Physique mesoscopique des electrons et des photons -
Structures fractales et quasi-periodiques

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ISRAEL SCIENCE FOUNDATION

Aux frontieres de la physique mesoscopique,
Mont Orford Quebec, Canada,

Part 2

Towards a quantitative description :
the tools of quantum mesoscopic physics

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1. More details on diffusion and quantum crossings

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3. How to relate **local** quantum crossings to the **global** scattering approach ?

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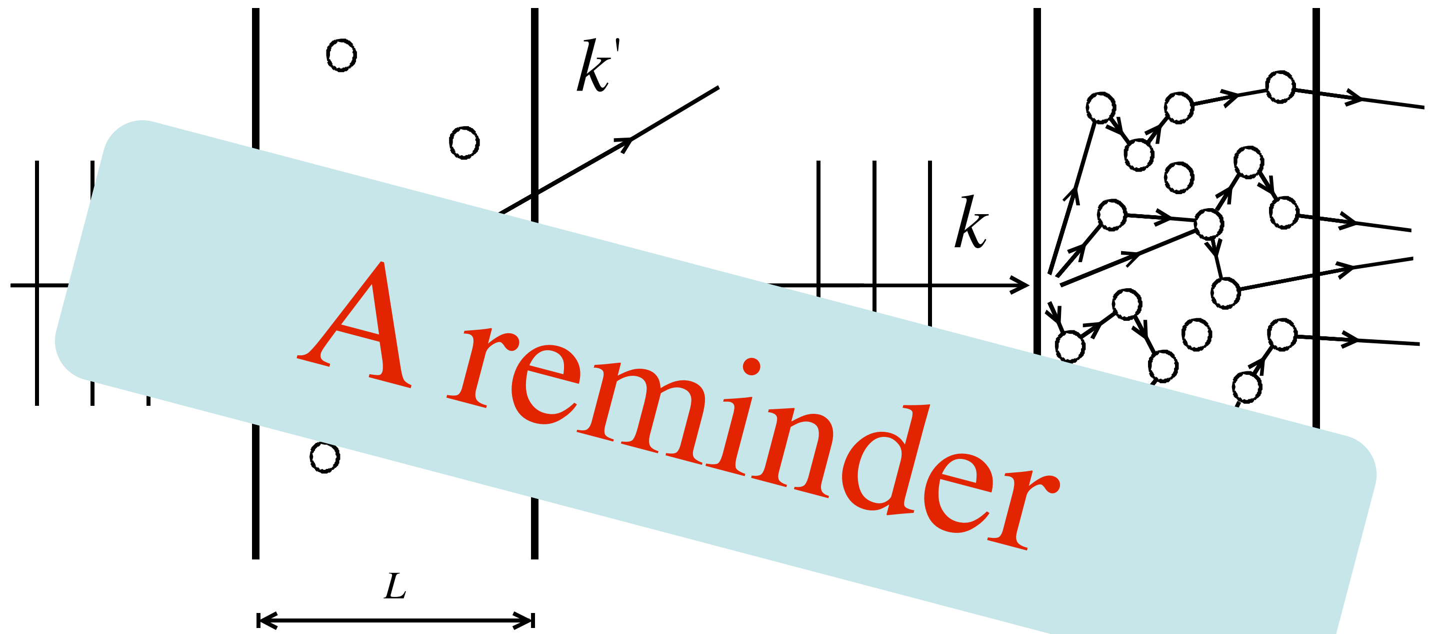
3. How to relate **local** quantum crossings to the **global** scattering approach ?

4. A brief overview on Anderson localization phase transition

Multiple scattering of electrons



A reminder

2 characteristic lengths:

Wavelength: $\lambda_F = k_F^{-1}$

Elastic mean free path: l (Disorder - Origin ?)

Weak disorder $\lambda_F \ll l$: independent scattering events

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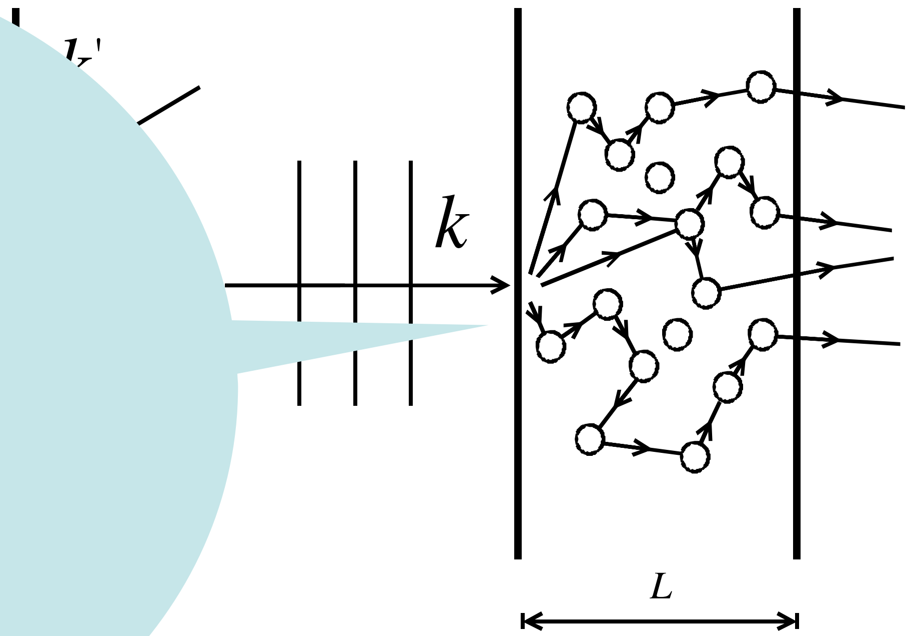

We shall be interested only by this limit

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Probability of quantum diffusion

Propagation of a wavepacket centered at energy ϵ between any two points.

It is obtained from the probability amplitude (Green's function for the afficionados !) :

$$G_\epsilon(\mathbf{r},\mathbf{r}') = \sum_j A_j(\mathbf{r},\mathbf{r}')$$

Superposition of amplitudes associated to all multiple scattering trajectories that relate $\mathbf{r}$ and $\mathbf{r}'$.

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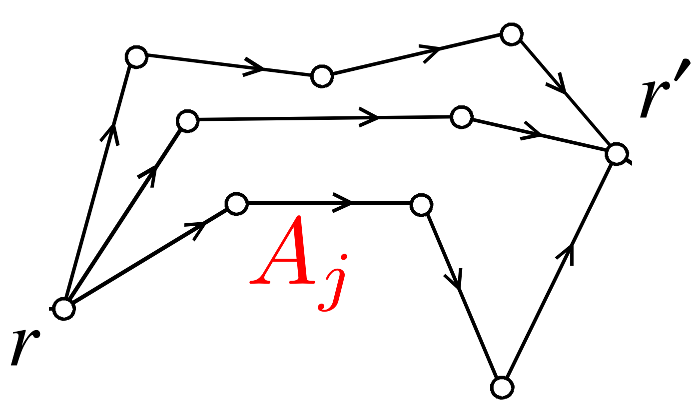
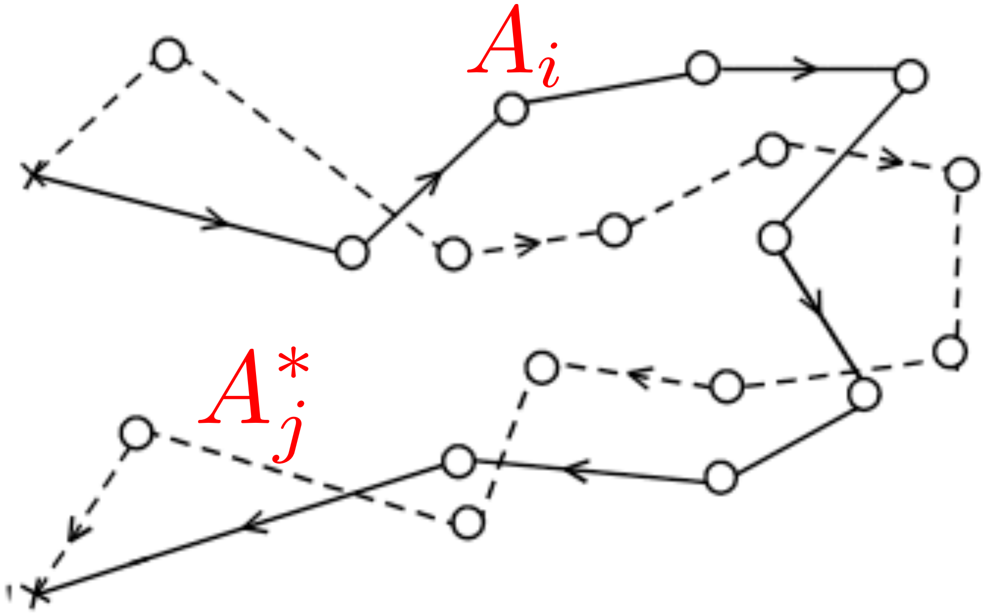
The probability of quantum diffusion averaged over disorder is:

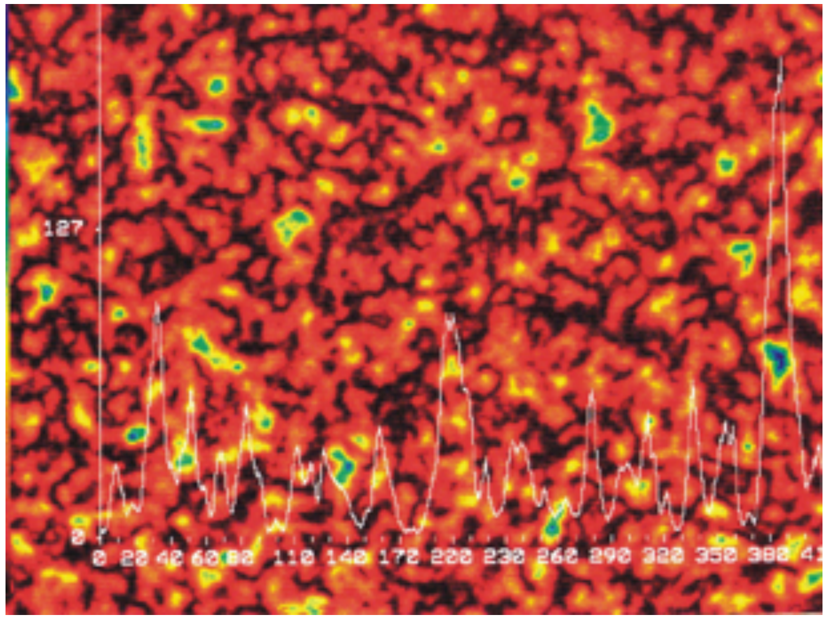
$$P(\mathbf{r},\mathbf{r}') \propto \overline{|G_\epsilon(\mathbf{r},\mathbf{r}')|^2} = \overline{\sum_j |A_j(\mathbf{r},\mathbf{r}')|^2} + \overline{\sum_{i\neq j} A_i^*(\mathbf{r},\mathbf{r}')A_j(\mathbf{r},\mathbf{r}')}$$



classical term

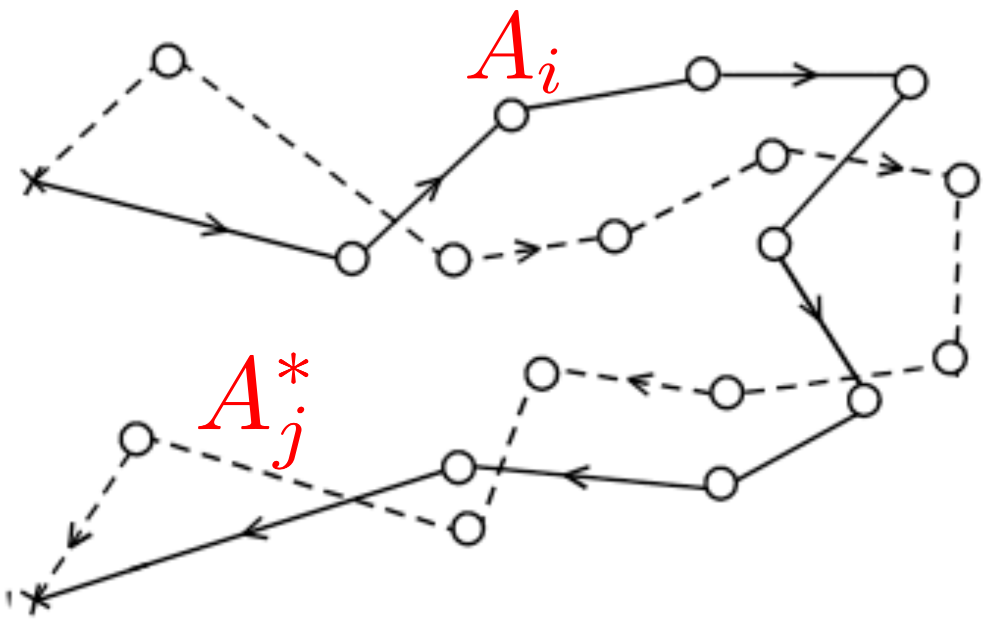
interference between distinct trajectories: vanishes upon averaging

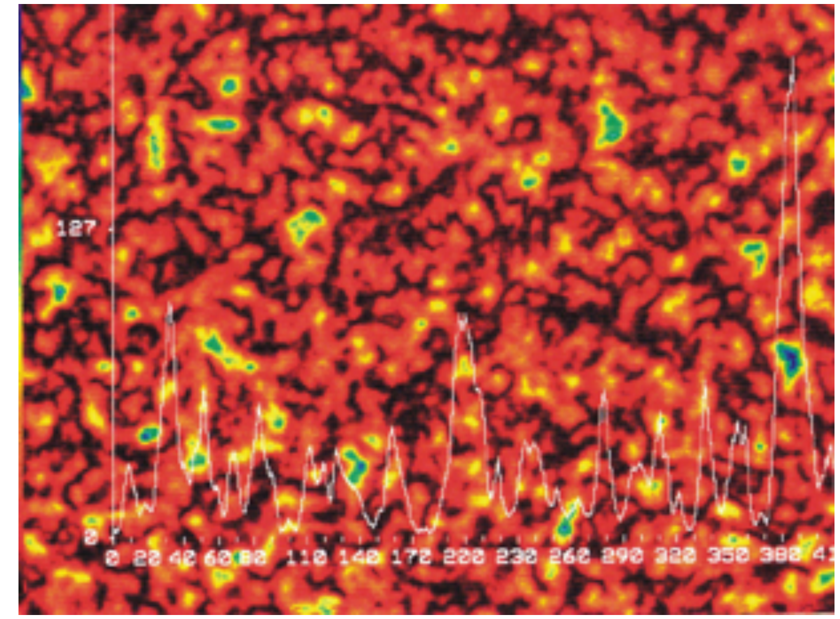




Before averaging : speckle pattern (full coherence)

Configuration average: most of the contributions vanish because of large phase differences.





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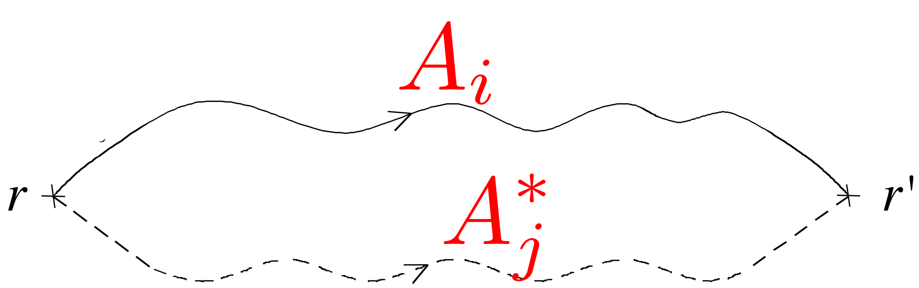
A new design !



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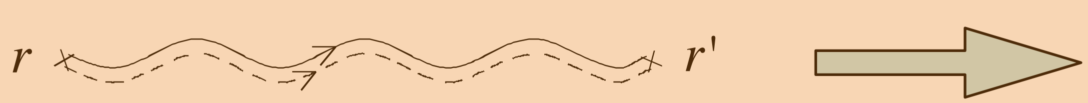


$$P_{cl}(\mathbf{r},\mathbf{r}') = \overline{\sum_j |A_j(\mathbf{r},\mathbf{r}')|^2}$$

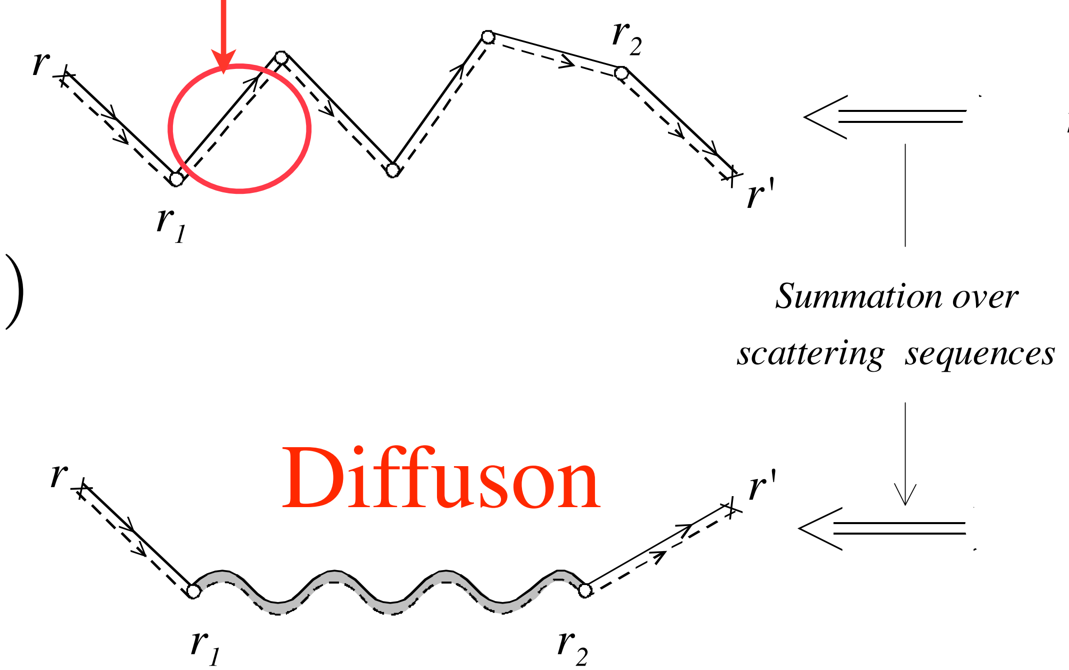
Diffuson

The diffusion approximation:

How to calculate $P_{cl}(\mathbf{r},\mathbf{r}')$? It may be obtained as an iteration equation

Iteration of the Drude-Boltzmann term $\boxed{P_0(r,r') = \bar{G}(r,r')\bar{G}^*(r',r) \propto \frac{e^{-R/l_e}}{R^2}}$

$$P_{cl}(\mathbf{r},\mathbf{r}') = P_0(\mathbf{r},\mathbf{r}') + \frac{1}{\tau_e}\int d\mathbf{r}'' P_{cl}(\mathbf{r},\mathbf{r}'')P_0(\mathbf{r}'',\mathbf{r}')$$



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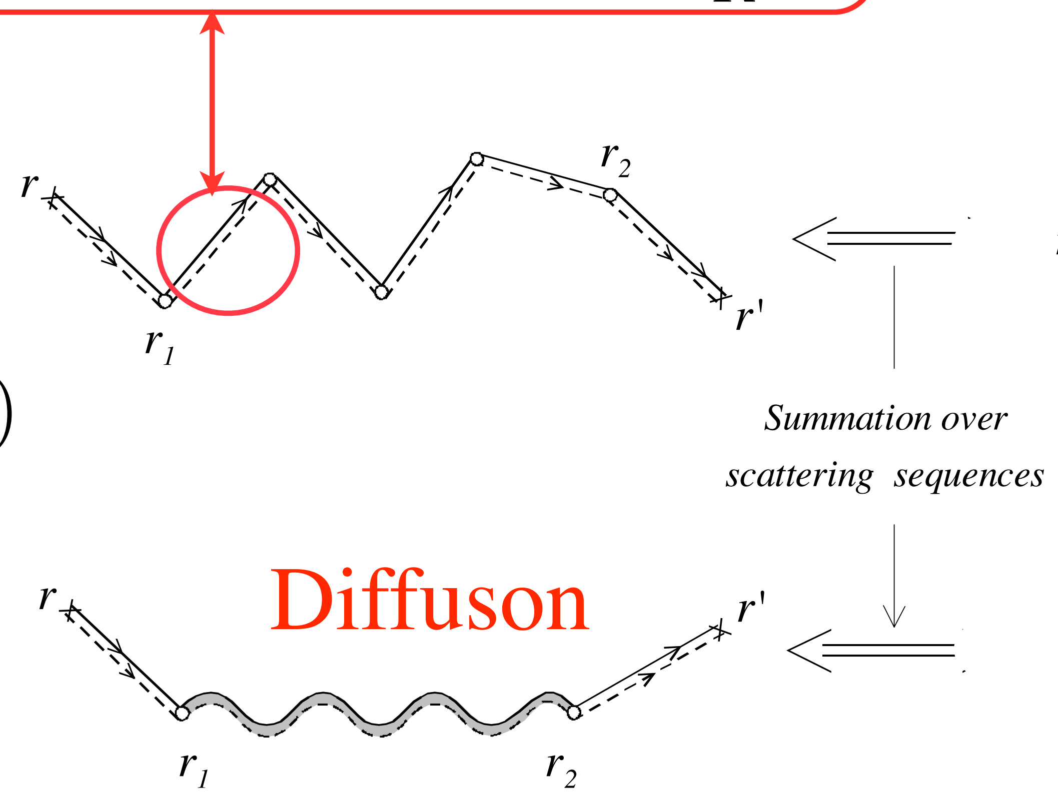

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In the limit of slow spatial and temporal variations, $|\mathbf{r}-\mathbf{r}'| \gg l_e$ and $t \gg \tau_e$

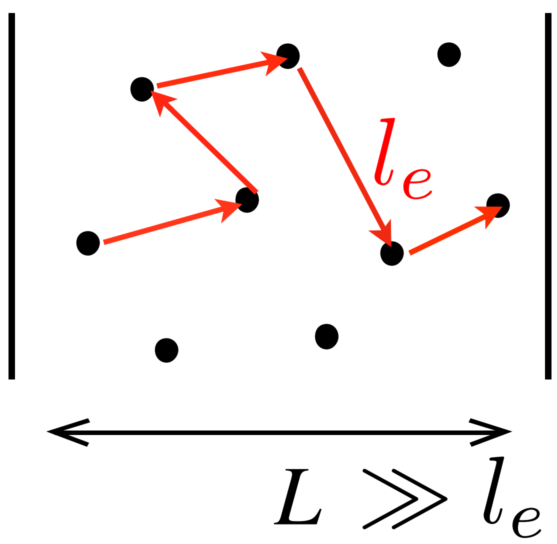
$$\left[\frac{\partial}{\partial t} - D\Delta\right]P_{cl}(\mathbf{r},\mathbf{r}',t) = \delta(\mathbf{r}-\mathbf{r}')\delta(t)$$

with $D = \dfrac{v_g l_e}{3}$

(diffusion equation)

Mesoscopic limit: characteristic length scales

The diffusion motion is characterized by its elementary step, the elastic mean free path l_e related to the elastic collision time by $l_e = v_g \tau_e$



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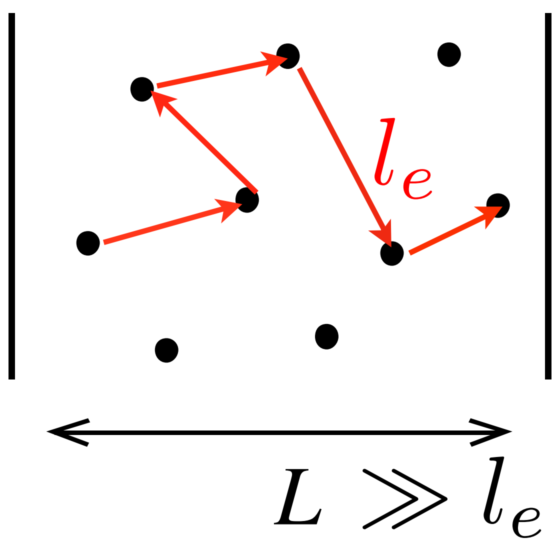

