol{S}}|1\rangle_{\boldsymbol{k}, \pm}=\hbar \lambda|1\rangle_{\boldsymbol{k}, \pm} \tag{3.82}
\end{equation*}
$$

Note that we cannot define a photon without its dependence on the $\boldsymbol{k}$ vector.
There are two states of polarization because the photon is a massless spin-1 particle. It means in doesn't have helicity 0 . If it had mass, it would mean we could find a rest reference frame; thus, it would have had 3 states.

Inside matter, all this derivation wouldn't work so cleanly. We wouldn't find simple expressions for $\boldsymbol{L}$ and $\boldsymbol{S}$.

## 4 Quantum Electrodynamics - Quantum Vacuum Effects ${ }^{16}$

### 4.1 Introduction - Technical Intermezzo

The main purpose of this chapter is to combine Hilbert (atom) and Fock (photons) spaces

$$
\begin{equation*}
\underbrace{\text { Hilbert }}_{\text {(atom) }} \otimes \underbrace{\text { Fock }}_{\text {(photons) }} . \tag{4.1}
\end{equation*}
$$

Or, in the Hamiltonian language,

$$
\begin{equation*}
\hat{H}=\hat{H}_{\mathrm{A}}+\hat{H}_{\mathrm{rad}}+\hat{H}_{\mathrm{int}} . \tag{4.2}
\end{equation*}
$$

We shall use the quantum electrodynamics (QED) language in this chapter.

### 4.1.1 Reminder - How to Calculate Relaxations in QED

In the framework of Fermi golden rule

$$
\begin{equation*}
P(t) \simeq 1-\Gamma t \tag{4.3}
\end{equation*}
$$

The relaxations were given phenomenologically. We shall below derive relaxations in a more systematic way.

In simple words, we need to find the evolution operator $\hat{U}\left(t, t^{\prime}\right)$. Its Fourier transform is given by

$$
\begin{equation*}
\hat{G}_{\mathrm{ill}}(E)=\frac{1}{\mathbb{1} E-\hat{H}} . \tag{4.4}
\end{equation*}
$$

However, at the eigenenergies (eigenvalues of $\hat{H}$ ) it is ill-defined. Therefore, we would move to the complex plane in order to avoid this issue,

$$
\begin{equation*}
E \rightarrow z=E+\mathrm{i} \eta, \quad \eta \in \mathbb{R} . \tag{4.5}
\end{equation*}
$$

Hence, we define the resolvant operator

$$
\begin{equation*}
\hat{G}(z)=\frac{1}{\mathbb{1} z-\hat{H}}, \quad z \in \mathbb{C} . \tag{4.6}
\end{equation*}
$$

Fig. 4.1: The resolvant operator in the complex plane.

### 4.1.2 Resolvant Operator Approach

Let us look on the Hamiltonian

$$
\begin{equation*}
\hat{H}=\hat{H}_{0}+\hat{V}, \tag{4.7}
\end{equation*}
$$

where

$$
\begin{equation*}
\hat{H}_{0}=\hat{H}_{\mathrm{A}}+\hat{H}_{\mathrm{rad}}, \quad \hat{V}=\hat{H}_{\mathrm{int}} . \tag{4.8}
\end{equation*}
$$

[^14]Write the associated resolvants,

$$
\begin{equation*}
\hat{G}(z)=\frac{1}{z-\hat{H}}, \quad \hat{G}_{0}(z)=\frac{1}{z-\hat{H}_{0}} . \tag{4.9}
\end{equation*}
$$

Using these definitions one can deduce

$$
\begin{equation*}
\hat{G}(z)=\hat{G}_{0}(z)+\hat{G}_{0} \hat{V} \hat{G} \tag{4.10}
\end{equation*}
$$

Thus, substituting this equation into itself, one has

$$
\begin{equation*}
\hat{G}(z)=\hat{G}_{0}(z)+\hat{G}_{0} \hat{V} \hat{G}_{0}+\hat{G}_{0} \hat{V} \hat{G}_{0} \hat{V} \hat{G}_{0}+\ldots \tag{4.11}
\end{equation*}
$$

Note that this is not a perturbative series. Usually, one says that $\hat{V}$ is small, and truncates all the elements $\propto \hat{V}^{2}$. However, it is a poor approximation, and we avoid it.

The matrix elements of $\hat{G}(z)$ in the set of eigenstates of $\hat{H}_{0}:\left\{\left|\phi_{k}\right\rangle\right\}$

$$
\begin{align*}
\hat{G}_{k l}(z) & \equiv\left\langle\phi_{k}\right| \hat{G}(z)\left|\phi_{l}\right\rangle \\
& =\frac{1}{z-E_{k}} \delta_{k l}+\frac{1}{z-E_{k}} V_{k l} \frac{1}{z-E_{l}}+\sum_{i} \frac{1}{z-E_{k}} V_{k i} \frac{1}{z-E_{i}} V_{i l} \frac{1}{z-E_{l}}+\ldots \tag{4.12}
\end{align*}
$$

Let us inspect the case with $k=l=b$,

$$
\begin{equation*}
\hat{G}_{b}(z)=\frac{1}{z-E_{b}}+\overbrace{\frac{1}{z-E_{b}} V_{b b} \frac{1}{z-E_{b}}+\sum_{i} \frac{1}{z-E_{b}} V_{b i} \frac{1}{z-E_{i}} V_{i b} \frac{1}{z-E_{b}}+\ldots}^{\propto 1 /\left(z-E_{b}\right)^{2}} \tag{4.13}
\end{equation*}
$$

Or, pictorially,

Where we have associated the connection $\left.\right|_{b} \equiv\left|\phi_{b}\right\rangle$ with $\circ \equiv V_{b b}$.
We can thus write the value $R_{b}(z)$ (without the outward $b$ connections)

$$
\begin{equation*}
R_{b}(z)=\square=\circ+\underset{\circ}{\circ}+\underset{\vdots}{\circ}+\stackrel{\circ}{\vdots i}+\ldots \tag{4.15}
\end{equation*}
$$

Or, explicitly,

$$
\begin{equation*}
R_{b}(z)=V_{b b}+\sum_{i} V_{b i} \frac{1}{z-E_{i}} V_{i b}+\ldots \tag{4.16}
\end{equation*}
$$

Higher powers of $\frac{1}{z-E_{b}}$ are proportional to $\left(\frac{1}{z-E_{b}}\right)^{3} R_{b}^{2}(z)$. Thus, we shall neglect them for now.

All this was to say that

$$
\begin{aligned}
G_{b}(z)=\sum_{n=1}^{\infty} \frac{\left(R_{b}(z)\right)^{n-1}}{\left(z-E_{b}\right)^{n}} & =\frac{1}{z-E_{b}} \sum_{n=0}^{\infty}\left(\frac{R_{b}(z)}{z-E_{b}}\right)^{n} \\
& =\frac{1}{z-E_{b}} \cdot \frac{1}{1-\frac{R_{b}(z)}{z-E_{b}}},
\end{aligned}
$$

without worrying too much about radius of convergence. Thus,

$$
\begin{equation*}
G_{b}(z)=\frac{1}{z-E_{b}-R_{b}(z)} . \tag{4.17}
\end{equation*}
$$

This $R_{b}(z)$ is sometimes called the self-energy.
Remark 4.1. This calculation is general for many-body physics problems, and standard in books. We follow Chap. 3 of Cohen-Tannoudji, Dupont-Roc, and Grynberg [2008].

### 4.1.3 Standard Approximations ${ }^{17}$

Let us introduce two standard approximations.

1. For $V \ll H_{0}$ :

Very often (controllably), the series $R_{b}(z)$ converges. It is often appropriate to keep a finite number of terms in $R_{b}(z)$. Define

$$
\begin{equation*}
\widetilde{R_{b}}(z) \simeq V_{b b}+\sum_{i \neq b} V_{b i} \frac{1}{z-E_{i}} V_{i b}, \tag{4.18}
\end{equation*}
$$

such that

$$
\begin{equation*}
\widetilde{G_{b}}(z)=\frac{1}{z-E_{b}-\widetilde{R_{b}}(z)} . \tag{4.19}
\end{equation*}
$$

Diagrammatically, it reads

$$
\begin{equation*}
\Delta \equiv 0+\stackrel{\circ}{i}_{i}^{\circ} \tag{4.20}
\end{equation*}
$$

But, the infinite series does not truncates; there is still an infinite number of terms,

$$
\begin{equation*}
\left|+\underset{\left.\right|^{b}}{\left.\right|_{b}}+\frac{\square_{\mid b}^{b}}{\square^{b}}+\cdots \simeq\right|+\underset{\left.\right|^{b}}{\left.\right|_{b} ^{b}}+\underset{\Delta_{\mid b}^{\Delta}}{\stackrel{\Delta}{b}_{i}^{b}}+\cdots \tag{4.21}
\end{equation*}
$$

2. Markov approximation.

Consider (4.17). Rewrite it to be

$$
\begin{equation*}
G_{b}(z)=\frac{1}{E \pm \mathrm{i} \eta-E_{b}-R_{b}(E \pm \mathrm{i} \eta)}, \tag{4.22}
\end{equation*}
$$

[^15]with $z=E \pm \mathrm{i} \eta$. Now, consider the limit $\eta \rightarrow 0$. In this case, $R_{b}(E \pm \mathrm{i} \eta)$ has a much "smoother" behavior close to $E \simeq E_{b}$ than $G_{0}\left(E \simeq E_{b}\right)$.

We therefore replace $E$ with $E_{b}$ and consider $R_{b}\left(E_{b} \pm \mathrm{i} \eta\right)$. This is the Markov approximation. The physical meaning is that the system looses its memory (in time).

### 4.2 Proper Theoretical Description of Relaxation

In this section we will describe properly the theory of relaxation. We shall also describe other things, such as energy shift, Lamb shift, etc.

### 4.2.1 Back to the Atom+Photon Problem

Let us recall the initial problem we sought to solve. The Hamiltonian reads

$$
\begin{equation*}
\hat{H}_{0}=\underbrace{\hbar \omega_{0}|b\rangle\langle b|}_{\text {atomic } \hat{H}_{\mathrm{at}}}+\underbrace{\sum_{\boldsymbol{k}, T} \hbar \omega_{\boldsymbol{k}, T}\left(\hat{a}_{\boldsymbol{k}, T} \hat{a}_{\boldsymbol{k}, T}^{\dagger}+\frac{1}{2}\right)}_{\text {quantum e.m. }} . \tag{4.23}
\end{equation*}
$$

The spectrum of $\hat{H}_{0}$ is a continuum with discrete state at $E_{g}=0$ and $E_{e}=E_{b}$.


Fig. 4.2: The spectrum of $\hat{H}_{0}$.

We want to calculate the relaxation rates. Introduce the interaction Hamiltonian

$$
\begin{equation*}
\hat{H}_{\mathrm{int}} \equiv \hat{V}=-\hat{\boldsymbol{p}} \cdot \hat{\boldsymbol{A}}_{\perp}=-\hat{\boldsymbol{d}} \cdot \hat{\boldsymbol{E}} . \tag{4.24}
\end{equation*}
$$

Let the initial state be $|b ; 0\rangle$ and ask ourselves, what is the probability to stay in it. It is given by

$$
\begin{equation*}
\left.P_{b}(\tau)=|\langle b ; 0| \hat{U}(\tau)| b ; 0\right\rangle\left.\right|^{2} \tag{4.25}
\end{equation*}
$$

with $\hat{U}(\tau)$ the evolution operator. We know that

$$
\begin{equation*}
U_{b}(\tau) \equiv\langle b ; 0| \hat{U}(\tau)|b ; 0\rangle=\frac{1}{2 \pi \mathrm{i}} \int_{C} \mathrm{~d} z \mathrm{e}^{-\mathrm{i} z \tau / \hbar}\langle b ; 0| \frac{1}{z-\hat{H}}|b ; 0\rangle . \tag{4.26}
\end{equation*}
$$

Implementing the First Approximation
Let us start with the first approximation $V \ll H_{0}$ to have

$$
\begin{equation*}
\widetilde{R_{b}}(z)=\underbrace{\langle b ; 0| \hat{V}|b ; 0\rangle}_{\equiv 0}+\sum_{a} \sum_{\boldsymbol{k}, T} \frac{\langle b ; 0| \hat{V}|a ; \boldsymbol{k}, T\rangle\langle a ; \boldsymbol{k}, T| \hat{V}|b ; 0\rangle}{z-E_{a}-\hbar \omega_{\boldsymbol{k}, T}} . \tag{4.27}
\end{equation*}
$$

Or, pictorially,

$$
\begin{equation*}
\widetilde{R_{b}}(z)=0+\widetilde{\Omega} \tag{4.28}
\end{equation*}
$$

Hence, we can write

This is the "Ladder Diagrams" approximation, or the random phase approximation (RPA).

Note that we have neglected several diagrams,

and


The last one belongs to the class of cross diagrams.
Finally, we can write

$$
\begin{equation*}
\widetilde{R_{b}}(E \pm \mathrm{i} \eta)=\sum_{a} \sum_{\boldsymbol{k}, T} \frac{|\langle a ; \boldsymbol{k}, T| \hat{V}| b ; 0\rangle\left.\right|^{2}}{E \pm \mathrm{i} \eta-E_{a}-\hbar \omega_{\boldsymbol{k}, T}} \tag{4.31}
\end{equation*}
$$

Note that

$$
\begin{equation*}
\frac{1}{x \pm \mathrm{i} \eta}=\frac{x}{x^{2}+\eta^{2}} \mp \frac{\mathrm{i} \eta}{x^{2}+\eta^{2}} \xrightarrow{\eta \rightarrow 0} \mathcal{P}\left(\frac{1}{x}\right) \mp \mathrm{i} \pi \delta(x), \tag{4.32}
\end{equation*}
$$

where $\mathcal{P}(\cdot)$ is the Cauchy principal value. Hence,

$$
\begin{equation*}
\widetilde{R_{b}}(E \pm \mathrm{i} \eta) \equiv \hbar \widetilde{\Delta_{b}}(E) \mp \mathrm{i} \hbar \frac{\widetilde{\Gamma_{b}}(E)}{2} \tag{4.33}
\end{equation*}
$$

with

$$
\left\{\begin{array}{l}
\widetilde{\Delta_{b}}(E) \equiv \frac{1}{\hbar} \mathcal{P}\left(\sum_{a} \sum_{\boldsymbol{k}, T} \frac{|\langle a ; \boldsymbol{k}, T| \hat{V}| b ; 0\rangle\left.\right|^{2}}{E-E_{a}-\hbar \omega_{\boldsymbol{k}, T}}\right)  \tag{4.34}\\
\left.\widetilde{\Gamma_{b}}(E) \equiv \frac{2 \pi}{\hbar} \sum_{a} \sum_{\boldsymbol{k}, T}|\langle a ; \boldsymbol{k}, T| \hat{V}| b ; 0\right\rangle\left.\right|^{2} \delta\left(E-E_{a}-\hbar \omega\right)
\end{array}\right.
$$

Therefore,

$$
\begin{equation*}
\widetilde{G_{b}}(E \pm \mathrm{i} \eta)=\frac{1}{E \pm \mathrm{i} \eta-E_{b}-\hbar \widetilde{\Delta_{b}}(E) \pm \mathrm{i} \frac{\hbar}{2} \widetilde{\Gamma_{b}}(E)} \tag{4.35}
\end{equation*}
$$

These are the spectral functions.

Remark 4.2. The two spectral functions $\Delta$ and $\Gamma$ are not independent.

- Causality (physics language).
- Kramers-Krönig relations (electrical engineering language).
- Hilbert transforms (mathematics language)

$$
\begin{equation*}
\widetilde{\Delta_{b}}(E)=\frac{1}{2 \pi} \mathcal{P} \int \mathrm{~d} E^{\prime}\left(\widetilde{\bar{\Gamma}_{b}\left(E^{\prime}\right)} \underset{E-E^{\prime}}{)} .\right. \tag{4.36}
\end{equation*}
$$

Remark 4.3. $\widetilde{\Gamma_{b}}(E) \geq 0$. Therefore, if $\widetilde{\Gamma_{b}}(E) \neq 0$ then $\widetilde{\Gamma_{b}}\left(E_{a}\right)=0$ (due to the $\delta$ function).

### 4.2.2 Implementing the Second Approximation

Take $E \rightarrow E_{b}$ to have

$$
\left\{\begin{array}{l}
\widetilde{\Delta_{b}}\left(E_{b}\right) \equiv \frac{1}{\hbar} \mathcal{P}\left(\sum_{a} \sum_{\boldsymbol{k}, T} \frac{|\langle a ; \boldsymbol{k}, T| \hat{V}| b ; 0\rangle\left.\right|^{2}}{E_{b}-E_{a}-\hbar \omega_{\boldsymbol{k}, T}}\right)  \tag{4.37}\\
\left.\widetilde{\Gamma_{b}}\left(E_{b}\right) \equiv \frac{2 \pi}{\hbar} \sum_{a} \sum_{\boldsymbol{k}, T}|\langle a ; \boldsymbol{k}, T| \hat{V}| b ; 0\right\rangle\left.\right|^{2} \delta\left(E_{b}-E_{a}-\hbar \omega\right)
\end{array}\right.
$$

Hence, one has

$$
\begin{equation*}
\widetilde{G_{b}}(E \pm \mathrm{i} \eta)=\frac{1}{E \pm \mathrm{i} \eta-E_{b}-\hbar \widetilde{\Delta_{b}} \pm \mathrm{i} \frac{\hbar}{2} \widetilde{\Gamma_{b}}} \tag{4.38}
\end{equation*}
$$

Note that $\widetilde{\Delta_{b}}$ and $\widetilde{\Gamma_{b}}$ are now constants. Now, since

$$
\begin{equation*}
\langle b ; 0| \hat{U}(\tau)|b ; 0\rangle=\frac{1}{2 \pi \mathrm{i}} \int_{C} \mathrm{~d} z \mathrm{e}^{-\mathrm{i} z \tau / \hbar}\langle b ; 0| G_{b}(z)|b ; 0\rangle, \tag{4.39}
\end{equation*}
$$

with $z=E \pm \mathrm{i} \eta$. Notice it is independent with $z$ due to approximation 2. Hence, the probability amplitude reads

$$
\begin{equation*}
\langle b ; 0| \hat{U}(\tau)|b ; 0\rangle=\mathrm{e}^{-\mathrm{i}\left(E_{b}+\hbar \Delta_{b}\right) \tau / \hbar} \mathrm{e}^{-\frac{1}{2} \Gamma \tau} \tag{4.40}
\end{equation*}
$$

Therefore, one gets the probability

$$
\begin{equation*}
\left.P_{b}(\tau)=|\langle b ; 0| \hat{U}(\tau)| b ; 0\right\rangle\left.\right|^{2}=\mathrm{e}^{-\Gamma \tau} . \tag{4.41}
\end{equation*}
$$

This is the Wigner-Weisskopf expression.
Remark 4.4. The shift $\mathrm{e}^{-\mathrm{i} \Delta_{b} \tau / \hbar}$ is systematic. If one has $\Gamma_{b}$ then $\Delta_{b}$ must exist too. This is the Lamb shift.

### 4.2.3 The Wigner-Weisskopf Derivation ${ }^{18}$

Let us derive the same result, but in a different, simpler, way. Let us start with the initial state $|b ; 0\rangle$. This state evolves due to interaction with the quantum electromagnetic field,

$$
\begin{equation*}
|\psi(t)\rangle=\gamma_{0}(t)|b ; 0\rangle+\sum_{\ell} \gamma_{\ell}(t)\left|a ; 1_{\ell}\right\rangle \tag{4.42}
\end{equation*}
$$

[^16]with $\ell \equiv\{\boldsymbol{k}, T\}$. This is a Schrödinger-like picture of evolution. Thus, we have the following coupled equations of evolution
\[

\left\{$$
\begin{array}{l}
\mathrm{i} \hbar \frac{\mathrm{~d} \gamma_{0}}{\mathrm{~d} t}=\hbar \omega_{0} \gamma_{0}+\sum_{\ell} \gamma_{\ell}(t)  \tag{4.43}\\
\mathrm{i} \hbar \frac{\mathrm{~d} \gamma_{\ell}}{\mathrm{d} t}=\hbar \omega_{\ell} \gamma_{\ell}+\gamma_{0} V_{\ell}^{*}
\end{array}
$$\right.
\]

with

$$
\begin{equation*}
V_{\ell}=\langle b ; 0| \hat{V}\left|a ; 1_{\ell}\right\rangle=-\mathrm{i} E_{\ell}\left(\hat{\epsilon}_{\ell} \cdot \hat{e}_{z}\right) d_{a b} \tag{4.44}
\end{equation*}
$$

where we have used the usual notations for the last equality (coming from $\hat{\boldsymbol{d}} \cdot \hat{\boldsymbol{E}}$ ). Next, we introduce the boundary conditions,

$$
\left\{\begin{array}{l}
\gamma_{0}(0)=1  \tag{4.45}\\
\gamma_{1}(0)=0
\end{array}\right.
$$

After integration one has

$$
\begin{equation*}
\gamma_{\ell}(t)=\frac{V_{\ell}^{*}}{\mathrm{i} \hbar} \int_{0}^{t} \mathrm{~d} t^{\prime} \gamma_{0}\left(t^{\prime}\right) \mathrm{e}^{\mathrm{i} \omega_{\ell}\left(t-t^{\prime}\right)} \tag{4.46}
\end{equation*}
$$

Plugging it into (4.43) one has

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t} \gamma_{0}(t)=-\mathrm{i} \omega_{0} \gamma_{0}-\sum_{\ell} \frac{\left|V_{\ell}\right|^{2}}{\hbar^{2}} \int_{0}^{t} \mathrm{~d} t^{\prime} \gamma_{0}\left(t^{\prime}\right) \mathrm{e}^{\mathrm{i} \omega_{\ell}\left(t-t^{\prime}\right)} \tag{4.47}
\end{equation*}
$$

Define the following unitary transformation

$$
\left\{\begin{array}{l}
\gamma_{0}(t)=\alpha(t) \mathrm{e}^{-\mathrm{i} \omega_{0} t}  \tag{4.48}\\
\mathcal{N}(t)=\frac{1}{\hbar^{2}} \sum_{\ell}\left|V_{\ell}\right|^{2} \mathrm{e}^{\mathrm{i}\left(\omega_{0}-\omega_{\ell}\right) t}
\end{array}\right.
$$

one obtains the formal solution

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t} \alpha(t)=-\int_{0}^{t} \mathrm{~d} \tau \mathcal{N}(\tau) \alpha(t-\tau) \tag{4.49}
\end{equation*}
$$

with $\tau=t-t^{\prime}$.
Notice that in order to know $\alpha$, one must know it in all prior times. Therefore, $\mathcal{N}(\tau)$ is called a memory function. Let us calculate it explicitly,

$$
\begin{equation*}
\mathcal{N}(\tau)=\frac{d_{a b}^{2}}{\hbar^{2}} \underbrace{\left(\sum_{\ell}\left(\hat{\epsilon}_{\ell} \cdot \hat{e}_{z}\right)^{2} E_{\ell}^{2} \mathrm{e}^{-\mathrm{i} \omega_{\ell} \tau}\right)}_{\equiv G(\tau)} \mathrm{e}^{\mathrm{i} \omega_{0} \tau} \tag{4.50}
\end{equation*}
$$

Note that $G(\tau)$ depends only on the quantum electromagnetic field. Thus, we have decoupled the electromagnetic field from the atom. Next, take correlation function of the $\hat{z}$ component of the electromagnetic field with respect to the vacuum, and obtain

$$
\begin{equation*}
G(\tau)=\langle 0| \hat{E}_{z}(t) \hat{E}_{z}^{\dagger}(t-\tau)|0\rangle \text {. } \tag{4.51}
\end{equation*}
$$

Corollary 4.1. We see that the memory function - describing the evolution of the atom - is dependent on the quantum correlations of the vacuum field only (and not the atom). This is what drives the atom in an excited state to emit a photon and decay to the ground state.