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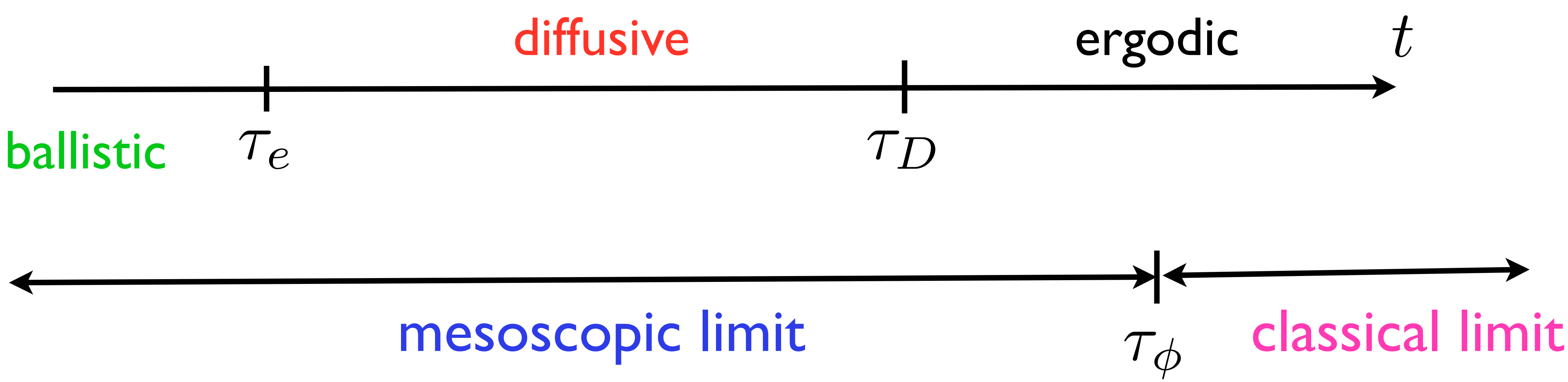

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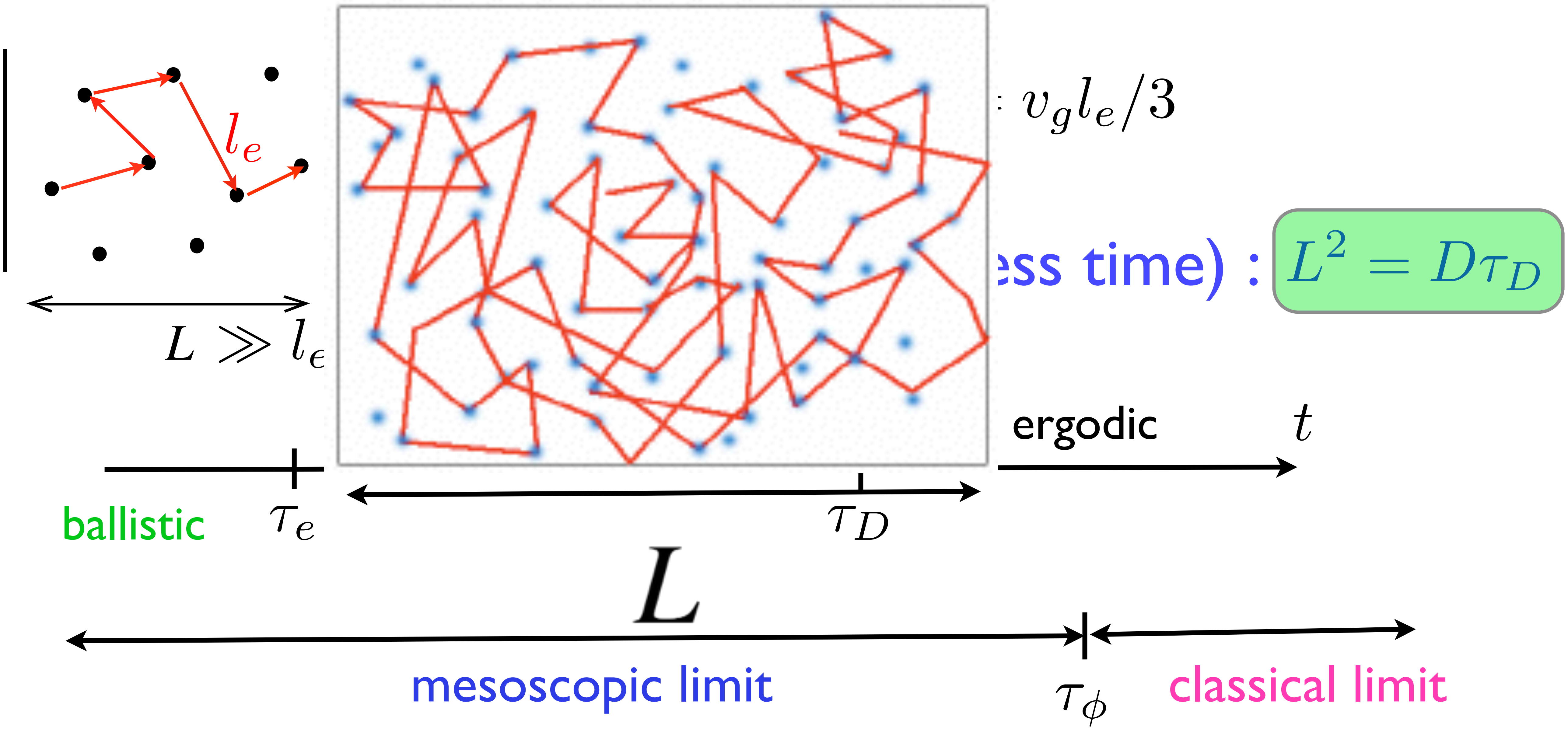

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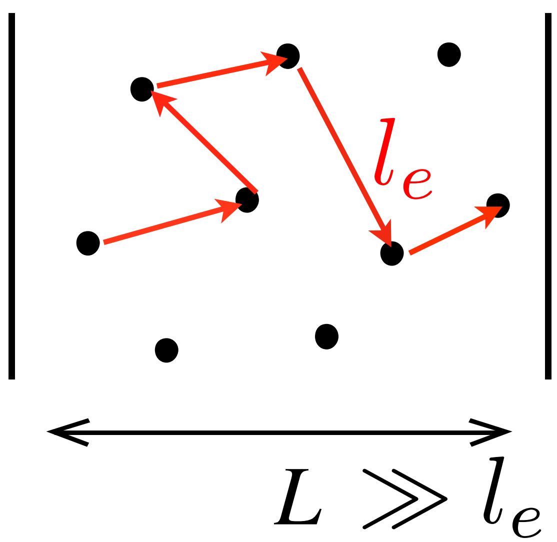

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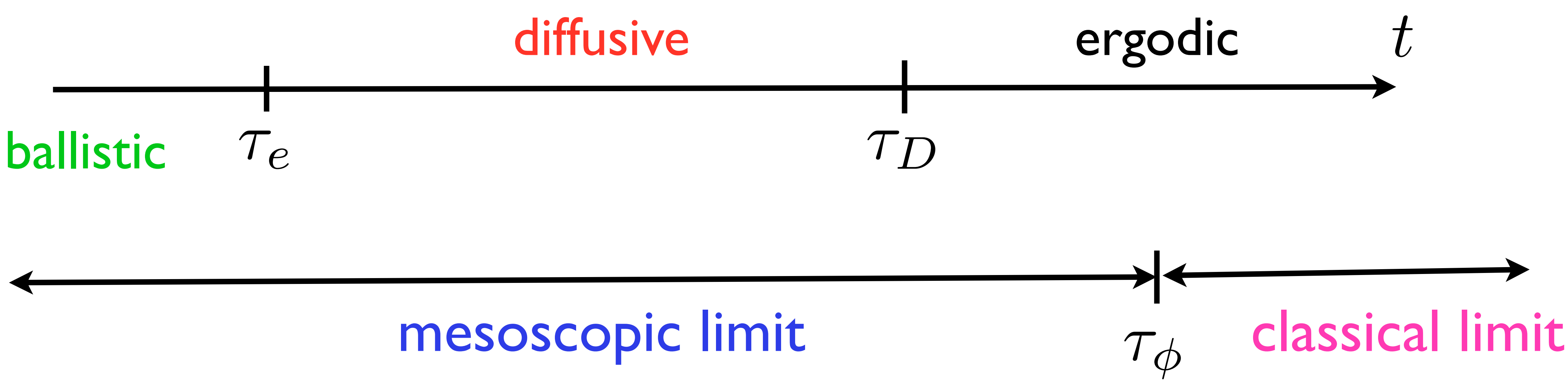

Did we miss something ?

Normalization of the probability

The probability of quantum diffusion must be *normalized*,

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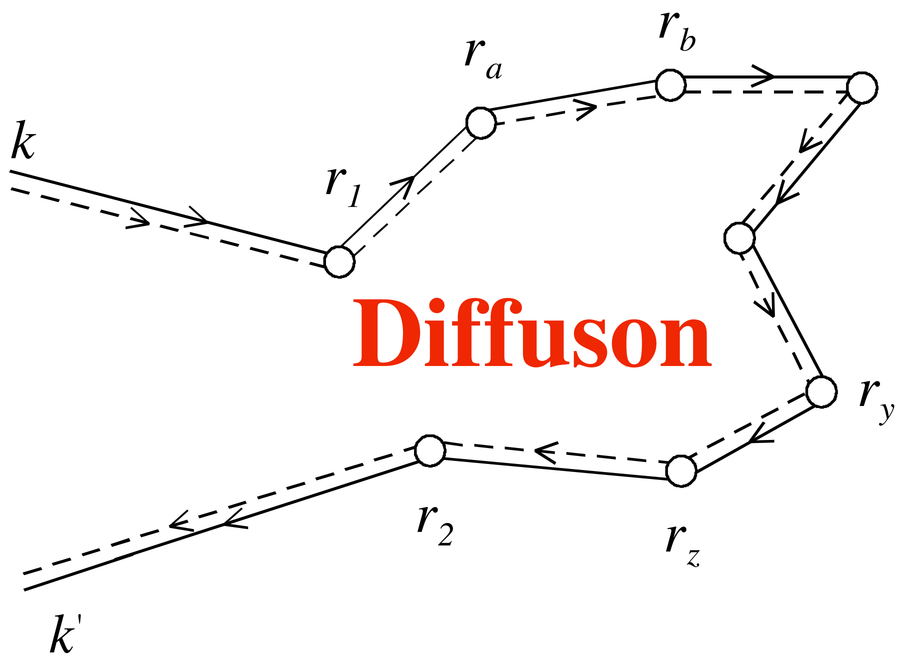
The Diffuson approx. does not take into account all contributions to the probability.

(a)



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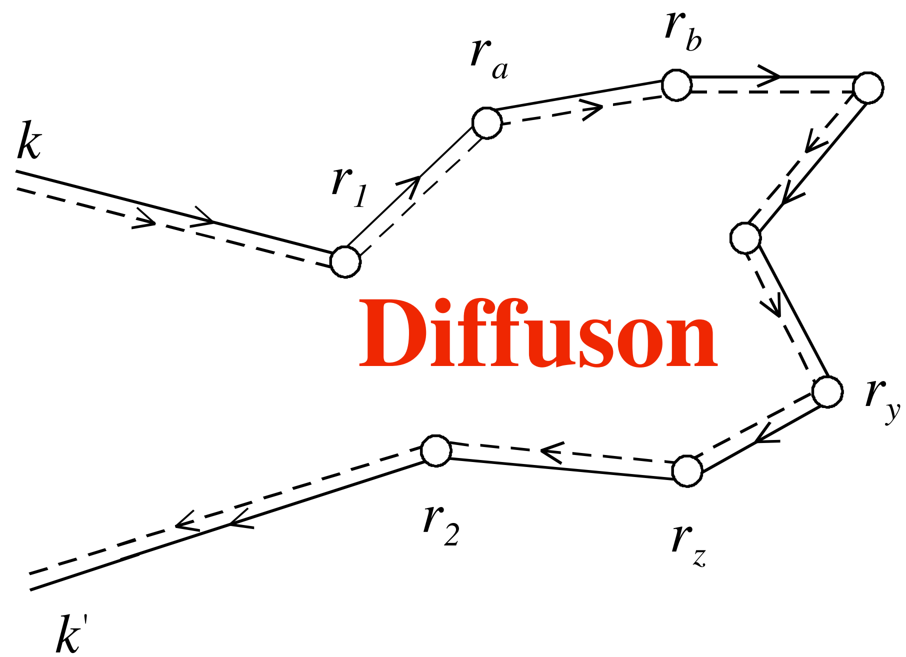
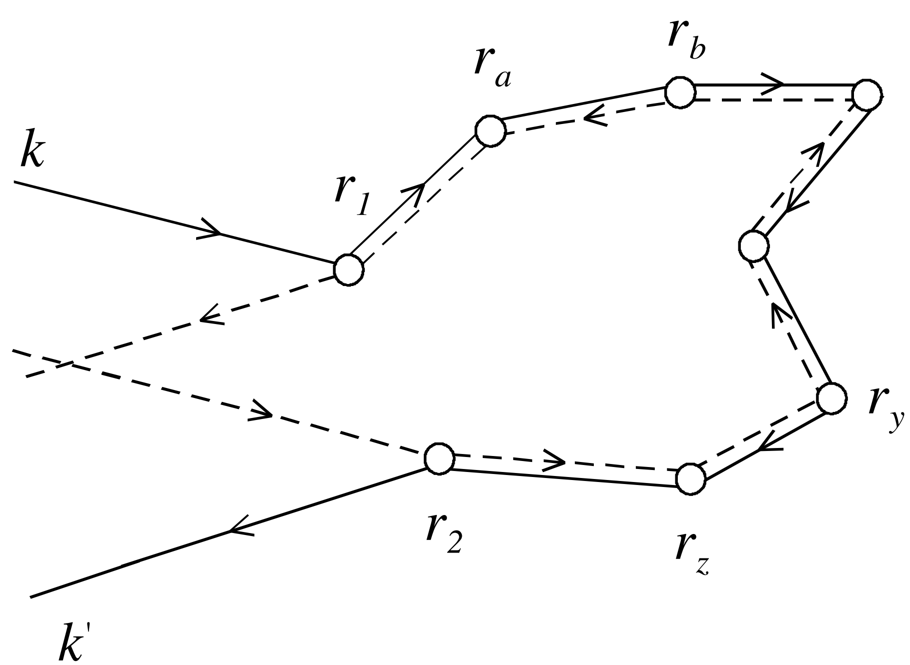
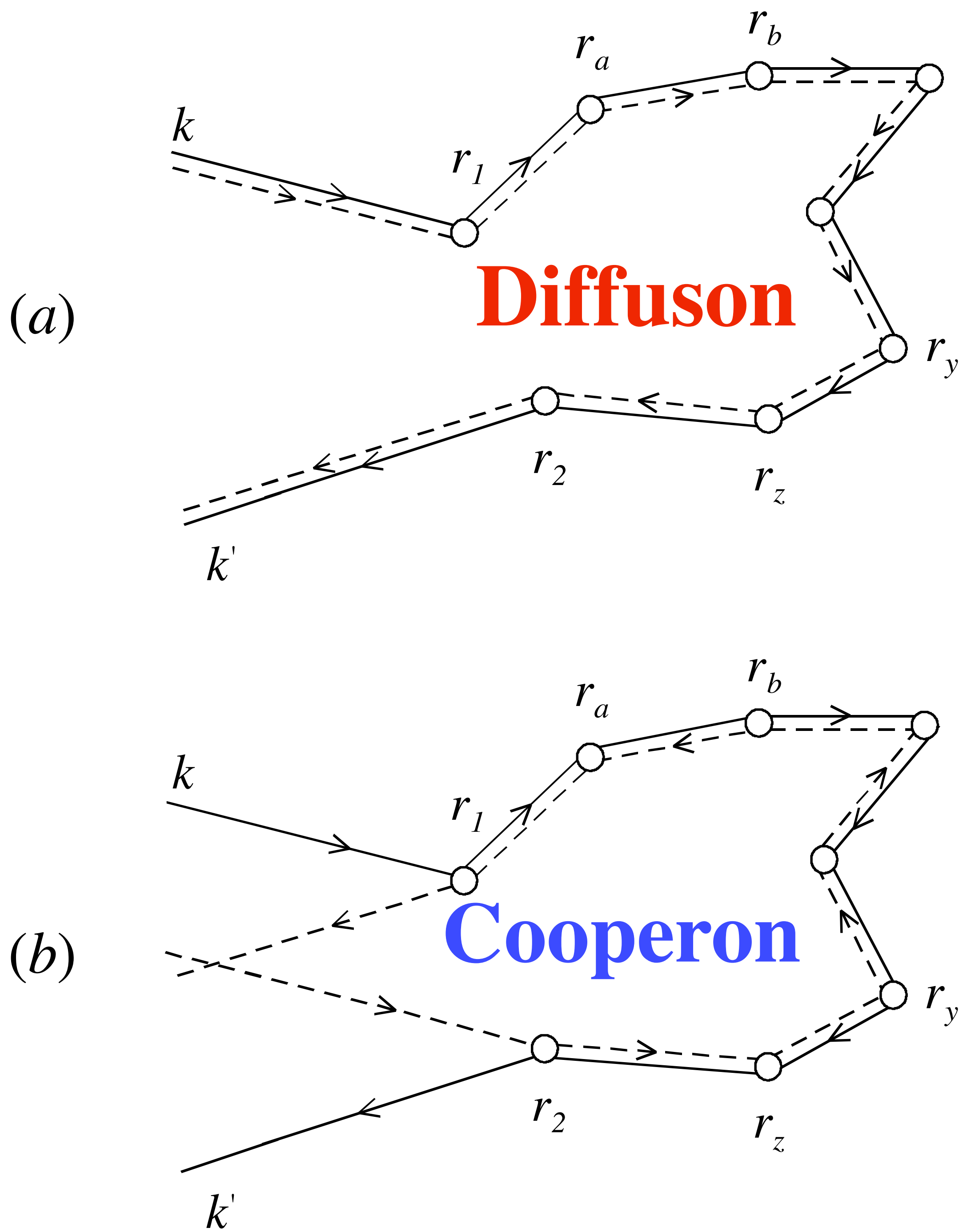


$$\mathbf{r_1 \longrightarrow r_a \longrightarrow r_b \cdots \longrightarrow r_y \longrightarrow r_z \longrightarrow r_2}$$

(b)



$$\mathbf{r_2 \longrightarrow r_z \longrightarrow r_y \cdots \longrightarrow r_b \longrightarrow r_a \longrightarrow r_1}$$



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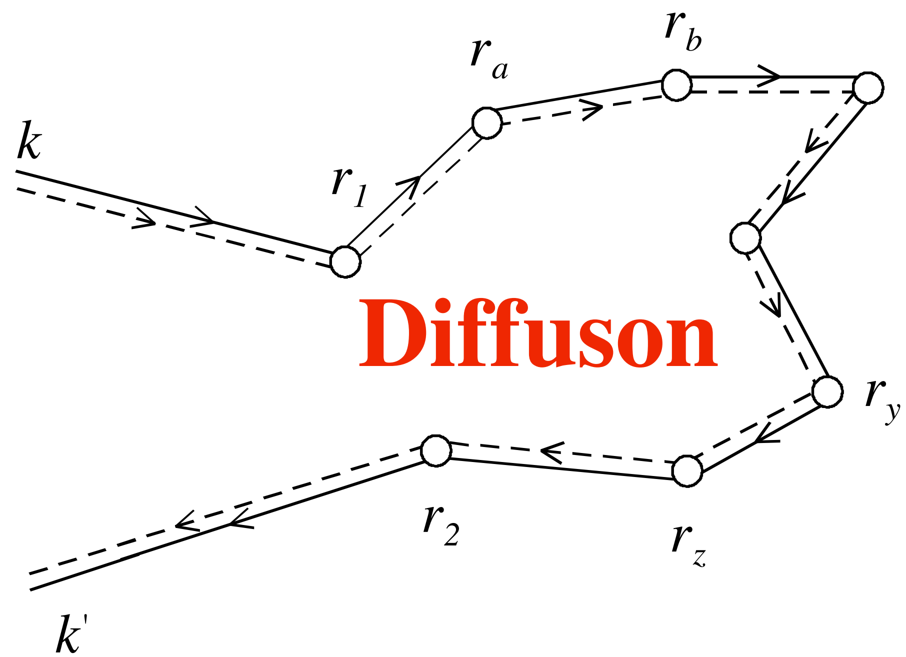

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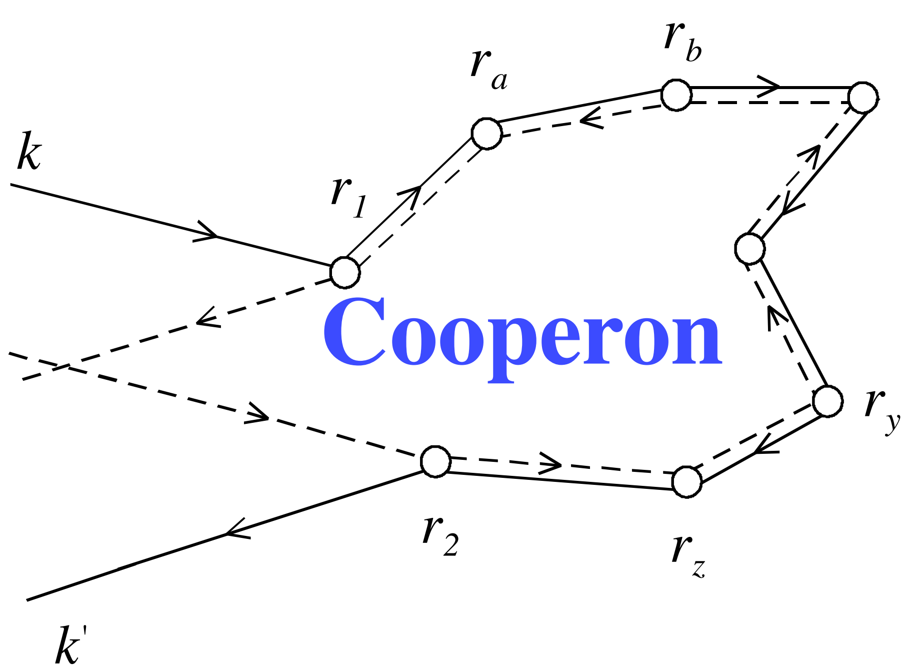

$$\mathbf{r_2} \longrightarrow \mathbf{r_z} \longrightarrow \mathbf{r_y} \cdots \longrightarrow \mathbf{r_b} \longrightarrow \mathbf{r_a} \longrightarrow \mathbf{r_1}$$

incoherent classical term

interference term

The total average intensity is:

$$\overline{|A(\mathbf{k},\mathbf{k}')|^2} = \overline{\sum_{\mathbf{r_1},\mathbf{r_2}} |f(\mathbf{r_1},\mathbf{r_2})|^2 \left[1 + e^{i(\mathbf{k}+\mathbf{k}').(\mathbf{r_1}-\mathbf{r_2})}\right]}$$

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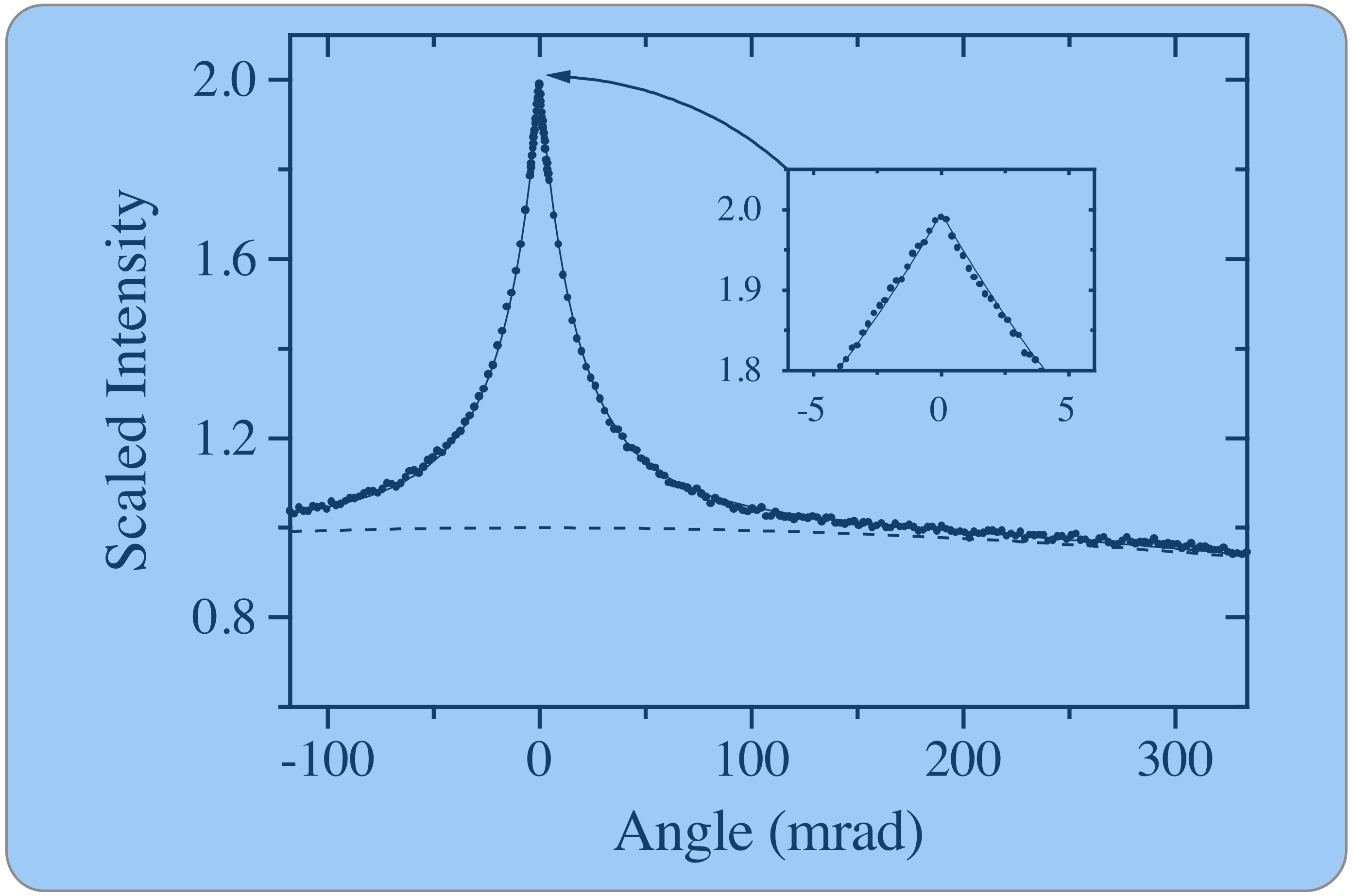
$\mathbf{k}+\mathbf{k}' \simeq 0$: Coherent backscattering



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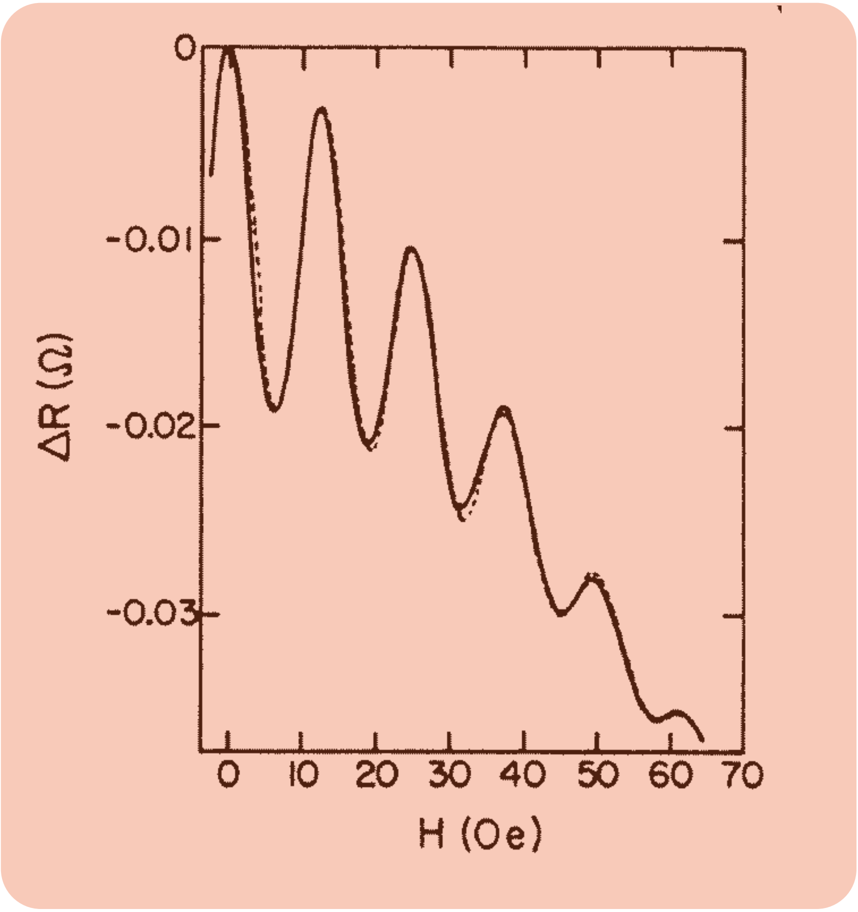
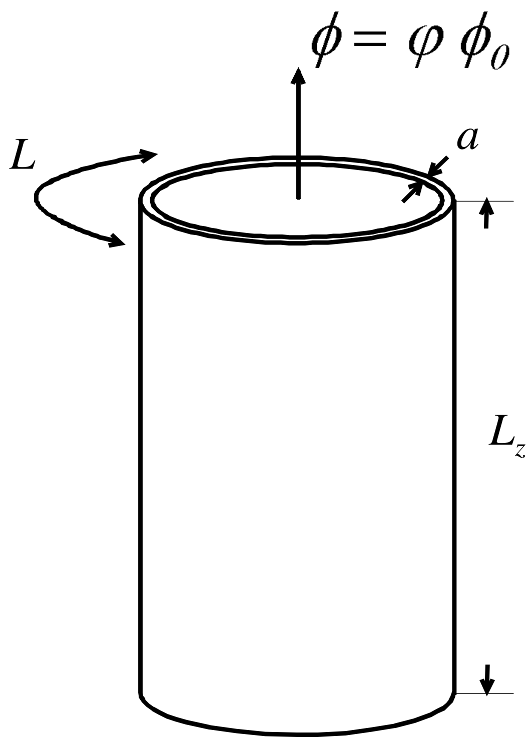
$\mathbf{r_1} - \mathbf{r_2} \simeq 0$: closed loops, weak localization and $\phi_0/2$ periodicity of the Sharvin effect.





Quantum crossings

A Diffuson is the product of 2 complex amplitudes: it can be viewed as a" diffusive trajectory with a phase". Coherent effects result from the Cooperon which can be viewed as a self-crossing

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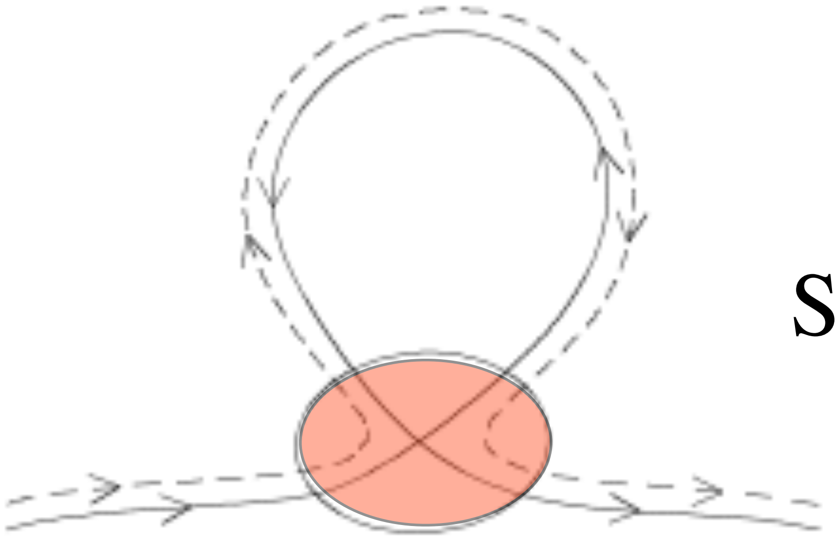

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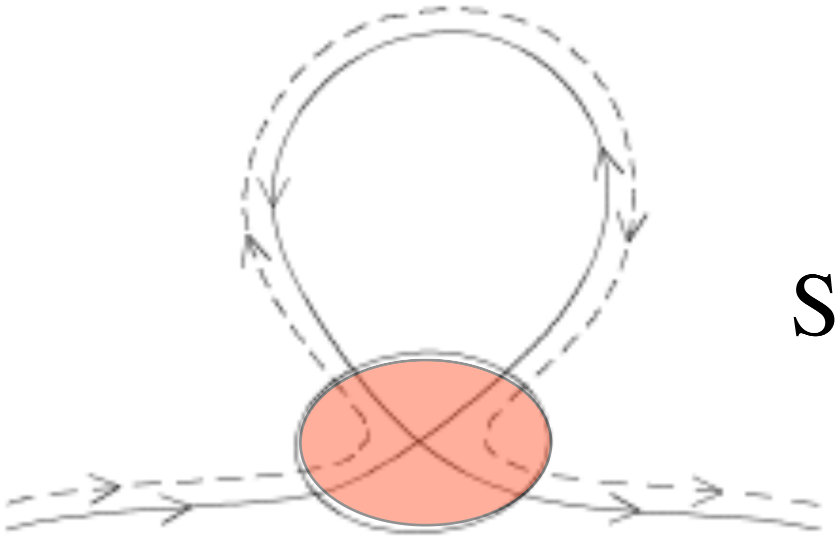

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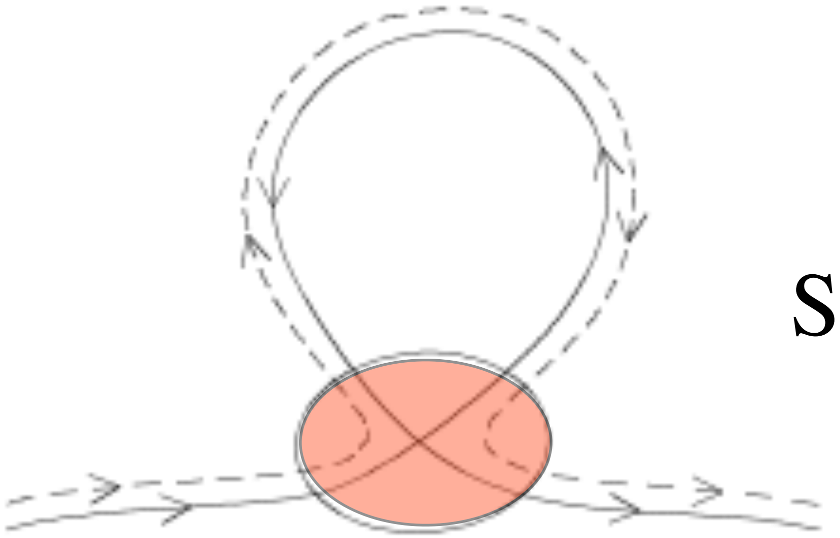

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Crossing probability of 2 diffusons:

$$p_{\times} = \int_0^{\tau_D} \frac{\lambda^{d-1} v_g \, dt}{L^d} = \frac{1}{g}$$

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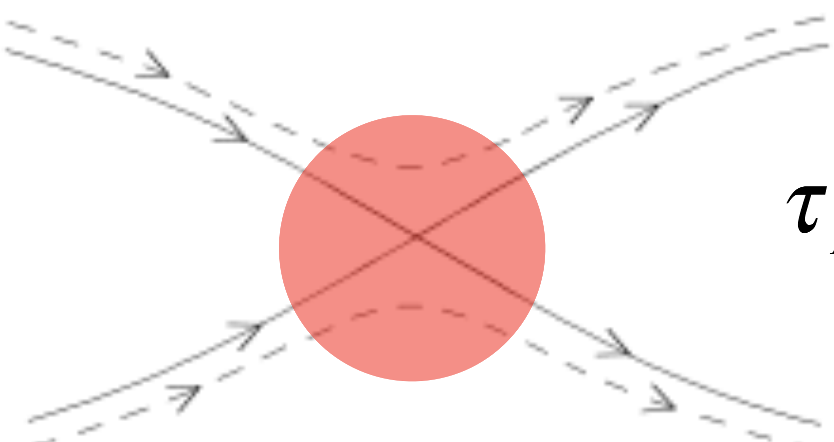

$\tau_D = L^2 / D$

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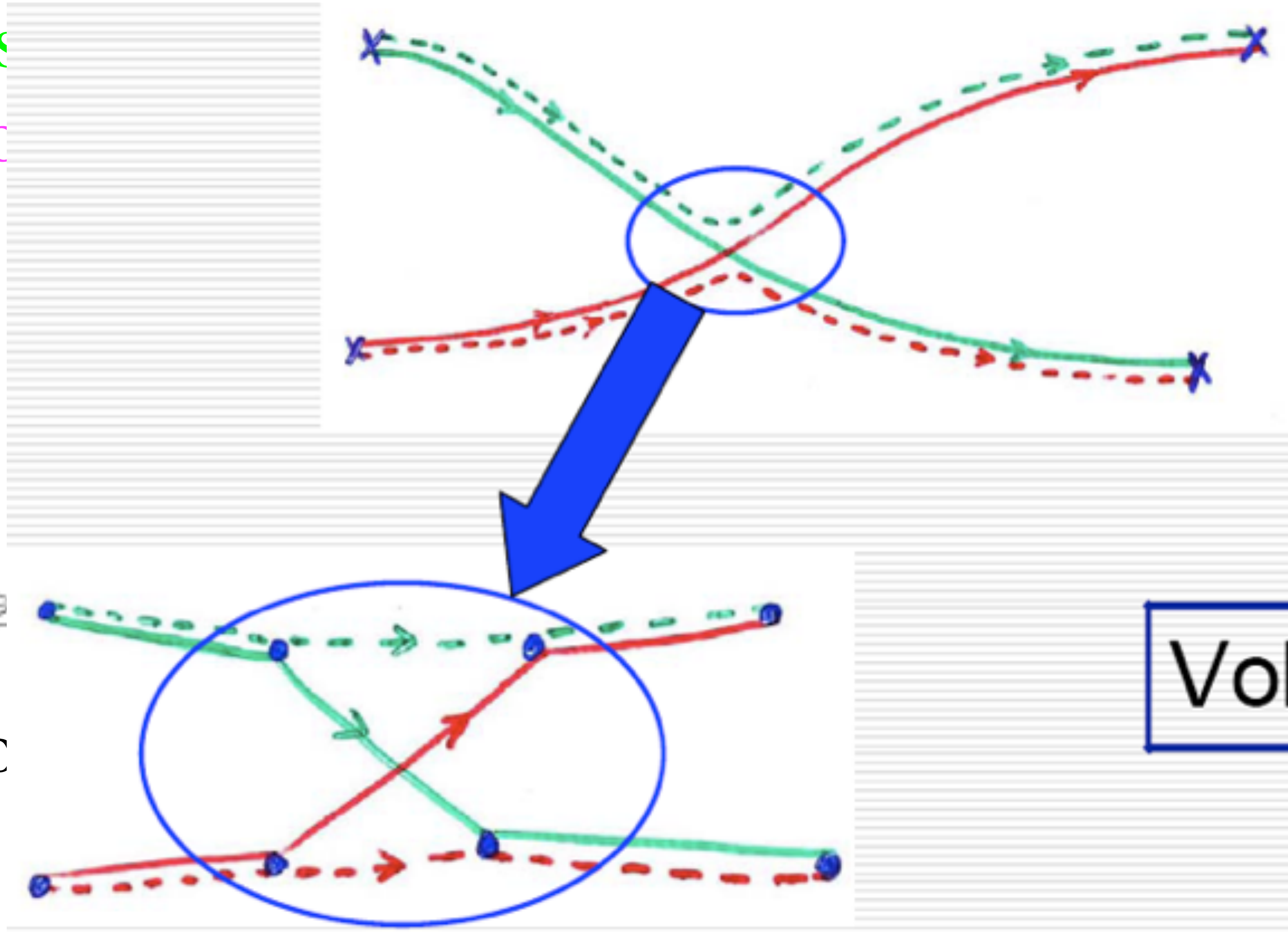

Volume $\lambda^{d-1} l_e$

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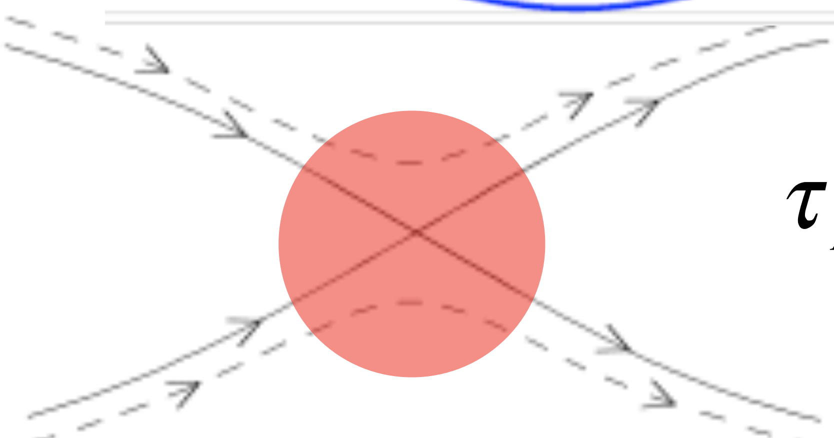

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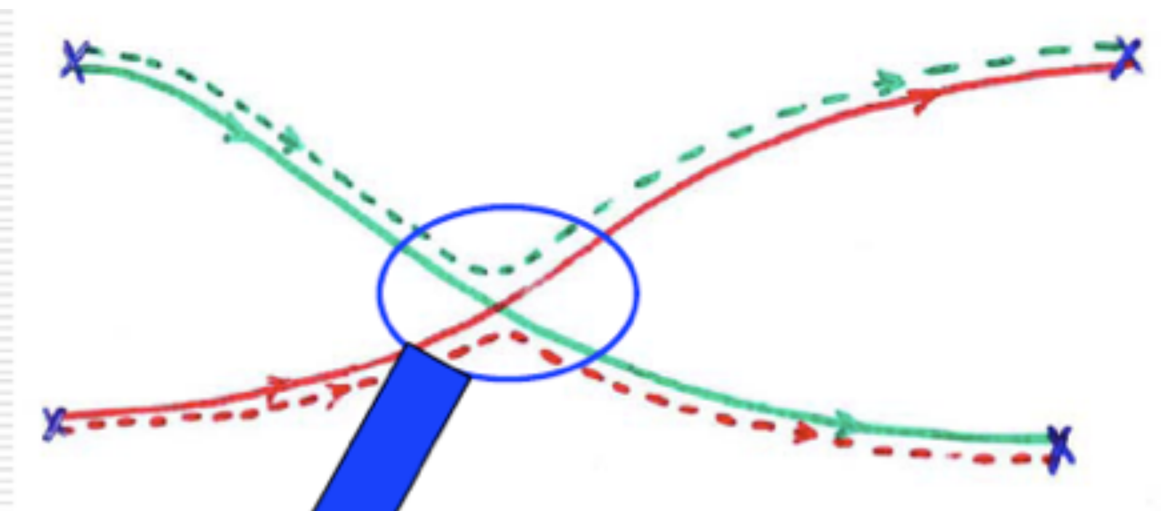

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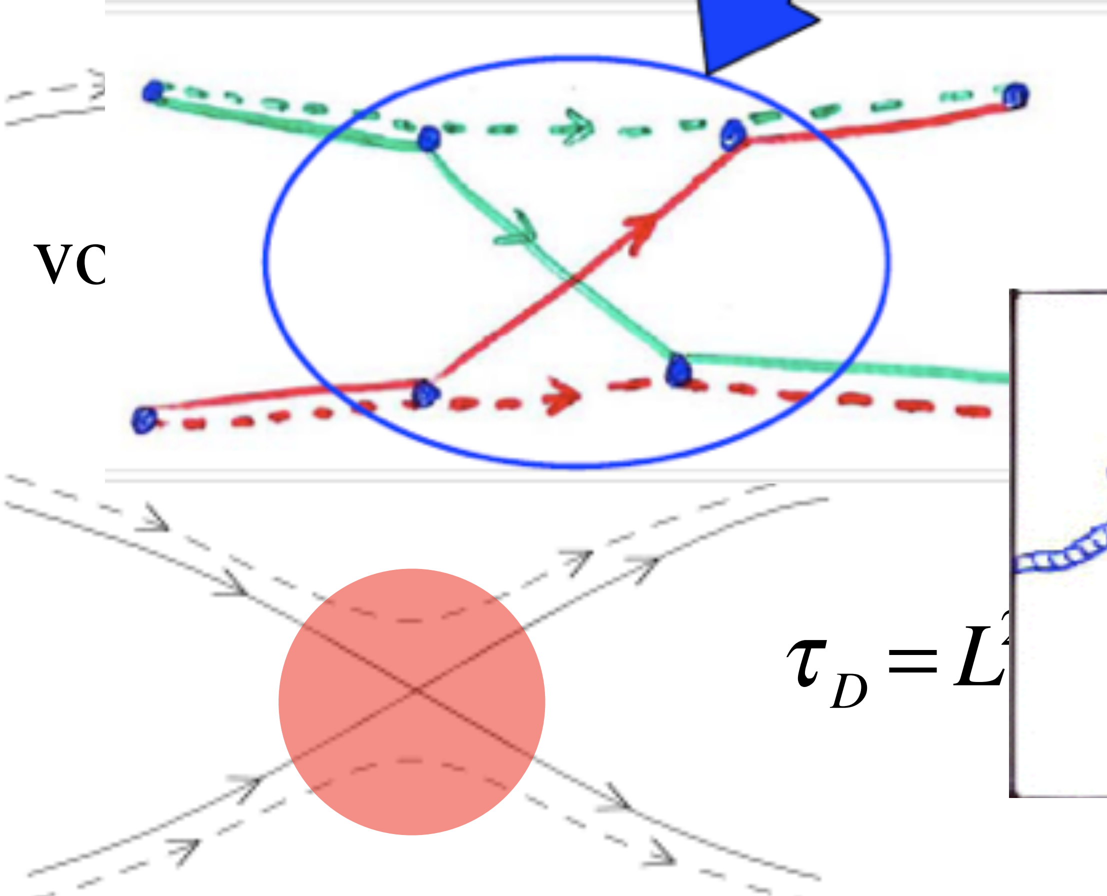

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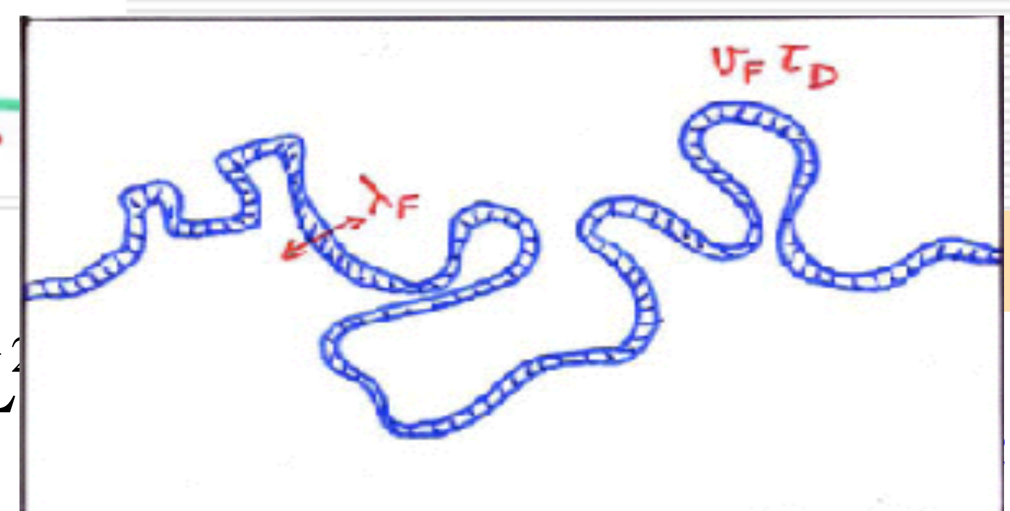
vc

$\tau_D = L$



$\gg 1$

$3\lambda^{d-1}$

Weak disorder physics

Weak disorder limit: $\lambda << l \Rightarrow g >> 1$

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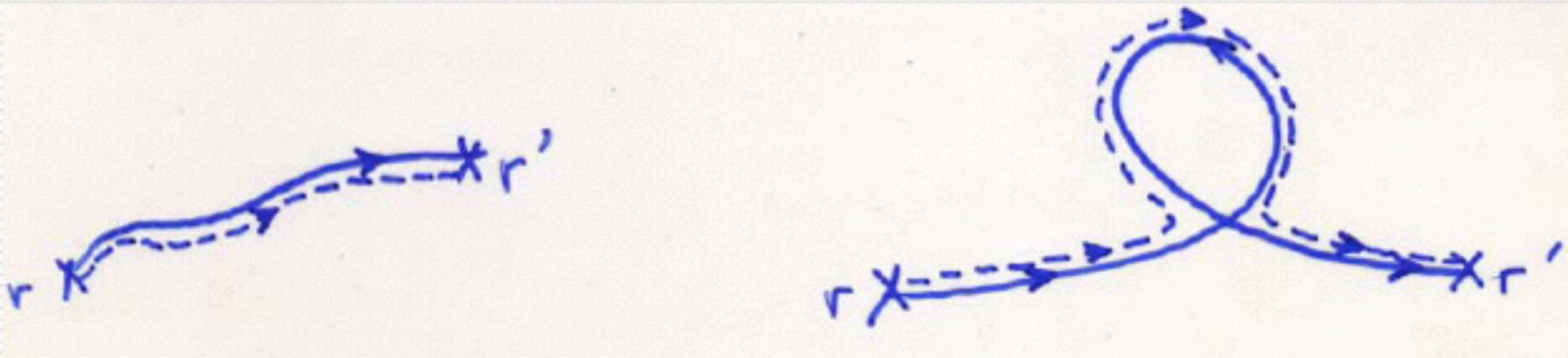
Quantum crossings are independently distributed :

We can generate higher order corrections to the Diffuson as an expansion in powers of $1/g$

How to calculate $P_{\text{int}}(t)$?