## Limits.

- For $\tau=0$ one has a full interference.
- For $\tau \neq 0$ and $\tau$ large enough, $G(\tau), \mathcal{N}(\tau) \rightarrow 0$. Typically, the correlation time is $\tau_{c} \propto 10^{-15}[\mathrm{sec}]$.
Let us continue the derivation of Eq. (4.49). Due to the large correlation time we can decouple $\mathcal{N}(\tau)$ from $\alpha(t)$ (approximation \#2), and take the limit to infinity (approximation $\# 1$ ) to have

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t} \alpha(t) \simeq-\alpha(t) \int_{0}^{\infty} \mathrm{d} \tau \mathcal{N}(\tau) \tag{4.52}
\end{equation*}
$$

Now, write formally

$$
\begin{equation*}
\int_{0}^{\infty} \mathrm{d} \tau \mathcal{N}(\tau) \equiv \frac{1}{2} \Gamma_{b}+\mathrm{i} \Delta_{b} . \tag{4.53}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
\gamma_{0}(t)=\mathrm{e}^{-\frac{1}{2} \Gamma_{b} t} \mathrm{e}^{-\mathrm{i}\left(\omega_{0}+\Delta_{b}\right) t} \tag{4.54}
\end{equation*}
$$

as before.
Remark 4.5. One could write a Lindblad equation for this problem. Since we use the Markov approximation, it will have exactly the same results up to $\omega_{0} \rightarrow$ $\omega_{0}+\Delta_{b}$.
Remark 4.6. We could also calculate the $T_{1}$ and $T_{2}$ times in this derivation. Recall the factor 2 between $T_{1}$ and $T_{2}$; it comes from the factor $\frac{1}{2}$ in $\Gamma_{b}$.

### 4.3 Calculation of the Lamb Shift

What we want to do next is the following. Let us calculate the Lamb shift $\Delta_{b}$; it will not be as trivial as seems. We would like to start from the diagram of $R_{b}(z)$ and get the number of $\Delta_{b} \simeq 1040[\mathrm{MHz}]$.

### 4.3.1 A Bit of History

First, note that if the shift is the same for all the levels, then we cannot measure it. However, since we calculate the matrix elements of $\Delta_{b}$ and $\Gamma_{b}$, we have selection rules. Thus, we can measure it.

Now, consider a hydrogen atom. One makes the calculation (Bohr model) that the states of $2 s$ and $2 p$ are degenerate


Dirac had later shown that these states were not degenerate. The shift to the next ${ }^{2} p_{\frac{3}{2}}$ level is $10950[\mathrm{MHz}]$ due to fine structure:


It was later discovered this shift of ${ }^{2} p_{\frac{3}{2}}$ is due to quantum vacuum fluctuations.
The states ${ }^{2} s_{\frac{1}{2}}$ and ${ }^{2} p_{\frac{1}{2}}$ are degenerate (from Schrödinger equation + fine structure). People had believed that there was a shift between the ${ }^{2} s_{\frac{1}{2}}$ and ${ }^{2} p_{\frac{1}{2}}$ states. However, the experiments at that time were not conclusive. In 1947, Lamb and his student Retherford [Lamb and Retherford, 1947] have measured the shift with microwave radiation to be $\sim 1058[\mathrm{MHz}]$ as shown below:

$$
\begin{gather*}
\overline{\underline{\underline{I} 1058[\mathrm{MHz}]}}{ }^{2} s_{\frac{1}{2}} p_{\frac{1}{2}}  \tag{4.57}\\
{ }^{1} s_{\frac{1}{2}}
\end{gather*}
$$

In 1947, there was a conference to which Lamb was invited to show his experiment results. On the way to the conference Hans Bethe has calculated on the back of an envelope the value of $\Delta_{b} \simeq 1040[\mathrm{MHz}]$, thus coining the term. We will show this calculation below.

Today we measure experimentally $\Delta_{b}^{\exp } \simeq 1057.893[\mathrm{MHz}]$, and calculate theoretically (with divergent series) $\Delta_{b}^{\text {theo }} \simeq 1057.864[\mathrm{MHz}]$. This is an excellent result demonstrating the power of QED. Note that this shift is very small compared to the fine structure.

### 4.3.2 Direct Calculation ${ }^{19}$

In this part we calculate the quantity $\Delta_{b}$ starting from Eq. (4.37). Note that all quantities involving such equations diverge - and very rapidly. We will introduce the notion of renormalization to tackle this problem.

We want to calculate the diagram


Let us start from Eq. (4.37). Take the volume $V \rightarrow \infty$ such that $\frac{1}{V} \sum_{\boldsymbol{k}} \rightarrow \int \mathrm{d}^{3} \boldsymbol{k}$ and choose the potential $\hat{V}=\frac{q}{m} \boldsymbol{p} \cdot \boldsymbol{A}$. Hence, (4.37) transforms to

$$
\begin{equation*}
\left.\hbar \Delta_{b}=\left.\mathcal{P}\left(\sum_{a} \frac{1}{V} \int \mathrm{~d}^{3} \boldsymbol{k} \sum_{T}\left|\langle a ; \boldsymbol{k}, T| \frac{q}{m} \boldsymbol{p} \cdot \boldsymbol{A}\right| b ; 0\right\rangle\right|^{2} \frac{1}{E_{b}-E_{a}-\hbar \omega_{\boldsymbol{k}, T}}\right) \tag{4.59}
\end{equation*}
$$

We saw that the vector potential is

$$
\begin{equation*}
\left|\hat{\boldsymbol{A}}_{\boldsymbol{k}, T}\right|^{2}=\frac{\hbar}{2 \varepsilon_{0} \omega} \hat{n} . \tag{4.60}
\end{equation*}
$$

Also, write explicitly

$$
\begin{equation*}
\int \mathrm{d}^{3} \boldsymbol{k}=\int_{0}^{\infty} \mathrm{d} \omega \int \mathrm{~d} \hat{\Omega} \rho(\omega) \tag{4.61}
\end{equation*}
$$

where $\omega=c|\boldsymbol{k}|$ the angular frequency, and

$$
\begin{equation*}
\rho(\omega)=\frac{\omega^{2}}{c^{3}} \frac{V}{(2 \pi)^{3}} . \tag{4.62}
\end{equation*}
$$

[^17]Put everything together to obtain

$$
\begin{align*}
\hbar \Delta_{b}=\mathcal{P}( & \sum_{a} \sum_{T} \frac{q^{2}}{m^{2}} \frac{1}{(2 \pi c)^{3}} \times  \tag{4.63}\\
& \left.\left.\int_{0}^{\infty} \mathrm{d} \omega \int \mathrm{~d} \hat{\Omega} \frac{\omega^{2}}{E_{b}-E_{a}-\hbar \omega} \cdot \frac{\hbar}{2 \varepsilon_{0} \omega}\left|\langle a ; \boldsymbol{k}, T| \boldsymbol{p} \cdot \hat{\epsilon}_{\boldsymbol{k}, T}\right| b ; 0\right\rangle\left.\right|^{2}\right) .
\end{align*}
$$

Collecting all constant terms together, and noting that the fine-structure constant is

$$
\begin{equation*}
\alpha_{\mathrm{fs}} \equiv \frac{q^{2} / 4 \pi \varepsilon_{0}}{\hbar c} \simeq \frac{1}{137} \tag{4.64}
\end{equation*}
$$

one has

$$
\begin{equation*}
\frac{q^{2}}{m^{2}} \frac{1}{2 c^{3} \varepsilon_{0} \hbar} \frac{1}{(2 \pi)^{3}}=\frac{\alpha_{\mathrm{fs}}}{4 \pi^{2}(m c)^{2}} \tag{4.65}
\end{equation*}
$$

Also,

$$
\begin{equation*}
\left.\left.\sum_{T=1}^{2} \int \mathrm{~d} \hat{\Omega}\left|\langle a ; \boldsymbol{k}, T| \boldsymbol{p} \cdot \hat{\epsilon}_{\boldsymbol{k}, T}\right| b ; 0\right\rangle\left.\right|^{2}=\frac{8 \pi}{3}|\langle a| \boldsymbol{p}| b\right\rangle\left.\right|^{2} . \tag{4.66}
\end{equation*}
$$

Combine all to have

$$
\begin{equation*}
\hbar \Delta_{b}=\frac{2 \alpha_{\mathrm{fs}}}{3 \pi} \sum_{a} \frac{|\langle a| \boldsymbol{p}| b\rangle\left.\right|^{2}}{(m c)^{2}} \mathcal{P}\left(\int_{0}^{\infty} \mathrm{d} E \frac{E}{E_{b}-E_{a}-E}\right), \tag{4.67}
\end{equation*}
$$

with $E=\hbar \omega$.
Remark 4.7. The dipole approximation $\boldsymbol{d} \cdot \boldsymbol{E}=\frac{q}{m} \boldsymbol{p} \cdot \boldsymbol{A}$ is not valid in this case, since it works only when the wavelength is larger than the atom size, but the integration is from 0 to $\infty$. It can be justified, but requires more gymnastics.

### 4.3.3 Dealing with Infinities

The principal value in the previous equation is justified when the integral is finite. When it is infinite, we must introduce cutoffs. Introducing a cutoff $E_{c}$, one has

$$
\begin{equation*}
\int_{0}^{E_{c}} \frac{E}{E_{b}-E_{a}-E} \mathrm{~d} E \sim-E_{c} . \tag{4.68}
\end{equation*}
$$

But the physics must not be dependent on the cutoff! Thus, this method is bad.
Let us use another approach by Bethe [1947]. If the calculation is wrong we have forgotten something. Let us inspect the electron mass. Classically, a charge interacting with an electromagnetic field introduces a change in mass, but this value is negligible. Nevertheless, let us assume, that in Lamb case, the change in mass

$$
\begin{equation*}
m_{e} \equiv m+\delta m \tag{4.69}
\end{equation*}
$$

is important. The kinetic energy changes as

$$
\begin{equation*}
\frac{p^{2}}{2 m}=\frac{p^{2}}{2 m_{e}}+\left(\frac{p^{2}}{2 m}-\frac{p^{2}}{2 m_{e}}\right) \tag{4.70}
\end{equation*}
$$

with

$$
\begin{equation*}
\frac{1}{m_{e}}=\frac{1}{m+\delta m} \simeq\left(1-\frac{\delta m}{m}\right) \tag{4.71}
\end{equation*}
$$

Therefore, one has a change in energy corresponding to an additional potential

$$
\begin{equation*}
\frac{p^{2}}{2 m}=\frac{p^{2}}{2 m_{e}}+\frac{p^{2}}{2 m} \frac{\delta m}{m} \tag{4.72}
\end{equation*}
$$

### 4.3.4 Intermezzo - Change of Mass of a Free Electron Coupled to a QED Vacuum

In a similar argument, a change of mass of a free electron corresponds to a change in the potential. Denote by $\hbar \Delta$ the change of kinetic energy of the free electron coupled to a QED vacuum. The relevant diagram is


Explicitly,

$$
\begin{equation*}
\hbar \Delta=\frac{2 q^{2}}{m^{2}} \frac{\hbar}{2 \varepsilon_{0} V \omega} \frac{1}{\hbar^{2}} \mathcal{P}\left(\sum_{\boldsymbol{p}^{\prime}, \boldsymbol{k}, T} \int_{0}^{\infty} \frac{\langle\boldsymbol{p} ; 0| \hat{\epsilon}_{\boldsymbol{k}, T} \cdot \hat{\boldsymbol{p}} \mathrm{e}^{\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{r}}\left|\boldsymbol{p}^{\prime} ; 1_{\boldsymbol{k}, T}\right\rangle}{E_{p}-E_{p-\hbar k}-\hbar \omega} \mathrm{d} E\right) \tag{4.74}
\end{equation*}
$$

Now, since

$$
\begin{equation*}
\langle\boldsymbol{p}| \hat{\boldsymbol{p}} \mathrm{e}^{\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{r}}\left|\boldsymbol{p}^{\prime}\right\rangle=\int_{V} \mathrm{~d}^{3} \boldsymbol{r} \frac{\mathrm{e}^{-\mathrm{i} \boldsymbol{p} \cdot \boldsymbol{r} / \hbar}}{\sqrt{V}} \frac{\hbar}{\mathrm{i}} \boldsymbol{\nabla}\left(\frac{\mathrm{e}^{\mathrm{i}\left(\boldsymbol{p}^{\prime} \cdot \boldsymbol{r}+\hbar \boldsymbol{k} \cdot \boldsymbol{r}\right) / \hbar}}{\sqrt{V}}\right)=\boldsymbol{p} \delta_{\boldsymbol{p}, \boldsymbol{p}^{\prime}+\hbar \boldsymbol{k}} \tag{4.75}
\end{equation*}
$$

Thus,

$$
\begin{equation*}
\hbar \Delta=\frac{2 \alpha_{\mathrm{fs}}}{3 \pi} \frac{p^{2}}{(m c)^{2}} \int_{0}^{\infty} \mathrm{d} E \frac{E}{E_{p}-E_{p-\hbar k}-E} \tag{4.76}
\end{equation*}
$$

with $E=\hbar \omega$. Note that it is almost identical to the atomic case, but with $|\langle a| \boldsymbol{p}| b\rangle\left.\right|^{2} \rightarrow p^{2}$ and the loss of the principal part $\mathcal{P}(\cdot)$.

We can omit the integrand, since

$$
\begin{align*}
-E+E_{p}-E_{p-\hbar k} & =\frac{1}{2 m}\left(p^{2}-(\boldsymbol{p}-\hbar \boldsymbol{k})^{2}\right)-E \\
& =-E+\frac{\hbar \boldsymbol{k} \cdot \boldsymbol{p}}{m}-\frac{\hbar^{2} k^{2}}{2 m} \\
& =-E(1-\underbrace{\boldsymbol{k} \cdot \frac{\boldsymbol{p}}{m c}}_{\ll 1}+\underbrace{\frac{E}{2 m c^{2}}}_{<1}) \simeq-E \tag{4.77}
\end{align*}
$$

and the $\ll 1$ part is because our electrons are not relativistic. Hence,

$$
\begin{equation*}
\hbar \Delta=-\frac{2 \alpha_{\mathrm{fs}}}{3 \pi} \frac{p^{2}}{(m c)^{2}} \int_{0}^{E_{c}} \mathrm{~d} E \simeq-\frac{2 \alpha_{\mathrm{fs}}}{3 \pi} \frac{p^{2}}{(m c)^{2}} E_{c} \tag{4.78}
\end{equation*}
$$

and we have the same cutoff problem. Yet, it hints us how to renormalize $\hbar \Delta_{b}$.

### 4.3.5 Back to the Case of an Electron Bound to an Atom

We see that it either case, we must renormalize the mass. Define

$$
\begin{equation*}
\hbar \Delta_{\mathrm{mass}}=-\frac{2 \alpha_{\mathrm{fs}}}{3 \pi} \int_{0}^{E_{c}} \mathrm{~d} E \frac{\langle b| \hat{\boldsymbol{p}}^{2}|b\rangle}{(m c)^{2}} . \tag{4.79}
\end{equation*}
$$

Now, replace

$$
\begin{equation*}
\hbar \Delta_{b} \rightarrow \hbar \Delta_{b}-\hbar \Delta_{\mathrm{mass}} \equiv \hbar \Delta_{\mathrm{eff}} \tag{4.80}
\end{equation*}
$$

And obtain
$\left.\left.\hbar \Delta_{\mathrm{eff}}=\left.\frac{2 \alpha_{\mathrm{fs}}}{3 \pi} \frac{1}{(m c)^{2}} \mathcal{P}\left[\left.\int_{0}^{E_{c}} \mathrm{~d} E\left(\sum_{a}|\langle b| \hat{\boldsymbol{p}}| a\right\rangle\right|^{2} \frac{E}{E_{b}-E_{a}-E}+\sum_{a}|\langle b| \hat{\boldsymbol{p}}| a\right\rangle\right|^{2}\right)\right]$.
Hence,

$$
\begin{equation*}
\left.\hbar \Delta_{\mathrm{eff}}=\left.\frac{2 \alpha_{\mathrm{fs}}}{3 \pi} \frac{1}{(m c)^{2}} \mathcal{P}\left(\sum_{a} \int_{0}^{E_{c}} \mathrm{~d} E|\langle b| \hat{\boldsymbol{p}}| a\right\rangle\right|^{2} \frac{E_{b}-E_{a}}{E_{b}-E_{a}-E}\right) \tag{4.82}
\end{equation*}
$$

The divergence is now logarithmic with $E_{c}$.
Corollary 4.2. In order to deal with diverging quantities, learn how to subtract to infinities.

Adapt to the hydrogen atom,

$$
\begin{equation*}
\left.\hbar \Delta_{\mathrm{eff}}=\left.\frac{2 \alpha_{\mathrm{fs}}}{3 \pi} \frac{1}{(m c)^{2}} \mathcal{P}\left(\sum_{I} \int_{0}^{E_{c}} \mathrm{~d} E|\langle I| \hat{\boldsymbol{p}}| b\right\rangle\right|^{2} \frac{E_{b}-E_{I}}{E_{b}-E_{I}-E}\right) \tag{4.83}
\end{equation*}
$$

with $\{I\}$ denoting the hydrogen states. The principal value reads

$$
\begin{align*}
& \mathcal{P}\left(\sum_{I} \int_{0}^{E_{c}} \frac{\mathrm{~d} E}{E_{b}-E_{I}-E}\right)= \lim _{\varepsilon \rightarrow 0}\left(-\left.\ln \left|E_{b}-E_{I}-E\right|\right|_{0} ^{E_{b}-E_{I}-\varepsilon}\right. \\
&\left.-\left.\ln \left|E_{b}-E_{I}-E\right|\right|_{E_{b}-E_{I}+\varepsilon} ^{E_{c}}\right) \\
&=-\ln \left|\frac{\left(E_{b}-E_{I}\right)-E_{c}}{E_{b}-E_{I}}\right| \\
& \simeq-\ln \left|\frac{E_{c}}{E_{b}-E_{I}}\right| \tag{4.84}
\end{align*}
$$

The last approximation holds, since $E_{b}-E_{I} \simeq[\mathrm{eV}]$ and $E_{c} \sim m c^{2} \simeq \frac{1}{2}[\mathrm{MeV}]$. In other words, we work in the limit $E_{c} \gg\left|E_{b}-E_{I}\right|$, which is the flat part of the $\ln$. Thus,

$$
\begin{equation*}
\left.\hbar \Delta_{\mathrm{eff}}=\frac{2 \alpha_{\mathrm{fs}}}{3 \pi} \frac{1}{(m c)^{2}} \sum_{I}|\langle I| \hat{\boldsymbol{p}}| b\right\rangle\left.\right|^{2}\left(E_{b}-E_{I}\right) \ln \frac{E_{c}}{\langle | E_{b}-E_{I}| \rangle} . \tag{4.85}
\end{equation*}
$$

In the flat-part approximation, one has

$$
\begin{equation*}
\left.\hbar \Delta_{\mathrm{eff}}=\frac{2 \alpha_{\mathrm{fs}}}{3 \pi} \frac{1}{(m c)^{2}} \ln \frac{E_{c}}{\langle | E_{b}-E_{I}| \rangle} \sum_{I}|\langle I| \hat{\boldsymbol{p}}| b\right\rangle\left.\right|^{2}\left(E_{b}-E_{I}\right) \tag{4.86}
\end{equation*}
$$

### 4.3.6 More Calculations

We want to calculate the quantity

$$
\begin{equation*}
\left[\hat{\boldsymbol{p}}, \hat{H}_{\mathrm{at}}\right]=\left[\hat{\boldsymbol{p}}, \frac{p^{2}}{2 m}+\hat{V}(\boldsymbol{r})\right]=\frac{\hbar}{\mathrm{i}} \boldsymbol{\nabla} \hat{V} . \tag{4.87}
\end{equation*}
$$

Since

$$
\begin{equation*}
\left(E_{I}-E_{b}\right)\langle b| \hat{\boldsymbol{p}}|I\rangle=\frac{\hbar}{\mathrm{i}}\langle b| \nabla \hat{V}|I\rangle, \tag{4.88}
\end{equation*}
$$

one has

$$
\begin{align*}
\left.\sum_{I}\left(E_{I}-E_{b}\right)|\langle b| \hat{\boldsymbol{p}}| I\right\rangle\left.\right|^{2} & =\frac{\hbar}{\mathrm{i}} \sum_{I}\langle b| \nabla V|I\rangle\langle I| \hat{\boldsymbol{p}}|b\rangle \\
& =\frac{\hbar}{\mathrm{i}}\langle b|(\boldsymbol{\nabla} V) \cdot \hat{\boldsymbol{p}}|b\rangle \\
& =-\frac{\hbar}{\mathrm{i}}\langle b| \hat{\boldsymbol{p}} \cdot(\nabla V)|b\rangle \\
& =-\frac{\hbar}{\mathrm{i}} \frac{1}{2}\langle b|\left[p_{j}, \partial_{j} V\right]|b\rangle \\
& =\frac{\hbar^{2}}{2}\langle b| \Delta V|b\rangle . \tag{4.89}
\end{align*}
$$

Hence,

$$
\begin{equation*}
\hbar \Delta_{\mathrm{eff}}=\frac{\alpha_{\mathrm{fs}}}{3 \pi} \frac{1}{(m c)^{2}} \ln \frac{E_{c}}{\langle | E_{b}-E_{I}| \rangle}\left(\frac{\hbar}{m c}\right)^{2}\langle b| \Delta V|b\rangle . \tag{4.90}
\end{equation*}
$$

Now, take the Coulomb potential

$$
\begin{equation*}
V(\boldsymbol{r})=-e \phi(\boldsymbol{r}) . \tag{4.91}
\end{equation*}
$$

The potential $\phi$ obeys the Poisson equation

$$
\begin{equation*}
\Delta \phi(\boldsymbol{r})=-\frac{1}{\varepsilon_{0}} \rho(\boldsymbol{r}) \tag{4.92}
\end{equation*}
$$

with

$$
\begin{equation*}
\rho(\boldsymbol{r})=e \delta(\boldsymbol{r}) \tag{4.93}
\end{equation*}
$$

Thus,

$$
\begin{align*}
\langle b| \Delta V|b\rangle & =\frac{e}{\varepsilon_{0}} \int \mathrm{~d}^{3} \boldsymbol{r}\left|\psi_{b}(\boldsymbol{r})\right|^{2} \rho(\boldsymbol{r}) \\
& =\frac{e^{2}}{\varepsilon_{0}}\left|\psi_{b}(\boldsymbol{r}=\mathbf{0})\right|^{2} . \tag{4.94}
\end{align*}
$$

Corollary 4.3. One has 0 Lamb-shift for the p-states, since $\psi_{p}(\boldsymbol{r}=\mathbf{0})=0$.
The $s$-states at 0 read

$$
\begin{equation*}
\psi_{n s}(\mathbf{0})=\frac{1}{\sqrt{4 \pi}} \frac{2}{\left(n a_{0}\right)^{3 / 2}}, \quad a_{0}=\frac{\hbar^{2}}{m} \frac{4 \pi \varepsilon_{0}}{e^{2}} \tag{4.95}
\end{equation*}
$$

Thus,

$$
\begin{equation*}
\hbar \Delta_{\mathrm{eff}}=\frac{8}{3 \pi} \frac{\alpha_{\mathrm{fs}}^{3}}{n^{3}} R_{\infty} \ln \frac{E_{c}}{\langle | E_{b}-E_{I}| \rangle}, \tag{4.96}
\end{equation*}
$$

with $R_{\infty} \simeq \frac{13.6[\mathrm{eV}]}{h c}$ the Rydberg constant and $a_{0} \simeq 0.529[\AA]$ the Bohr radius. The factor $\alpha_{\mathrm{fs}}^{3}$ is interpreted as the coupling between the atom and the quantum vacuum. If there is no coupling $\left(\alpha_{\mathrm{fs}} \rightarrow 0\right)$, there is no Lamb shift $\left(\Delta_{\text {eff }} \rightarrow 0\right)$.

For $n=2$,

$$
\begin{equation*}
\frac{\alpha_{\mathrm{fs}}^{3}}{3 \pi} R_{\infty}=135.549[\mathrm{MHz}] \tag{4.97}
\end{equation*}
$$

and [Bethe, Brown, and Stehn, 1950],

$$
\begin{equation*}
\langle | E_{b}-E_{I}| \rangle \simeq 20 R_{\infty} \tag{4.98}
\end{equation*}
$$

Plug in $E_{c}=m c^{2}=\frac{1}{2}[\mathrm{MeV}]$ to have

$$
\begin{equation*}
\Delta_{\mathrm{eff}}=1042[\mathrm{MHz}] . \tag{4.99}
\end{equation*}
$$

The approximation we have used works well and does not change much in the choice of cutoff.

This process - of changing finite amount of quantities in order to remove infinities - is called "renormalization". Note that it works for gauge field theories, and may not work in other cases.