$$P_{\text{int}}(t) = \int P_{\text{int}}(r,r,t)\, d^d r$$

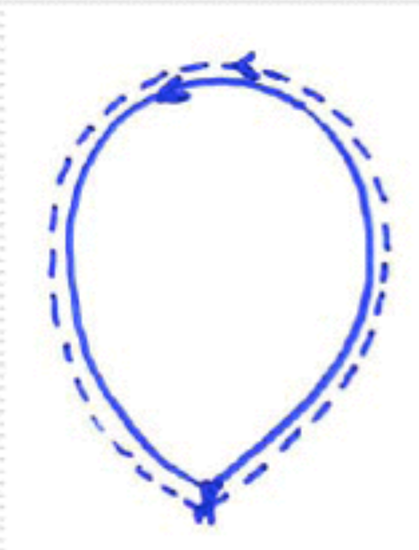


$P_{cl}(r,r',t)$

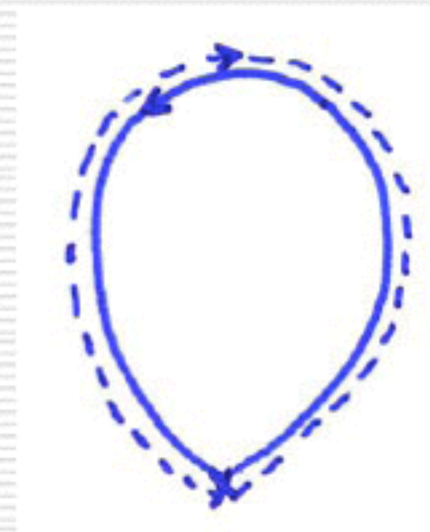
$P_{\text{int}}(r,r',t) \sim \dfrac{1}{g}$ if $r \neq r'$

If $r = r'$

Diffuson



$=$



Cooperon

Classical return probability

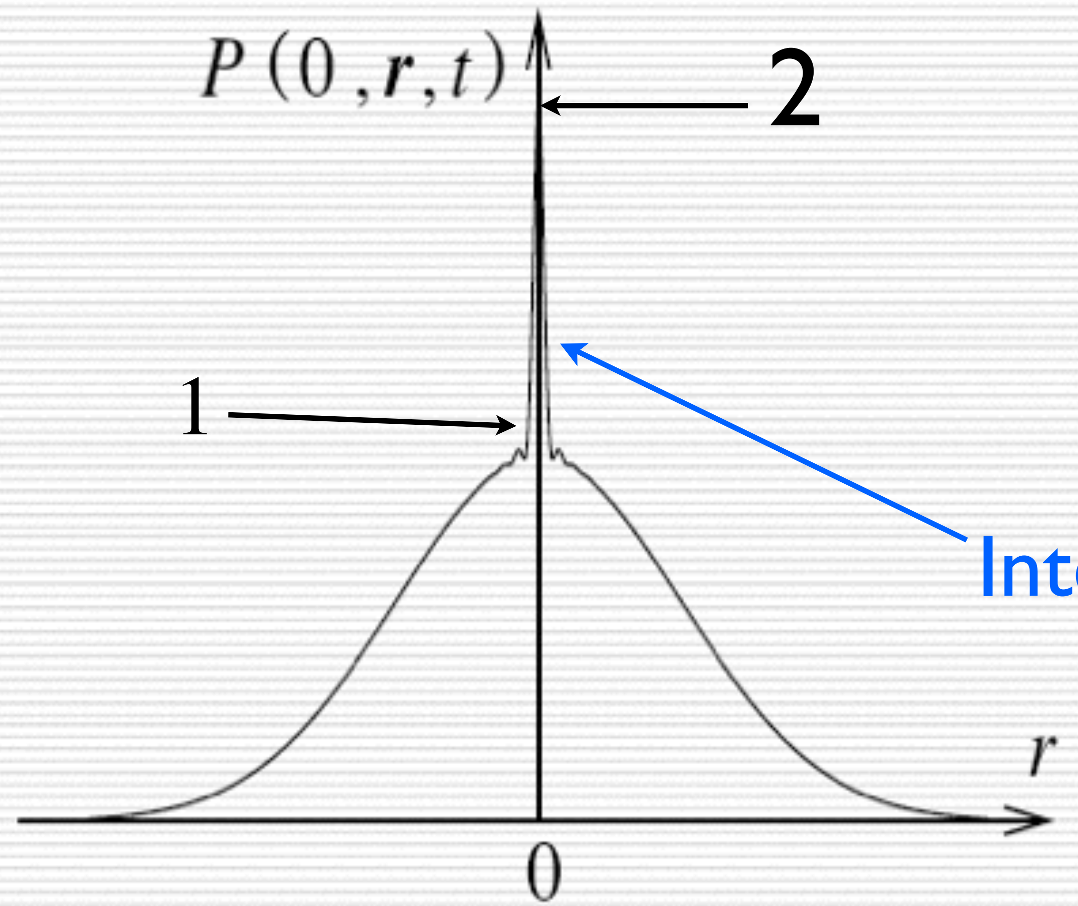
Interference term

$$P_{cl}(r,r,t) = P_{\text{int}}(r,r,t)$$

Return probability is doubled !!

If time reversal invariance

Probability $P_{cl} + P_{int}$



Interference correction

$$P(0,r,t) = \frac{e^{-\frac{r^2}{4Dt}}}{(4\pi Dt)^{d/2}}\left[1+\left(\frac{\sin k_F r}{k_F r}\right)^2\right]$$

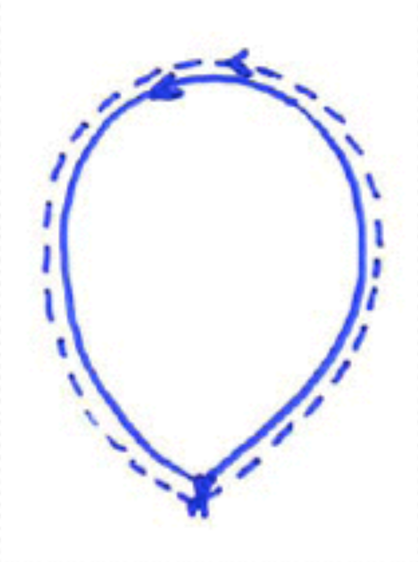
Classical diffusion

Important difference :

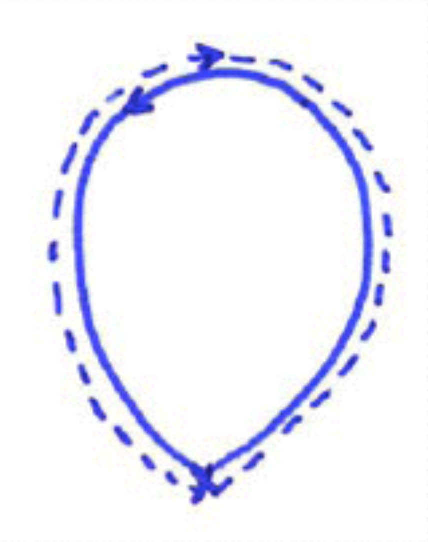
$P_{cl}(r,r',t) \Rightarrow$ paired trajectories follow the same direction

$P_{\text{int}}(r,r',t) \Rightarrow$ paired trajectories follow opposite directions

Diffuson

A_j  A_j^*

Cooperon

A_j  A_j^{*T}

$A_j \quad A_j^T$ have the same phase $\int \vec{p}.\vec{dl}$

$$P_{\text{int}}(r,r,t) = P_{cl}(r,r,t)$$

If time reversal invariance

If <u>phase coherence</u> between the reversed trajectories is preserved

In the presence of a dephasing mechanism that breaks time coherence, only trajectories with $t < \tau_\phi$ contribute.

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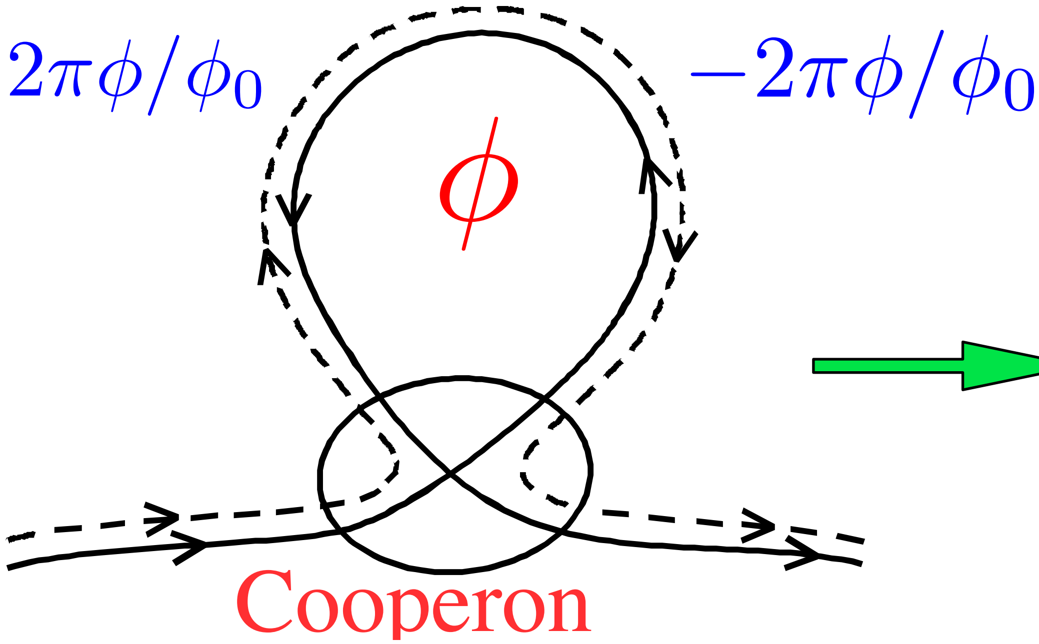

the phase difference becomes: $4\pi\phi/\phi_0$

$\phi_0/2$ periodicity of the Sharvin effect

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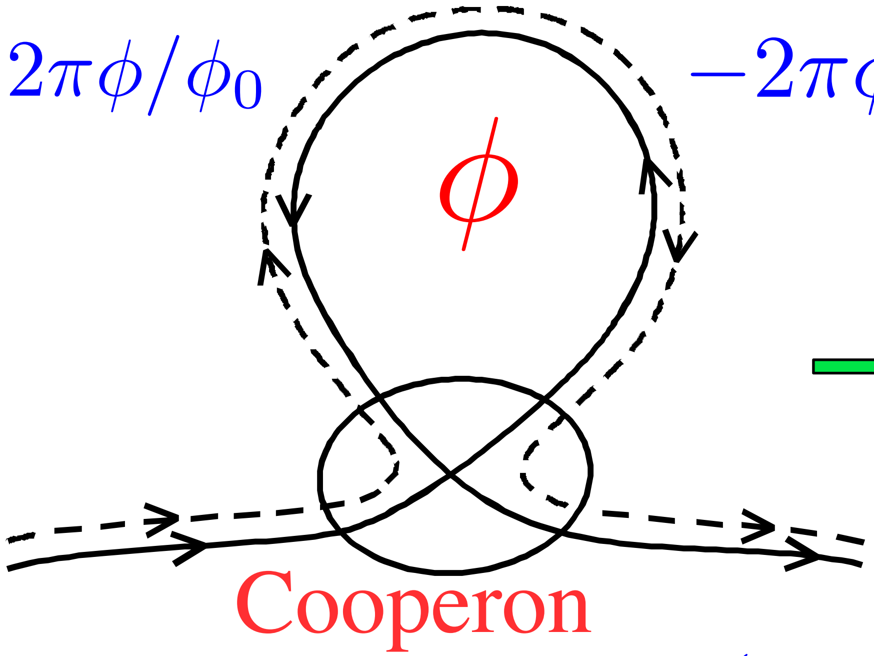

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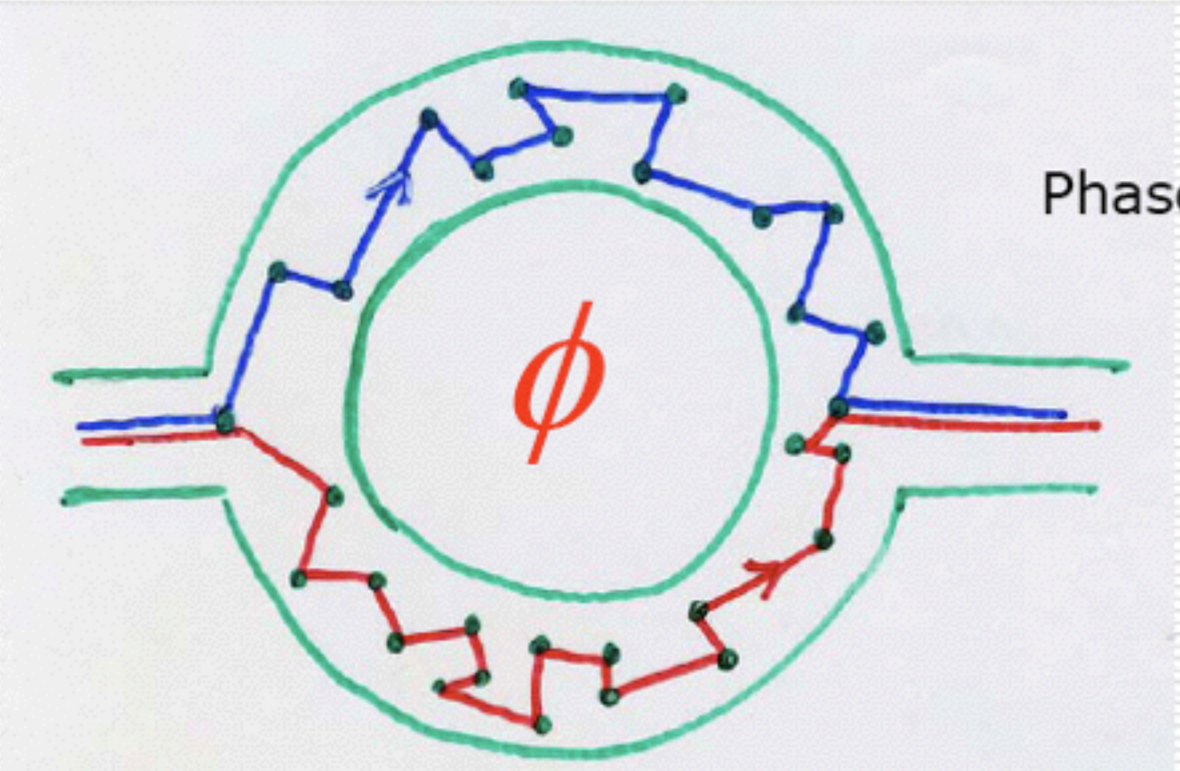
$\phi_0/2$ periodicity of the Sharvin effect

$P_{int}(r, r', t)$ is obtained from the *covariant* diffusion equation

effective charge 2e

$$\left(\frac{1}{\tau_\phi} + \frac{\partial}{\partial t} - D\left[\nabla_{r'} + i\frac{2e}{\hbar}\mathbf{A}(r')\right]^2\right) P_{int}(r, r', t) = \delta(r - r')\delta(t)$$

Webb

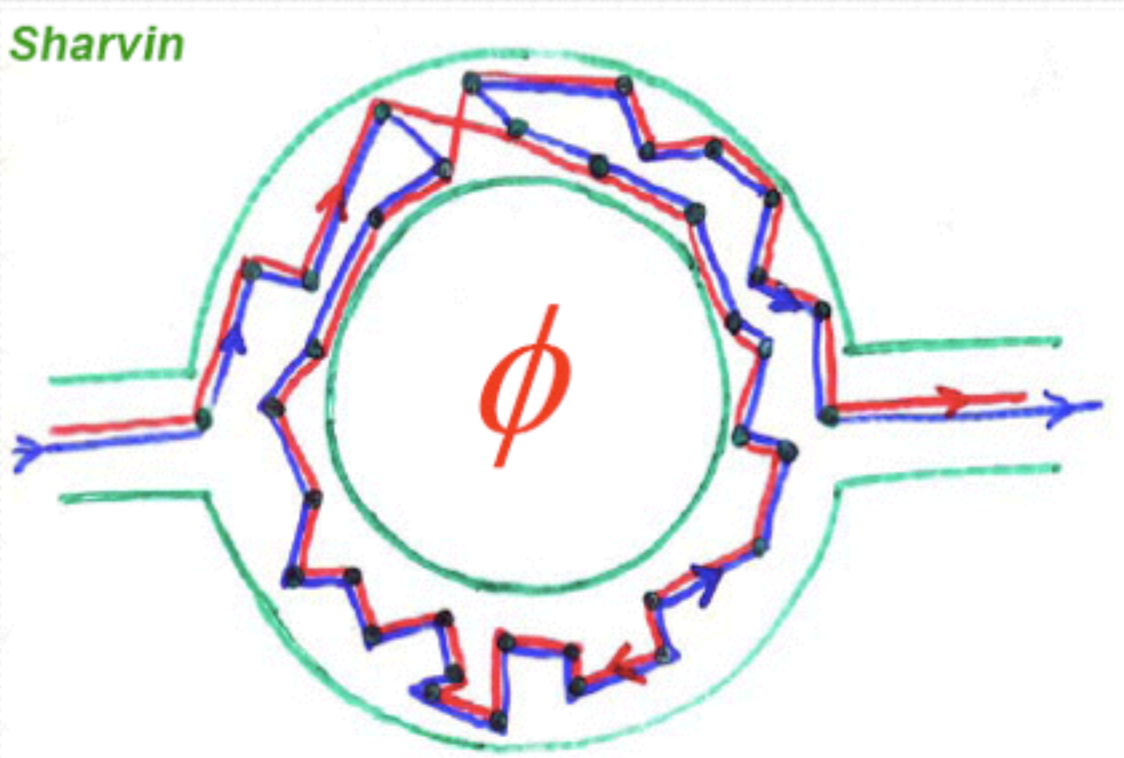


Sample specific interference

Phase difference $2\pi \frac{\phi}{\phi_0}$

Oscillates with period h/e

Sharvin, Sharvin



Survives disorder average

Phase difference $4\pi \frac{\phi}{\phi_0}$

Oscillates with period $h/2e$

Weak localization- Electronic transport

To the classical probability corresponds the Drude conductance G_{cl}

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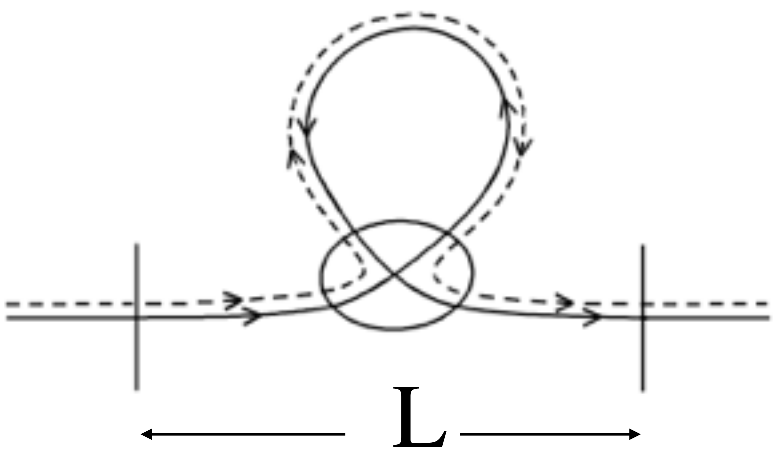


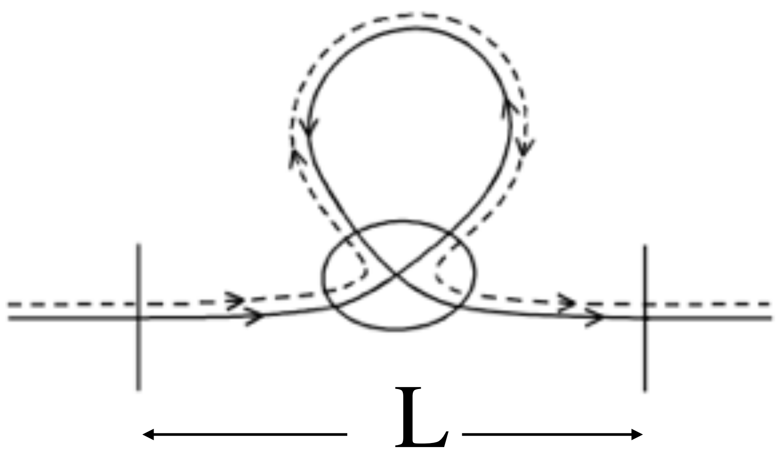
First correction $(\propto 1/g)$ involves one quantum crossing and the probability $p_o(\tau_D)$ to have a closed loop:

$$\boxed{\frac{\Delta G}{G_{cl}} = -\,p_o(\tau_D)}$$

$$\tau_D = L^2/D$$

Weak localization- Electronic transport

$\Longrightarrow$ To the classical probability corresponds the Drude conductance G_{cl}



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$$\tau_D = L^2/D$$

$$p_o(\tau_D) = \frac{1}{g}\int_0^{\tau_D} Z(t)\frac{dt}{\tau_D}$$

quantum correction decreases the conductance: weak localization

Return probability $Z(t) = \int dr P_{int}(r,r,t) = \left(\frac{\tau_D}{4\pi t}\right)^{d/2}$

Quantum mesoscopic physics :

the global scattering approach

(Landauer-Schwinger)

An Intermezzo !

global

Aim of the intermezzo:

to present in general terms, a **global (i.e. non local)** approach to account for both the <u>thermodynamic</u> and the <u>non equilibrium</u> behavior of **quantum complex systems**

Elastic di… rder does not break phase coherence

and … duce irreversibility

D … ness and

A reminder

All symmetries are … are no good quantum numbers.

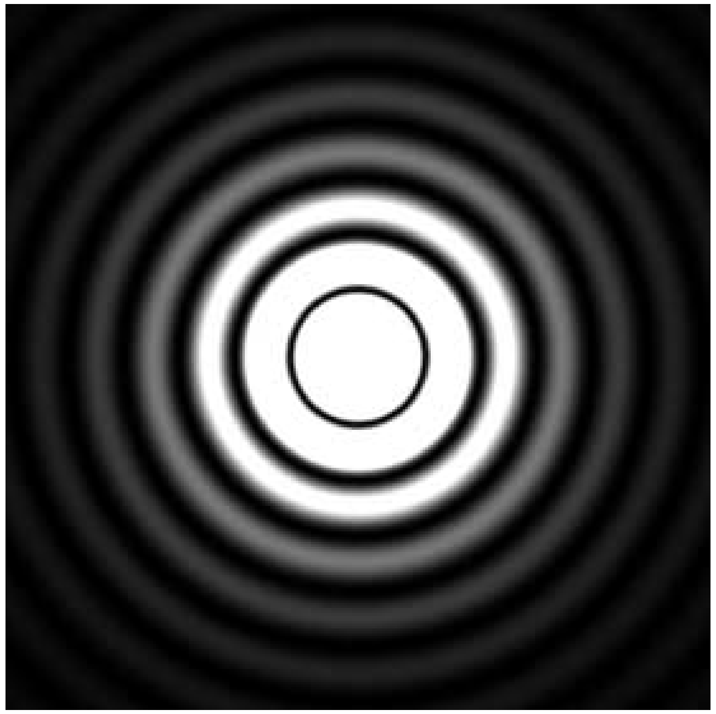
Elastic disorder does not break phase coherence and it does not introduce irreversibility

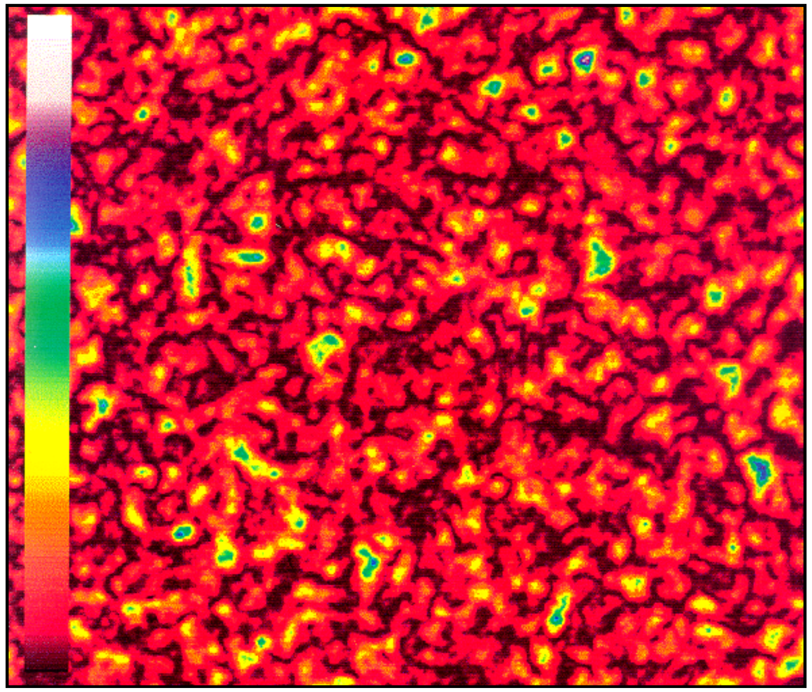
Disorder introduces randomness and complexity:

All symmetries are lost, there are no good quantum numbers.

Exemple: speckle patterns in optics

Diffraction through a circular aperture: order in interference





Transmission of light through a disordered suspension: complex system

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In complex systems (metals, dielectrics, ...), it is difficult to obtain local quantities and sometimes it is even impossible. But in many cases, **it is also not necessary.**

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Use a global description : Landauer-Schwinger approach

Basics: Usually we start from local differential equations and try to solve them with appropriate boundary conditions.

Express local physical quantities, e.g. electrical conductivity, dielectric function in terms of **local Green's functions** for the quantum coherent matter field (electrons)

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$$\sigma_{xx}(\omega) = s\frac{\hbar}{\pi\Omega}\,\mathrm{Tr}\left[\hat{j}_x\,\mathrm{Im}\hat{G}^R_{\epsilon_F}\,\hat{j}_x\,\mathrm{Im}\hat{G}^R_{\epsilon_F-\omega}\right]$$

$$\sigma_{\alpha\beta}(\boldsymbol{r},\boldsymbol{r}') = -s\frac{e^2\hbar^3}{2\pi m^2}\left[\partial_\alpha \mathrm{Im}G^R_\epsilon(\boldsymbol{r},\boldsymbol{r}')\partial'_\beta \mathrm{Im}G^R_\epsilon(\boldsymbol{r}',\boldsymbol{r}) - \mathrm{Im}G^R_\epsilon(\boldsymbol{r},\boldsymbol{r}')\partial_\alpha\partial'_\beta \mathrm{Im}G^R_\epsilon(\boldsymbol{r}',\boldsymbol{r})\right]$$

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1. local divergences of the Green's functions close to a boundary

PHYSICAL REVIEW D VOLUME 20, NUMBER 12 15 DECEMBER 1979

Boundary effects in quantum field theory

D. Deutsch and P. Candelas
Center for Theoretical Physics, Department of Physics, The University of Texas at Austin, Austin, Texas 78712
(Received 15 September 1978)

Electromagnetic and scalar fields are quantized in the region near an arbitrary smooth boundary, and the renormalized expectation value of the stress-energy tensor is calculated. The energy density is found to diverge as the boundary is approached. For nonconformally invariant fields it varies, to leading order, as the inverse fourth power of the distance from the boundary. For conformally invariant fields the coefficient of this leading term is zero, and the energy density varies as the inverse cube of the distance. An asymptotic series for the renormalized stress-energy tensor is developed as far as the inverse-square term in powers of the distance. Some criticisms are made of the usual approach to this problem, which is via the "renormalized mode sum energy," a quantity which is generically infinite. Green's-function methods are used in explicit calculations, and an iterative scheme is set up to generate asymptotic series for Green's functions near a smooth boundary. Contact is made with the theory of the asymptotic distribution of eigenvalues of the Laplacian operator. The method is extended to nonflat space-times and to an example with a nonsmooth boundary.

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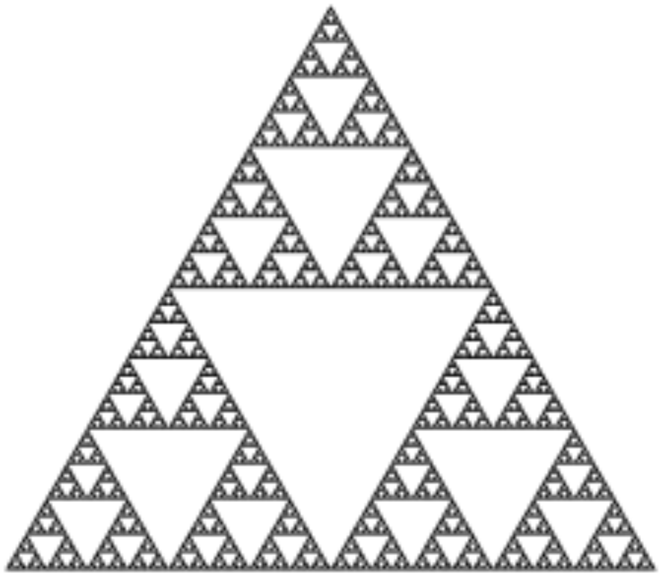
2. average over existing intrinsic disorder : no analytic known solution of the Anderson problem either for weak or strong disorder.

3. It can be also because we simply do not have local differential eqs., e.g. on fractals



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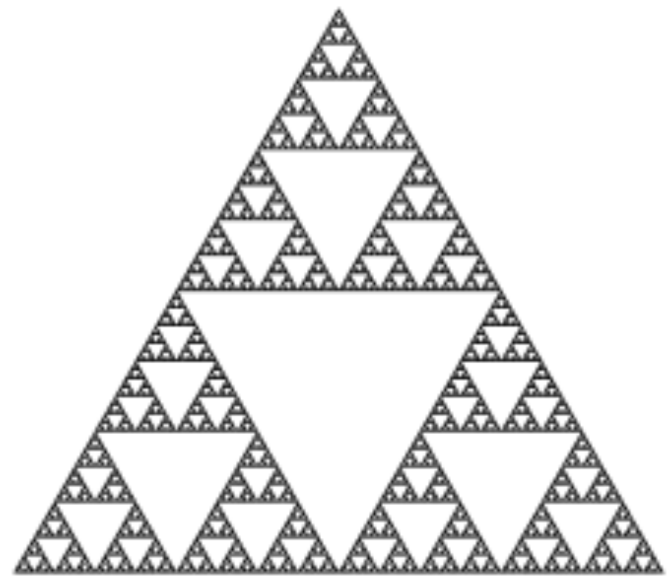

4. Or because the physical quantity we wish to calculate does not have a local description : for instance there exists a local wave eq. but we do not have a (local) Kubo formula for the diffusion coefficient.

Transport in a metal : Landauer approach

1. Electric transport:

<u>Local</u> Kubo formulation for the electric current:

$$j(x)=\int dx'\,\sigma(x,x')\,E(x') \Rightarrow j(x)=\sigma E(x)$$

where $\sigma(x,x')$ is the local conductivity (response) expressed in terms of local solutions (Green's functions).

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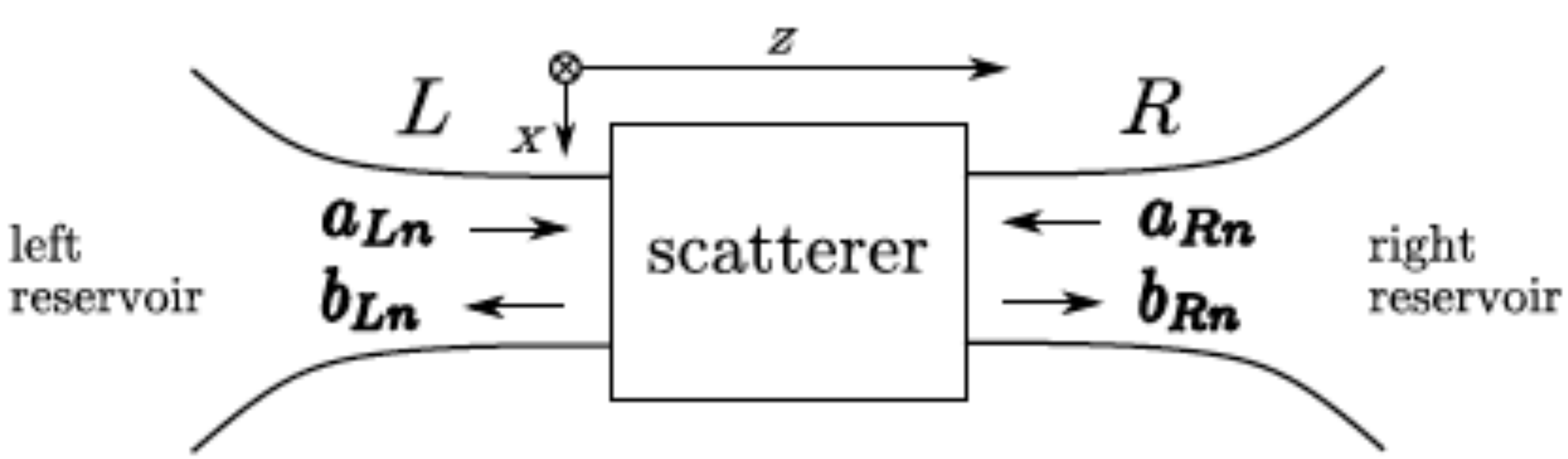

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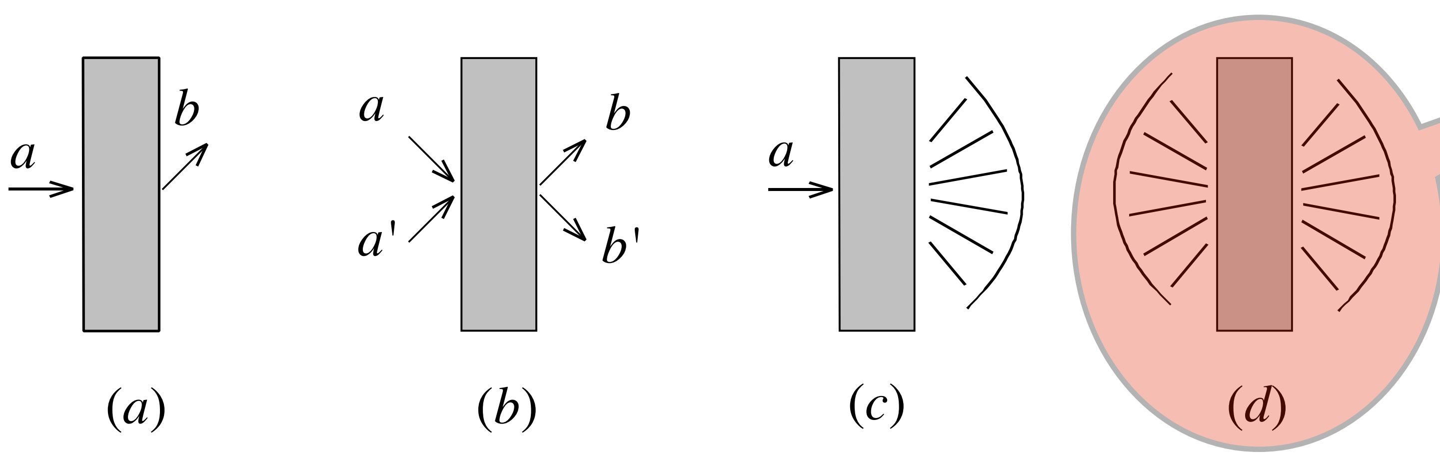


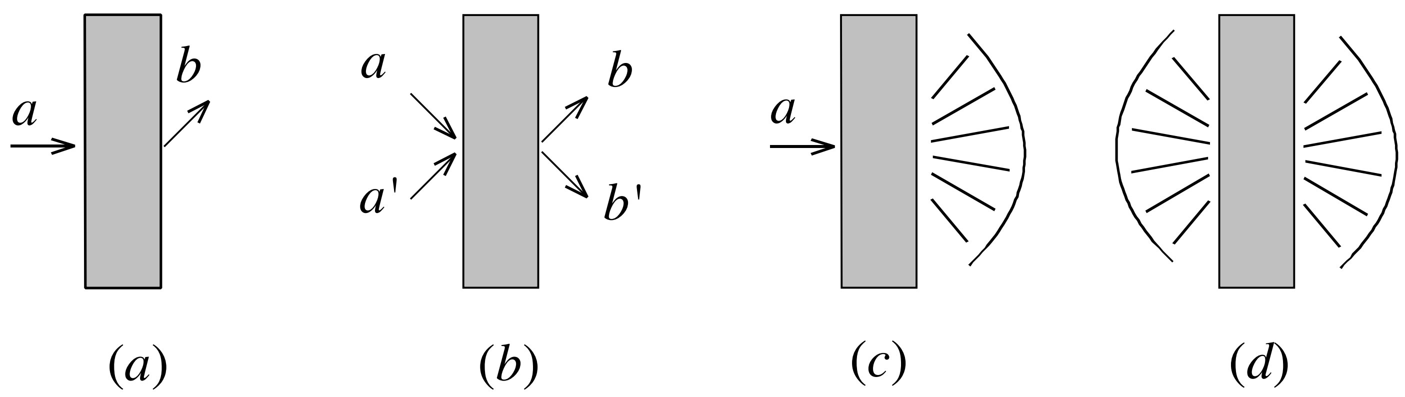
$$G=\frac{e^2}{h}\sum_{a,b}T_{ab}$$

2. Waves through complex disordered/chaotic media:

for instance there exists a local wave eq. but we do not have a (local) Kubo formula for the diffusion coefficient.

But there is a well defined Landauer description based on the

Scattering matrix-Transmission coefficient, etc.



Spectral properties-Thermodynamics : Krein-Schwinger formula

Waves in free space : Density of states $\rho_0(\omega)$ per unit volume.

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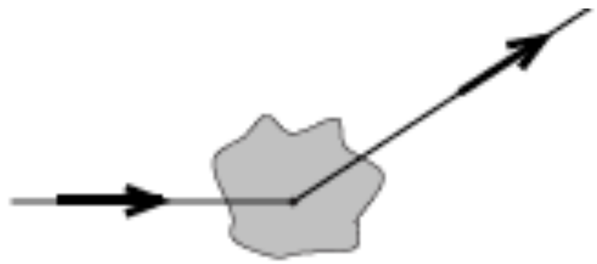
Scatterer:



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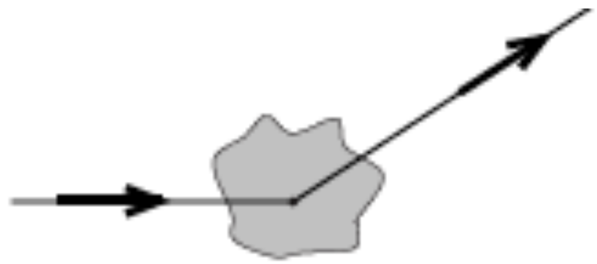

The **S-matrix** accounts for all relevant changes : e.g. DOS $\rho(\omega)$ of the waves in the presence of the scatterer is:

$$\rho(\omega) - \rho_0(\omega) = -\frac{1}{\pi} \Im m \frac{d}{d\omega} \ln Det\, S(\omega)$$

Krein formula

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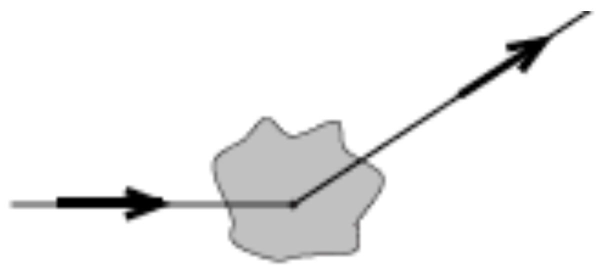

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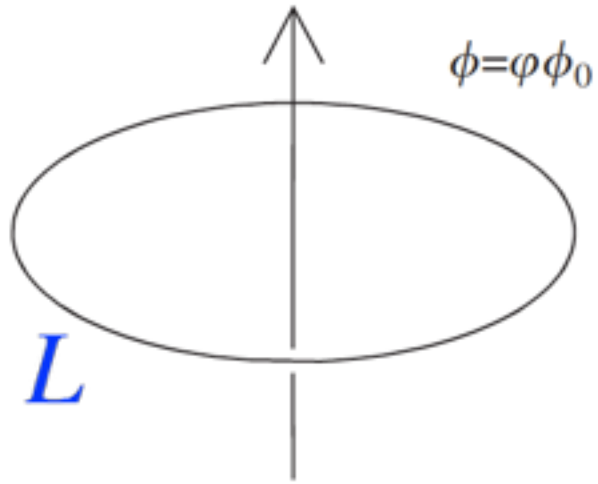
Krein formula

Thermodynamic changes can be deduced from this formula:

Variation of the partition function (Dashen,Ma,Bernstein):

$$Tr\,e^{-\beta H}-Tre^{-\beta H_0}=-\frac{1}{\pi}\int d\omega\,e^{-\beta\omega}\,\Im m\frac{d}{d\omega}\ln Det\,S(\omega) \qquad H=H_0+V$$

<u>Thermodynamics :</u> persistent current in a mesoscopic ring submitted to a Aharonov-Bohm flux



$$\phi_0 \equiv \frac{h}{e}$$

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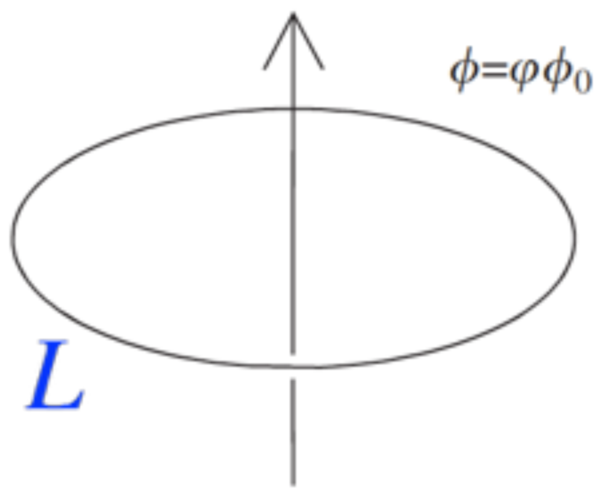

$$\phi_0 \equiv \frac{h}{e}$$

Energy spectrum of an electron in a Aharonov-Bohm magnetic flux

$$\epsilon_n = \frac{\hbar^2}{2m}\left(\frac{2\pi}{L}\right)^2 (n-\varphi)^2$$

Easy !

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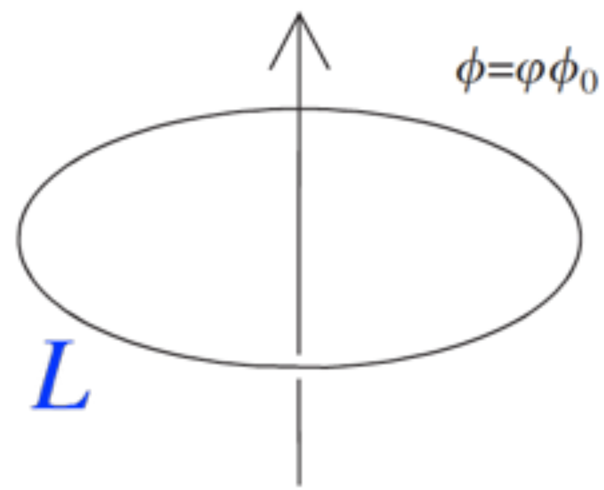

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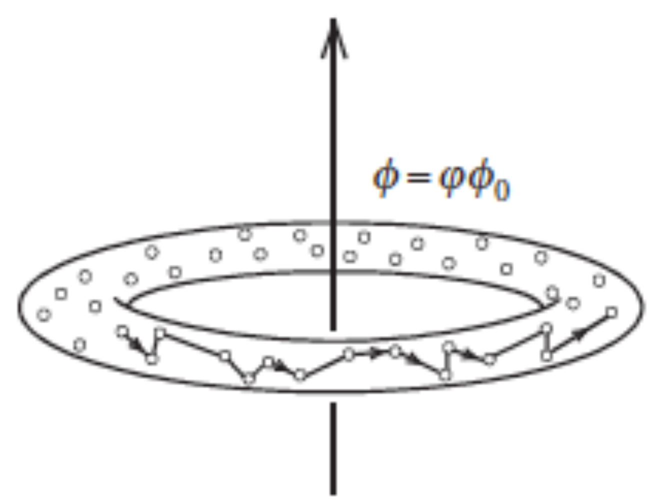
Disordered metal



Less easy !

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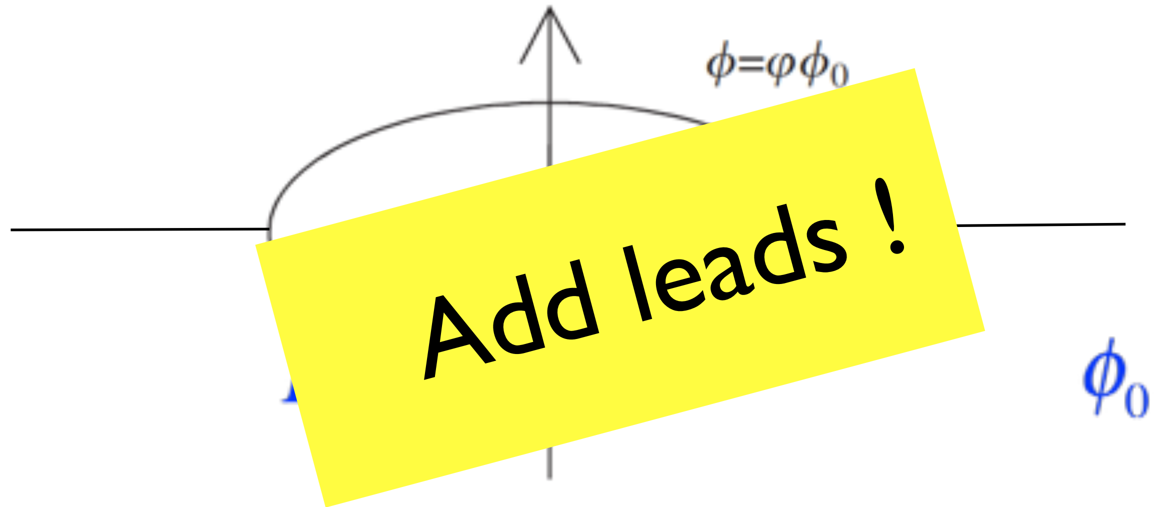

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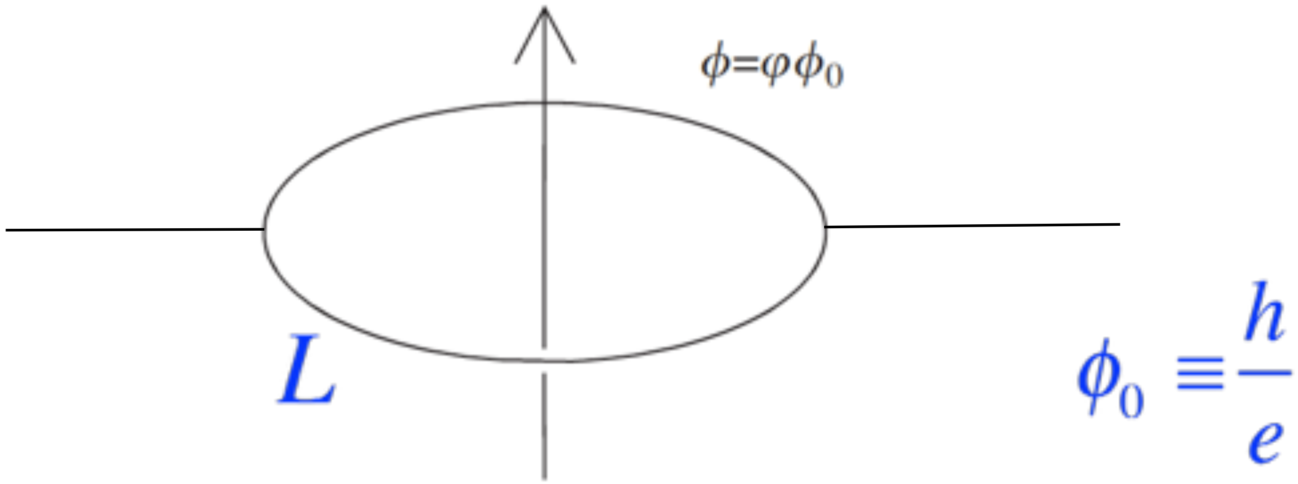

$$I(\phi) = \frac{1}{2i\pi} \int dE \frac{\partial}{\partial \phi} \ln Det\, S(E,\phi)$$

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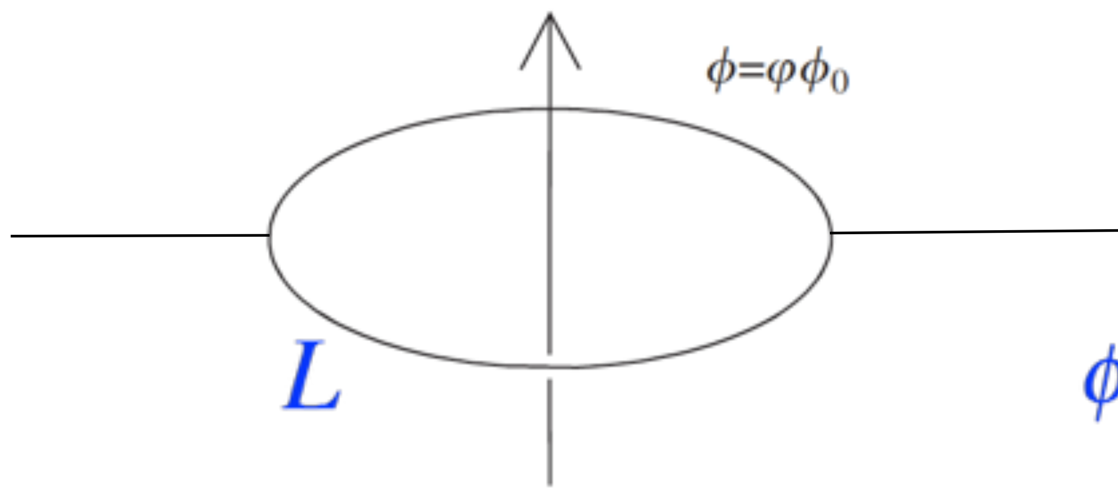

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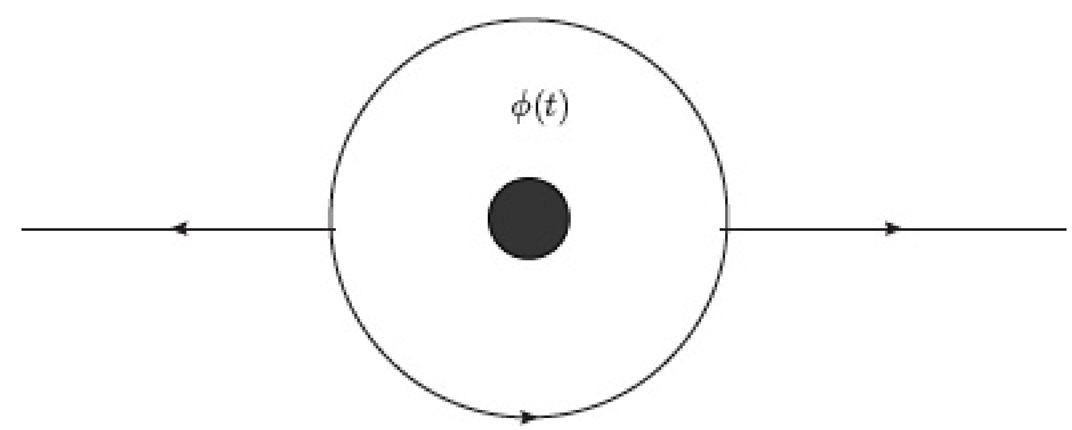
Electrical conductance G (**out of equilibrium**)

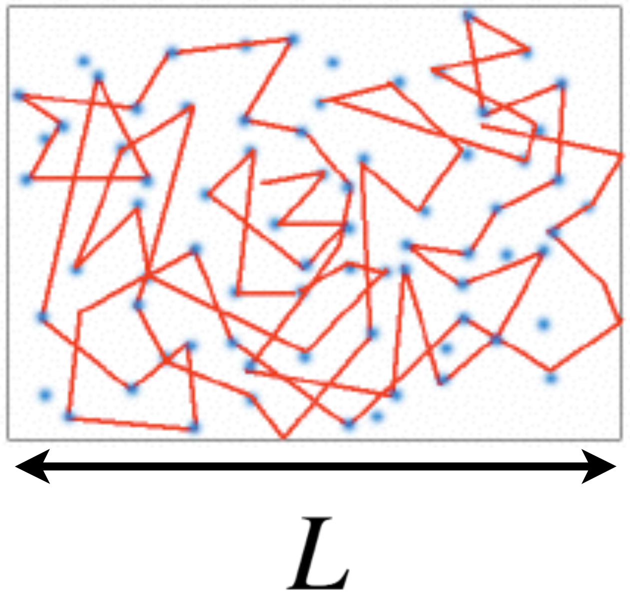
$$G = \frac{2e^2}{\pi\hbar}\left(\Im m\, \frac{\partial}{\partial\phi} \ln Det\, S(E_F, \phi(0))\right)^2$$



Equivalent to the Landauer formula.