### 4.3.7 A Different Point of View ${ }^{20}$

This is contributed to Feynman in 1961 (in proceedings; no paper). Let us inspect the vacuum energy of the electromagnetic field,

$$
\begin{equation*}
E_{0}=\sum_{\boldsymbol{k}, T} \frac{\hbar \omega_{\boldsymbol{k}, T}}{2} \tag{4.100}
\end{equation*}
$$

Now, ask ourselves what can be perturbed. Since $\hbar$ cannot be changes, then the refractive index of the vacuum $n$ changes in the presence of a perturbation atom,

$$
\begin{equation*}
\omega \lambda=\frac{c}{n} . \tag{4.101}
\end{equation*}
$$

Thus,

$$
\begin{equation*}
\Delta E=\sum_{\boldsymbol{k}, T} \frac{\hbar}{2}\left(\frac{\omega_{\boldsymbol{k}, T}}{n}-\omega_{\boldsymbol{k}, T}\right) . \tag{4.102}
\end{equation*}
$$

For $n \simeq 1$

$$
\begin{equation*}
n(\omega) \simeq 1+2 \pi N \alpha(\omega) \tag{4.103}
\end{equation*}
$$

with $N$ the number of atoms, and $\alpha(\omega)$ the polarizability. Hence,

$$
\begin{equation*}
\Delta E=\sum_{\boldsymbol{k}, T}(-2 \pi) \frac{\hbar}{2} \alpha\left(\omega_{\boldsymbol{k}, T}\right) \omega_{\boldsymbol{k}, T}=\ldots=\hbar \Delta_{\mathrm{eff}} \tag{4.104}
\end{equation*}
$$

The (...) will be calculated by us in homework $\# 2$.
The change of energy of the vacuum by perturbation - resulting in a force is the Casimir effect. We shall describe it in details in the next section.

[^18]
### 4.4 The Casimir Effect

### 4.4.1 History

The Casimir effect was found by the Dutch physicist H. Casimir (see [Casimir and Polder, 1948]). The idea of Casimir was to calculate the van der Waals interactions in the vacuum. In fact, the van der Waals interactions are longrange, and therefore a manifestation of the Casimir effect. The first time it was measured by Lamoreaux [1997].

Today, it is popular as an explanation to the dark energy, whose main contributions are the vacuum energy. This depiction is wrong, as the Casimir force is calculated in presence of something else, and never a measure of the vacuum energy itself.

### 4.4.2 Basics

Consider two infinite perfect metallic plates separated by distance $L$ between them. Start with a classical electromagnetic field,


Fig. 4.3: The Casimir effect setup.

$$
\begin{equation*}
\left(\nabla^{2}-\frac{1}{c^{2}} \frac{\partial^{2}}{\partial t^{2}}\right) \boldsymbol{E}(\boldsymbol{r}, t) \tag{4.105}
\end{equation*}
$$

with the boundary conditions (BCs) of perfect plates

$$
\begin{cases}\hat{n} \times \boldsymbol{E}=\mathbf{0}, & \text { (Dirichlet BCs) }  \tag{4.106}\\ \hat{n} \cdot \boldsymbol{B}=0, & \text { (Neumann BCs) }\end{cases}
$$

Assume scalar fields (no polarizations) to replace the electric and magnetic fields with the scalar field $\varphi(\boldsymbol{r}, t)$. Let $\omega_{\boldsymbol{k}}=c|\boldsymbol{k}|$ to have

$$
\begin{equation*}
\varphi(x, \boldsymbol{\rho}, t)=\mathrm{e}^{\mathrm{i} \boldsymbol{k}_{\perp} \cdot \boldsymbol{\rho}} \mathrm{e}^{\mathrm{i} \omega t} \varphi(x) \tag{4.107}
\end{equation*}
$$

where $\rho$ is in the $\hat{y}, \hat{z}$ directions. The boundary conditions thus read

$$
\begin{cases}\varphi^{D}(x=0)=\varphi^{D}(x=L)=0, & \text { (Dirichlet BCs) }  \tag{4.108}\\ \partial_{n} \varphi^{N}(x=0)=\partial_{n} \varphi^{N}(x=L)=0 . & \text { (Neumann BCs) }\end{cases}
$$

Solving the equations, the eigenfrequency spectrum reads

$$
\begin{cases}\omega_{k_{\perp}, n}^{D}=c \sqrt{\left(\frac{\pi n}{L}\right)^{2}+k_{\perp}^{2}}, & n=1,2,3, \ldots  \tag{4.109}\\ \omega_{k_{\perp}, n}^{N}=c \sqrt{\left(\frac{\pi n}{L}\right)^{2}+k_{\perp}^{2}}, & n=0,1,2,3, \ldots\end{cases}
$$

### 4.4.3 Calculating the Energy

Let us calculate the energy between the plates. It reads,

$$
\begin{align*}
E_{0}(L) & =\frac{\hbar}{2} \sum_{\boldsymbol{k}} \omega_{\boldsymbol{k}} \\
& =\sum_{\boldsymbol{k}}\left(E_{\boldsymbol{k}}^{2}+c^{2} B_{k}^{2}\right) \\
& =\frac{\hbar}{2} \sum_{\boldsymbol{k}}\left(\omega_{\boldsymbol{k}}^{D}+\omega_{\boldsymbol{k}}^{N}\right) \\
& =\frac{\hbar c}{2}\left[\sum_{n=1}^{\infty} S \int \frac{\mathrm{~d}^{2} \boldsymbol{k}_{\perp}}{(2 \pi)^{2}} \sqrt{\left(\frac{\pi n}{L}\right)^{2}+k_{\perp}^{2}}+\sum_{n=0}^{\infty} S \int \frac{\mathrm{~d}^{2} \boldsymbol{k}_{\perp}}{(2 \pi)^{2}} \sqrt{\left(\frac{\pi n}{L}\right)^{2}+k_{\perp}^{2}}\right] . \tag{4.110}
\end{align*}
$$

With $S$ the area of the plates. Now, let us calculate its difference from the vacuum energy,

$$
\begin{align*}
\mathcal{E}_{0}(L) & =E_{0}(L)-\frac{\hbar c}{2} S \int \frac{\mathrm{~d}^{2} \boldsymbol{k}_{\perp}}{(2 \pi)^{2}} k_{\perp} \\
& =S \hbar c \sum_{n=1}^{\infty} \underbrace{\int \frac{\mathrm{d}^{2} \boldsymbol{k}_{\perp}}{(2 \pi)^{2}} \sqrt{\left(\frac{\pi n}{L}\right)^{2}+k_{\perp}^{2}}}_{\equiv I_{n}} \tag{4.111}
\end{align*}
$$

Let us calculate $I_{n}$. We employ the Feynman trick:

$$
\begin{equation*}
\frac{1}{\left(\lambda^{2}+a^{2}\right)^{s}}=\frac{1}{\Gamma(s)} \int_{0}^{\infty} \mathrm{d} t t^{s-1} \mathrm{e}^{-\left(\lambda^{2}+a^{2}\right) t} \tag{4.112}
\end{equation*}
$$

Now, substitute $\lambda^{2}=\left(\frac{\pi n}{L}\right)^{2}, a^{2}=k_{\perp}^{2}$ and $s=-\frac{1}{2}$ to obtain

$$
\begin{align*}
I_{n} & =\frac{1}{\Gamma\left(-\frac{1}{2}\right)} \int_{0}^{\infty} \mathrm{d} t t^{-\frac{3}{2}} \mathrm{e}^{-\lambda^{2} t} \int \frac{\mathrm{~d}^{2} \boldsymbol{k}_{\perp}}{(2 \pi)^{2}} \mathrm{e}^{-k_{\perp}^{2} t} \\
& =\frac{1}{\Gamma\left(-\frac{1}{2}\right) 4 \pi} \int_{0}^{\infty} \mathrm{d} t t^{-\frac{5}{2}} \mathrm{e}^{-\lambda^{2} t} . \tag{4.113}
\end{align*}
$$

Substitute $x=t \lambda^{2}$ to have

$$
\begin{equation*}
I_{n}=\frac{\lambda^{3}}{\Gamma\left(-\frac{1}{2}\right)} \frac{1}{4 \pi} \int_{0}^{\infty} \mathrm{d} x x^{-\frac{5}{2}} \mathrm{e}^{-x} \tag{4.114}
\end{equation*}
$$

Now, since the Euler gamma function is defined by

$$
\begin{equation*}
\Gamma(s)=\int_{0}^{\infty} \mathrm{d} t t^{s-1} \mathrm{e}^{-t} \tag{4.115}
\end{equation*}
$$

one obtains

$$
\begin{equation*}
I_{n}=\left(\frac{\pi n}{L}\right)^{3} \frac{1}{4 \pi} \frac{\Gamma\left(-\frac{3}{2}\right)}{\Gamma\left(-\frac{1}{2}\right)} \tag{4.116}
\end{equation*}
$$

Since $\Gamma\left(-\frac{1}{2}\right)=-2 \sqrt{\pi}$, one has

$$
\begin{equation*}
I_{n}=-\left(\frac{\sqrt{\pi} n}{2 L}\right)^{3} \Gamma\left(-\frac{3}{2}\right) . \tag{4.117}
\end{equation*}
$$

Thus,

$$
\begin{equation*}
\frac{\mathcal{E}_{0}(L)}{S L}=-\frac{\hbar c}{L} \frac{\pi^{\frac{3}{2}}}{8 L^{3}} \Gamma\left(-\frac{3}{2}\right) \sum_{n=1}^{\infty} n^{3} . \tag{4.118}
\end{equation*}
$$

### 4.4.4 Zeta Function Regularization

The sum $\sum_{n=1}^{\infty} n^{3}$ is ill-defined. We therefore must employ some dirty tricks.
Introduce the Riemann $\zeta$-function defined by

$$
\begin{equation*}
\zeta(s)=\sum_{n=1}^{\infty} n^{-s}, \quad \operatorname{Re} s>1 \tag{4.119}
\end{equation*}
$$

This function is analytic. Therefore, we can name an analytic continuation to the whole complex plane $s$. Several properties arise such as

$$
\begin{equation*}
\Gamma\left(\frac{s}{2}\right) \zeta(s)=\pi^{s-\frac{1}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s) \tag{4.120}
\end{equation*}
$$

Thus, we have a recipe to convert negative $s$ to positive.
For $s=-3$ one has

$$
\begin{equation*}
\Gamma\left(-\frac{3}{2}\right) \zeta(-3)=\pi^{-\frac{7}{2}} \Gamma(2) \zeta(4) \tag{4.121}
\end{equation*}
$$

with

$$
\begin{equation*}
\Gamma(2)=1, \quad \zeta(4)=\sum_{n=1}^{\infty} \frac{1}{n^{4}}=\frac{\pi^{4}}{90} \tag{4.122}
\end{equation*}
$$

Therefore, the Casimir energy reads

$$
\begin{equation*}
\frac{\mathcal{E}_{0}(L)}{S L}=-\frac{\hbar c \pi^{2}}{720 L^{4}} \tag{4.123}
\end{equation*}
$$

Remark 4.8. This gives us the first and simplest theory of $\zeta$-function regularization. Today, it is a widely used tool in quantum field theory.

One can calculate the Casimir force

$$
\begin{equation*}
F=\frac{\partial \mathcal{E}_{0}(L)}{\partial L}=-\frac{\hbar c \pi^{2} S}{240 L^{4}} \tag{4.124}
\end{equation*}
$$

Note this force it attractive. What was actually measured by Lamoreaux was the pressure

$$
\begin{equation*}
P=\frac{F}{S}=-\frac{\hbar c \pi^{2}}{240 L^{4}} \tag{4.125}
\end{equation*}
$$

Note that it reduces as $\propto L^{-4}$, which is very small. Also note he had to deal with corrugations on the surface, and thus used a bit different geometry $\mid \leftrightarrow($.

Remark 4.9. Considering $U(L, T)$ the universal energy function we have calculated, we note that $U(L, T)=U\left((T / L)^{4}\right)$. In the high temperature limit one recovers the Stefan Boltzmann law $U(T)=\sigma T^{4}$; in the low temperature limit one has the Casimir energy.
Remark 4.10. Note that $\alpha_{\mathrm{fs}}$ absent here. In fact, one can show that by demanding perfect metallic plates one obtains $\alpha_{\mathrm{fs}} \rightarrow \infty$ (see [Jaffe, 2005]).

## 5 Statistical Properties of Photons ${ }^{21}$

This chapter is all about clicks - how to design, experiment and measure clicks in a detector. We shall find the conditions when a click is a photon and when it is not. These arguments also true for other systems such as electrons.

### 5.1 Shot Noise and Poisson Distribution

We will describe beams of indiscernible particles (bosons, fermions, photons, light, etc.). Note that this section is classical.

Definition 5.1 (Beams). Systems prepared in a stationary state but out of thermal equilibrium.

Out of equilibrium there exists a finite average current. We shall describe several types of beams:

- Light beams (photons);
- Currents in conductors (electrons, shot noise).


### 5.1.1 Basic Intuition - Detection Probabilities

Consider the case of current fluctuations for a classical beam of independent particles, for example, rain on a roof. Here a beam is the probability $I(t) \mathrm{d} t$ to detect a particle between $t$ and $\mathrm{d} t$.


Fig. 5.1: A $1 D$ beam between a source and a detector.
We ask what is $I(t)$ for light beams. It is the flux of the Poynting vector

$$
\begin{equation*}
\boldsymbol{S}=\frac{1}{\mu_{0}} \boldsymbol{E} \times \boldsymbol{B} \tag{5.1}
\end{equation*}
$$

- If $I(t)$ does not fluctuate, it is described by a shot noise.
- If $I(t)$ fluctuates, it is described by shot noise + Hanbury-Brown \& Twiss correlations.

Define $P(n, t)$ the probability to detect $n$ particles during the interval $[0, t]$. Expand to obtain

$$
\begin{equation*}
P(n, t+\mathrm{d} t)=(1-I(t) \mathrm{d} t) P(n, t)+I(t) \mathrm{d} t P(n-1, t)+O\left(\mathrm{~d} t^{2}\right) . \tag{5.2}
\end{equation*}
$$

[^19]Here $O\left(\mathrm{~d} t^{2}\right)$ is the probability to detect 2 particles or more. We use the initial conditions

$$
\begin{equation*}
P(n=-1, t)=0, \quad \forall t . \tag{5.3}
\end{equation*}
$$

One can thus write a differential equation

$$
\begin{equation*}
\frac{\partial P(n, t)}{\partial t}=-I(t) P(n, t)+I(t) P(n-1, t) \text {. } \tag{5.4}
\end{equation*}
$$

### 5.1.2 Generating Function

To solve this equation, we use the generating function method. Consider the moment generating function,

$$
\begin{equation*}
M(\xi, t) \equiv\left\langle\mathrm{e}^{\xi n}\right\rangle_{t}=\sum_{n=0}^{\infty} \mathrm{e}^{\xi n} P(n, t) \tag{5.5}
\end{equation*}
$$

It is just the Laplace transform of $P(n, t)$, which we used to disentangle $n$ from $t$. To find the $p$-th moment, consider the derivatives

$$
\begin{equation*}
\frac{\partial^{p} M}{\partial \xi^{p}}=\sum_{n=0}^{\infty} n^{p} \mathrm{e}^{\xi n} P(n, t)=\left\langle n^{p} \mathrm{e}^{\xi n}\right\rangle_{t} \tag{5.6}
\end{equation*}
$$

thus, the moments read

$$
\begin{equation*}
\left.\frac{\partial^{p} M}{\partial \xi^{p}}\right|_{\xi=0}=\left\langle n^{p}\right\rangle_{t} \tag{5.7}
\end{equation*}
$$

Now, use Eq. (5.4) to obtain the derivatives of $M$ :

$$
\begin{equation*}
\frac{\partial}{\partial t} M(\xi, t)=\left(\mathrm{e}^{\xi}-1\right) I(t) M(\xi, t) \tag{5.8}
\end{equation*}
$$

The solution reads

$$
\begin{equation*}
M(\xi, t)=\exp \left[\left(\mathrm{e}^{\xi}-1\right) \int_{0}^{t} I\left(t^{\prime}\right) \mathrm{d} t^{\prime}\right]+M(\xi, 0) \tag{5.9}
\end{equation*}
$$

with the chosen boundary condition

$$
\begin{equation*}
M(\xi, 0)=P\left(0, t_{0}=0\right)=1 \tag{5.10}
\end{equation*}
$$

To calculate the average number of particles, plug in Eq. (5.7) to obtain

$$
\begin{equation*}
\langle n(t)\rangle=\left.\frac{\partial M}{\partial \xi}\right|_{\xi=0}=\int_{0}^{t} I\left(t^{\prime}\right) \mathrm{d} t^{\prime} \tag{5.11}
\end{equation*}
$$

For the variance, one has

$$
\begin{equation*}
\left\langle n^{2}(t)\right\rangle=\left.\frac{\partial^{2} M}{\partial \xi^{2}}\right|_{\xi=0}=\left(\int_{0}^{t} I\left(t^{\prime}\right) \mathrm{d} t^{\prime}\right)^{2}+\int_{0}^{t} I\left(t^{\prime}\right) \mathrm{d} t^{\prime} \tag{5.12}
\end{equation*}
$$

Thus,

$$
\begin{equation*}
\underbrace{\left\langle n^{2}(t)\right\rangle-\langle n(t)\rangle^{2}}_{\equiv\left\langle\Delta n(t)^{2}\right\rangle}=\langle n(t)\rangle, \tag{5.13}
\end{equation*}
$$

so that

$$
\begin{equation*}
\frac{\left\langle\Delta n(t)^{2}\right\rangle^{\frac{1}{2}}}{\langle n(t)\rangle}=\frac{1}{\langle n(t)\rangle^{\frac{1}{2}}} \tag{5.14}
\end{equation*}
$$

Hence,

$$
\begin{align*}
M(\xi, t) & =\exp \left[\left\langle n(t)\left(\mathrm{e}^{\xi}-1\right)\right\rangle\right] \\
& =\mathrm{e}^{-\langle n(t)\rangle} \sum_{p=0}^{\infty} \mathrm{e}^{p \xi} \frac{\langle n(t)\rangle^{p}}{p!} \tag{5.15}
\end{align*}
$$

And we have the detection probability

$$
\begin{equation*}
P(m, t)=\mathrm{e}^{-\langle n(t)\rangle} \frac{\langle n(t)\rangle^{m}}{m!} . \tag{5.16}
\end{equation*}
$$

This is the Poisson distribution.

### 5.1.3 Remarks

Remark 5.1. What we have just described is actually the shot noise. We will show a full derivation later.
Remark 5.2. For quantum entangled particles, this calculation does not work.

### 5.2 Noise in Light Beams - The Semiclassical Approach

In this section, we add fluctuations to $I(t)$. Take, for example, an ideal laser - it does not fluctuate. Now, take the other limit - black body. The light, as we have seen in Section 1.3, admits thermal fluctuations. This section is also classical.

Hanbury Brown and Twiss [1956] sought to measure photons from stars (and wrongly concluded so). Consider then light emitted by a black body (with thermal fluctuations). Inspect then

$$
\begin{equation*}
\overline{P(m, t)}=\overline{\mathrm{e}^{-\langle n(t)\rangle} \frac{\langle n(t)\rangle^{m}}{m!}} \tag{5.17}
\end{equation*}
$$

with $\cdots$ a thermal average.
Remark 5.3 (Intermezzo). Consider a thermodynamic system. In order to characterize this thermodynamic system, one has to calculate the partition function. Any quantity is calculated by expression of this sort $\left\langle\mathrm{e}^{-\beta H}\right\rangle$. Now, consider disorder (say of magnetic spins on a lattice). This is calculated by an average over configurations $\left\langle\mathrm{e}^{-\beta \bar{H}}\right\rangle$. Note the abuse of notations. It is important to notice that

$$
\begin{equation*}
\overline{\left\langle\mathrm{e}^{-\beta H}\right\rangle} \neq\left\langle\mathrm{e}^{-\beta \bar{H}}\right\rangle \tag{5.18}
\end{equation*}
$$

but one problem is hard whereas the other is trivial.

### 5.2.1 Detector Response

For each realization of $I(t)$, calculate

$$
\begin{equation*}
\overline{\left\langle n^{2}\right\rangle}-\overline{\langle n\rangle^{2}}=\overline{\langle n\rangle} . \tag{5.19}
\end{equation*}
$$

Now, since

$$
\begin{equation*}
\overline{\left\langle n^{2}\right\rangle}-(\overline{\langle n\rangle})^{2}=\underbrace{\overline{\langle n\rangle}}_{\text {Shot noise }}+(\underbrace{\overline{\langle n\rangle\langle n\rangle}-(\overline{\langle n\rangle})^{2}}_{\text {Intensity fluctuations }}) \tag{5.20}
\end{equation*}
$$

One can identify the distributions of the shot noise we have calculated and additional (intensity) fluctuations. This new contribution is essential for our later discussion.