From Kubo to Landauer

$$j(x)=\int dx'\,\sigma(x,x')E(x')\Rightarrow j(x)=\sigma E(x)$$



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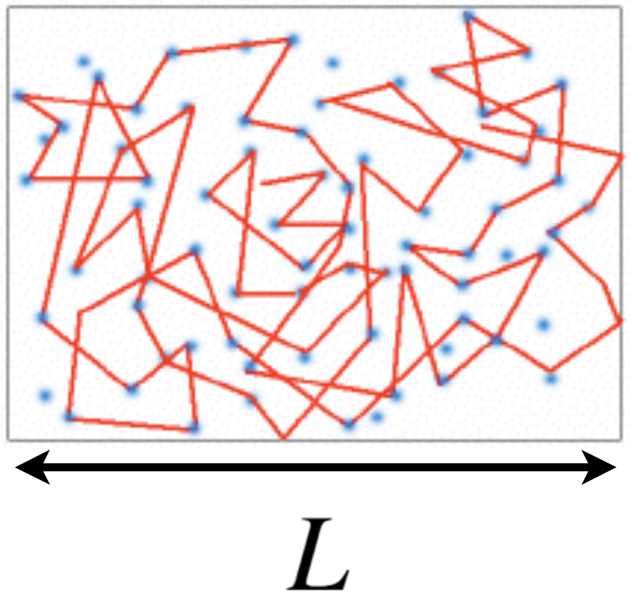

$$\sigma_{\alpha\beta}(\boldsymbol{r},\boldsymbol{r}')=-s\frac{e^2\hbar^3}{2\pi m^2}\left[\partial_\alpha \mathrm{Im}G^R_\epsilon(\boldsymbol{r},\boldsymbol{r}')\partial'_\beta \mathrm{Im}G^R_\epsilon(\boldsymbol{r}',\boldsymbol{r})-\mathrm{Im}G^R_\epsilon(\boldsymbol{r},\boldsymbol{r}')\partial_\alpha\partial'_\beta \mathrm{Im}G^R_\epsilon(\boldsymbol{r}',\boldsymbol{r})\right]$$

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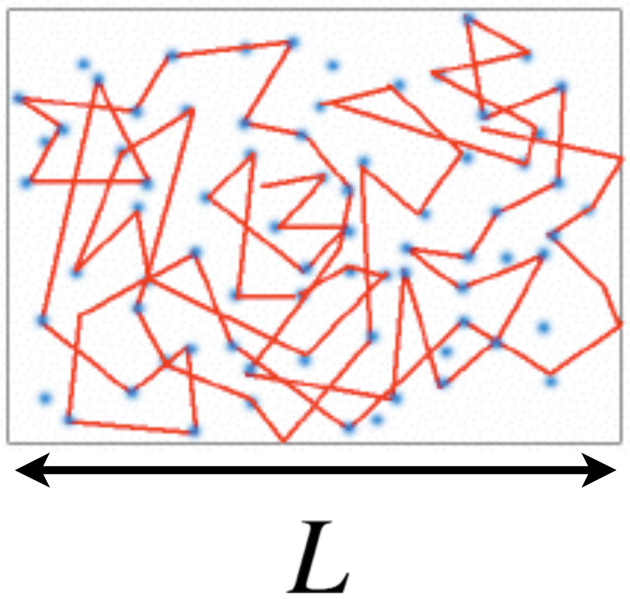
$$j(x) = \int dx' \, \sigma(x,x') E(x') \Rightarrow j(x) = \sigma E(x)$$

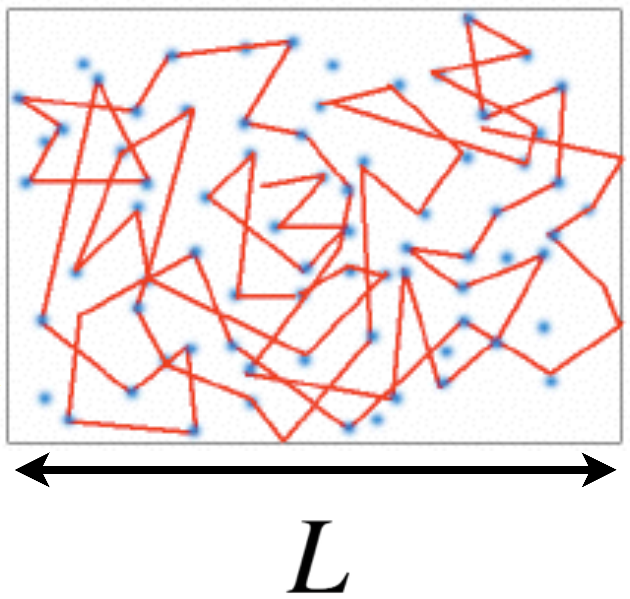


$$\sigma_{\alpha\beta}(\boldsymbol{r},\boldsymbol{r}') = -s\frac{e^2\hbar^3}{2\pi m^2}\left[\partial_\alpha \mathrm{Im}G^R_\epsilon(\boldsymbol{r},\boldsymbol{r}')\partial'_\beta \mathrm{Im}G^R_\epsilon(\boldsymbol{r}',\boldsymbol{r}) - \mathrm{Im}G^R_\epsilon(\boldsymbol{r},\boldsymbol{r}')\partial_\alpha\partial'_\beta \mathrm{Im}G^R_\epsilon(\boldsymbol{r}',\boldsymbol{r})\right]$$

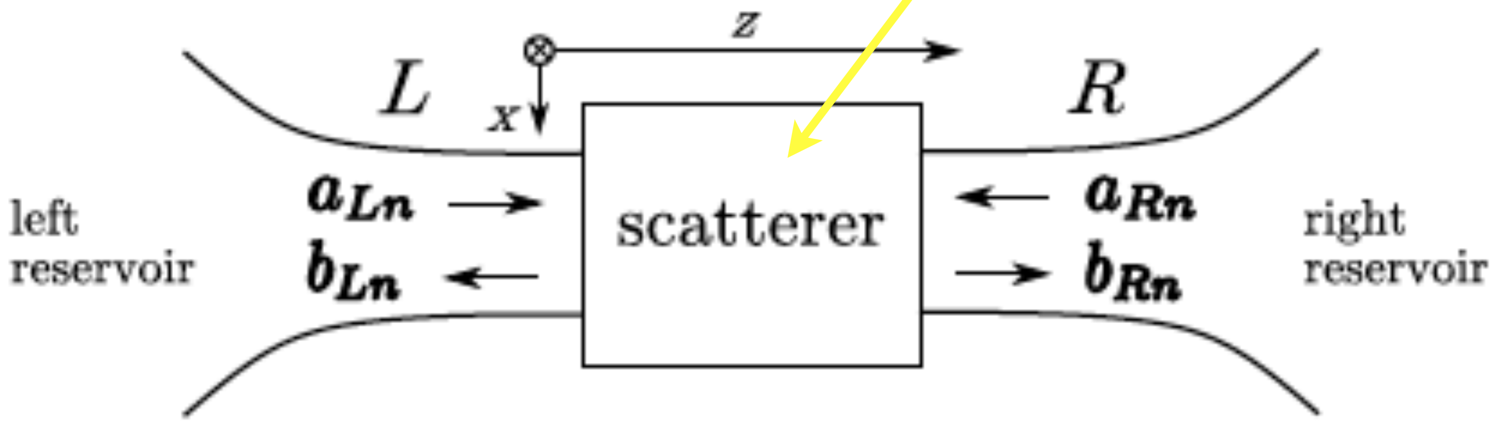
Add leads !

From Kubo to Landauer

$$j(x) = \int dx'\, \sigma(x,x')\, E(x') \Rightarrow j(x) = \sigma E(x)$$



$$\sigma_{\alpha\beta}(\boldsymbol{r},\boldsymbol{r}') = -s\frac{e^2\hbar^3}{2\pi m^2}\left[\partial_\alpha \mathrm{Im}G^R_\epsilon(\boldsymbol{r},\boldsymbol{r}')\partial'_\beta \mathrm{Im}G^R_\epsilon(\boldsymbol{r}',\boldsymbol{r}) - \mathrm{Im}G^R_\epsilon(\boldsymbol{r},\boldsymbol{r}')\partial_\alpha\partial'_\beta \mathrm{Im}G^R_\epsilon(\boldsymbol{r}',\boldsymbol{r})\right]$$



From Kubo to Landauer

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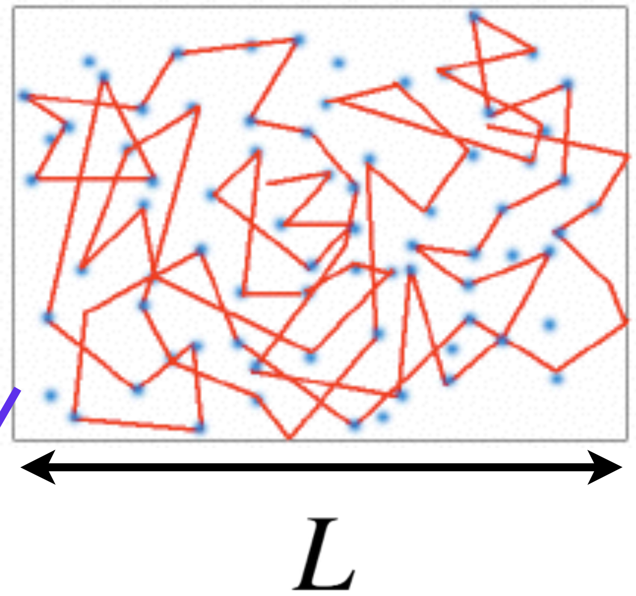
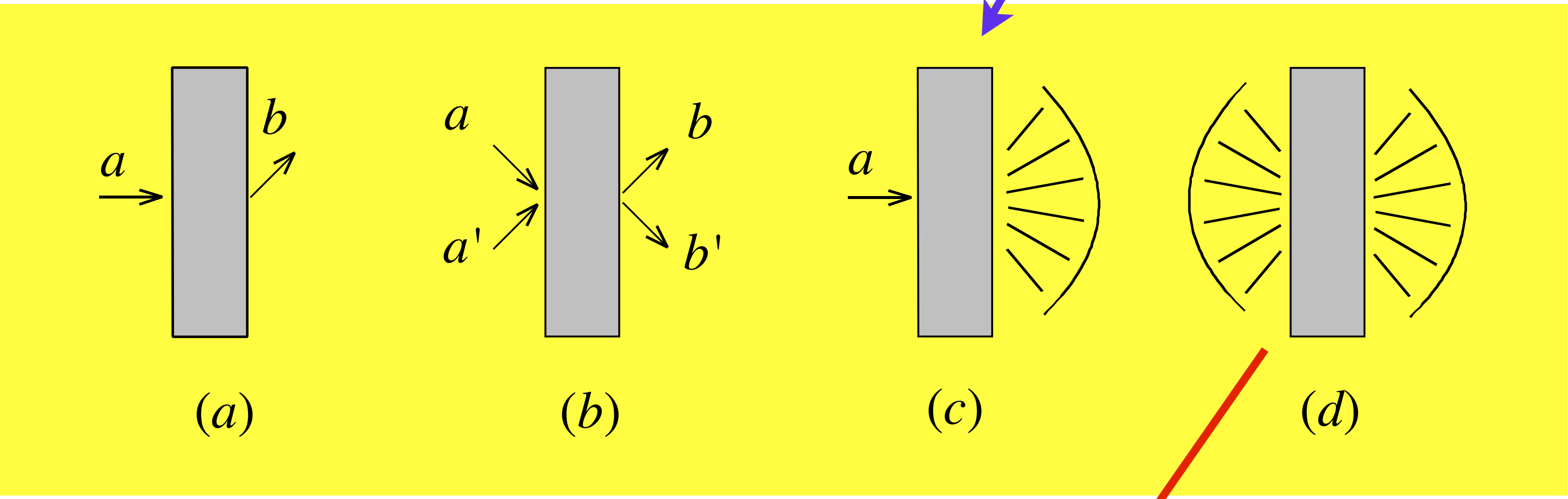


$$T_{ab} = |t_{ab}|^2$$



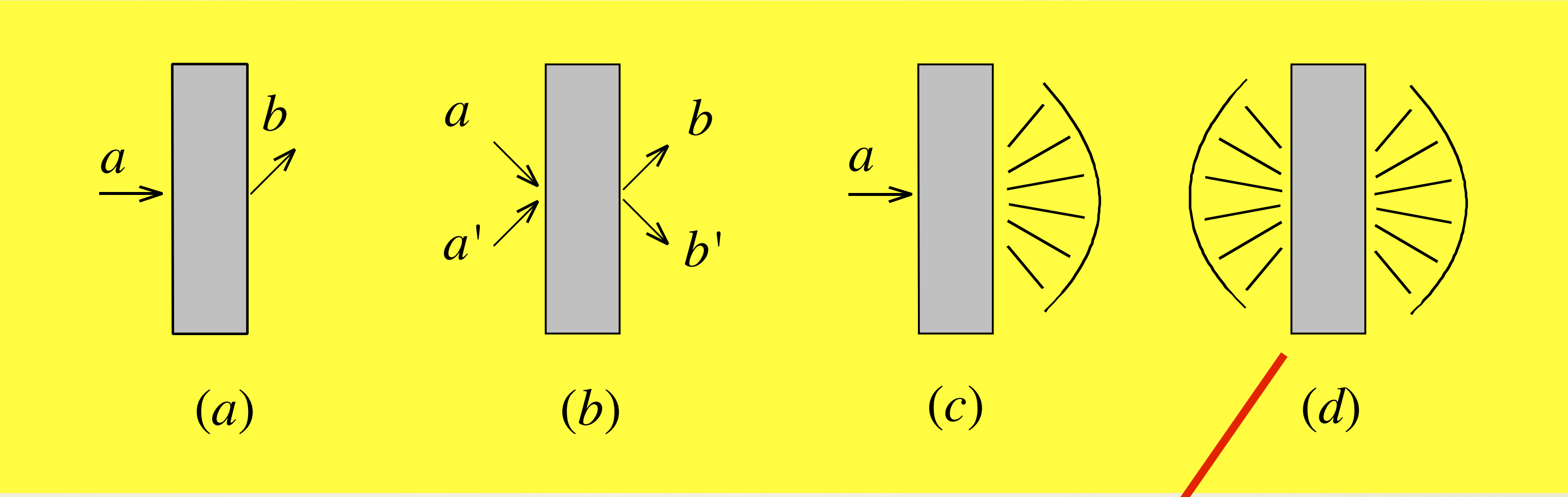
Landauer formula

$$G = \frac{e^2}{h} Tr\, t\, t^{\dagger}$$

Quantum conductance and shot noise

Slab geometry - two-terminal conductors

$$T_{ab} = \left| t_{ab} \right|^2$$



Landauer formula

$$G = \frac{e^2}{h} Tr\, t\, t^\dagger$$

Noise power is defined as the symmetric current-current correlation function

$$S(\omega,V)=\int dt\, e^{i\omega t}\left\langle \delta\hat{I}(t)\delta\hat{I}(0)+\delta\hat{I}(0)\delta\hat{I}(t)\right\rangle$$

where $\delta\hat{I}(t)=\hat{I}(t)-\langle I\rangle$ are electronic current operators

Equilibrium noise (V=0)

$$S(\omega,0)=2G\omega\ \coth\left(\frac{\omega}{2T}\right)$$

(Nyquist fluctuation-dissipation)

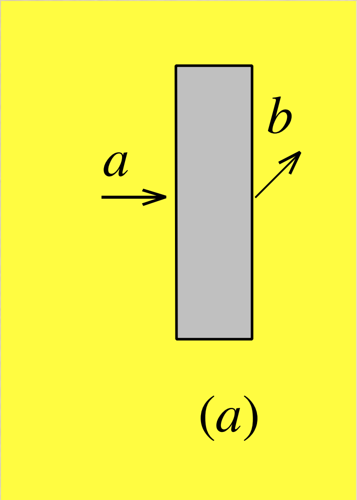
Non-equilibrium noise $V \neq 0$ at $T = 0$

$$S(0,V) - S(0,0) = \frac{e^2}{h} |2eV| Tr\ tt^\dagger \left(1 - tt^\dagger\right)$$

Excess noise measures the second cumulant of charge fluctuations :

$$S(0,V) - S(0,0) \propto \left\langle Q_t^2 \right\rangle - \left\langle Q_t \right\rangle^2$$

The Fano factor



$$F = \frac{S(0,V) - S(0,0)}{e\,I} = \frac{\sum_{ab} T_{ab}(1 - T_{ab})}{\sum_{ab} T_{ab}}$$

T_{ab} is the transmission coefficient along the channel ab

F takes a universal value 1/3 for weakly disordered "one-dimensional" metals

To end this intermezzo :

Well known examples (Landauer-Schwinger approach).

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Physically relevant quantities of a system are expressed in terms of in-out signals, including correlations.

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This idea has been successfully used in Quantum mesoscopic physics, quantum optics, quantum field theory,...

To end this intermezzo :

Well known examples (Landauer-Schwinger approach).

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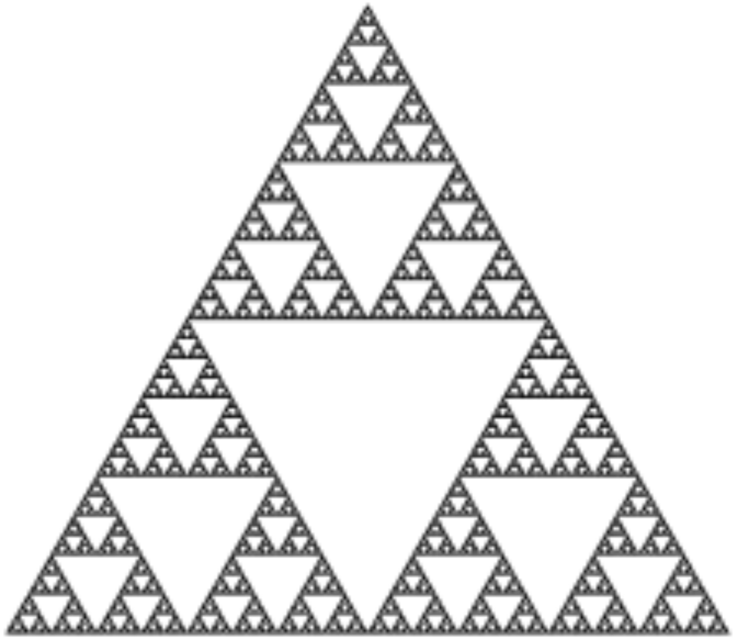
Physically relevant quantities of a system are expressed in terms of in-out signals, including correlations.

This idea has been successfully used in Quantum mesoscopic physics, quantum optics, quantum field theory,...

It is relatively new and promising in other fields:

1. Shannon information theory- MIMO (Multiple input-Multiple output)
2. Full counting statistics and shot noise (quantum mesoscopic physics)
3. Out of equilibrium quantum systems- Wigner time delay
4. Casimir effects
5. Non-perturbative effects (Unruh effects, Hawking radiation, Schwinger pair production,...)
6. Waves and quantum mechanics on fractal structures.

Energy spectrum - Thermodynamics - Transport ?



Energy spectrum - Thermodynamics - Transport ?

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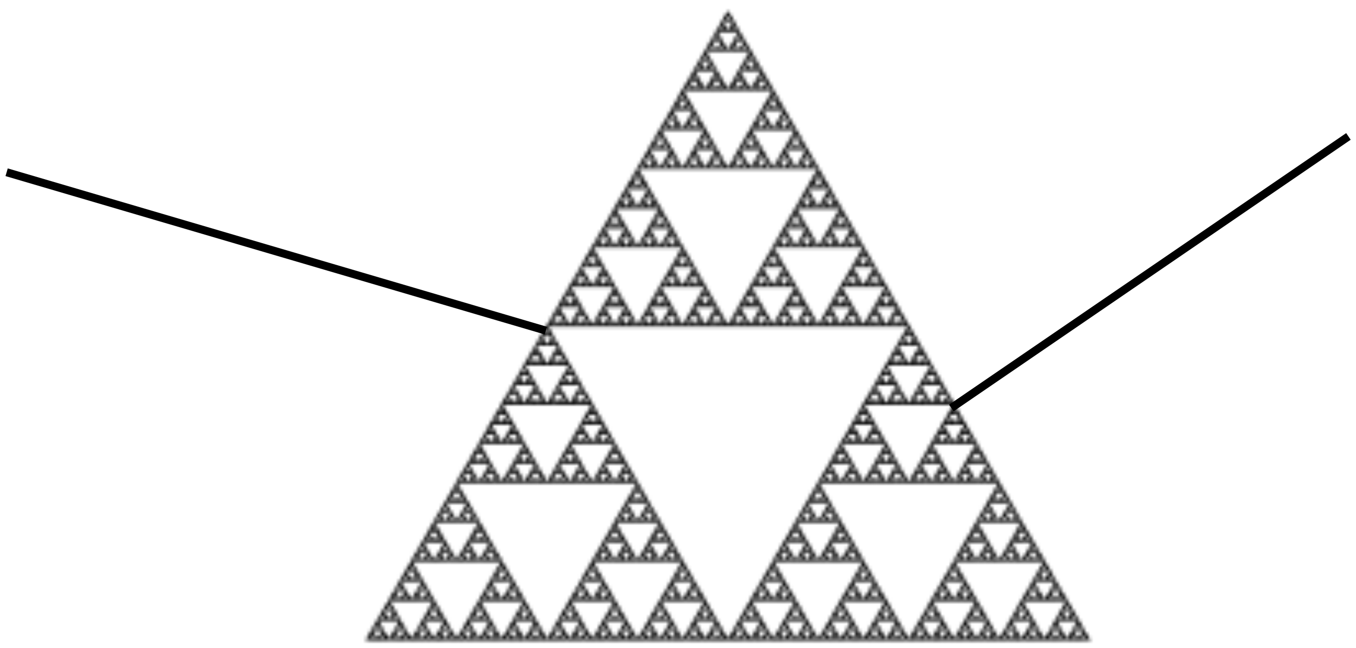
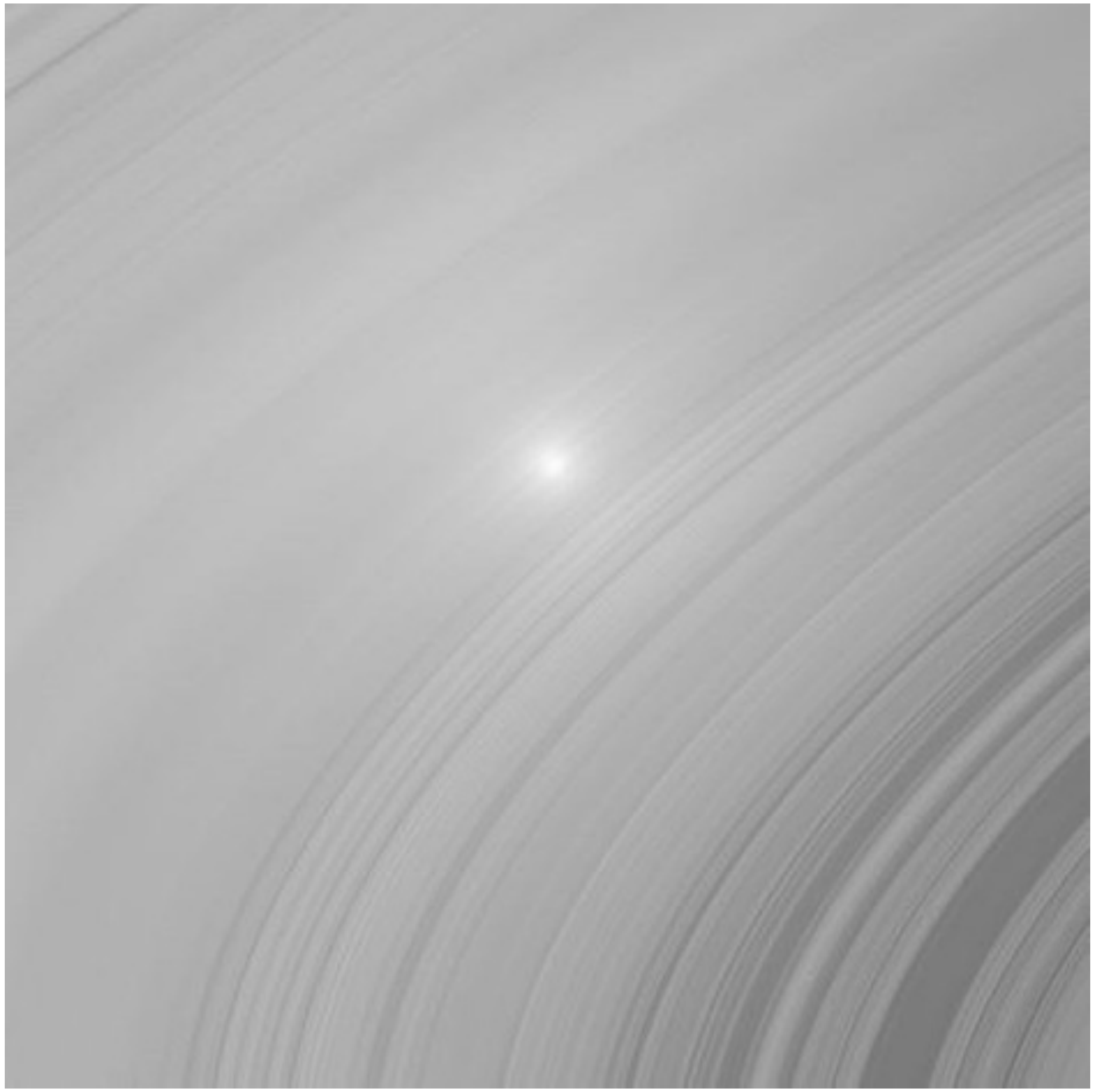