In reality, detectors $D$ do not have an instantaneous response. The current reads

$$
\begin{equation*}
J(t)=\sum_{m} k\left(t-t_{m}\right), \tag{5.21}
\end{equation*}
$$

with $k(t)$ the detector response for a single particle. Define the local particle density

$$
\begin{equation*}
\rho(t)=\sum_{m} \delta\left(t-t_{m}\right), \tag{5.22}
\end{equation*}
$$

so that

$$
\begin{equation*}
J(t)=\int_{-\infty}^{+\infty} \mathrm{d} t^{\prime} k\left(t-t^{\prime}\right) \rho\left(t^{\prime}\right) \tag{5.23}
\end{equation*}
$$

The measured quantity is thus,

$$
\begin{align*}
& \overline{\langle\delta J(t+\tau) \delta J(t)\rangle} \equiv \overline{\langle J(t+\tau) J(t)\rangle}-\overline{\langle J(t+\tau)\rangle\langle J(t)\rangle} \\
&=\int \mathrm{d} t_{1} \mathrm{~d} t_{2} k\left(t+\tau-t_{1}\right) k\left(t-t_{2}\right) \times \\
& \quad\left[\overline{\left\langle\rho\left(t_{1}\right) \rho\left(t_{2}\right)\right\rangle}-\overline{\left\langle\rho\left(t_{1}\right)\right\rangle\left\langle\rho\left(t_{2}\right)\right\rangle}\right] . \tag{5.24}
\end{align*}
$$

### 5.2.2 Generating Function

Recall the definition of the generating function. In the new notations it is written as

$$
\begin{equation*}
M(\xi, t)=\left\langle\exp \left[\xi \int_{0}^{t} \mathrm{~d} t_{1} \rho\left(t_{1}\right)\right]\right\rangle \tag{5.25}
\end{equation*}
$$

with

$$
\begin{equation*}
n(t)=\int_{0}^{t} \mathrm{~d} t_{1} \rho\left(t_{1}\right) \tag{5.26}
\end{equation*}
$$

Now, take into account the fluctuations of $I(t)$ by promoting the fluctuating source to be a function of time $\xi(t)$. Thus, we have to generalize $M(\xi, t)$ by taking realizations $\xi(t)$ such that

$$
\begin{equation*}
M(\{\xi(t)\}, t)=\left\langle\exp \left[\int_{0}^{t} \mathrm{~d} t_{1} \xi\left(t_{1}\right) \rho\left(t_{1}\right)\right]\right\rangle . \tag{5.27}
\end{equation*}
$$

Here, $\{\xi(t)\}$ means that for each time interval $\left[t_{i}, t_{i+1}\right]$ we have a different realization of $\xi$ :

$$
\begin{equation*}
+\underset{0}{\frac{\xi\left(t_{1}\right)}{t_{1}}}+\frac{\xi\left(t_{2}\right)}{t_{2}}+-+-+\cdots+{ }_{t}+- \tag{5.28}
\end{equation*}
$$

Note that this entire discussion is classical.
Let us continue. ${ }^{22}$ Suppose $\xi(t)$ is a stair function constant over the intervals $I_{p} \equiv\left[\frac{p t}{N}, \frac{(p+1) t}{N}\right]$. Therefore,

$$
\begin{equation*}
M(\{\xi(t)\}, t)=\left\langle\exp \left[\sum_{p=0}^{N-1} \xi_{p} \int_{\frac{p t}{N}}^{\frac{(p+1) t}{N}} \mathrm{~d} t_{1} \rho\left(t_{1}\right)\right]\right\rangle \tag{5.29}
\end{equation*}
$$

Here $\int_{\frac{p t}{N}}^{\frac{(p+1) t}{N}} \mathrm{~d} t_{1} \rho\left(t_{1}\right)$ is the number of particles detected during $I_{p}$. We assume that the particles are independent; thus contributions of 2 distinct intervals are independent and one obtains

$$
\begin{align*}
M(\{\xi(t)\}, t) & =\prod_{p=0}^{N-1}\left\langle\exp \left[\xi_{p} \int_{\frac{p t}{N}}^{\frac{(p+1) t}{N}} \mathrm{~d} t_{1} \rho\left(t_{1}\right)\right]\right\rangle \\
& =\prod_{p=0}^{N-1} \exp \left[\left(\mathrm{e}^{\xi_{p}}-1\right) \int_{\frac{p t}{N}}^{\frac{(p+1) t}{N}} \mathrm{~d} t_{1} \rho\left(t_{1}\right)\right] \\
& =\exp \left[\sum_{p=0}^{N-1}\left(\mathrm{e}^{\xi_{p}}-1\right) \int_{\frac{p t}{N}}^{\frac{(p+1) t}{N}} \mathrm{~d} t_{1} \rho\left(t_{1}\right)\right] \tag{5.30}
\end{align*}
$$

Here we used Eq. (5.15) to proceed from the 1st to the 2nd rows. In the limit $N \rightarrow \infty$ one has

$$
\begin{equation*}
M(\xi(t), t)=\exp \left[\int_{0}^{t} \mathrm{~d} t_{1}\left(\mathrm{e}^{\xi(t)}-1\right) I\left(t_{1}\right)\right] . \tag{5.31}
\end{equation*}
$$

Expand both sides of this expression up to the second order in $\xi$ to have

$$
\begin{align*}
1+\int_{0}^{t} \mathrm{~d} t_{1} \xi\left(t_{1}\right)\left\langle\rho\left(t_{1}\right)\right\rangle & +\frac{1}{2} \int_{0}^{t} \mathrm{~d} t_{1} \int_{0}^{t} \mathrm{~d} t_{2} \xi\left(t_{1}\right) \xi\left(t_{2}\right)\left\langle\rho\left(t_{1}\right) \rho\left(t_{2}\right)\right\rangle+\ldots \\
=1+\int_{0}^{t} \mathrm{~d} t_{1} \xi\left(t_{1}\right) I\left(t_{1}\right) & +\frac{1}{2} \int_{0}^{t} \mathrm{~d} t_{1} \xi^{2}\left(t_{1}\right) I\left(t_{1}\right) \\
& +\frac{1}{2} \int_{0}^{t} \mathrm{~d} t_{1} \int_{0}^{t} \mathrm{~d} t_{2} \xi\left(t_{1}\right) \xi\left(t_{2}\right) I\left(t_{1}\right) I\left(t_{2}\right)+\ldots \tag{5.32}
\end{align*}
$$

Compare term-by-term to have

$$
\left\{\begin{array}{l}
\left\langle\rho\left(t_{1}\right)\right\rangle=I\left(t_{1}\right)  \tag{5.33}\\
\left\langle\rho\left(t_{1}\right) \rho\left(t_{2}\right)\right\rangle=I\left(t_{1}\right) \delta\left(t_{1}-t_{2}\right)+I\left(t_{1}\right) I\left(t_{2}\right)
\end{array}\right.
$$

With thermal average one can make the identification

$$
\frac{\left\{\begin{array}{l}
\overline{\left\langle\rho\left(t_{1}\right)\right\rangle}=\overline{I\left(t_{1}\right)}  \tag{5.34}\\
\left\langle\rho\left(t_{1}\right) \rho\left(t_{2}\right)\right\rangle
\end{array}=\right.}{\text { @hot noise }} \overline{\frac{e^{\prime\left(t_{1}\right)}}{05 / 06 / 2017} \delta\left(t_{1}-t_{2}\right)+\underbrace{\overline{I\left(t_{1}\right) I\left(t_{2}\right)}}_{\text {intensity fluc. }}}
$$

Remark 5.4. Up to the ' 50 s people were aware only about the shot noise. The first that have shown the 2nd term were Hanbury Brown and Twiss [1956].
Remark 5.5. The calculation-by Einstein-we will show shortly tells us that for a classical field there is no 2nd term, but for a photon source the fluctuation term also exist. This is (another) Einstein relation. By this analogy Hanbury Brown and Twiss claimed that their measuring of light from stars shows a photon field. We will deconstruct this claim below.

### 5.3 Quantum Theory of Photodetection

Consider a fluctuating source of light (stars, blackbody radiation, lasers, etc.).

### 5.3.1 Fluctuations of the Number of Photons in a Finite Box

In a semi-classical picture of detection one calculates the total number of photons in a finite box, as in Fig. 5.2. The photon number reads


Fig. 5.2: Photons in a box.

$$
\begin{equation*}
N_{\mathrm{tot}}=\int \frac{\mathrm{d}^{3} \boldsymbol{k}}{(2 \pi)^{3}} \hat{a}^{\dagger}(\boldsymbol{k}) \hat{a}(\boldsymbol{k}), \tag{5.35}
\end{equation*}
$$

for scalar waves. Define the field operators

$$
\left\{\begin{array}{l}
\hat{\psi}(\boldsymbol{r}, t) \equiv \int \frac{\mathrm{d}^{3} \boldsymbol{k}}{(2 \pi)^{3}} \mathrm{e}^{\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r}-\omega t)} \hat{a}(\boldsymbol{k})  \tag{5.36}\\
\hat{\psi}^{\dagger}(\boldsymbol{r}, t) \equiv \int \frac{\mathrm{d}^{3} \boldsymbol{k}}{(2 \pi)^{3}} \mathrm{e}^{-\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r}-\omega t)} \hat{a}^{\dagger}(\boldsymbol{k}),
\end{array}\right.
$$

with the commutation relations

$$
\begin{equation*}
\left[\hat{\psi}(\boldsymbol{r}, t), \hat{\psi}^{\dagger}\left(\boldsymbol{r}^{\prime}, t\right)\right]=\delta^{(3)}\left(\boldsymbol{r}-\boldsymbol{r}^{\prime}\right) \tag{5.37}
\end{equation*}
$$

We dropped the polarization for the sake of convenience. The number of photons in a box reads

$$
\begin{equation*}
\hat{N}_{B}=\int_{B} \mathrm{~d}^{3} \boldsymbol{r} \hat{\psi}^{\dagger}\left(\boldsymbol{r}^{\prime}, t\right) \hat{\psi}\left(\boldsymbol{r}^{\prime}, t\right) \tag{5.38}
\end{equation*}
$$

### 5.3.2 Probabilities and Distributions

Now, ask what is the probability that $N_{B}=n$ for $n \in \mathbb{Z}$. Start with the case of a single boson operator ( $\hat{a}, \hat{a}^{\dagger}$ ) such that

$$
\begin{equation*}
\left[\hat{a}, \hat{a}^{\dagger}\right]=1 . \tag{5.39}
\end{equation*}
$$

For a state $|\psi\rangle$, the probability to have $n$ is

$$
\begin{equation*}
P_{n}=|\langle\psi \mid n\rangle|^{2} \tag{5.40}
\end{equation*}
$$

for the state

$$
\begin{equation*}
|n\rangle=\frac{\left(a^{\dagger}\right)^{n}}{\sqrt{n!}}|0\rangle \tag{5.41}
\end{equation*}
$$

Introduce the generating function,

$$
\begin{equation*}
M(\xi) \equiv\left\langle\mathrm{e}^{\xi \hat{n}}\right\rangle=\langle\psi| \mathrm{e}^{\xi \hat{a}^{\dagger} \hat{a}}|\psi\rangle=\sum_{n=0}^{\infty} \mathrm{e}^{\xi n} P_{n} \tag{5.42}
\end{equation*}
$$

We want $|n\rangle$ to admit the Poisson distribution. There exists such (complex valued) states, called "coherent states" with $P_{n}$ the Poisson dist.:

$$
\begin{equation*}
|z\rangle \equiv \mathrm{e}^{-\frac{1}{2}|z|^{2}} \mathrm{e}^{z \hat{a}^{\dagger}}|0\rangle=\sum_{n=0}^{\infty} \mathrm{e}^{-\frac{1}{2}|z|^{2}} \frac{z^{n}}{\sqrt{n!}}|n\rangle . \tag{5.43}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
P_{n}(z)=|\langle z \mid n\rangle|^{2}=\mathrm{e}^{-|z|^{2}} \frac{z^{2 n}}{n!} \tag{5.44}
\end{equation*}
$$

is Poisson distribution.
These states $|z\rangle$ are the eigenstates of

$$
\begin{equation*}
a|z\rangle=z|z\rangle \tag{5.45}
\end{equation*}
$$

Now, since we know $|z\rangle$, we can calculate the generating function

$$
\begin{equation*}
M_{z}(\xi)=\exp \left[|z|^{2}\left(\mathrm{e}^{\xi}-1\right)\right]=\langle z| \mathrm{e}^{\xi^{\dagger} \hat{a}}|z\rangle . \tag{5.46}
\end{equation*}
$$

### 5.3.3 Methodological Intermezzo - Normal Ordering

Now we will show another derivation of Eq. (5.46). We will also show-by the way - that the $\frac{1}{2}$ in the harmonic oscillator eigenenergy $E_{n}=\hbar \omega\left(n+\frac{1}{2}\right)$ has no physical meaning.

Consider the generating function

$$
\begin{equation*}
\langle z| \mathrm{e}^{\xi \hat{a}^{\dagger} \hat{a}}|z\rangle=\sum_{n=0}^{\infty} \frac{\xi^{n}}{n!}\langle z|\left(\hat{a}^{\dagger} \hat{a}\right)^{n}|z\rangle \tag{5.47}
\end{equation*}
$$

Now,

- For $n=1$ :

$$
\begin{equation*}
\langle z| \hat{a}^{\dagger} \hat{a}|z\rangle=|z|^{2}=\langle n\rangle \tag{5.48a}
\end{equation*}
$$

- For $n=2$ :

$$
\begin{equation*}
\langle z|\left(\hat{a}^{\dagger} \hat{a}\right)^{2}|z\rangle=\langle z| \hat{a}^{\dagger} \hat{a}^{\dagger} \hat{a} \hat{a}+\hat{a}^{\dagger} \hat{a}|z\rangle=|z|^{4}+|z|^{2} \tag{5.48b}
\end{equation*}
$$

because $\hat{a}^{\dagger} \hat{a} \hat{a}^{\dagger} \hat{a}=\hat{a}^{\dagger} \hat{a}^{\dagger} \hat{a} \hat{a}+\hat{a}^{\dagger} \hat{a}$ has a normal order.

- For $n=3$ :

$$
\begin{equation*}
\langle z|\left(\hat{a}^{\dagger} \hat{a}\right)^{3}|z\rangle=|z|^{6}+3|z|^{4}+|z|^{2} . \tag{5.48c}
\end{equation*}
$$

Generally, we need the terms that involve

$$
\begin{equation*}
\langle z|\left(\hat{a}^{\dagger}\right)^{p}(\hat{a})^{p}|z\rangle=|z|^{2 p} . \tag{5.49}
\end{equation*}
$$

Therefore, we want the product of all $\hat{a}$ and $\hat{a}^{\dagger}$ s such that all $\hat{a}^{\dagger}$ s are on the left. This is the normal ordering, which we denote by :(...) : with the operators inside the brackets.

Let us give a few examples

$$
\begin{align*}
: \hat{a}^{\dagger} \hat{a}: & =\hat{a}^{\dagger} \hat{a}  \tag{5.50a}\\
: \hat{a}^{\dagger} \hat{a} \hat{a}^{\dagger}: & =\hat{a}^{\dagger} \hat{a}^{\dagger} \hat{a}=: \hat{a} \hat{a}^{\dagger} \hat{a}^{\dagger}:  \tag{5.50b}\\
:\left(\hat{a}^{\dagger} \hat{a}\right)^{p}: & =\left(\hat{a}^{\dagger}\right)^{p} \hat{a}^{p}  \tag{5.50c}\\
: \exp \left(y \hat{a}^{\dagger} \hat{a}\right): & =\sum_{p=0}^{\infty} \frac{y^{p}}{p!}\left(\hat{a}^{\dagger}\right)^{p} \hat{a}^{p} . \tag{5.50d}
\end{align*}
$$

Let us expand the last equation.

$$
\begin{equation*}
: \exp \left(y \hat{a}^{\dagger} \hat{a}\right):|n\rangle=\sum_{p=0}^{\infty} \frac{n!}{(n-p)!p!} y^{p}|n\rangle=(1+y)^{n}|n\rangle . \tag{5.51}
\end{equation*}
$$

This is since the terms with $p>n$ have $\hat{a}^{p}|n\rangle=0$ so that

$$
\begin{equation*}
\left(\hat{a}^{\dagger}\right)^{p} \hat{a}^{p}|n\rangle=n(n-1) \ldots(n-p+1)|n\rangle=\frac{n!}{(n-p)!}|n\rangle . \tag{5.52}
\end{equation*}
$$

Continue and have

$$
\begin{equation*}
: \exp \left(y \hat{a}^{\dagger} \hat{a}\right):|n\rangle=\mathrm{e}^{n \ln (1+y)}|n\rangle=\mathrm{e}^{\ln (1+y) \hat{a}^{\dagger} \hat{a}}|n\rangle \tag{5.53}
\end{equation*}
$$

This is true $\forall n$. Thus,

$$
\begin{equation*}
: \exp \left(y \hat{a}^{\dagger} \hat{a}\right):=\exp \left[\ln (1+y) \hat{a}^{\dagger} \hat{a}\right] \tag{5.54}
\end{equation*}
$$

### 5.3.4 Back to the Characteristic Function

Rewrite the characteristic function in terms of the normal ordering to have

$$
\begin{aligned}
M(\xi) & =\langle z|: \exp \left(y \hat{a}^{\dagger} \hat{a}\right):|z\rangle \\
& =\sum_{m, n} \mathrm{e}^{-|z|^{2}} \frac{\bar{z}^{m}}{\sqrt{m!}} \frac{z^{n}}{\sqrt{n!}} \underbrace{\langle m|: \exp \left(y \hat{a}^{\dagger} \hat{a}\right):|n\rangle}_{(1+y)^{n}\langle m \mid n\rangle=(1+y)^{n} \delta_{m n}} \\
& =\sum_{n=0}^{\infty} \mathrm{e}^{-|z|^{2}} \frac{|z|^{2 n}}{n!}(1+y)^{n} \\
& =\mathrm{e}^{-|z|^{2}} \sum_{n=0}^{\infty} \frac{1}{n!}\left(|z|^{2}(1+y)\right)^{n}
\end{aligned}
$$

$$
\begin{align*}
& =\mathrm{e}^{-|z|^{2}} \mathrm{e}^{|z|^{2}(1+y)} \\
& =\mathrm{e}^{y|z|^{2}} \tag{5.55}
\end{align*}
$$

Plug in $y=\mathrm{e}^{\xi}-1$ to have

$$
\begin{align*}
M(\xi)=\langle z| \mathrm{e}^{\xi \hat{a}^{\dagger} \hat{a}}|z\rangle & =\langle z|: \exp \left(\left(\mathrm{e}^{\xi}-1\right) \hat{a}^{\dagger} \hat{a}\right):|z\rangle \\
& =\exp \left[|z|^{2}\left(\mathrm{e}^{\xi}-1\right)\right] \tag{5.56}
\end{align*}
$$

Remark 5.6. The detection procedure uses generating function with normal ordering to count clicks. Since we can put whatever value in $z$, the value of $\frac{1}{2}$ in quantum Harmonic oscillator does not matter.

### 5.3.5 Thermal (Blackbody) Light - Einstein Relation

As everything else with Einstein, this formula work for blackbody (thermal light). Let us return to

$$
\begin{equation*}
\hat{N}_{B}=\int_{B} \mathrm{~d}^{3} \boldsymbol{r} \hat{\psi}^{\dagger}\left(\boldsymbol{r}^{\prime}, t\right) \hat{\psi}\left(\boldsymbol{r}^{\prime}, t\right) \tag{5.57}
\end{equation*}
$$

The expectation value of the number of particles is given by the Bose-Einstein distribution,

$$
\left\{\begin{array}{l}
\left\langle\hat{a}^{\dagger}(\boldsymbol{k}) \hat{a}(\boldsymbol{k})\right\rangle=(2 \pi)^{3} \delta\left(\boldsymbol{k}-\boldsymbol{k}^{\prime}\right) n_{\mathrm{BE}}\left(\frac{\hbar \omega(\boldsymbol{k})}{k_{\mathrm{B}} T}\right)  \tag{5.58}\\
n_{\mathrm{BE}}(x) \equiv \frac{1}{\mathrm{e}^{x}-1}
\end{array}\right.
$$

Thus,

$$
\begin{align*}
\left\langle\hat{N}_{B}\right\rangle & =\int_{B} \mathrm{~d}^{3} \boldsymbol{r}\left\langle\hat{\psi}^{\dagger}\left(\boldsymbol{r}^{\prime}, t\right) \hat{\psi}\left(\boldsymbol{r}^{\prime}, t\right)\right\rangle \\
& =\int_{B} \mathrm{~d}^{3} \boldsymbol{r} \int \frac{\mathrm{~d}^{3} \boldsymbol{k}}{(2 \pi)^{3}} \int \frac{\mathrm{~d}^{3} \boldsymbol{k}^{\prime}}{(2 \pi)^{3}} \mathrm{e}^{-\mathrm{i}\left(\boldsymbol{k}-\boldsymbol{k}^{\prime}\right) \cdot \boldsymbol{r}}\left\langle\hat{a}^{\dagger}(\boldsymbol{k}) \hat{a}\left(\boldsymbol{k}^{\prime}\right)\right\rangle \tag{5.59}
\end{align*}
$$

Hence,

$$
\begin{equation*}
\left\langle\hat{N}_{B}\right\rangle=(\operatorname{Vol}(B)) \int \frac{\mathrm{d}^{3} \boldsymbol{k}}{(2 \pi)^{3}} n_{\mathrm{BE}}\left(\frac{\hbar \omega(\boldsymbol{k})}{k_{\mathrm{B}} T}\right)=C_{1} \frac{\operatorname{Vol}(B)}{\lambda_{T}^{3}} \tag{5.60}
\end{equation*}
$$

with

$$
\begin{equation*}
C_{1}=4 \pi \int_{0}^{\infty} \mathrm{d} x \frac{x^{2}}{\mathrm{e}^{x}-1} \tag{5.61}
\end{equation*}
$$

and

$$
\begin{equation*}
\lambda_{T}=\frac{h c}{k_{\mathrm{B}} T} \tag{5.62}
\end{equation*}
$$

the thermal wavelength.
Let us continue. ${ }^{23}$ The second moment of $\hat{N}_{B}$ reads

$$
\begin{equation*}
\left\langle\hat{N}_{B}^{2}\right\rangle=\int_{B} \mathrm{~d}^{3} \boldsymbol{r} \mathrm{~d}^{3} \boldsymbol{r}^{\prime} \int \prod_{i=1}^{4} \frac{\mathrm{~d}^{3} \boldsymbol{k}_{i}}{(2 \pi)^{3}} \mathrm{e}^{\mathrm{i}\left(\boldsymbol{k}_{2}-\boldsymbol{k}_{1}\right) \cdot \boldsymbol{r}+\mathrm{i}\left(\boldsymbol{k}_{4}-\boldsymbol{k}_{3}\right) \cdot \boldsymbol{r}^{\prime}}\left\langle\hat{a}^{\dagger}\left(\boldsymbol{k}_{1}\right) \hat{a}\left(\boldsymbol{k}_{2}\right) \hat{a}^{\dagger}\left(\boldsymbol{k}_{3}\right) \hat{a}\left(\boldsymbol{k}_{4}\right)\right\rangle . \tag{5.63}
\end{equation*}
$$

[^20]According to Wick theorem, the product of $\langle\ldots\rangle$ can be decomposed into two contractions

$$
\begin{equation*}
{\hat{\hat{a}^{\dagger}}\left(\boldsymbol{k}_{1}\right) \hat{a}\left({ }^{(1)} \boldsymbol{k}_{2}\right) \hat{a}^{\dagger}\left(\boldsymbol{k}_{3}\right)}_{\hat{a}}^{\left(\boldsymbol{k}_{4}\right)} \tag{5.64}
\end{equation*}
$$

They explicitly read

$$
\begin{align*}
\left\langle\hat{a}^{\dagger}\left(\boldsymbol{k}_{1}\right) \hat{a}\left(\boldsymbol{k}_{2}\right) \hat{a}^{\dagger}\left(\boldsymbol{k}_{3}\right) \hat{a}\left(\boldsymbol{k}_{4}\right)\right\rangle & =\overbrace{\left\langle\hat{a}^{\dagger}\left(\boldsymbol{k}_{1}\right) \hat{a}\left(\boldsymbol{k}_{2}\right)\right\rangle\left\langle\hat{a}^{\dagger}\left(\boldsymbol{k}_{3}\right) \hat{a}\left(\boldsymbol{k}_{4}\right)\right\rangle}^{\left\langle\hat{N}_{B}\right\rangle^{2}} \\
& +\left\langle\hat{a}^{\dagger}\left(\boldsymbol{k}_{1}\right) \hat{a}\left(\boldsymbol{k}_{4}\right)\right\rangle\left\langle\hat{a}\left(\boldsymbol{k}_{2}\right) \hat{a}^{\dagger}\left(\boldsymbol{k}_{3}\right)\right\rangle, \tag{5.65}
\end{align*}
$$

with the first line corresponding to (1) and the second to (2). Therefore, after some algebra one obtains

$$
\begin{equation*}
\left\langle\hat{N}_{B}^{2}\right\rangle-\left\langle\hat{N}_{B}\right\rangle^{2}=\int \frac{\mathrm{d}^{3} \boldsymbol{k}_{1}}{(2 \pi)^{3}} \int \frac{\mathrm{~d}^{3} \boldsymbol{k}_{2}}{(2 \pi)^{3}} n_{\mathrm{BE}}\left(\frac{\hbar \omega\left(\boldsymbol{k}_{1}\right)}{k_{\mathrm{B}} T}\right)\left(1+n_{\mathrm{BE}}\left(\frac{\hbar \omega\left(\boldsymbol{k}_{2}\right)}{k_{\mathrm{B}} T}\right)\right) F\left(\boldsymbol{k}_{1}-\boldsymbol{k}_{2}\right) \tag{5.66}
\end{equation*}
$$

with

$$
\begin{equation*}
F(\boldsymbol{k})=\left|\int_{B} \mathrm{~d}^{3} \boldsymbol{r} \exp (\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{r})\right|^{2} \tag{5.67}
\end{equation*}
$$

the structure factor. For a spherical box $B$ of radius $R$ one has that

$$
\begin{equation*}
F(\boldsymbol{q})=16 \pi^{2} \frac{R^{2}}{q^{4}}\left(\cos q R-\frac{\sin q R}{q R}\right)^{2}, \quad \boldsymbol{q}=\boldsymbol{k}_{1}-\boldsymbol{k}_{2} \tag{5.68}
\end{equation*}
$$

The Bose-Einstein distribution is dependent on the thermal wavelength

$$
\begin{equation*}
n_{\mathrm{BE}}\left(k \lambda_{T}\right)=\frac{1}{\mathrm{e}^{\lambda_{T} k}-1} \tag{5.69}
\end{equation*}
$$

but $F(q)$ varies on the scale of $\frac{1}{R}$.


Fig. 5.3: Structure factor $F(q)$ for $R=1$.
Now, for $\frac{1}{R} \ll \frac{1}{\lambda_{T}} \Leftrightarrow \lambda_{T} \ll R$ (big box or large temperature). In this limit, one takes $F(\boldsymbol{q} \rightarrow \mathbf{0})$, where the change is minimal, and obtains

$$
\begin{equation*}
\left\langle\hat{N}_{B}^{2}\right\rangle-\left\langle\hat{N}_{B}\right\rangle^{2}=\int \frac{\mathrm{d}^{3} \boldsymbol{k}}{(2 \pi)^{3}} n_{\mathrm{BE}}\left(\frac{\hbar \omega(\boldsymbol{k})}{k_{\mathrm{B}} T}\right)\left(1+n_{\mathrm{BE}}\left(\frac{\hbar \omega(\boldsymbol{k})}{k_{\mathrm{B}} T}\right)\right) \underbrace{\int \frac{\mathrm{d}^{3} \boldsymbol{q}}{(2 \pi)^{3}} F(\boldsymbol{q})}_{\operatorname{Vol}(B)} . \tag{5.70}
\end{equation*}
$$

Therefore,

$$
\left\langle\hat{N}_{B}^{2}\right\rangle-\left\langle\hat{N}_{B}\right\rangle^{2}=\underbrace{\operatorname{Vol}(B) \int \frac{\mathrm{d}^{3} \boldsymbol{k}}{(2 \pi)^{3}} n_{\mathrm{BE}}\left(\frac{\hbar \omega(\boldsymbol{k})}{k_{\mathrm{B}} T}\right)}_{\text {shot noise }}+\underbrace{\operatorname{Vol}(B) \int \frac{\mathrm{d}^{3} \boldsymbol{k}}{(2 \pi)^{3}} n_{\mathrm{BE}}^{2}\left(\frac{\hbar \omega(\boldsymbol{k})}{k_{\mathrm{B}} T}\right)}_{\text {fluctuations of intensity }} .
$$

This is the Einstein formula [Einstein, 1917]. By this Einstein claimed that the fluctuations of intensity are proof of photons. Yet, there is nothing quantum here, since its blackbody radiation. That's why the explanation of Hanbury Brown and Twiss [1956] is wrong.
Remark 5.7. There is another way to obtain this result without the approximation $\lambda_{T} \ll R$ we made. It is still classical.

### 5.4 Photons vs. Semiclassical Theories

In this section we consider the question, when do we really need photons ("to photon or not to photon"). We shall do it via photodetection with and without photons.

### 5.4.1 Photodetection

Let us consider $H_{A}$ an atomic Hamiltonian. The interaction is ex dipolar with


Fig. 5.4: Level scheme for the photodetection experiment.
classical radiation $\mathcal{E} \sin (\omega t)$ so that the potential reads

$$
\begin{equation*}
V(t)=-e x \mathcal{E} \sin (\omega t) \tag{5.72}
\end{equation*}
$$

The probability for the detector to do its job reads

$$
\begin{equation*}
P(t)=\sum_{a} R_{a} P_{g \rightarrow a}(t) \tag{5.73}
\end{equation*}
$$

where $R_{a}$ is the collection efficiency with

- $R_{a}=\delta\left(E_{a}-E\right)$ collect only electrons of energy $E$.
- $R_{a}=1$ (broadband) collects everything.

The probability to go from the ground state to $a$ is

$$
\begin{equation*}
P_{g \rightarrow a}(t)=\frac{e^{2} \mathcal{E}^{2}}{4}\left|x_{a g}\right|^{2} \underbrace{\left(\frac{\sin \frac{\omega_{a g}-\omega}{2} t}{\frac{\omega_{a g}-\omega}{2}}\right)^{2}}_{F\left(t, \omega_{a g}-\omega\right)} \tag{5.74}
\end{equation*}
$$

with $\omega_{a g}=E_{a}-E_{g}$ and $x_{a g}=\langle a| \hat{x}|g\rangle$ the matrix element. Define

$$
\begin{equation*}
S=\frac{e^{2} \mathcal{E}^{2}}{4} \sum_{a} R_{a}\left|x_{a g}\right|^{2} \tag{5.75}
\end{equation*}
$$

be the detector collection efficiency.


Fig. 5.5: Spectral detection bands. In blue, the $F\left(t, \omega_{a g}-\omega\right)$ emission factor, with $t$ given in dotted purple. In orange, the detector collection efficiency $S$, with bandwidth $\Delta$ given in dashed yellow line.

When the time is much larger than the energy separation $t \gg \frac{1}{\Delta}$ (with $\Delta$ the detector bandwidth in $S$, meaning we have a broadband detector) one has

$$
\begin{equation*}
F=2 \pi t \delta\left(\omega-\omega_{a g}\right) \tag{5.76}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
P(t)=\underbrace{2 \pi \sum_{a} R_{a} \frac{e^{2} \mathcal{E}^{2}}{4}\left|x_{a g}\right|^{2} \delta\left(\omega-\omega_{a g}\right)}_{W} t=W t \tag{5.77}
\end{equation*}
$$

It is simply photoelectric effect given by Fermi golden rule.

### 5.4.2 Generalization to a Classical Wave

Let the electric field be

$$
\begin{equation*}
\mathcal{E}(r, t)=\sum_{k} \alpha_{k} \mathrm{e}^{\mathrm{i}(k r-\omega t)}+\alpha_{k}^{*} \mathrm{e}^{-\mathrm{i}(k r-\omega t)} \equiv \mathcal{E}^{(+)}(r, t)+\mathcal{E}^{(-)}(r, t) \tag{5.78}
\end{equation*}
$$

The intensity reads

$$
\begin{equation*}
I(r, t)=\left|\mathcal{E}^{(+)}(r, t)\right|^{2} \tag{5.79}
\end{equation*}
$$

Fluctuations are given by some distribution function on $\alpha_{k}, P\left(\left\{\alpha_{k}\right\}\right)$.
Let us calculate the photoionization. The probability to go to the excited state reads

$$
\begin{equation*}
\overline{P_{g \rightarrow a}(t)}=e^{2}\left|x_{a g}\right|^{2} \int_{0}^{t} \mathrm{~d} t^{\prime} \int_{0}^{t} \mathrm{~d} t^{\prime \prime} \mathrm{e}^{\mathrm{i} \omega_{a g}\left(t^{\prime}-t^{\prime \prime}\right)} \overline{\mathcal{E}^{(-)}\left(r, t^{\prime \prime}\right) \mathcal{E}^{(+)}\left(r, t^{\prime}\right)} \tag{5.80}
\end{equation*}
$$

Here we have repeated the calculation before. To make measurement we take the average over fluctuation distributions $\ldots$.. Let us define

$$
\begin{equation*}
\mathcal{G}^{(1)}\left(r, t^{\prime \prime}, r, t^{\prime}\right)=\overline{\mathcal{E}^{(-)}\left(r, t^{\prime \prime}\right) \mathcal{E}^{(+)}\left(r, t^{\prime}\right)} \tag{5.81}
\end{equation*}
$$

The detection probability thus reads

$$
\begin{equation*}
P(t)=\frac{1}{2 \pi} \iint_{0}^{t} \mathrm{~d} t^{\prime} \mathrm{d} t^{\prime \prime} \int_{-\infty}^{+\infty} S(\omega) \mathrm{e}^{\mathrm{i} \omega\left(t^{\prime}-t^{\prime \prime}\right)} \mathcal{G}^{(1)}\left(r, t^{\prime \prime}, r, t^{\prime}\right) \tag{5.82}
\end{equation*}
$$

where

$$
\begin{equation*}
S(\omega)=2 \pi e^{2} \sum_{a}\left|x_{a g}\right|^{2} R_{a} \delta\left(\omega-\omega_{a g}\right) \tag{5.83}
\end{equation*}
$$

Now, let us assume that $S(\omega)$ varies slowly with $\omega$. We get

$$
\begin{equation*}
P(t)=S \int_{0}^{t} \mathrm{~d} t^{\prime} \mathcal{G}^{(1)}\left(r, t^{\prime}, r, t^{\prime}\right) \tag{5.84}
\end{equation*}
$$

Therefore, we have

$$
\begin{equation*}
W_{1}(t) \equiv \frac{\mathrm{d} P}{\mathrm{~d} t}=S \mathcal{G}^{(1)}\left(r, t^{\prime}, r, t^{\prime}\right)=S \overline{I(r, t)} \tag{5.85}
\end{equation*}
$$

Notice that $\overline{I(r, t)}$ is the shot noise. Therefore, ionization only measures the shot noise.

### 5.4.3 Quantum Calculation

Let us consider a quantum electrical field

$$
\begin{equation*}
\hat{E}(r)=\sum_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{r}}+\hat{a}_{\boldsymbol{k}}^{\dagger} \mathrm{e}^{-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{r}} \tag{5.86}
\end{equation*}
$$

Take the Hamiltonian

$$
\begin{equation*}
H=H_{A}+H_{R}+V=H_{A}+\sum_{\boldsymbol{k}} \hbar \omega\left(\hat{a}_{\boldsymbol{k}}^{\dagger} \hat{a}_{\boldsymbol{k}}+\frac{1}{2}\right)-e \hat{x} \hat{E}(\boldsymbol{r}) . \tag{5.87}
\end{equation*}
$$

Here we use the interaction picture

$$
\begin{equation*}
\hat{E}(\boldsymbol{r}, t)=\mathrm{e}^{\mathrm{i} H_{R} t} \hat{E}(\boldsymbol{r}) \mathrm{e}^{-\mathrm{i} H_{R} t} \tag{5.88}
\end{equation*}
$$

In the quantum picture, in order to skip all the boring calculations, we have the following transformation

$$
\begin{equation*}
\mathcal{E}^{(+)}(r, t) \rightarrow\langle f| \hat{E}^{(+)}(\boldsymbol{r}, t)|i\rangle \tag{5.89}
\end{equation*}
$$

where $|i\rangle$ is the initial state of the radiation, and $|f\rangle$ is the final. Also,

$$
\begin{equation*}
\mathcal{E}^{(-)}\left(r, t^{\prime \prime}\right) \mathcal{E}^{(+)}\left(r, t^{\prime}\right)=\langle i| \hat{E}^{(-)}\left(\boldsymbol{r}, t^{\prime \prime}\right)|f\rangle\langle f| \hat{E}^{(+)}\left(\boldsymbol{r}, t^{\prime}\right)|i\rangle \tag{5.90}
\end{equation*}
$$

Averaging over the classical field will translate to summing over all the final fields (we measure the initial state and the ionization of the atom). Therefore,

$$
\begin{align*}
\overline{\mathcal{E}^{(-)}\left(r, t^{\prime \prime}\right) \mathcal{E}^{(+)}\left(r, t^{\prime}\right)} & =\sum_{f}\langle i| \hat{E}^{(-)}\left(\boldsymbol{r}, t^{\prime \prime}\right)|f\rangle\langle f| \hat{E}^{(+)}\left(\boldsymbol{r}, t^{\prime}\right)|i\rangle \\
& =\langle i| \hat{E}^{(-)}\left(\boldsymbol{r}, t^{\prime \prime}\right) \hat{E}^{(+)}\left(\boldsymbol{r}, t^{\prime}\right)|i\rangle \tag{5.91}
\end{align*}
$$

But this is the normal ordering.

Corollary 5.1. Measurement always leads to normal ordering of operators by default.

Consider now initial states $|i\rangle$. Generally, one has a mixed state

$$
\begin{equation*}
\rho_{R}=\sum_{i} \pi_{i}|i\rangle\langle i| . \tag{5.92}
\end{equation*}
$$

When we talk about the averages, we take trace over the operators

$$
\begin{equation*}
G^{(1)}\left(r, t^{\prime \prime}, r, t^{\prime}\right)=\operatorname{Tr}_{R}\left(\rho_{R} \hat{E}^{(-)}\left(\boldsymbol{r}, t^{\prime \prime}\right) \hat{E}^{(+)}\left(\boldsymbol{r}, t^{\prime}\right)\right) \tag{5.93}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
W_{1}(t)=S G^{(1)}\left(r, t^{\prime}, r, t^{\prime}\right) \tag{5.94}
\end{equation*}
$$

Corollary 5.2. There are no quantum effects in the correlation function of a single atom.

### 5.4.4 Reminder ${ }^{24}$

Working in the scheme of Figure 5.4, we have $H_{A}$ the atomic Hamiltonian, $e \hat{x}$ the dipolar interaction with $V(t)=-e \hat{x}(\mathcal{E} \sin \omega t)$ with the radiation of a classical wave $\mathcal{E} \sin \omega t$. We have also defined $P(t)$ the probability to excite one particle in the continuum.

Assuming broad-band detector, we saw that

$$
\begin{equation*}
P(t)=s I(r, t)=s|\mathcal{E}|^{2} \tag{5.95}
\end{equation*}
$$

with

$$
\begin{equation*}
\mathcal{E}(r, t)=\sum_{\boldsymbol{k}}\left(\alpha_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r}-\omega t)}+\text { c.c. }\right), \tag{5.96}
\end{equation*}
$$

and the classical amplitudes $\alpha_{k}$ fluctuate. We saw that

$$
\begin{equation*}
P(t)=s \int_{0}^{t} \mathrm{~d} t^{\prime} \mathcal{G}^{(1)}\left(r, t^{\prime}, r, t\right) \tag{5.97}
\end{equation*}
$$

such that

$$
\begin{equation*}
w_{1}(t) \equiv \frac{\mathrm{d} P}{\mathrm{~d} t}=s \overline{\mathcal{E}^{(-)}(r, t) \mathcal{E}^{(+)}(r, t)}=s \overline{I(r, t)} \tag{5.98}
\end{equation*}
$$

Using quantum radiation field we saw that

$$
\begin{equation*}
\hat{E}(r)=\sum_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{r}}+\hat{a}_{\boldsymbol{k}}^{\dagger} \mathrm{e}^{-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{r}}, \tag{5.99}
\end{equation*}
$$

with $V=-e \hat{x} \cdot \hat{E}(r)$. Similarly, we calculated

$$
\begin{equation*}
w_{1}(t)=s\left\langle\hat{E}^{(-)}(r, t) \hat{E}^{(+)}(r, t)\right\rangle=s G^{(1)}\left(r, t^{\prime}, r, t\right) \tag{5.100}
\end{equation*}
$$

In conclusion, we saw that

1. There is no way to detect quantum from classical calculations using a single particle.
2. The normal ordering comes naturally for measurements.
[^21]
### 5.5 Correlation between Two Photoionizations

In this section we will show that a two-particle correlation $G^{(2)}$ can distinguish between classical and quantum correlations. We will show that bunching does not distinguish between them, whereas anti-bunching does.

### 5.5.1 The Problem Description

- Two atoms $(1,2)$ placed at $r_{1}$ and $r_{2}$.
- Neglect all interactions between them.
- Probability: for atom 1 to be ionized between $t_{1}$ and $t_{1}+\mathrm{d} t_{1}$ and for atom 2 to be ionized between $t_{2}$ and $t_{2}+\mathrm{d} t_{2}$, which is denoted $w_{2}\left(r_{1}, t_{1} ; r_{2}, t_{2}\right) \mathrm{d} t_{1} \mathrm{~d} t_{2}$ (note that $w_{2}$ is not 2 nd atom but both).

The Hamiltonian thus reads

$$
\begin{equation*}
\hat{H}=H_{A_{1}}+H_{A_{2}}+H_{R}-e \hat{x}_{1} \cdot \hat{E}\left(r_{1}\right)-e \hat{x}_{2} \cdot \hat{E}\left(r_{2}\right) \tag{5.101}
\end{equation*}
$$

First, let us calculate the amplitude to be in the ground state of both atoms and some field $\left|g_{1} g_{2} i\right\rangle$ to the evolution to some state

$$
\begin{equation*}
A=\left\langle a_{1} a_{2} f\right| \tilde{U}(0, t)\left|g_{1} g_{2} i\right\rangle \tag{5.102}
\end{equation*}
$$

What is important are the interaction terms $\hat{V}_{i}=-e \hat{x}_{i} \cdot \hat{E}\left(r_{i}\right)$. To first order is not interesting, since we want to ionize both atoms.

### 5.5.2 Second Order in Perturbation

Let us then calculate the second order in perturbation. It reads

$$
\begin{equation*}
A \rightarrow\left\langle a_{1} a_{2} f\right| \int_{t^{\prime}}^{t} \mathrm{~d} t^{\prime \prime} \int_{0}^{t} \mathrm{~d} t^{\prime} \tilde{V}\left(t^{\prime \prime}\right) \tilde{V}\left(t^{\prime}\right)\left|g_{1} g_{2} i\right\rangle \tag{5.103}
\end{equation*}
$$

where $\tilde{V}=\tilde{V}_{1}+\tilde{V}_{2}$. To avoid double interaction with the same atom, replace $\tilde{V}\left(t^{\prime \prime}\right) \tilde{V}\left(t^{\prime}\right)$ by $\tilde{V}_{2}\left(t^{\prime \prime}\right) \tilde{V}_{1}\left(t^{\prime}\right)+\tilde{V}_{1}\left(t^{\prime \prime}\right) \tilde{V}_{2}\left(t^{\prime}\right)$. Therefore,

$$
\begin{align*}
A \rightarrow & -e^{2} \int_{t^{\prime}}^{t} \mathrm{~d} t^{\prime \prime} \int_{0}^{t} \mathrm{~d} t^{\prime}\left(\hat{x}_{1}\right)_{a_{1} g_{1}}\left(\hat{x}_{2}\right)_{a_{2} g_{2}} \mathrm{e}^{\mathrm{i} \omega_{a_{1} g_{1}} t^{\prime}} \mathrm{e}^{\mathrm{i} \omega_{a_{2} g_{2}} t^{\prime \prime}}\langle f| \hat{E}^{(+)}\left(r_{2}, t^{\prime \prime}\right) \hat{E}^{(-)}\left(r_{1}, t^{\prime}\right)|i\rangle \\
& +(\text { terms where } 1 \leftrightarrows 2) \tag{5.104}
\end{align*}
$$

Approximation: Drop the anti-resonant terms $\omega_{a_{i} g_{i}}+\omega$; keep resonant terms

$$
\omega_{a_{i} g_{i}}-\omega
$$

After some tedious calculations, obtain the probability $P=|A|^{2}$,

$$
\begin{align*}
P_{i g_{1} g_{2} \rightarrow f a_{1} a_{2}}(t) & =e^{4} \int_{0}^{t} \mathrm{~d} t^{\prime} \int_{0}^{t} \mathrm{~d} t^{\prime \prime} \int_{0}^{t} \mathrm{~d} t^{(3)} \int_{0}^{t} \mathrm{~d} t^{(4)}\left|\left(\hat{x}_{1}\right)_{a_{1} g_{1}}\right|^{2}\left|\left(\hat{x}_{2}\right)_{a_{2} g_{2}}\right|^{2} \\
& \times \mathrm{e}^{\mathrm{i} \omega a_{1} g_{1}\left(t^{\prime}-t^{(3)}\right)} \mathrm{e}^{\mathrm{i} \omega a_{2} g_{2}\left(t^{\prime \prime}-t^{(4)}\right)} \\
& \times\langle i| \hat{E}^{(-)}\left(r_{1}, t^{(3)}\right) \hat{E}^{(-)}\left(r_{2}, t^{(4)}\right)|f\rangle\langle f| \hat{E}^{(+)}\left(r_{2}, t^{\prime \prime}\right) \hat{E}^{(+)}\left(r_{1}, t^{\prime}\right)|i\rangle \tag{5.105}
\end{align*}
$$

Now, assume in the measurement, we do not measure the state $|f\rangle$ of the field. Thus, we need to trace it out $\sum_{|f\rangle}$. Therefore, one replaces the field operators with

$$
\begin{equation*}
\sum_{|f\rangle}\langle i| \hat{E}^{(-)} \hat{E}^{(-)}|f\rangle\langle f| \hat{E}^{(+)} \hat{E}^{(+)}|i\rangle=\langle i| E^{(-)} E^{(-)} E^{(+)} E^{(+)}|i\rangle, \tag{5.106}
\end{equation*}
$$

which is in the normal ordering of the field.
In the classical description of the field, one has the correlation of two advanced and two retarded fields,

$$
\begin{equation*}
\mathcal{G}^{(2)}(\ldots)=\overline{\mathcal{E}^{(-)} \mathcal{E}^{(-)} \mathcal{E}^{(+)} \mathcal{E}^{(+)}} . \tag{5.107}
\end{equation*}
$$

Return to the quantum field. After summation over $a_{1}$ and $a_{2}$

$$
\begin{align*}
& \sum_{a_{1}} R_{a_{1}} \rightarrow s \delta\left(t^{\prime}-t^{(3)}\right)  \tag{5.108}\\
& \sum_{a_{2}} R_{a_{2}} \rightarrow s \delta\left(t^{\prime \prime}-t^{(4)}\right)
\end{align*}
$$

the probability reads

$$
\begin{equation*}
P(t)=s^{2} \int_{0}^{t} \mathrm{~d} t^{\prime} \int_{0}^{t} \mathrm{~d} t^{\prime \prime} G^{(2)}\left(r_{1}, t^{\prime}, r_{2}, t^{\prime \prime} ; r_{2}, t^{\prime}, r_{1}, t^{\prime \prime}\right) \tag{5.109}
\end{equation*}
$$

Now, distinguish $t_{1}$ from $t_{2}$ to have

$$
\begin{equation*}
P\left(t_{1}, t_{2}\right)=s^{2} \int_{0}^{t_{1}} \mathrm{~d} t^{\prime} \int_{0}^{t_{2}} \mathrm{~d} t^{\prime \prime} G^{(2)}\left(r_{1}, t^{\prime}, r_{2}, t^{\prime \prime} ; r_{2}, t^{\prime}, r_{1}, t^{\prime \prime}\right) \tag{5.110}
\end{equation*}
$$

Define

$$
\begin{equation*}
w_{2}\left(r_{1}, t_{1}, r_{2}, t_{2}\right) \equiv \frac{\mathrm{d}^{2} P}{\mathrm{~d} t_{1} \mathrm{~d} t_{2}}=s^{2} G^{(2)}\left(r_{1}, t_{1}, r_{2}, t_{2} ; r_{2}, t_{2}, r_{1}, t_{1}\right) . \tag{5.111}
\end{equation*}
$$

For the classical field one has classical correlations with quantum atoms

$$
\begin{equation*}
w_{2}\left(r_{1}, t_{1}, r_{2}, t_{2}\right)=s^{2} \overline{\mathcal{E}^{(-)} \mathcal{E}^{(-)} \mathcal{E}^{(+)} \mathcal{E}^{(+)}}=\overline{I\left(r_{2}, t_{2}\right) I\left(r_{1}, t_{1}\right)} . \tag{5.112}
\end{equation*}
$$

If one has anything quantum, then it is seen in $w_{2}$ - the correlations of intensities.

### 5.6 Classical and Quantum Correlation Functions - Analogies and Differences

### 5.6.1 Classical Case

Here $\mathcal{E}(r, t)=\sum_{\boldsymbol{k}}\left(\alpha_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r}-\omega t)}+\right.$ c.c. $)$, and $\alpha_{\boldsymbol{k}}$ complex random variables.
Denote $\{\alpha\}=\left\{\alpha_{1}, \alpha_{2}, \ldots\right\}$ a realization of the field. Denote $P(\{\alpha\})$ the probability of $\{\alpha\} . P(\{\alpha\}) \geq 0$ and real. Now,

$$
\begin{equation*}
1=\int P(\{\alpha\}) \mathrm{d}^{2}\{\alpha\}=\int P\left(\alpha_{1}, \alpha_{2}, \ldots\right) \mathrm{d}^{2} \alpha_{1} \mathrm{~d}^{2} \alpha_{2} \ldots \tag{5.113}
\end{equation*}
$$

The correlation function then reads

$$
\begin{equation*}
\mathcal{G}^{(2)}\left(r_{1}, t_{1}, r_{2}, t_{2}\right)=\int \mathrm{d}^{2}\{\alpha\} P(\{\alpha\}) \mathcal{E}^{(-)}\left(r_{2}, t_{2},\{\alpha\}\right) \mathcal{E}^{(+)}\left(r_{1}, t_{1},\{\alpha\}\right) \tag{5.114}
\end{equation*}
$$

### 5.6.2 Quantum Fields

Here $\hat{E}(r)=\sum_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{r}}+\hat{a}_{\boldsymbol{k}}^{\dagger} \mathrm{e}^{-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{r}}$. Let $\left|\alpha_{\boldsymbol{k}}\right\rangle$ be the eigenstates of $\hat{a}_{\boldsymbol{k}}$ with eigenvalues $\alpha_{\boldsymbol{k}} \in \mathbb{C}$ (coherent states), such that

$$
\begin{equation*}
\hat{a}_{\boldsymbol{k}}\left|\alpha_{\boldsymbol{k}}\right\rangle=\alpha_{\boldsymbol{k}}\left|\alpha_{\boldsymbol{k}}\right\rangle . \tag{5.115}
\end{equation*}
$$

Describe the state of the quantum field $\hat{E}$ as a product of coherent state for each mode $\boldsymbol{k}$,

$$
\begin{equation*}
|\{\alpha\}\rangle=\left|\alpha_{1}\right\rangle\left|\alpha_{2}\right\rangle\left|\alpha_{3}\right\rangle \ldots\left|\alpha_{k}\right\rangle \ldots \tag{5.116}
\end{equation*}
$$

Explicitly, we have the correspondence,

$$
\begin{equation*}
\hat{E}^{(+)}\left(r_{1}, t_{1},\{\alpha\}\right)|\{\alpha\}\rangle=\mathcal{E}^{(+)}\left(r_{1}, t_{1},\{\alpha\}\right)|\{\alpha\}\rangle \tag{5.117}
\end{equation*}
$$

Calculating the correlation, one has for a single coherent state,

$$
\begin{equation*}
\langle\{\alpha\}| \hat{E}^{(-)}\left(r_{2}, t_{2},\{\alpha\}\right) \hat{E}^{(+)}\left(r_{1}, t_{1},\{\alpha\}\right)|\{\alpha\}\rangle=\mathcal{E}^{(-)}\left(r_{2}, t_{2},\{\alpha\}\right) \mathcal{E}^{(+)}\left(r_{1}, t_{1},\{\alpha\}\right) \tag{5.118}
\end{equation*}
$$

Usually, the quantum field is a statistical superposition described with a density matrix $\hat{\rho}$ :

$$
\begin{equation*}
\hat{\rho}=\int \mathrm{d}^{2}\{\alpha\} P_{Q}(\{\alpha\})|\{\alpha\}\rangle\langle\{\alpha\}|, \tag{5.119}
\end{equation*}
$$

where $P_{Q}(\{\alpha\})$ is a quantum distribution. There are several properties of $\hat{\rho}$ :

$$
\left\{\begin{array}{l}
\hat{\rho}=\hat{\rho}^{\dagger}  \tag{5.120}\\
\operatorname{Tr} \hat{\rho}=1 \\
\operatorname{diag} \hat{\rho} \geq 0
\end{array}\right.
$$

Therefore,

$$
\begin{equation*}
P_{Q}(\{\alpha\}) \in \mathbb{R} \Longrightarrow 1=\int \mathrm{d}^{2}\{\alpha\} P_{Q}(\{\alpha\}) \tag{5.121}
\end{equation*}
$$

Remark 5.8. Unlike the classical case, $P_{Q}(\{\alpha\})$ does not have to be positive (e.g., Wigner distribution).