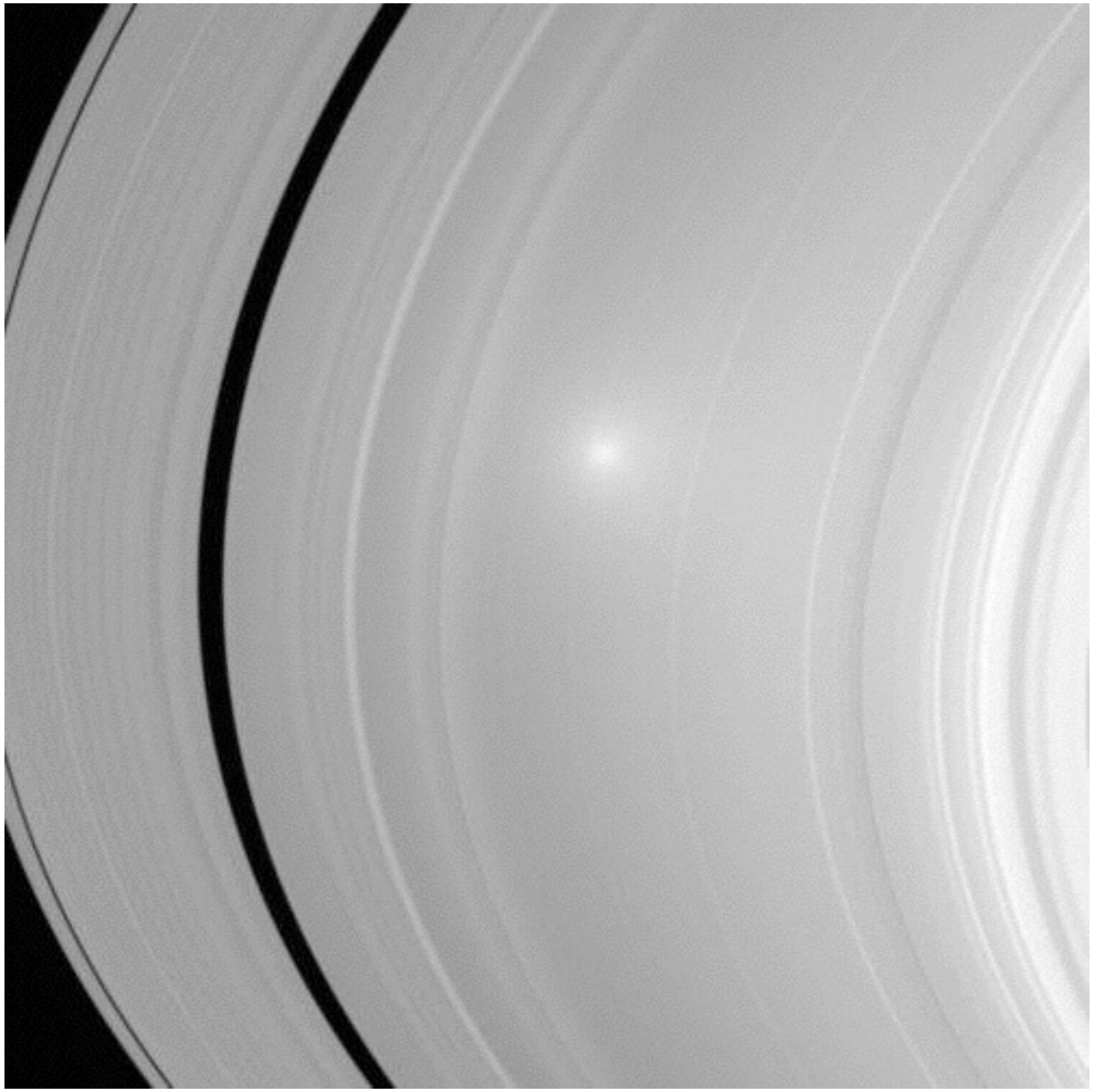
and calculate the S-matrix : possible

How to connect the 2 previous approaches:

* Local quantum crossings
* Global Landauer scattering formalism

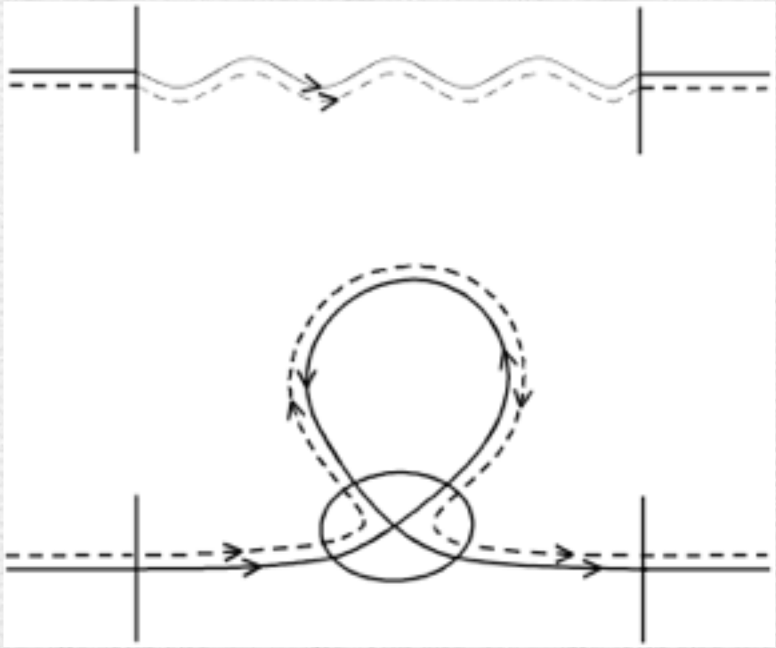
Coherent backscattering and Saturn rings





Cassini mission 2006

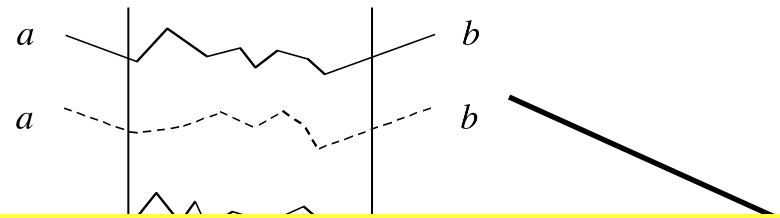
Beyond the conductance

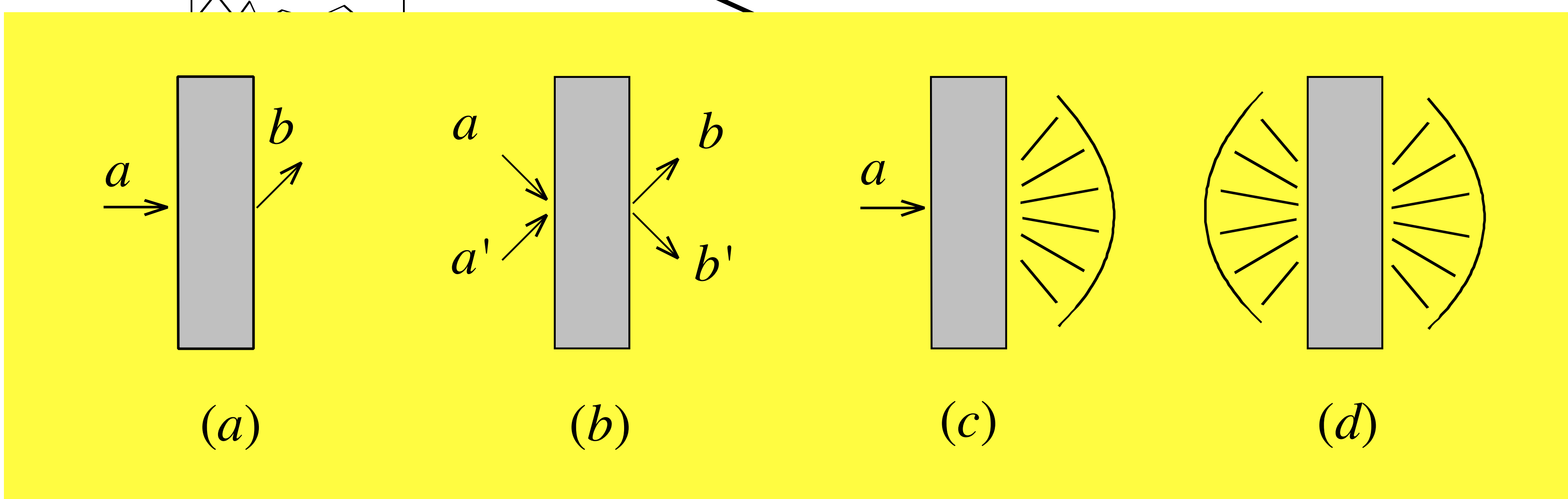


Fluctuations and correlations

transmission coefficient

$$T_{ab} = \left| t_{ab} \right|^2$$





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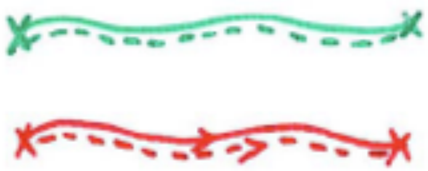
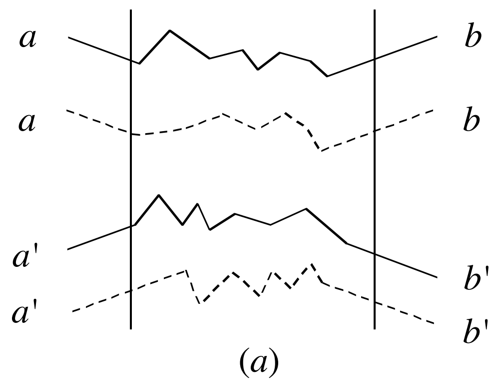
transmission coefficient :

$$C_{aba'b'} = \frac{\overline{\delta T_{ab} \delta T_{a'b'}}}{\bar{T}_{ab} \bar{T}_{a'b'}}$$

Slab geometry



Fluctuations and correlations



(a)

Slab geometry

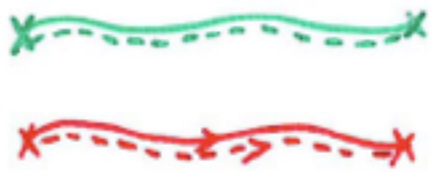
transmission coefficient

$$T_{ab} = \left| t_{ab} \right|^2$$

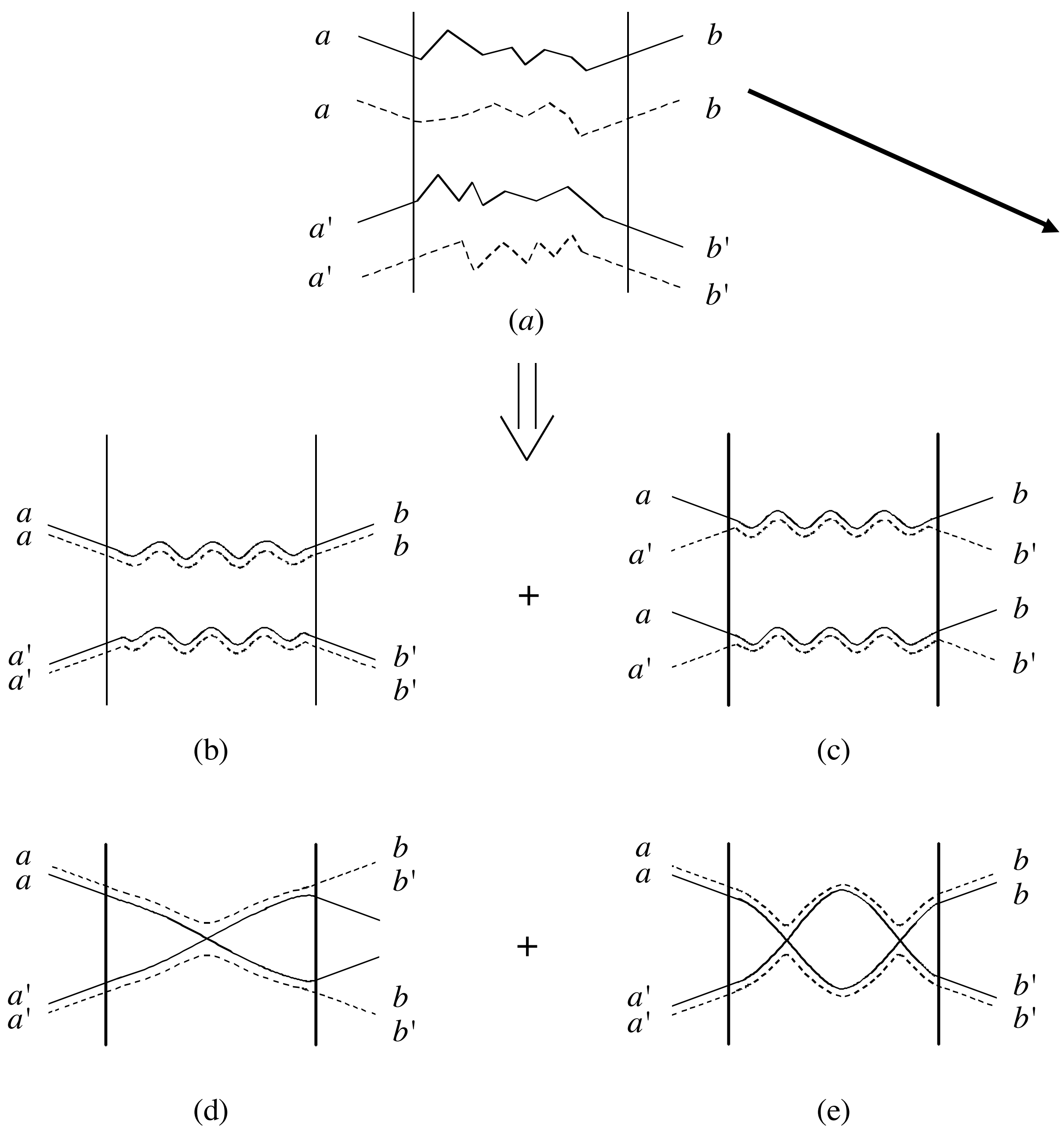
correlations involve the product of 4 complex amplitudes with or without quantum crossings

Correlation function of the transmission coefficient :

$$C_{aba'b'} = \frac{\overline{\delta T_{ab} \delta T_{a'b'}}}{\bar{T}_{ab} \bar{T}_{a'b'}}$$



Fluctuations and correlations



Slab geometry

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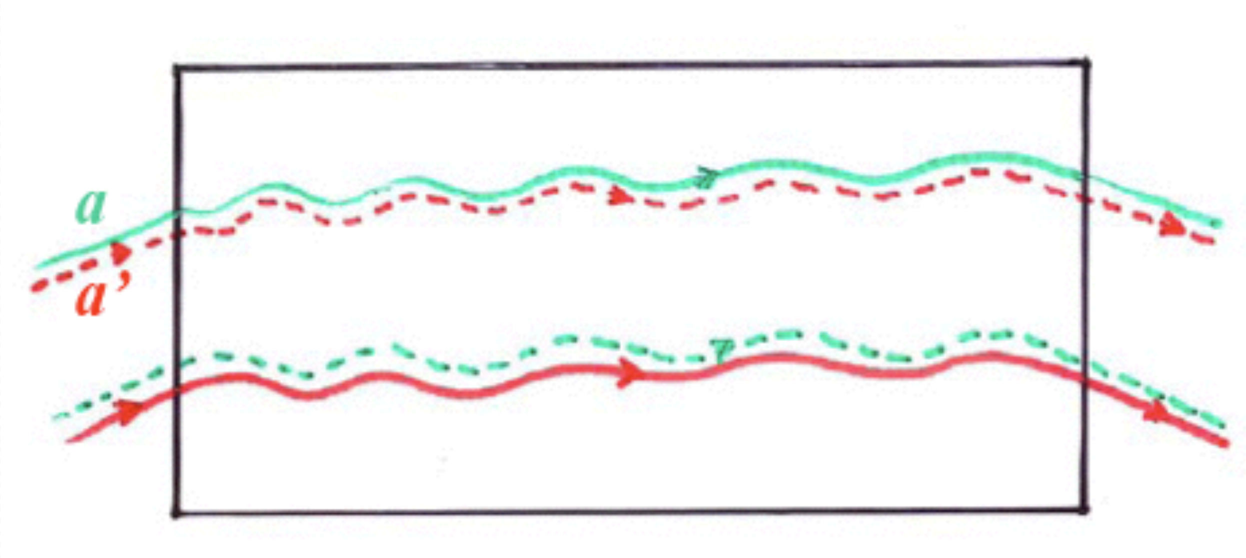
Speckle fluctuations *vs* conductance fluctuations

Landauer description : $G = \frac{e^2}{h}\sum_{ab} T_{ab}$

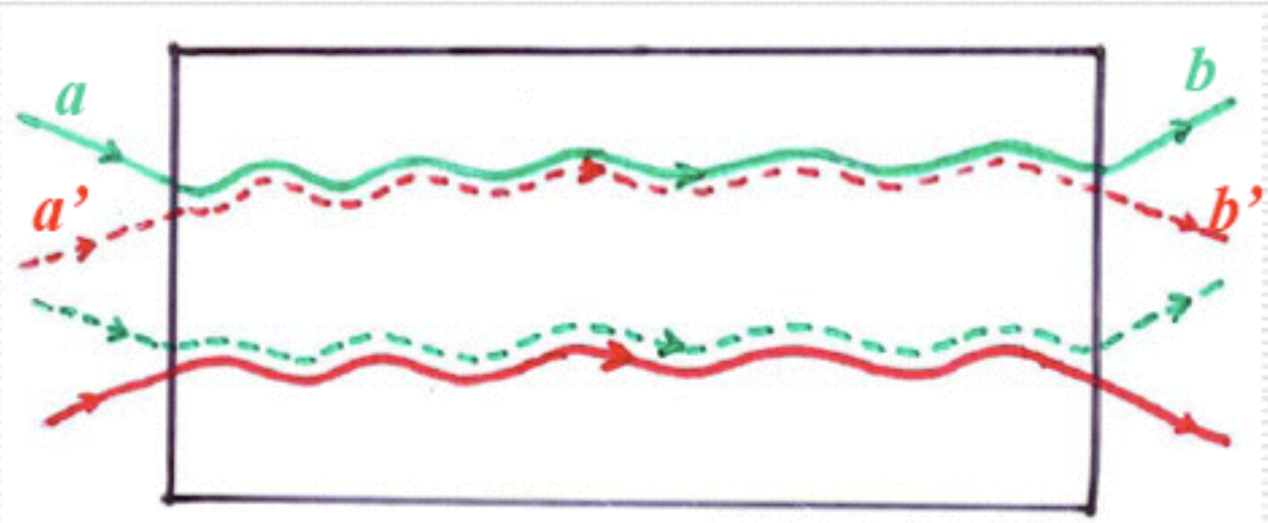
$a' = a$ $\quad$ $b' = b$



$$\overline{\delta T_{ab}\delta T_{ab}} = \overline{T_{ab}}^2$$

Rayleigh law

$$\overline{\delta g^2} = \sum_{a,a',b,b'} \overline{\delta T_{ab}\delta T_{a'b'}}$$

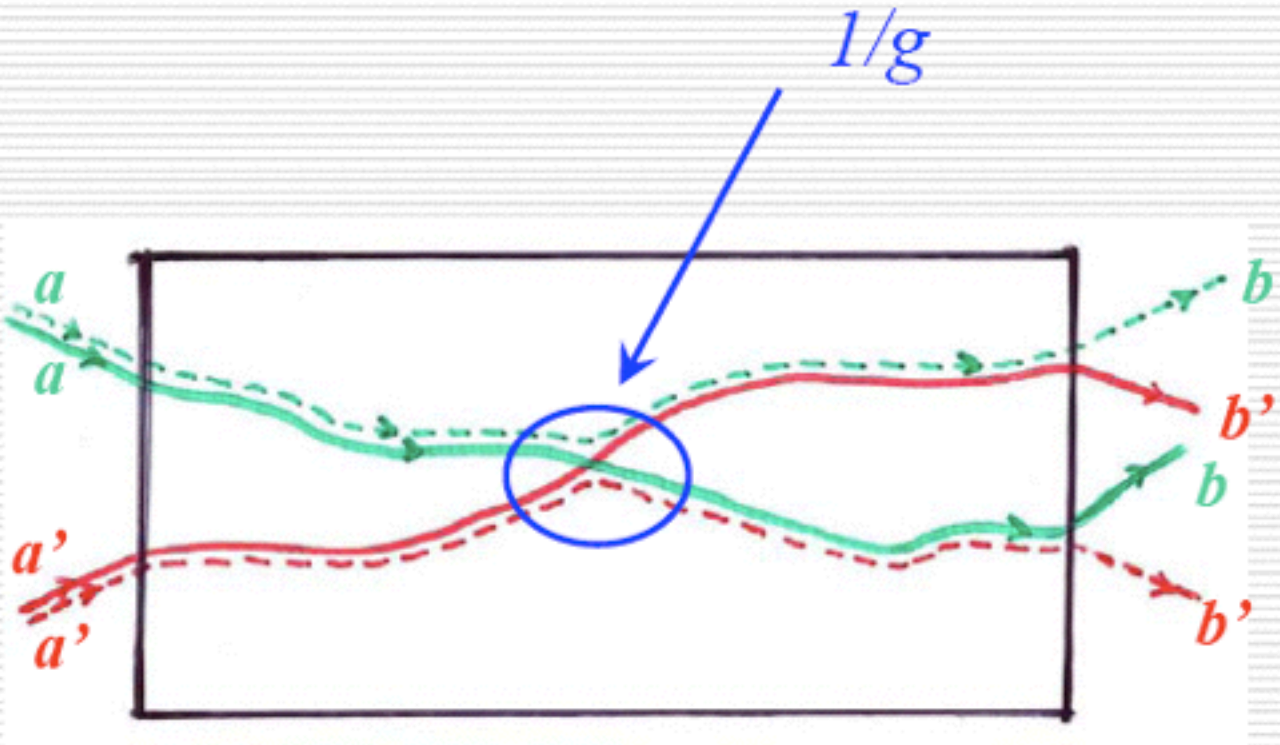


$$\overline{\delta g^2} = \sum_{a,a',b,b'} \overline{T_{ab}}\ \overline{T_{a'b'}}\ f(a,a',b,b') : 0$$

No conductance fluctuations !