### 5.6.3 Correlations

The quantum correlation reads

$$
\begin{align*}
G^{(1)}\left(r_{1}, t_{1}, r_{2}, t_{2}\right) & =\operatorname{Tr}\left(\hat{\rho} \hat{E}^{(-)}\left(r_{2}, t_{2}\right) \hat{E}^{(+)}\left(r_{1}, t_{1}\right)\right) \\
& =\int \mathrm{d}^{2}\{\alpha\} P_{Q}(\{\alpha\}) \mathcal{E}^{(-)}\left(r_{2}, t_{2}\right) \mathcal{E}^{(+)}\left(r_{1}, t_{1}\right) \tag{5.122}
\end{align*}
$$

Corollary 5.3. If $P_{Q}(\{\alpha\})$ is positive, there is no way to distinguish between classical and quantum fields.

Corollary 5.4. Moreover, even if the radiation is quantum, but $P_{Q}(\{\alpha\})$ is positive, one can find a classical systems that has exactly the same correlations.


Fig. 5.6: The Hanburry-Brown \& Twiss setups. (A) time-independent setup; (B) time-dependent setup. Incident field is indicated by red arrows; split field by yellow arrows; signal by gray arrows. PM is photomultiplier (detector); C is a correlator.

### 5.7 The Hanburry-Brown \& Twiss Effect

Consider ${ }^{25}$ the two setups as in Figure 5.6.
(A) The spatial setup. Here $r_{1} \neq r_{2}$ but $t_{1}=t_{2}$. Here, the spatial difference can be very large.
(B) Time dependent setup. Here one interested in the time difference $\tau$.

### 5.7.1 Bunching of "Photons" - Simple Explanation

Let us present the bunching of "photons" for a classical and fluctuating wave HBT without photons.

Let us inspect $\mathcal{G}^{(2)}\left(r_{1}, r_{2}, \tau\right)$ for two cases: $\tau=0$ and $\tau \rightarrow \infty$. Both of these are classical; a simple explanation is as follows.

$$
\begin{align*}
\mathcal{G}^{(2)}\left(r_{1}, r_{2}, \tau=0\right) & =\overline{I(r, t) I(r, t)}=\overline{I^{2}},  \tag{5.123a}\\
\mathcal{G}^{(2)}\left(r_{1}, r_{2}, \tau=\infty\right) & =\lim _{\tau \rightarrow \infty} \overline{I(r, t) I(r, t+\tau)} \\
& =\overline{I(r, t)} \cdot \overline{I(r, t+\tau)}=\overline{I(r, t)}^{2} . \tag{5.123b}
\end{align*}
$$

However, we always have

$$
\begin{equation*}
\overline{I^{2}}-\bar{I}^{2}=\overline{(I-\bar{I})^{2}} \geq 0 \tag{5.124}
\end{equation*}
$$

so that

$$
\begin{equation*}
\mathcal{G}^{(2)}\left(r_{1}, r_{2}, 0\right) \geq \mathcal{G}^{(2)}\left(r_{1}, r_{2}, \infty\right) \tag{5.125}
\end{equation*}
$$

This is the bunching of photons at $\tau=0$.
For the spatial correlation one has

$$
\begin{equation*}
\mathcal{G}^{(2)}\left(r_{1}=r_{2}\right)=\overline{I^{2}} \geq \mathcal{G}^{(2)}\left(\left|r_{1}-r_{2}\right| \rightarrow \infty\right) \simeq \overline{I\left(r_{1}\right)} \cdot \overline{I\left(r_{2}\right)}=\overline{I(r)}^{2} \tag{5.126}
\end{equation*}
$$

[^22]What Hanbury Brown and Twiss claimed that since the photons are bosons, this bunching shows their bosonic nature, thus proving them being particles. We will show why it is wrong.

### 5.7.2 Bunching of "Photons" - Fano Factor

The right quantity to study, as we will see, it is the Fano factor. Inspect the time-dependent setup. Let us define

$$
\begin{align*}
& g_{1}(\tau)=\frac{\mathcal{G}^{(1)}(r, \tau)}{\mathcal{G}^{(1)}(r, 0)}  \tag{5.127a}\\
& g_{2}(\tau)=\frac{\mathcal{G}^{(2)}(r, r, \tau)}{\left|\mathcal{G}^{(1)}(r, 0)\right|^{2}} \tag{5.127b}
\end{align*}
$$

The normalization in the last equation is due to

$$
\begin{equation*}
\mathcal{G}^{(2)}(r, r, \infty)=\bar{I}^{2}=\left|\mathcal{G}^{(1)}(r, 0)\right|^{2} \tag{5.128}
\end{equation*}
$$

Now, the difference reads

$$
\begin{align*}
g_{2}(0)-g_{2}(\infty)=g_{2}(0)-1 & =\frac{\mathcal{G}^{(2)}(r, r, 0)-\left|\mathcal{G}^{(1)}(r, 0)\right|^{2}}{\left|\mathcal{G}^{(1)}(r, 0)\right|^{2}} \\
& =\frac{\overline{I^{2}}-\bar{I}^{2}}{\bar{I}^{2}} \tag{5.129}
\end{align*}
$$

This is the Fano factor.

### 5.7.3 Bunching of "Photons" - Probability Distributions

Let us use the language of probabilities we defined in the previous section in order to show interesting results. The $\mathcal{G}$-factors can be written as

$$
\begin{align*}
\mathcal{G}^{(2)}(r, r, 0) & =\int \mathrm{d}^{2}\{\alpha\} P(\{\alpha\})\left|\mathcal{E}^{(+)}(r, t,\{\alpha\})\right|^{4}  \tag{5.130a}\\
\mathcal{G}^{(1)}(r, 0) & =\int \mathrm{d}^{2}\{\alpha\} P(\{\alpha\})\left|\mathcal{E}^{(+)}(r, t,\{\alpha\})\right|^{2} \tag{5.130b}
\end{align*}
$$

Since $\{\alpha\}$ is not the right variable to use, we shall change the variables $P(\{\alpha\}) \rightarrow$ $P(J)$ such that $J \equiv\left|\mathcal{E}^{(+)}(r, t,\{\alpha\})\right|^{2}$. In other words,

$$
\begin{equation*}
P(J)=\int \mathrm{d}^{2}\{\alpha\} P(\{\alpha\}) \delta\left(J-\left|\mathcal{E}^{(+)}(r, t,\{\alpha\})\right|^{2}\right) \tag{5.131}
\end{equation*}
$$

Note that $P(J) \geq 0$ and normalized.
With this change of variables, our equations read

$$
\begin{align*}
\mathcal{G}^{(2)}(r, r, 0) & =\int \mathrm{d} J J^{2} P(J)  \tag{5.132a}\\
\mathcal{G}^{(1)}(r, 0) & =\int \mathrm{d} J J P(J) \tag{5.132b}
\end{align*}
$$

Therefore, the Fano factor reads

$$
\begin{equation*}
g_{2}(0)-g_{2}(\infty)=\frac{\int \mathrm{d} J P(J)\left(J-\int \mathrm{d} J^{\prime} J^{\prime} P\left(J^{\prime}\right)\right)^{2}}{\left(\int \mathrm{~d} J J P(J)\right)^{2}} \tag{5.133}
\end{equation*}
$$

Corollary 5.5. Since $P(J) \geq 0$ one has $g_{2}(0)-g_{2}(\infty) \geq 0$. This is bunching, which is the result of $P(J)$ being positive, and nothing else.

### 5.7.4 Quantum Distribution

In the quantum case, one has $P_{Q}\{\alpha\}$. Therefore, the equations change to

$$
\begin{align*}
\mathcal{G}^{(2)}(r, r, 0) & =\int \mathrm{d}^{2}\{\alpha\} P_{Q}(\{\alpha\})\left|\mathcal{E}^{(+)}(r, t,\{\alpha\})\right|^{4}  \tag{5.134a}\\
\mathcal{G}^{(1)}(r, 0) & =\int \mathrm{d}^{2}\{\alpha\} P_{Q}(\{\alpha\})\left|\mathcal{E}^{(+)}(r, t,\{\alpha\})\right|^{2} \tag{5.134b}
\end{align*}
$$

Change the variables $P_{Q}(\{\alpha\}) \rightarrow P_{Q}(J)$ such that $J \equiv\left|\mathcal{E}^{(+)}(r, t,\{\alpha\})\right|^{2}$. In other words,

$$
\begin{equation*}
P_{Q}(J)=\int \mathrm{d}^{2}\{\alpha\} P_{Q}(\{\alpha\}) \delta\left(J-\left|\mathcal{E}^{(+)}(r, t,\{\alpha\})\right|^{2}\right) \tag{5.135}
\end{equation*}
$$

Note that $P_{Q}$ is normalized, but not necessarily positive.
With this change of variables, our equations read

$$
\begin{align*}
\mathcal{G}^{(2)}(r, r, 0) & =\int \mathrm{d} J J^{2} P_{Q}(J),  \tag{5.136a}\\
\mathcal{G}^{(1)}(r, 0) & =\int \mathrm{d} J J P_{Q}(J) . \tag{5.136b}
\end{align*}
$$

Therefore, the Fano factor reads

$$
\begin{equation*}
g_{2}(0)-g_{2}(\infty)=\frac{\int \mathrm{d} J P_{Q}(J)\left(J-\int \mathrm{d} J^{\prime} J^{\prime} P_{Q}\left(J^{\prime}\right)\right)^{2}}{\left(\int \mathrm{~d} J J P_{Q}(J)\right)^{2}} \tag{5.137}
\end{equation*}
$$

Corollary 5.6. For $P_{Q} \leq 0$ one has $g_{2}(0)-g_{2}(\infty) \leq 0$. This is anti-bunching.
Remark 5.9. If we get a positive Fano factor, it doesn't mean that our system is definitely classical. Rather, the system can be quantum, but we could always find a classical system with the same results.

### 5.8 Gaussian Classical Fields

In this section we describe the case of Gaussian classical fields - practical applications of HBT.

### 5.8.1 Chaotic Sources

By this we mean stars, blackbodies, etc. In this case, we can calculate $g_{2}\left(r_{1}, r_{2}, \tau\right)$ and $g_{1}\left(r_{1}, r_{2}, \tau\right)$. Recall that

Wick's theorem allows us to reduce this product into a sum of products,

$$
\begin{equation*}
\mathcal{G}^{(2)}=\underbrace{\overline{\mathcal{E E}} \cdot \overline{\overline{\mathcal{E}}}}_{\text {first contraction }}+\underbrace{\overline{\mathcal{E E} \cdot \overline{\mathcal{E E}}}}_{\text {second contraction }} \text {. } \tag{5.139}
\end{equation*}
$$

In the case of Gaussian fields, this can be written as

$$
\begin{equation*}
\mathcal{G}^{(2)}=\mathcal{G}^{(1)}\left(r_{1}, t_{1}, r_{1}, t_{1}\right) \mathcal{G}^{(1)}\left(r_{2}, t_{2}, r_{2}, t_{2}\right)+\left|\mathcal{G}^{(1)}\left(r_{1}, t_{1}, r_{2}, t_{2}\right)\right|^{2} \tag{5.140}
\end{equation*}
$$

thus,

$$
\begin{equation*}
g_{2}\left(r_{1}, r_{2}, \tau\right)=1+\left|g_{1}\left(r_{1}, r_{2}, \tau\right)\right|^{2}, \tag{5.141}
\end{equation*}
$$

with $t_{1}$ and $t_{2}=t_{1}+\tau$ (Siegert formula). This is Einstein's formula in disguise; purely classical.

### 5.8.2 Applications

1. First application: measure correlation times $\tau_{c}$. The idea is as follows. Make an HBT measurement with $r_{1}=r_{2}$; plot $g_{2}(r, r, \tau)$ vs. $\tau$ (see Fig. 5.7A).
2. Second application: measure spatial correlations $L$. Here, $t_{1}=t_{2}$ such that $\tau=0$ (see Fig. 5.7B). The $L$ one finds is the diameter of the measured star.



Fig. 5.7: Normalized $g_{2}$ plots. (A) the first application of correlation times; (B) the second application of spatial correlations.

### 5.9 Quantum Behavior of the Radiation

Consider a simple experiment, as in Figure 5.8.

1. Classical wave: $A$ and $B$ are reached. A source with $N \gg 1$ emitting atoms.
2. Quantum $(S, A, B):\left|e_{s} g_{A} g_{B}\right\rangle \rightarrow\left\{\begin{array}{l}\left|g_{s} e_{A} g_{B}\right\rangle \\ \left|g_{s} g_{A} e_{B}\right\rangle\end{array}\right.$


Fig. 5.8: Quantum radiation experiment. $S$ is a source (classical or quantum); $A$ and $B$ are detectors.

In such a setup, it is conceptually easy to check the quantum character of the system: an atom emits a photon, which is then detected. In practice, there are many atoms, so that one has interference with other sources of light, and the quantum effects become obscured.

### 5.9.1 Hanburry-Brown \& Twiss as an Interference Effect

This description is due to Fano [1961]. Consider $N$ independent atoms $i=$ $1,2, \ldots, N$. Diagrammatically, denote

$$
\begin{array}{cc}
\mathcal{E}_{i}^{(+)}\left(r_{A}, t_{A}\right) & \mathcal{E}_{i}^{(-)}\left(r_{B}, t_{B}\right)  \tag{5.142}\\
r_{A}, t_{A} & \\
r_{B}, t_{B}
\end{array}
$$

Here, above is the field $\mathcal{E}^{( \pm)}$and below is the detector $\left(r_{A}, t_{A}\right)$. Now, the different statistical averages read

$$
\left\{\begin{array}{l}
\overline{\mathcal{E}_{i} \mathcal{E}_{j}}=\overline{\mathcal{E}_{i}} \cdot \overline{\mathcal{E}_{j}}  \tag{5.143}\\
\overline{\mathcal{E}_{i} \mathcal{E}_{j} \mathcal{E}_{i}}=\overline{\mathcal{E}_{i}^{2}} \cdot \overline{\mathcal{E}_{j}} \\
\overline{\mathcal{E}_{i}}=0 \quad \text { (random phases) } .
\end{array}\right.
$$

### 5.9.2 Structure of $\mathcal{G}^{(1)}$

Let us inspect

$$
\begin{equation*}
\mathcal{G}^{(1)}\left(r_{A}, t_{A}, r_{B}, t_{B}\right)=\overline{\left(\sum_{i} \mathcal{E}_{i}^{(-)}\left(r_{A}, t_{A}\right)\right)\left(\sum_{i} \mathcal{E}_{i}^{(+)}\left(r_{B}, t_{B}\right)\right)} . \tag{5.144}
\end{equation*}
$$

Diagrammatically it reads


A specific case of $A=B$ reads


Now, we want a closed loop from a source to a target. Thus, $\alpha=\alpha^{\prime}=0$ and

$$
\begin{equation*}
\beta^{\prime}=\sum_{i} \overline{I_{i}(r, t)}, \tag{5.146}
\end{equation*}
$$

and

$$
\begin{equation*}
\beta=\sum_{i} \overline{\mathcal{E}_{i}^{(-)}\left(r_{A}, t_{A}\right) \mathcal{E}_{i}^{(+)}\left(r_{B}, t_{B}\right)} \xrightarrow[\left|r_{A}-r_{B}\right| \gg L]{\left|t_{A}-t_{B}\right| \gg r_{c}} 0 . \tag{5.147}
\end{equation*}
$$

Hence, only $\beta^{\prime} \neq 0$.

### 5.9.3 Structure of $\mathcal{G}^{(2)}$

We want to inspect the structure of $\mathcal{G}^{(2)}\left(r_{A}, t_{A}, r_{B}, t_{B}\right)$, which shows nontrivial correlations. Inspect

( $\gamma^{\prime}$ )
$\left(\delta^{\prime}\right)$



and
( $\varepsilon$

$(\zeta)$

( $\eta$ )


In this case, we want fully connected diagrams from two distinct sources to two distinct targets; thus, $\gamma=\gamma^{\prime}=\delta^{\prime}=0$.

Now, note that $\delta, \varepsilon, \zeta, \eta$ result from interference between two amplitudes

such that

$$
(\kappa+\lambda)\left(\kappa^{*}+\lambda^{*}\right)=\underbrace{\kappa \kappa^{*}}_{\delta}+\underbrace{\kappa \lambda^{*}}_{\zeta}+\underbrace{\kappa^{*} \lambda}_{\eta}+\underbrace{\lambda \lambda^{*}}_{\varepsilon} .
$$

All that is left to do is to calculate the appropriate values from these Feynman diagrams.

Let us continue. ${ }^{26}$

- For $A=B$ that is $\left(r_{A}, t_{A}\right)=\left(r_{B}, t_{B}\right)$ the diagrams collapse. In this case, $\kappa=\lambda$ so that

$$
\begin{equation*}
\delta+\varepsilon+\zeta+\eta=|2 \kappa|^{2}=4|\kappa|^{2} . \tag{5.151}
\end{equation*}
$$

- For $A \neq B$ we expect interference effects, that is

$$
\begin{equation*}
|\kappa+\lambda|^{2} \leq 4|\kappa|^{2} . \tag{5.152}
\end{equation*}
$$



Fig. 5.9: Normalized $\mathcal{G}^{(2)}$ as an interference.

For a strong dephasing,

$$
\begin{equation*}
|\kappa+\lambda|^{2}=|\kappa|^{2}+|\lambda|^{2}=2|\kappa|^{2} . \tag{5.153}
\end{equation*}
$$

Corollary 5.7. HBT is an interference effect between 2 amplitudes


This is a classical phenomenon.

### 5.9.4 Quantum "Photons"

Here,


Here, $e_{i}$ are the atoms and $g_{I}$ are the detectors.

### 5.9.5 Summary

1. Bunching of HB\&T is an interference effect resulting from 2 atom processes.
2. It can be interpreted classically (without photons) or quantum-mechanically (with photons).
3. Distinction between photons and no photons arises for single atomic sources:

- Photons $=$ antibunching;
- No photons = bunching.

4. Must get rid of HB\&T bunching to observe quantum effects.
[^23]
## 6 Atoms in Cavities (Cavity QED) ${ }^{27}$

In this chapter, we will show how to observe quantum phenomena using cavities.

### 6.1 Basic Properties of Optical Cavities

So far, we have only seen photons in an open space. The idea now is to put atom in a finite volume.

### 6.1.1 Planar Cavity

Inspect Fig. 6.1. Let us calculate the phase shift between (1) and (2). Let $\delta$ be


Fig. 6.1: Planar cavity scheme. (A) The setup. (B) The resonator scheme. the length difference given by

$$
\begin{align*}
\delta & =\left[J_{1} I_{2} J_{2}-J_{1} H\right] \\
& =\frac{2 L_{\mathrm{cav}}}{\cos i}-\left(2 L_{\mathrm{cav}} \tan i\right) \sin i \\
& =\frac{2 L_{\mathrm{cav}}}{\cos i}\left(1-\sin ^{2} i\right)=2 L_{\mathrm{cav}} \cos i . \tag{6.1}
\end{align*}
$$

Hence, for $i=0$ one has the phase difference

$$
\begin{equation*}
\varphi=\frac{2 \pi}{\lambda} 2 L_{\mathrm{cav}} . \tag{6.2}
\end{equation*}
$$

Let $a_{0}$ be the amplitude of the incident wave. Let $r$ and $t$ be the reflection and transmission coefficients, respectively

$$
\begin{align*}
r^{2} & =R \\
t^{2} & =T  \tag{6.3}\\
R+T & =1
\end{align*}
$$

The total transmitted amplitude reads

$$
\begin{aligned}
A & =a_{0} \mathrm{e}^{\mathrm{i} \varphi} t^{2}+a_{0} \mathrm{e}^{\mathrm{i} \varphi} t^{2} r^{2} \mathrm{e}^{\mathrm{i} \varphi}+\ldots \\
& =a_{0} \mathrm{e}^{\mathrm{i} \varphi} T\left(1+R \mathrm{e}^{\mathrm{i} \varphi}+R^{2} \mathrm{e}^{2 \mathrm{i} \varphi}+\ldots\right)
\end{aligned}
$$

[^24]\[

$$
\begin{equation*}
=a_{0} \mathrm{e}^{\mathrm{i} \varphi} \frac{T}{1-R \mathrm{e}^{\mathrm{i} \varphi}} . \tag{6.4}
\end{equation*}
$$

\]

The transmitted intensity

$$
\begin{align*}
I & =I_{0} \frac{T^{2}}{(1+R)^{2}-2 R \cos \varphi} \\
& =I_{0}\left(\frac{T}{1-R}\right)^{2} \frac{1}{1+\frac{4 R}{(1-R)^{2}} \sin ^{2} \frac{\varphi}{2}}, \tag{6.5}
\end{align*}
$$

so that

$$
\begin{equation*}
I=I_{0} \frac{1}{1+\frac{4 F^{2}}{\pi^{2}} \sin ^{2} \frac{\varphi}{2}} . \tag{6.6}
\end{equation*}
$$

The finesse of the cavity is defined by

$$
\begin{equation*}
F \equiv \frac{\pi \sqrt{R}}{1-R} \tag{6.7}
\end{equation*}
$$

The intensity $I$ is maximal for $\sin ^{2} \frac{\varphi}{2}=0$ or $\frac{\varphi}{2}=\pi n, n \in \mathbb{Z}$. Hence,

$$
\begin{equation*}
4 \pi \frac{L_{\mathrm{cav}}}{\lambda}=2 \pi n \Longrightarrow \frac{L_{\mathrm{cav}}}{\lambda}=\frac{n}{2} . \tag{6.8}
\end{equation*}
$$

### 6.1.2 Finesse



Fig. 6.2: Maximal Intensity vs. $\varphi$.

The finesse is also given in terms of angle full-width half-max (FWHM)

$$
\begin{equation*}
F=\frac{2 \pi}{\Delta \varphi} \tag{6.9}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
I=\frac{I_{\max }}{2} \quad \text { for } \quad \varphi=2 \pi n \pm \frac{\Delta \varphi}{2} . \tag{6.10}
\end{equation*}
$$

In other words,

$$
\frac{1}{2}=\frac{1}{1+\frac{4 R}{(1-R)^{2}} \sin ^{2}\left(\frac{2 \pi n \pm \frac{\Delta \varphi}{2}}{2}\right)}
$$

$$
\begin{align*}
& =\frac{1}{1+m \sin ^{2}\left(\frac{\Delta \varphi}{4}\right)} \\
& \simeq \frac{1}{1+m \frac{\Delta \varphi^{2}}{16}} \tag{6.11}
\end{align*}
$$

with $m=\frac{4 R}{(1-R)^{2}}$. Thus,

$$
\begin{equation*}
F=\frac{2 \pi}{\Delta \varphi}=\frac{\pi}{2} \sqrt{m} \tag{6.12}
\end{equation*}
$$

The angular frequency of each resonant mode is given by

$$
\begin{equation*}
\omega=\frac{2 \pi c}{\lambda}=\frac{2 \pi c}{2 L_{\mathrm{cav}}} n=\frac{\pi c}{L_{\mathrm{cav}}} n \tag{6.13}
\end{equation*}
$$

Define the spectral width to be $\Delta \omega$. Hence,

$$
\begin{align*}
\varphi=\frac{4 \pi}{\lambda} L_{\mathrm{cav}} & \Longrightarrow \varphi=\frac{2 L_{\mathrm{cav}}}{c} \omega \\
& \Longrightarrow \Delta \varphi=\frac{2 L_{\mathrm{cav}}}{c} \Delta \omega=\frac{2 \pi}{F} \tag{6.14}
\end{align*}
$$

Thus,

$$
\begin{equation*}
\Delta \omega=\frac{\pi c}{L_{\mathrm{cav}}} \tag{6.15}
\end{equation*}
$$

### 6.1.3 "Photon" Lifetime in a Cavity

The round-trip time $T$ is given by


Fig. 6.3: Photon lifetime in cavity.

$$
\begin{equation*}
t=\frac{T}{2}=\frac{L_{\mathrm{cav}}}{c} \tag{6.16}
\end{equation*}
$$

After $2 t$ there are $R^{2} N$ photon left from $N$. Therefore, we lose

$$
\begin{equation*}
\Delta N=N(1-R)=N T \tag{6.17}
\end{equation*}
$$

photons per $t=\frac{L_{\text {cav }}}{c}$ in time. Thus,

$$
\begin{equation*}
\frac{\mathrm{d} N}{\mathrm{~d} t}=-\frac{\Delta N}{L_{\mathrm{cav}} / c}=-c \frac{1-R}{L_{\mathrm{cav}}} N \tag{6.18}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
N(t)=N_{0} \mathrm{e}^{-t / \tau_{c \mathrm{cav}}} \tag{6.19}
\end{equation*}
$$

where

$$
\begin{equation*}
\tau_{\mathrm{cav}}=\frac{L_{\mathrm{cav}}}{c(1-R)} \equiv \frac{1}{\kappa} \tag{6.20}
\end{equation*}
$$

where $\kappa$ is the photon decay rate. Thus,

$$
\begin{equation*}
\Delta \omega=\frac{\pi c}{L_{\text {cav }}} \frac{1-R}{\pi \sqrt{R}} \stackrel{R \simeq 1}{\sim} \frac{c(1-R)}{L_{\mathrm{cav}}}=\frac{1}{\tau_{\text {cav }}}=\kappa . \tag{6.21}
\end{equation*}
$$

### 6.1.4 Summary

There are 2 key parameters: $\omega_{m}$ of resonant modes, and $F$ the finesse $(\Delta \omega)$. We thus define the quality factor

$$
\begin{equation*}
Q=\frac{\omega}{\Delta \omega} . \tag{6.22}
\end{equation*}
$$

### 6.2 Atom-Cavity Coupling

As usual, we would couple a two-level atom to a cavity.

### 6.2.1 Basics

The strength of atom-cavity coupling depends on

1. The decay rate $\kappa$;
2. Non-resonant decay rate $\Gamma$;
3. Atom-photon coupling parameter $g_{0}$.

- For $g_{0} \gg \kappa, \Gamma$ we are in the strong coupling regime.
- For $g_{0} \ll \kappa, \Gamma$ we are in the weak coupling regime.


### 6.2.2 Strong Coupling Limit

If the photon leaves the cavity, it is the old physics we know - an irreversible process. But, if the photon stays in a cavity, it can be reabsorbed by the atom; a reversible process. This is the new physics of the strong coupling limit. Therefore, one must build good cavities with excellent reflectivity. We shall show it below.

Recall that the interaction energy between field (vacuum) and atom is given by

$$
\begin{equation*}
\Delta E=\left|d E_{\mathrm{vac}}\right|, \quad E_{\mathrm{vac}}=\sqrt{\frac{\hbar \omega}{2 \varepsilon_{0} V}} \tag{6.23}
\end{equation*}
$$

Hence, the coupling reads

$$
\begin{equation*}
\Delta E \equiv \hbar g_{0}=\sqrt{\frac{d^{2} \hbar \omega}{2 \varepsilon_{0} V}} \tag{6.24}
\end{equation*}
$$

so that

$$
\begin{equation*}
g_{0}=\sqrt{\frac{d^{2} \omega}{2 \varepsilon_{0} \hbar V}} . \tag{6.25}
\end{equation*}
$$

The strong coupling condition reads

$$
\begin{equation*}
g_{0} \gg \frac{\omega}{Q} \tag{6.26}
\end{equation*}
$$

where $Q$ is the quality factor, so that

$$
\begin{equation*}
Q \gg \sqrt{\frac{2 \varepsilon_{0} \hbar \omega V}{d^{2}}} . \tag{6.27}
\end{equation*}
$$

Plugging in typical values results in $Q \gg 3 \cdot 10^{6}$. Hence, $1-R \ll 10^{-5}$. With such a high- $Q$ factor, S. Haroche made the experiments leading to his Nobel prize of 2012 (see Brune et al. [1996]).

### 6.2.3 Spontaneous Emission in a Cavity - Purcell Effect ${ }^{28}$

Let us inspect the spontaneous emission in a single-mode cavity; this is the Purcell effect. We would calculate it qualitatively.

Recall that the free-space spontaneous emission rate

$$
\begin{equation*}
\Gamma_{f}=\frac{2 \pi}{\hbar^{2}} d^{2} \rho(\omega) E_{\mathrm{vac}}^{2} \tag{6.28}
\end{equation*}
$$

Substituting all the values we have found results in

$$
\begin{equation*}
\Gamma_{f}=\frac{2 \pi}{\hbar^{2}} d^{2} \frac{\omega^{2} \nmid}{\pi^{2} c^{3}} \frac{\hbar \omega}{2 \varepsilon_{0} V} \tag{6.29}
\end{equation*}
$$

Note that it is independent on the volume.
Let us inspect a single-mode cavity. In this case, we are interested only in this cavity mode $\omega_{c}$. The density of states can be approximated to Lorentzian and normalized to 1 ,


Fig. 6.4: Cavity DOS.

$$
\begin{equation*}
\rho_{c}(\omega)=\frac{2}{\pi \Delta \omega_{c}} \frac{\Delta \omega_{c}^{2}}{4\left(\omega-\omega_{c}\right)^{2}+\Delta \omega_{c}^{2}} \tag{6.30}
\end{equation*}
$$

[^25]At resonance,

$$
\begin{equation*}
\rho_{c}\left(\omega_{c}\right)=\frac{2}{\pi \Delta \omega_{c}}=\frac{2}{\pi} \frac{Q}{\omega_{c}} . \tag{6.31}
\end{equation*}
$$

The cavity emission rate is thus,

$$
\begin{align*}
\Gamma_{c} & =\frac{2 \pi}{\hbar^{2}} \frac{Q}{\omega_{c}} d^{2} \frac{\hbar \omega}{2 \varepsilon_{0} V_{m}} \\
& =\frac{2 Q d^{2}}{\hbar \varepsilon_{0} V_{m}} \tag{6.32}
\end{align*}
$$

### 6.2.4 What is $V$ ?

In free space, it is innocuous quantization volume. In cavity, we take a single mode and quantize it $|n\rangle$ such that

$$
\left\{\begin{array}{l}
a|n\rangle=\sqrt{n}|n-1\rangle  \tag{6.33}\\
a^{\dagger}|n\rangle=\sqrt{n+1}|n+1\rangle .
\end{array}\right.
$$

The electric field operator reads,

$$
\begin{equation*}
\hat{\boldsymbol{E}}_{\mathrm{vac}}=\sqrt{\frac{\hbar \omega_{c}}{2 \varepsilon_{0} V_{m}}}\left(\hat{\epsilon} \hat{a} \hat{a} f(r)+\hat{\epsilon}^{*} \hat{a}^{\dagger} f^{*}(r)\right) \tag{6.34}
\end{equation*}
$$

where $f(r)$ is the spatial profile of the field.
Next, we want to calculate the volume $V_{m}$. Recall that the average of the stored energy in cavity is

$$
\begin{equation*}
\hbar \omega_{c}\left(n+\frac{1}{2}\right)=\varepsilon_{0} \int_{\text {cavity }} \mathrm{d}^{3} \boldsymbol{r}\langle n| \hat{\boldsymbol{E}}_{\text {vac }}^{2}|n\rangle \tag{6.35}
\end{equation*}
$$

Hence, the mode volume (modal volume) $V_{m}$ reads

$$
\begin{equation*}
V_{m}=\int_{\text {cavity }} \mathrm{d}^{3} \boldsymbol{r}|f(r)|^{2} \tag{6.36}
\end{equation*}
$$

We thus define the Purcell [1946] factor

$$
\begin{equation*}
F_{p}=\frac{\Gamma_{c}}{\Gamma_{f}}=\frac{Q \lambda^{3}}{4 \pi^{2} V_{m}} \tag{6.37}
\end{equation*}
$$

Corollary 6.1. In order to get strong coupling we could either improve the quality factor $Q$ - or, equivalently, reduce the mode volume $V_{m}$.

Corollary 6.2. Another way is to use quasiperiodic cavities.

### 6.3 CQED - Jaynes \& Cummings Hamiltonian

Here we would describe the Jaynes and Cummings [1963] model in cavity QED (CQED).

### 6.3.1 Single Model Cavity + Atom (Static)

Let us inspect

$$
\begin{equation*}
\hat{H}=\hat{H}_{\mathrm{at}}+\hat{H}_{\mathrm{field}}+\hat{H}_{\mathrm{int}} \tag{6.38}
\end{equation*}
$$

with

$$
\begin{align*}
\hat{H}_{\mathrm{at}} & =E_{b}|b\rangle\langle b|+E_{a}|a\rangle\langle a|  \tag{6.39a}\\
\hat{H}_{\mathrm{field}} & =\hbar \omega_{\ell}\left(\hat{a}_{\ell}^{\dagger} \hat{a}_{\ell}+\frac{1}{2}\right)  \tag{6.39b}\\
\hat{H}_{\mathrm{int}} & =-\mathrm{i} E_{\ell} \hat{\boldsymbol{D}} \cdot \hat{\epsilon}_{\ell}\left(\hat{a}_{\ell}-\hat{a}_{\ell}^{\dagger}\right) . \tag{6.39c}
\end{align*}
$$

Here, $\hat{\boldsymbol{D}}$ is the dipole operator. Taking $\hat{\epsilon}_{\ell} \| \hat{z}$ one has

$$
\begin{equation*}
\hat{H}_{\mathrm{int}} \propto(|a\rangle\langle b|+|b\rangle\langle a|)\left(\hat{a}_{\ell}-\hat{a}_{\ell}^{\dagger}\right) \tag{6.40}
\end{equation*}
$$

We thus have 4 processes

$$
\begin{array}{ccc} 
& \text { atom } & \text { field } \\
\hline \text { (i) } & b \rightarrow a & n \rightarrow n-1  \tag{6.41}\\
\text { (ii) } & b \rightarrow a & n \rightarrow n+1 \\
\text { (iii) } & a \rightarrow b & n \rightarrow n-1 \\
\text { (iv) } & a \rightarrow b & n \rightarrow n+1 \\
\hline
\end{array}
$$

Close to resonance, only (ii) and (iii) remain (recall RWA).
Setting $E_{a}=0$ and ignoring the constant term $\frac{1}{2} \hbar \omega_{\ell}$, one obtains the Jaynes and Cummings Hamiltonian

$$
\begin{equation*}
\hat{H}_{\mathrm{JC}}=\hbar \omega_{0}|b\rangle\langle b|+\hbar \omega_{\ell} \hat{a}_{\ell}^{\dagger} \hat{a}_{\ell}+\mathrm{i} \frac{\hbar \Omega_{v}}{2}\left(|a\rangle\langle b| \hat{a}_{\ell}^{\dagger}-|b\rangle\langle a| \hat{a}_{\ell}\right) \tag{6.42}
\end{equation*}
$$

with $\omega_{0}=E_{b}-E_{a}$. The Rabi frequency of the vacuum reads

$$
\begin{equation*}
\hbar \Omega_{v}=2 E_{\ell} d_{a b} \tag{6.43}
\end{equation*}
$$

This Hamiltonian can be shown to be strongly entangled.

### 6.3.2 Eigenstates and Eigenvalues ${ }^{29}$

The Jaynes-Cummings Hamiltonian (6.42) has a very nice property

$$
\left\{\begin{array}{c}
\hat{H}_{\mathrm{JC}}|a ; n\rangle=n \hbar \omega_{\ell}|a ; n\rangle-\mathrm{i} \hbar \Omega_{v} \sqrt{n}|b ; n-1\rangle  \tag{6.44}\\
\hat{H}_{\mathrm{JC}}|b ; n-1\rangle=\hbar\left(\omega_{0}+(n-1) \omega_{\ell}\right)|b ; n-1\rangle+\mathrm{i} \hbar \Omega_{v} \sqrt{n}|a ; n\rangle
\end{array}\right.
$$

Notice that $M_{n}$ the subspace $\{|a ; n\rangle,|b ; n-1\rangle\}$ is closed under the action of $\hat{H}_{\mathrm{JC}}$. We also need the initial condition

$$
\begin{equation*}
\hat{H}_{\mathrm{JC}}|a ; 0\rangle=0 \tag{6.45}
\end{equation*}
$$

Call $\left|\psi_{ \pm n}\right\rangle$ the 2 eigenvalues

$$
\begin{equation*}
\hat{H}_{\mathrm{JC}}\left|\psi_{ \pm n}\right\rangle=E_{ \pm n}\left|\psi_{ \pm n}\right\rangle \tag{6.46}
\end{equation*}
$$

[^26]A direct calculation reads

$$
\begin{equation*}
E_{ \pm n}=\hbar\left(n \omega_{\ell}-\frac{\delta}{2} \pm \frac{1}{2} \sqrt{n \Omega_{v}^{2}+\delta^{2}}\right), \quad \delta=\omega_{\ell}-\omega_{0} \tag{6.47}
\end{equation*}
$$

And the eigenstates read

$$
\left\{\begin{array}{l}
\left|\psi_{+n}\right\rangle=\cos \theta_{n}|a ; n\rangle+\mathrm{i} \sin \theta_{n}|b ; n-1\rangle  \tag{6.48}\\
\left|\psi_{-n}\right\rangle=\mathrm{i} \sin \theta_{n}|a ; n\rangle+\cos \theta_{n}|b ; n-1\rangle,
\end{array} \quad \tan 2 \theta_{n}=-\frac{\Omega_{v} \sqrt{n}}{\delta}\right.
$$

Remark 6.1. The eigenstates $\left|\psi_{ \pm n}\right\rangle$ cannot be factorized into $\left|\psi_{ \pm n}\right\rangle=\left|\psi_{\text {at }}\right\rangle \otimes$ $\left|\psi_{\mathrm{rad}}\right\rangle$. Meaning, there is a strong correlation between atom and field; in other words, this is entanglement.
Remark 6.2. The eigenstates $\left|\psi_{ \pm n}\right\rangle$ are called "dressed states". Cohen-Tannoudji has got his Nobel prize for them.

Example. For $\delta=0$ and $\theta_{n}=\frac{\pi}{4}$ one has

$$
\left\{\begin{array}{l}
\left|\psi_{+n}\right\rangle=\frac{1}{\sqrt{2}}(|a ; n\rangle+\mathrm{i}|b ; n-1\rangle)  \tag{6.49}\\
\left|\psi_{-n}\right\rangle=\frac{1}{\sqrt{2}}(\mathrm{i}|a ; n\rangle+|b ; n-1\rangle) .
\end{array}\right.
$$

These are maximally entangled states.

### 6.4 Spontaneous Emission of an Atom in a Cavity

With these dressed states of the Jaynes-Cummings model, we can calculate interesting physics. We shall inspect spontaneous emission.

### 6.4.1 Revisiting Previous Results

Consider an atom in an excited state without a photon

$$
\begin{equation*}
|\psi(0)\rangle=|b ; 0\rangle ; \quad t=0 \tag{6.50}
\end{equation*}
$$

And for simplicity, we assume $\delta=0$ (i.e., the radiation is at resonance with the atom energy).

This $|\psi(0)\rangle$ is not an eigenstate of the subspace $M_{1}=\{|b ; 0\rangle ;|a ; 1\rangle\}$. In terms of its eigenstates, it reads

$$
\begin{equation*}
|\psi(0)\rangle=\frac{1}{\sqrt{2} \mathrm{i}}\left(\left|\psi_{+1}\right\rangle+\mathrm{i}\left|\psi_{-1}\right\rangle\right) . \tag{6.51}
\end{equation*}
$$

The time evolution $\forall t$ thus reads

$$
\begin{equation*}
|\psi(t)\rangle=\frac{\mathrm{e}^{-\mathrm{i} \omega_{0} t}}{\sqrt{2} \mathrm{i}}\left(\mathrm{e}^{-\mathrm{i} \Omega_{v} \frac{t}{2}}\left|\psi_{+1}\right\rangle+\mathrm{i} \mathrm{e}^{+\mathrm{i} \Omega_{v} \frac{t}{2}}\left|\psi_{-1}\right\rangle\right), \tag{6.52}
\end{equation*}
$$

or

$$
\begin{equation*}
|\psi(t)\rangle=\mathrm{e}^{-\mathrm{i} \omega_{0} t}\left(\cos \left(\frac{1}{2} \Omega_{v} t\right)|b ; 0\rangle-\sin \left(\frac{1}{2} \Omega_{v} t\right)|a ; 1\rangle\right) . \tag{6.53}
\end{equation*}
$$

Since we are dealing with a single atom with a definite quantum level, we cannot apply the Fermi golden rule. The probability to stay in the excited state thus reads

$$
\begin{equation*}
P_{b}(t)=\sum_{n}\langle b ; n \mid \psi(t)\rangle\langle\psi(t) \mid b ; n\rangle=\cos ^{2}\left(\frac{1}{2} \Omega_{v} t\right) \tag{6.54}
\end{equation*}
$$

This probability oscillates between $|b\rangle$ and $|a\rangle$. It is called "Vacuum Rabi oscillations". It is a quantum squash between an atom and a photon.

If we release this constraint $\delta \neq 0$ and repeat the calculations, we get that the behavior will still be oscillatory, but with a smaller amplitude for $\delta \gg \Omega_{v}$ :

$$
\begin{equation*}
P_{b}(t)=1-\frac{\Omega_{v}^{2}}{\Omega_{v}^{2}+\delta^{2}} \sin ^{2}\left(\sqrt{\Omega_{v}^{2}+\delta^{2}} \cdot \frac{t}{2}\right) \simeq 1 \tag{6.55}
\end{equation*}
$$

Remark 6.3. Spontaneous emission is a characteristic of both the atom and the cavity (environment).

### 6.4.2 Quantum Evolution of the Atom under the Excitation of a Pure Fock State

Prepare the system at $t=0$ such that

$$
\begin{equation*}
|\psi(0)\rangle=|a ; n\rangle ; \quad t=0 \tag{6.56}
\end{equation*}
$$

After some algebra, one gets

$$
\begin{equation*}
|\psi(t)\rangle=\mathrm{e}^{-\mathrm{i} n \omega_{0} t}\left(\cos \left(\frac{1}{2} \sqrt{n} \Omega_{v} t\right)|a ; n\rangle-\sin \left(\frac{1}{2} \sqrt{n} \Omega_{v} t\right)|b ; n-1\rangle\right) \tag{6.57}
\end{equation*}
$$

The probability to stay in the excited state is

$$
\begin{equation*}
P_{b}(t)=\cos ^{2}\left(\frac{1}{2} \sqrt{n} \Omega_{v} t\right) \tag{6.58}
\end{equation*}
$$

This is the "Rabi oscillation" (no vacuum).
Example. For a $\frac{\pi}{2}$-pulse: $\sqrt{n} \Omega_{v} t_{\frac{\pi}{2}}=\frac{\pi}{2}$, the state reads

$$
\begin{equation*}
\left|\psi\left(t_{\frac{\pi}{2}}\right)\right\rangle=\frac{\mathrm{e}^{-\mathrm{i} n \omega_{0} t_{\frac{\pi}{2}}}}{\sqrt{2}}(|a ; n\rangle+|b ; n-1\rangle) \tag{6.59}
\end{equation*}
$$

This state is very different from

$$
\begin{equation*}
\left|\psi_{\mathrm{at}}\right\rangle=\frac{1}{\sqrt{2}}(|a\rangle+|b\rangle) . \tag{6.60}
\end{equation*}
$$

This is because the expectation value reads

$$
\begin{equation*}
\left\langle\psi_{\mathrm{at}}\right| \hat{\boldsymbol{D}}\left|\psi_{\mathrm{at}}\right\rangle \neq 0 \tag{6.61}
\end{equation*}
$$

but

$$
\left\{\begin{array}{l}
\left\langle\psi\left(t_{\frac{\pi}{2}}\right)\right| \hat{\boldsymbol{D}}\left|\psi\left(t_{\frac{\pi}{2}}\right)\right\rangle=0  \tag{6.62}\\
\langle\hat{\boldsymbol{E}}\rangle=0 .
\end{array}\right.
$$

The last result is due to entanglement.

### 6.4.3 Quantum Evolution under the Excitation by a Coherent State ${ }^{30}$

Recall that the Fock state is not a pure classical state. Set the state to a coherent state

$$
\begin{equation*}
|\psi(0)\rangle=|0\rangle \otimes|\alpha\rangle, \quad t=0 \tag{6.63}
\end{equation*}
$$

[^27]with
\[

$$
\begin{equation*}
|\alpha\rangle=\sum_{n} c_{n}|n\rangle, \quad c_{n}=\mathrm{e}^{-\frac{1}{2}|\alpha|^{2}} \frac{\alpha^{n}}{\sqrt{n!}} . \tag{6.64}
\end{equation*}
$$

\]

In other words,

$$
\begin{equation*}
|\psi(0)\rangle=\sum_{n} c_{n}|a ; n\rangle \tag{6.65}
\end{equation*}
$$

Since the equations are closed under the subspace $M_{n}=\{|a ; n\rangle,|b ; n-1\rangle\}$ for each $n$, we can write the evolution $\forall t$ as

$$
\begin{equation*}
|\psi(t)\rangle=\sum_{n} \mathrm{e}^{-\mathrm{i} n \omega_{0} t} c_{n}\left(\cos \left(\frac{1}{2} \sqrt{n} \Omega_{v} t\right)|a ; n\rangle-\sin \left(\frac{1}{2} \sqrt{n} \Omega_{v} t\right)|b ; n-1\rangle\right) . \tag{6.66}
\end{equation*}
$$

Thus, the probability $P_{a}(t)$ to find the atom in the initial state $|a\rangle$ is

$$
\begin{equation*}
P_{a}(t)=\sum_{n}\left|c_{n}\right|^{2} \cos ^{2}\left(\frac{1}{2} \sqrt{n} \Omega_{v} t\right) . \tag{6.67}
\end{equation*}
$$

These are called (see Figure 6.5) "quantum revivals". The pulse "dies" and then reappears. This is in contrast with Rabi oscillations, where the pulse envelope never decays.


Fig. 6.5: Quantum revivals. Here $\Omega_{v}=20$ and $\alpha=6$.

Remark 6.4. In a coherent state, $\alpha$ counts the number of photons. For $\alpha \sim 1$, we have the quantum revivals. For a large $\alpha \gg 1$, the state decomposes back to $|\psi(t)\rangle=\left|\psi_{\text {at }}\right\rangle \otimes\left|\psi_{\text {coh }}\right\rangle$, where $\left|\psi_{\text {coh }}\right\rangle$ is classical (usual Bloch eqs., etc.).
Remark 6.5. With non-perfect mirrors (leaking of the EM field outside), one doesn't have the nice property that the subspace is closed. What one gets is an apparition of Mollow triplet.

## 7 Cooperative Effects between Atoms ${ }^{31}$

This description is due to Dicke [1954]. Thus far, we have seen ensembles of atoms that do not interact coherently. This has given rise to effects such as blackbody radiation or the Casimir effect. In this chapter, we will inspect what happens when the atoms can speak coherently.

Consider, for example, a state of $N$ atoms, all of which are in the excited state and such that all of them are in a superradiant coherent state (Dicke state). When they emit, they can form a laser without mirrors. We shall inspect what is required to prepare such a state.

### 7.1 Collective Atomic States

### 7.1.1 Introduction

Consider $N$ identical atoms

placed at a point $\boldsymbol{r}_{i}$. Define the operators

$$
\begin{align*}
\hat{S}_{i}^{+} & \equiv\left|e_{i}\right\rangle\left\langle g_{i}\right|  \tag{7.2a}\\
\hat{S}_{i}^{-} & \equiv\left|g_{i}\right\rangle\left\langle e_{i}\right|  \tag{7.2b}\\
\hat{S}_{z i} & \equiv-\frac{1}{2}\left(\left|g_{i}\right\rangle\left\langle g_{i}\right|-\left|e_{i}\right\rangle\left\langle e_{i}\right|\right) . \tag{7.2c}
\end{align*}
$$

These operators $\left(\hat{S}_{i}^{ \pm}, \hat{S}_{z i}\right)$ satisfy the algebra of angular momenta

$$
\begin{align*}
{\left[\hat{S}_{i}^{+}, S_{j}^{-}\right] } & =2 \delta_{i j} S_{i z}  \tag{7.3a}\\
\hat{S}_{i}^{+} \hat{S}_{i}^{-}+\hat{S}_{i}^{-} \hat{S}_{i}^{+} & =\mathbb{1} \tag{7.3b}
\end{align*}
$$

This defines the atomic Hamiltonian

$$
\begin{equation*}
\hat{H}_{\mathrm{at}}=\sum_{i=1}^{N} \hbar \omega_{0}\left|e_{i}\right\rangle\left\langle e_{i}\right|=\sum_{i=1}^{N} \hbar \omega_{0} \hat{S}_{z i} . \tag{7.4}
\end{equation*}
$$

The last equation is because

$$
\begin{equation*}
\left|e_{i}\right\rangle\left\langle e_{i}\right|=\left|g_{i}\right\rangle\left\langle g_{i}\right|+2 \hat{S}_{z i}=1-\left|e_{i}\right\rangle\left\langle e_{i}\right|+2 \hat{S}_{z i}, \tag{7.5}
\end{equation*}
$$

so that

$$
\begin{equation*}
\left|e_{i}\right\rangle\left\langle e_{i}\right|=\frac{1}{2} \mathbb{1}+\hat{S}_{z i} . \tag{7.6}
\end{equation*}
$$

The dipole moment operator is defined by

$$
\begin{equation*}
\hat{\boldsymbol{d}}_{j}=\boldsymbol{d}_{j}\left(\left|e_{j}\right\rangle\left\langle g_{j}\right|+\text { h.c. }\right)=\boldsymbol{d}_{j}\left(\hat{S}_{j}^{+}+\hat{S}_{j}^{-}\right), \tag{7.7}
\end{equation*}
$$

with $\boldsymbol{d}_{j} \in \mathbb{R}^{3}$.

[^28]
### 7.1.2 Interaction Hamiltonian ${ }^{32}$

The interaction between atoms and radiation reads

$$
\begin{equation*}
\hat{H}_{\mathrm{int}}=-\sum_{i=1}^{N} \boldsymbol{d}_{i}\left(\hat{S}_{i}^{+}+\hat{S}_{i}^{-}\right) \cdot \hat{\boldsymbol{E}}\left(\boldsymbol{r}_{i}\right) \tag{7.8}
\end{equation*}
$$

with the electric field operator

$$
\begin{equation*}
\hat{\boldsymbol{E}}\left(\boldsymbol{r}_{i}\right)=\mathrm{i} \sum_{\boldsymbol{k}, \hat{\epsilon}} \hat{\epsilon}_{\boldsymbol{k}} \sqrt{\frac{2 \pi \hbar \omega_{\boldsymbol{k}}}{V}}\left(\hat{a}_{\boldsymbol{k} \hat{\epsilon}} \mathrm{e}^{\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{r}}-\hat{a}_{\boldsymbol{k} \hat{\epsilon}}^{\dagger} \mathrm{e}^{-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{r}}\right) \tag{7.9}
\end{equation*}
$$

Using the RWA, one obtains (as we have seen earlier),

$$
\begin{equation*}
\hat{H}_{\mathrm{int}}=\sum_{\boldsymbol{k}, \hat{\epsilon}} \sum_{i=1}^{N} g_{\boldsymbol{k} \hat{\epsilon}}\left(\hat{a}_{\boldsymbol{k} \hat{\epsilon}} \hat{S}_{i}^{+} \mathrm{e}^{\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{r}}+\hat{a}_{\boldsymbol{k} \hat{\epsilon}}^{\dagger} \hat{S}_{i}^{-} \mathrm{e}^{-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{r}}\right) \tag{7.10}
\end{equation*}
$$

with

$$
\begin{equation*}
g_{\boldsymbol{k} \hat{\epsilon}}=-\mathrm{i} \hat{\epsilon}_{\boldsymbol{k}} \cdot \boldsymbol{d} \sqrt{\frac{2 \pi \hbar \omega_{\boldsymbol{k}}}{V}} \tag{7.11}
\end{equation*}
$$

Now, define the Fourier transform of $\hat{S}_{i}^{ \pm}$,

$$
\begin{equation*}
\hat{S}_{\boldsymbol{k}}^{ \pm} \equiv \sum_{i=1}^{N} \hat{S}_{i}^{ \pm} \mathrm{e}^{ \pm \mathrm{i} \boldsymbol{k} \cdot \boldsymbol{r}} \tag{7.12}
\end{equation*}
$$

Thus,

$$
\begin{equation*}
\hat{H}_{\mathrm{int}}=\sum_{\boldsymbol{k}, \hat{\epsilon}}\left(g_{\boldsymbol{k} \hat{\epsilon}} \hat{a}_{\boldsymbol{k} \hat{\epsilon}} \hat{S}_{\boldsymbol{k}}^{+}+g_{\boldsymbol{k} \hat{\epsilon}}^{*} \hat{a}_{\boldsymbol{k} \hat{\epsilon}}^{\dagger} \hat{S}_{\boldsymbol{k}}^{-}\right) \tag{7.13}
\end{equation*}
$$

Similarly, define

$$
\begin{equation*}
\hat{S}_{z} \equiv \sum_{i=1}^{N} \hat{S}_{z i} \tag{7.14}
\end{equation*}
$$

so that the total Hamiltonian reads

$$
\begin{equation*}
\hat{H}=\hbar \omega_{0} \hat{S}_{z}+\sum_{\boldsymbol{k}, \hat{\epsilon}} \hbar \omega_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k} \hat{\epsilon}}^{\dagger} \hat{a}_{\boldsymbol{k} \hat{\epsilon}}+\hat{H}_{\mathrm{int}} \tag{7.15}
\end{equation*}
$$

### 7.1.3 Dicke Atomic States

Define the collective spin operators

$$
\begin{equation*}
\hat{S}^{ \pm} \equiv \sum_{i=1}^{N} \hat{S}_{i}^{ \pm}, \quad \hat{S}_{z} \tag{7.16}
\end{equation*}
$$

Notice that the algebra of those operators is the algebra of spins. Use two operators $\left(\hat{S}^{2}, \hat{S}_{z}\right)$ such that

$$
\begin{equation*}
\hat{S}^{2}|s, m\rangle=s(s+1)|s, m\rangle \tag{7.17a}
\end{equation*}
$$

[^29]\[

$$
\begin{align*}
\hat{S}_{z}|s, m\rangle & =m|s, m\rangle  \tag{7.17b}\\
\hat{S}^{ \pm}|s, m\rangle & =\sqrt{s(s+1)-m(m \pm 1)}|s, m \pm 1\rangle \tag{7.17c}
\end{align*}
$$
\]

with

$$
\left\{\begin{array}{l}
m=-s,-s+1, \ldots,+s  \tag{7.18}\\
s=0,1,2, \ldots \frac{N}{2}
\end{array}\right.
$$

These $|s, m\rangle$ are called Dicke states [Dicke, 1954]. They are degenerate.

### 7.1.4 Atomic Product States

Let us inspect another state of atomic states, called "atomic product states",

$$
\begin{equation*}
|\phi\rangle=\left|g_{1} g_{2} e_{3}, \ldots, g_{N}\right\rangle \tag{7.19}
\end{equation*}
$$

With the operator

$$
\begin{equation*}
\hat{S}_{z i}=\frac{1}{2}\left(\hat{S}_{i}^{+} \hat{S}_{i}^{-}-\hat{S}_{i}^{-} \hat{S}_{i}^{+}\right) \tag{7.20}
\end{equation*}
$$

one obtains

$$
\begin{equation*}
\hat{S}_{z}|\phi\rangle=\frac{1}{2}\left(N_{e}-N_{g}\right)|\phi\rangle, \tag{7.21}
\end{equation*}
$$

where $N_{e, g}$ is the total number of excited/ground states. Define

$$
\begin{equation*}
m=\frac{1}{2}\left(N_{e}-N_{g}\right) \tag{7.22}
\end{equation*}
$$

be the total atomic inversion.

### 7.1.5 Relation between Product and Dicke States

Let us inspect the case $N=2$. We have 4 product states

$$
\begin{equation*}
\left|g_{1} g_{2}\right\rangle,\left|g_{1} e_{2}\right\rangle,\left|e_{1} g_{2}\right\rangle,\left|e_{1} e_{2}\right\rangle \tag{7.23}
\end{equation*}
$$

The Dicke states $s=0,1$ read

$$
\left\{\begin{array}{lll}
S=0 \Longrightarrow m=0 & :|00\rangle & \text { Singlet state }  \tag{7.24}\\
S=1 \Longrightarrow m=-1,0,1 & :|1,-1\rangle,|1,0\rangle|1,1\rangle & \text { Triplet states. }
\end{array}\right.
$$

Now, to combine them have

- For $s=0$ :

$$
\begin{equation*}
|00\rangle=\frac{1}{\sqrt{2}}\left(\left|e_{1} g_{2}\right\rangle-\left|g_{1} e_{2}\right\rangle\right) \tag{7.25a}
\end{equation*}
$$

- For $s=1$ :

$$
\left\{\begin{align*}
|1,1\rangle & =\left|e_{1} e_{2}\right\rangle  \tag{7.25b}\\
|1,0\rangle & =\frac{1}{\sqrt{2}}\left(\left|e_{1} g_{2}\right\rangle+\left|g_{1} e_{2}\right\rangle\right) \\
|1,-1\rangle & =\left|g_{1} g_{2}\right\rangle
\end{align*}\right.
$$

Note that the triplet states are not degenerate:

$$
\begin{equation*}
2 \otimes 2=1 \oplus 3 \tag{7.26}
\end{equation*}
$$

### 7.1.6 Clebsch-Gordan Decomposition

We need a general rule to translate Dicke states to product states and vice versa. This is the Clebsch-Gordan decomposition

For instance, with $N=4$,

$$
\begin{align*}
2 \otimes 2 \otimes 2 \otimes 2 & =(3 \oplus 1) \otimes(3 \oplus 1) \\
& =(3 \otimes 3) \oplus(3 \otimes 1) \oplus(1 \otimes 3) \oplus(1 \otimes 1) \\
& =(5 \oplus 3 \oplus 1) \oplus(3) \oplus(3) \oplus(1) \tag{7.27}
\end{align*}
$$

We thus have 1 spin 2 state (5); 3 spin 1 states (3); and 2 spin 0 states (1).

### 7.2 Cooperative Spontaneous Emission of $N$ Atoms

For $N$ atoms, we have two possibilities: Dicke states and Product states. Let us calculate both.

### 7.2.1 Cooperative Spontaneous Emission in a Dicke State

Let us reinspect the interaction Hamiltonian

$$
\begin{equation*}
\hat{H}_{\mathrm{int}}=\hat{a}_{\boldsymbol{k}} \hat{S}_{\boldsymbol{k}}^{+}+\hat{a}_{\boldsymbol{k}}^{\dagger} \hat{S}_{\boldsymbol{k}}^{-} \tag{7.28}
\end{equation*}
$$

where

$$
\begin{equation*}
\hat{S}_{\boldsymbol{k}}^{ \pm} \equiv \sum_{i=1}^{N} \hat{S}_{i}^{ \pm} \mathrm{e}^{ \pm \mathrm{i} \boldsymbol{k} \cdot \boldsymbol{r}} \tag{7.29}
\end{equation*}
$$

The problem with this, is that there is nothing one can do with $N>2 .{ }^{33}$ We therefore make the following assumption to get rid of the $\mathrm{e}^{ \pm \mathrm{i} \boldsymbol{k} \cdot \boldsymbol{r}}$ factor:

Conjecture 7.1 (Dicke assumption). All atoms are withing a volume $\ll \lambda^{3}$.
Doing these assumptions accumulates to inspect the electric fields at $\hat{\boldsymbol{E}}\left(\boldsymbol{r}_{i}=\mathbf{0}\right)$. In other words, we can look on a single operator. The interaction Hamiltonian therefore reads,

$$
\begin{equation*}
\hat{H}_{\mathrm{int}}=-\hat{\boldsymbol{D}} \cdot \hat{\boldsymbol{E}}(\mathbf{0})=-\left(\langle g| \boldsymbol{d}|e\rangle \hat{S}^{-}+\langle e| \boldsymbol{d}|g\rangle \hat{S}^{+}\right) \cdot \hat{\boldsymbol{E}}(\mathbf{0}) . \tag{7.30}
\end{equation*}
$$

Here, $\langle e| \boldsymbol{d}|g\rangle$ is the transition rate for a single atom. Define

$$
\begin{equation*}
\hat{W}=\langle g| \boldsymbol{d}|e\rangle \cdot \hat{\boldsymbol{E}}(\mathbf{0}) \hat{S}^{-}, \tag{7.31}
\end{equation*}
$$

so that

$$
\begin{equation*}
\hat{H}_{\mathrm{int}}=-\left(\hat{W}+\hat{W}^{\dagger}\right) \tag{7.32}
\end{equation*}
$$

Consider the transition rate $\Gamma$ given by the Fermi golden rule,

$$
\begin{equation*}
\left.\Gamma \propto \sum_{f}|\langle f| \hat{W}| s, m ; 0\right\rangle\left.\right|^{2}, \tag{7.33}
\end{equation*}
$$

[^30]with the initial state of 0 photons $|s, m ; 0\rangle$. Thus,
\[

$$
\begin{align*}
\Gamma & =\Gamma_{0}\langle s, m| \hat{W} \hat{W}^{\dagger}|s, m\rangle \\
& =\Gamma_{0}\langle s, m| \hat{S}^{+} \hat{S}^{-}|s, m\rangle \tag{7.34}
\end{align*}
$$
\]

with $\Gamma_{0}$ the emission rate of a single atom. Hence,

$$
\begin{equation*}
\frac{\Gamma}{\Gamma_{0}}=(s+m)(s-m+1) . \tag{7.35}
\end{equation*}
$$

Corollary 7.1. All the cooperative part of the spontaneous emission in a Dicke state is in the $(s+m)(s-m+1)$ part.

### 7.2.2 Analyzing Interesting Cases for Dicke States

1. All the atoms are excited: $|s, m\rangle=\left|\frac{N}{2}, \frac{N}{2}\right\rangle$. In this case,

$$
\begin{equation*}
\frac{\Gamma}{\Gamma_{0}}=N \tag{7.36}
\end{equation*}
$$

so that there is nothing interesting.
2. Half of the atoms are in the excited states $|s, 0\rangle$. Here

$$
\begin{equation*}
\frac{\Gamma}{\Gamma_{0}}=s(s+1) \tag{7.37a}
\end{equation*}
$$

(a) If $s=\frac{N}{2}$ is maximum, then

$$
\begin{equation*}
\frac{\Gamma}{\Gamma_{0}}=\frac{N}{2}\left(\frac{N}{2}+1\right) \tag{7.37b}
\end{equation*}
$$

this is superradiance.
(b) If $s=0$ is minimum, then

$$
\begin{equation*}
\frac{\Gamma}{\Gamma_{0}}=0 \tag{7.37c}
\end{equation*}
$$

this is subradiance (atoms are "swinging": one up and its neighbor is down).
3. A single atom is excited among $N,\left|s, 1-\frac{N}{2}\right\rangle=|s,-s+1\rangle$. Here,

$$
\begin{equation*}
\frac{\Gamma}{\Gamma_{0}}=2 s \tag{7.38}
\end{equation*}
$$

Take, for example, $s=\frac{N}{2}$, then

$$
\begin{equation*}
\frac{\Gamma}{\Gamma_{0}}=N \tag{7.39}
\end{equation*}
$$

4. All atoms are in the ground state, $|s, m\rangle=|s,-s\rangle$, then

$$
\begin{equation*}
\frac{\Gamma}{\Gamma_{0}}=0 . \tag{7.40}
\end{equation*}
$$

Remark 7.1 (Important). We have studied the $N$-atoms dipole operator

$$
\begin{equation*}
\hat{\boldsymbol{D}}=\langle g| \boldsymbol{d}|e\rangle \hat{S}^{-}+\langle e| \boldsymbol{d}|g\rangle \hat{S}^{+} \tag{7.41}
\end{equation*}
$$

Then, calculating the strength of the dipole moment of $N$ atoms reads for Dicke states

$$
\begin{equation*}
\langle s, m| \hat{\boldsymbol{D}}|s, m\rangle=0 . \tag{7.42}
\end{equation*}
$$

Yet the strength of fluctuations of $\hat{\boldsymbol{D}}$ is not 0 ; this is the source of spontaneous emission for the Dicke states.

### 7.2.3 Cooperative Spontaneous Emission in a Product State

Let us inspect the following product state

$$
\begin{equation*}
|\phi\rangle=\bigotimes_{j=0}^{N}\left(c_{g}\left|g_{j}\right\rangle+c_{e}\left|e_{j}\right\rangle\right), \tag{7.43}
\end{equation*}
$$

with

$$
\begin{equation*}
\left|c_{g}\right|^{2}+\left|c_{e}\right|^{2}=1 \tag{7.44}
\end{equation*}
$$

Describe this state using the density matrix

$$
\begin{equation*}
\rho=|\phi\rangle\langle\phi| \otimes|0\rangle\langle 0| . \tag{7.45}
\end{equation*}
$$

Next, use the same assumption $\hat{\boldsymbol{E}}\left(\boldsymbol{r}_{i}=\mathbf{0}\right)$ for the same reasons. The interaction Hamiltonian reads

$$
\begin{equation*}
\hat{H}_{\mathrm{int}}=-\hat{\boldsymbol{D}} \cdot \hat{\boldsymbol{E}}(\mathbf{0}) \tag{7.46}
\end{equation*}
$$

so that

$$
\begin{align*}
\frac{\Gamma}{\Gamma_{0}} & =\langle\phi| \sum_{i=1}^{N} \sum_{j=1}^{N} \hat{S}_{i}^{+} \hat{S}_{j}^{-}|\phi\rangle \\
& =N\left|c_{e}\right|^{2}+N(N-1)\left|c_{e}\right|^{2}\left|c_{g}\right|^{2} . \tag{7.47}
\end{align*}
$$

### 7.2.4 Analyzing Interesting Cases for Product States

1. All atoms are in the excited state $c_{g}=0$, then

$$
\begin{equation*}
\frac{\Gamma}{\Gamma_{0}}=N . \tag{7.48}
\end{equation*}
$$

2. Half of the atoms are in the excited state, $c_{e}=c_{g}=\frac{1}{\sqrt{2}}$. Then,

$$
\begin{equation*}
\frac{\Gamma}{\Gamma_{0}}=\frac{N}{2}+\frac{N(N-1)}{2}=\frac{N(N+1)}{2} \sim \frac{N^{2}}{2} \tag{7.49}
\end{equation*}
$$

This case is often called "superradiance", but there is a definite distinction between it and the real superradiant case, since there is no subradiant case. This is actually superfluorescence.
3. All atoms are in the ground state $|g\rangle$ so that $c_{e}=0$. Then

$$
\begin{equation*}
\frac{\Gamma}{\Gamma_{0}}=0 \tag{7.50}
\end{equation*}
$$

Fact 7.1. There is no subradiance for product states.
Remark 7.2 (Important). Let us reinspect the $N$-atoms dipole operator $\hat{\boldsymbol{D}}$. Its strength is given by

$$
\begin{align*}
\langle\phi| \hat{\boldsymbol{D}}|\phi\rangle & =\langle g| \boldsymbol{d}|e\rangle\langle\phi| \hat{S}^{-}|\phi\rangle+\text { c.c. } \\
& =2 N \operatorname{Re}\left(\langle g| \boldsymbol{d}|e\rangle c_{e}^{*} c_{g}\right) \tag{7.51}
\end{align*}
$$

This is a huge factor proportional to $N$. The atomic correlations

$$
\begin{equation*}
\Delta \hat{S}^{ \pm}=\hat{S}^{ \pm}-\left\langle\hat{S}^{ \pm}\right\rangle \tag{7.52}
\end{equation*}
$$

read

$$
\begin{cases}\langle s, m| \Delta \hat{S}^{+} \Delta \hat{S}^{-}|s, m\rangle=s(s+1) \neq 0 & \text { Dicke states }  \tag{7.53}\\ \langle\phi| \Delta \hat{S}^{+} \Delta \hat{S}^{-}|\phi\rangle=0 . & \text { Product states }\end{cases}
$$

Note that product states have classical correlations.
Remark 7.3. Typically, when superradiance was reported, it was actually superfluorescence. To show real quantum effect, one have to show subradiance. The first to show that was Guerin, Araújo, and Kaiser [2016].

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[^0]:    ${ }^{1}$ Lesson \#1 @ 20/03/2017

[^1]:    ${ }^{2}$ Lesson \#2 @ 21/03/2017

[^2]:    3 "Radiative pressure" is a misleading term; the proper one is "radiative power". It has been used historically to describe radiative pressure forces, thus the emphasis on "pressure". We shall keep this inaccurate terminology nonetheless.
    ${ }^{4}$ Acknowledgments to Ariane Soret for summarizing this part.

[^3]:    ${ }^{5}$ Lesson \#3 @ 27/03/2017

[^4]:    ${ }^{6}$ Lesson \#4 @ 28/03/2017

[^5]:    ${ }^{7}$ Lesson \#5 @ 03/05/2017

[^6]:    ${ }^{8}$ Lesson \#5 @ 03/04/2017

[^7]:    ${ }^{9}$ Lesson \#6 @ 04/04/2017

[^8]:    10 Lesson \#7@18/04/2017

[^9]:    ${ }^{11}$ Lesson \#8 @ 24/04/2017

[^10]:    12 Lesson \#9 @ 25/04/2017

[^11]:    13 Lesson \#9 @ 25/04/2017

[^12]:    ${ }^{14}$ Lesson \#10@ 08/05/2017

[^13]:    15 Lesson \#11 @ 09/05/2017

[^14]:    16 Lesson \#11@ 09/05/2017

[^15]:    17 Lesson \#12 @ 15/05/2017

[^16]:    18 Lesson \#13 @ 16/05/2017

[^17]:    19 Lesson \#14@ 22/05/2017

[^18]:    ${ }^{20}$ Lesson \#15 @ 23/05/2017

[^19]:    ${ }^{21}$ Lesson \#16@ 29/05/2017

[^20]:    ${ }^{23}$ Lesson \#18@ 06/06/2017

[^21]:    ${ }^{24}$ Lesson \#19 @ 19/06/2017

[^22]:    ${ }^{25}$ Lesson \#20@ 20/06/2017

[^23]:    ${ }^{26}$ Lesson \#21@ 27/06/2017

[^24]:    ${ }^{27}$ Lesson \#21@ 27/06/2017

[^25]:    ${ }^{28}$ Lesson \#22@ 28/06/2017

[^26]:    ${ }^{29}$ Lesson \#23@ 03/07/2017

[^27]:    30 This section helps to solve homework assignment $\# 1$.

[^28]:    ${ }^{31}$ Lesson \#23@ 03/07/2017

[^29]:    32 Lesson \#24 (last) @ 04/07/2017

[^30]:    ${ }^{33}$ Say, the atoms are distributed randomly. Then the quantities one measures have to be averaged, which means that the phases wash out. It is unclear what is to be measured. Similar arguments hold for atoms on a lattice.