Speckle fluctuations *vs* conductance fluctuations

C_2



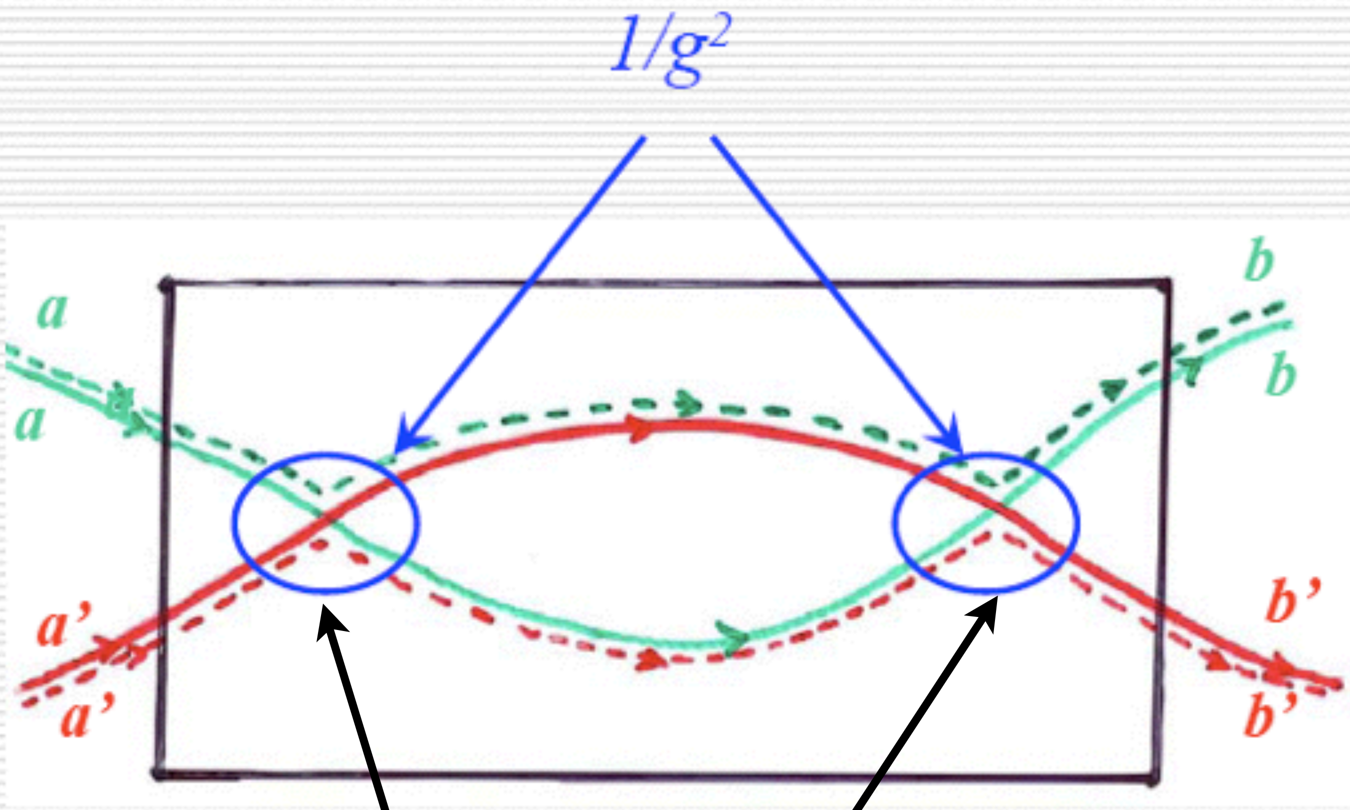
$$\overline{\delta T_{ab}\delta T_{a'b'}} = \frac{2}{3g}\overline{T_{ab}}\ \overline{T_{a'b'}}\ F(b,b')$$

Angular correlations of intermediate range

$$\overline{\delta g^2} = \frac{2}{3g}\sum_{a,a',b,b'}\overline{T_{ab}}\ \overline{T_{a'b'}}\ F(b,b') : 0$$

No conductance correlations !

Speckle fluctuations *vs* conductance fluctuations C_3

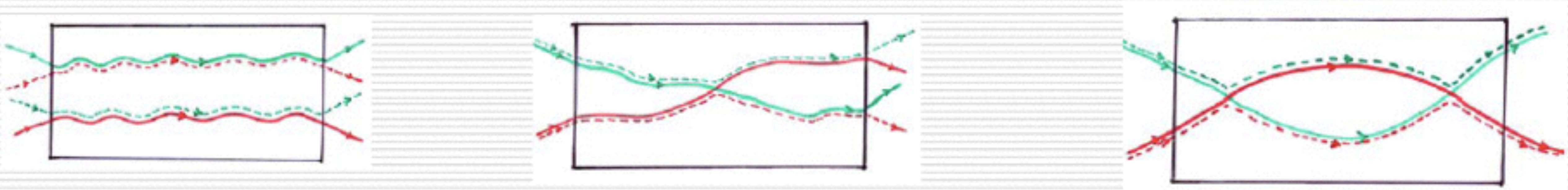


$$\overline{\delta T_{ab}\delta T_{a'b'}} = \frac{2}{15g^2}\overline{T_{ab}}\ \overline{T_{a'b'}}$$

Long-range angular correlations, with very weak amplitude

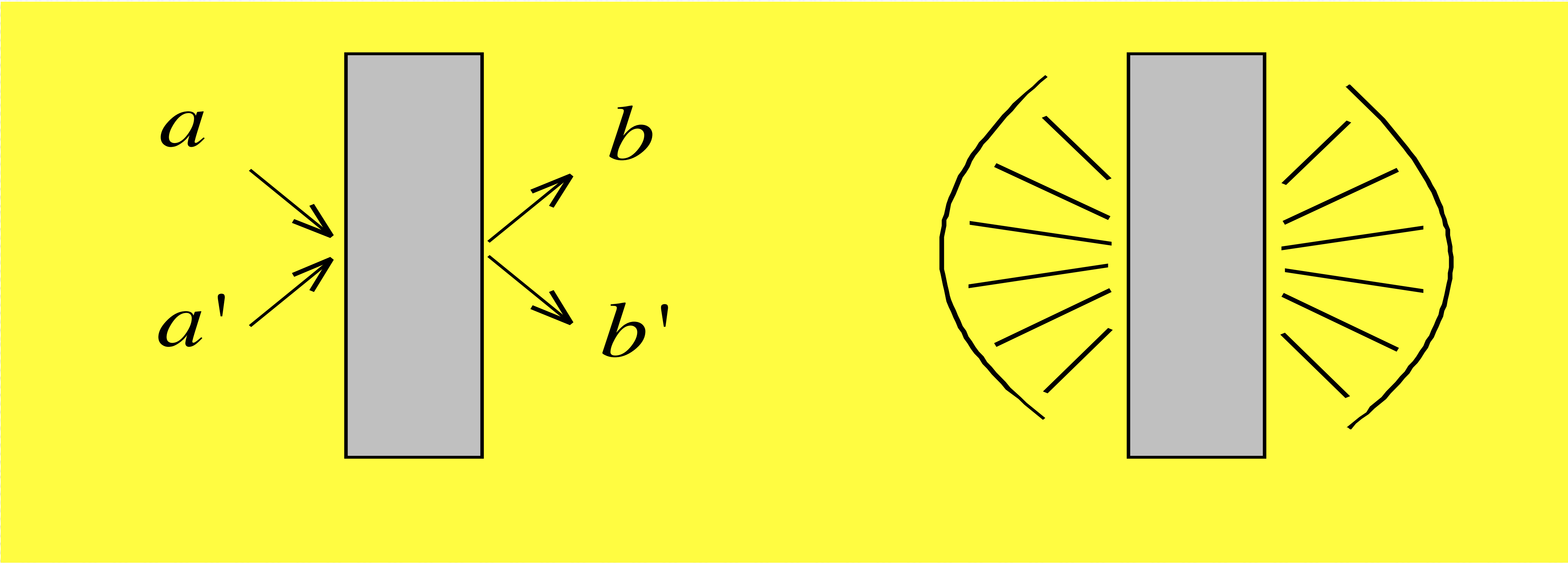
Local quantum effects (crossings) are propagated over long distances through classical diffusion

Fluctuations and correlations - Summary

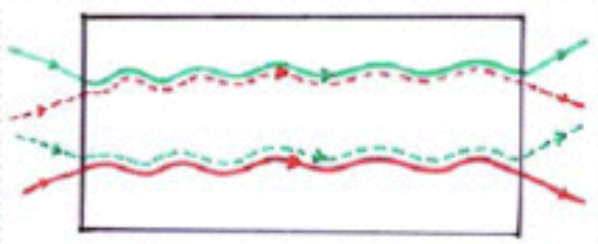
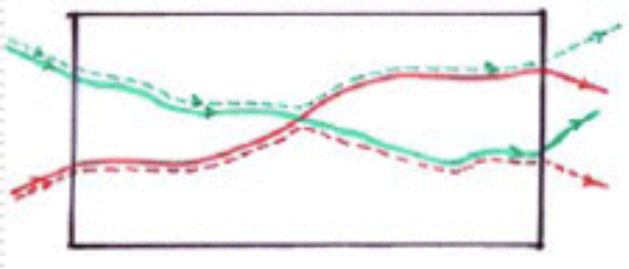
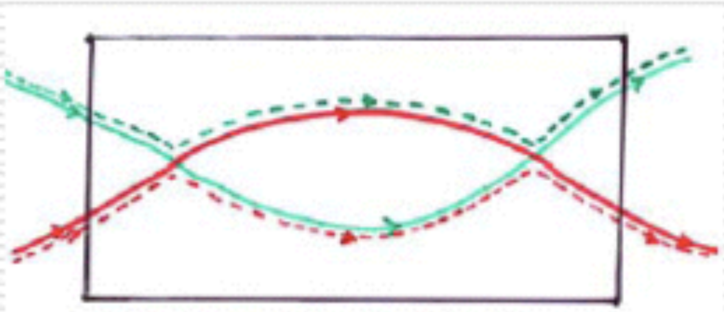


C_1 C_2 C_3

$$\overline{\delta T_{ab}\delta T_{a'b'}} = \overline{T_{ab}}\ \overline{T_{a'b'}}\left(f(a,a',b,b') + \frac{2}{3g}\left[F(a,a') + F(b,b')\right] + \frac{2}{15g^2}\right)$$



Fluctuations and correlations - Summary

C$_1$ C$_2$ C$_3$

$$\overline{\delta T_{ab}\delta T_{a'b'}} = \overline{T_{ab}}\ \overline{T_{a'b'}} \left(f(a,a',b,b') + \frac{2}{3g}\left[F(a,a') + F(b,b')\right] + \frac{2}{15g^2} \right)$$

$$\overline{\delta g^2} = \frac{2}{15g^2} \sum_{a,a',b,b'} \overline{T_{ab}}\ \overline{T_{a'b'}} = \frac{2}{15}$$

Universal conductance fluctuations

A direct consequence: quantum corrections to electrical transport

Cla… $G_{cl} = g \times \frac{e^2}{h}$ with $g \gg 1$

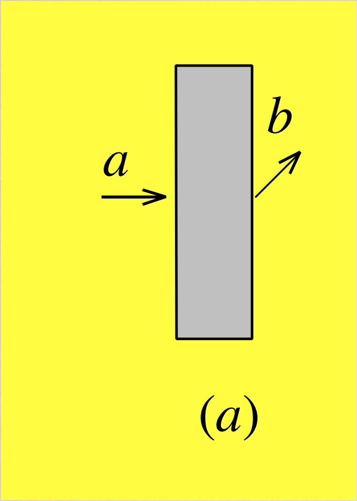
Quantum correc…

Not that simple !

numbers... Need to sum up Feynman diagrams.

so that $\boxed{\Delta G = \# \frac{e^2}{h}}$ is <u>universal</u>

The Fano factor



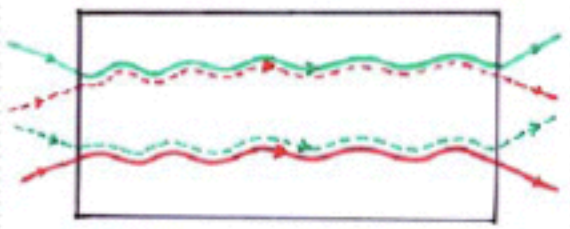
$$F = \frac{S(0,V) - S(0,0)}{e\,I} = \frac{\sum_{ab} T_{ab}(1 - T_{ab})}{\sum_{ab} T_{ab}}$$

T_{ab} is the trans … nt along the channel …

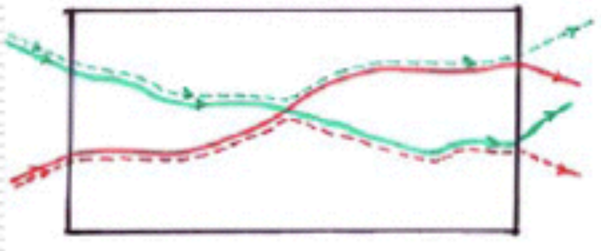
Since we know how to get numbers, what about that one ?

F takes a universal value 1/3 for weakly disordered "one-dimensional" metals

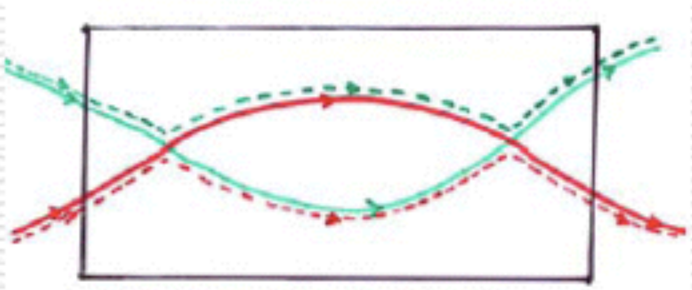
Shot noise and C_2- correlation - Is there a relation ?



C_1



C_2



C_3

$$\overline{\delta T_{ab}\delta T_{a'b'}} = \overline{T_{ab}}\ \overline{T_{a'b'}}\left(f(a,a',b,b') + \frac{2}{3g}\left[F(a,a') + F(b,b')\right] + \frac{2}{15g^2}\right)$$

Shot noise

$$\boxed{S = 2eI \times \frac{1}{3}}$$

Fano factor

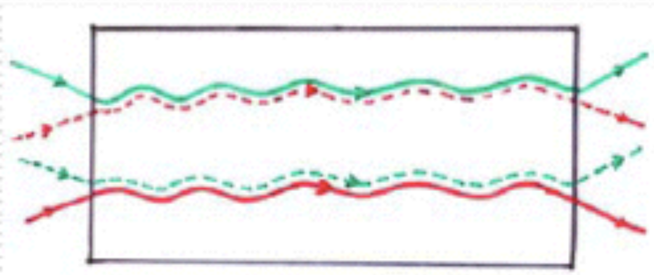
Shot noise and C_2- correlation - Is there a relation ?

$$G = \frac{e^2}{h} Tr\, tt^\dagger$$

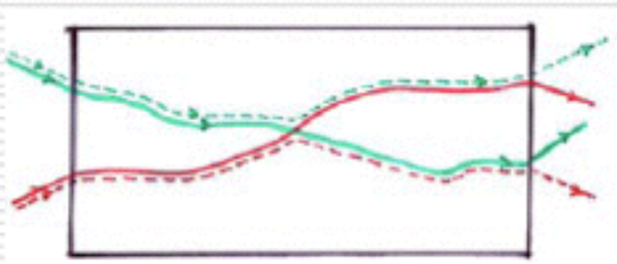
$$\left\langle \delta g^2 \right\rangle = \left\langle \left(Tr\, tt^\dagger \right)^2 \right\rangle - \left\langle Tr\, tt^\dagger \right\rangle^2 = \frac{2}{15}$$

$$\Downarrow$$

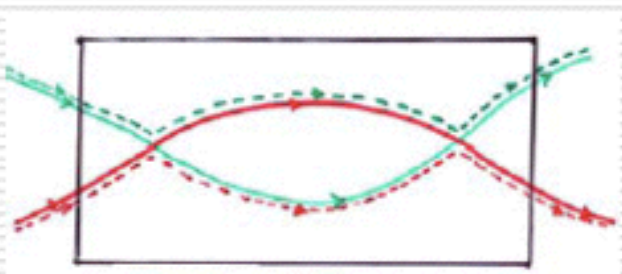
$$\left\langle \sum_{aba'b'} t_{ab} t^*_{ba}\, t_{a'b'} t^*_{b'a'} \right\rangle$$



$\Longrightarrow$ 0



$\Longrightarrow$ 0



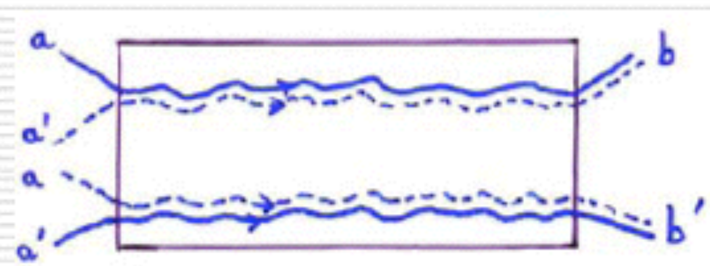
$\Longrightarrow \frac{2}{15}$

$$S = 2\frac{e^2}{h} eV\, Tr\, tt^\dagger (1 - tt^\dagger)$$

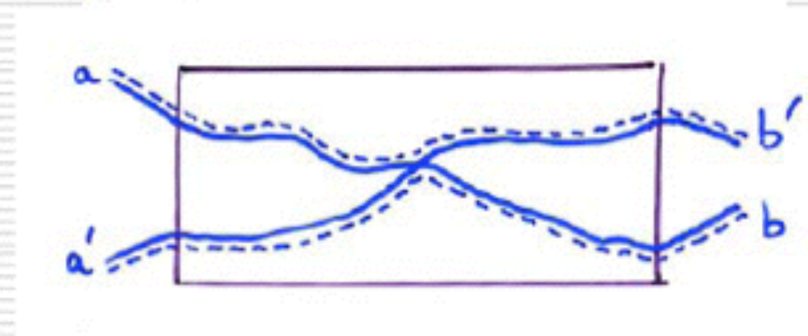
$$\left\langle Tr\, tt^\dagger tt^\dagger \right\rangle ?$$

$$\Downarrow$$

$$\left\langle \sum_{aba'b'} t_{ab} t^*_{ba'}\, t_{a'b'} t^*_{b'a} \right\rangle$$



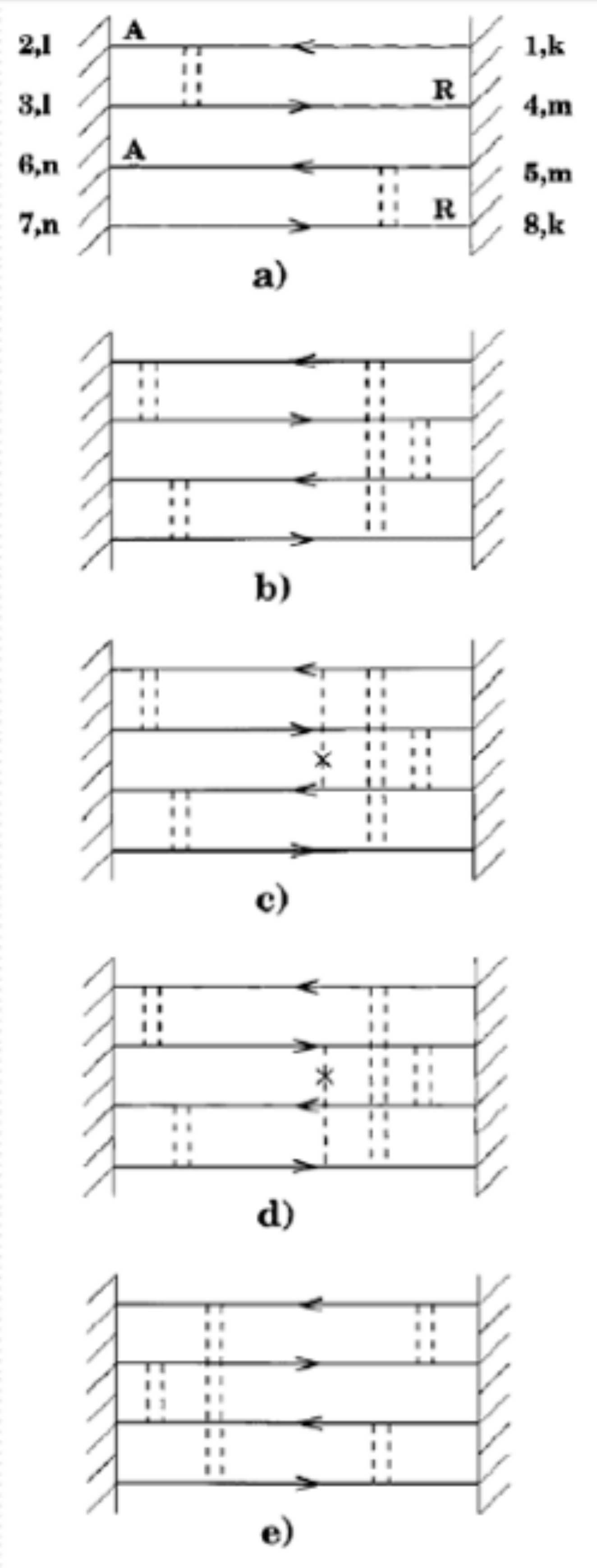
$\Longrightarrow$ 0

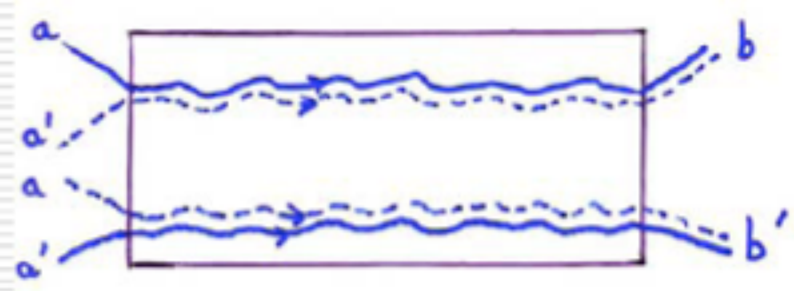


$\Longrightarrow g^2 \times \frac{2}{3g}$

$$\boxed{S = 2\frac{e^2}{h} eV\, g\left(1 - \frac{2}{3}\right) = 2eI \times \frac{1}{3}}$$

Blanter, Büttiker (97)





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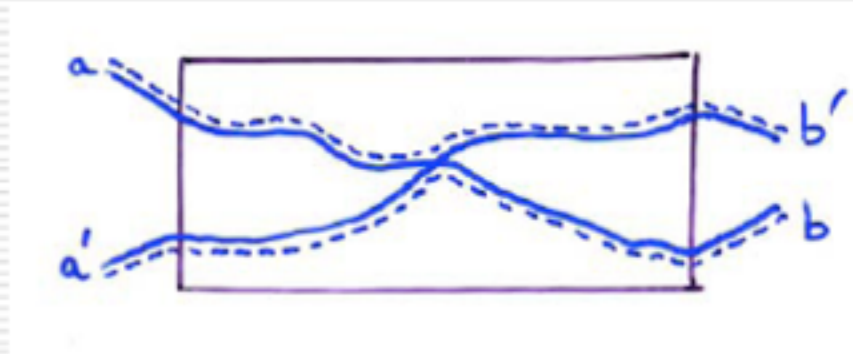
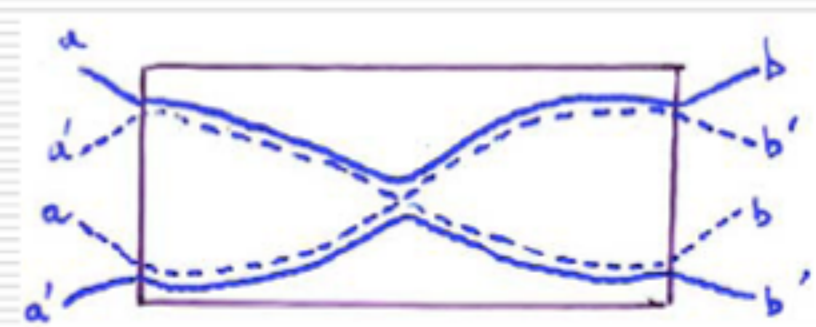


$g^2 \times \frac{2}{3g}$



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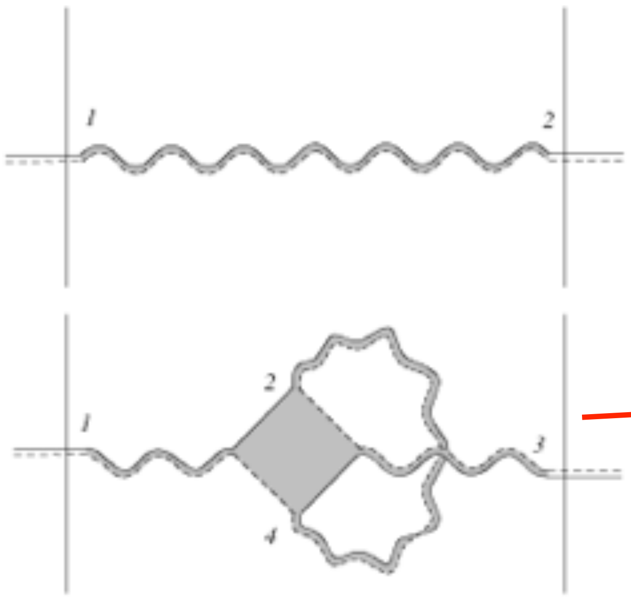
Summary ... and closed loops :

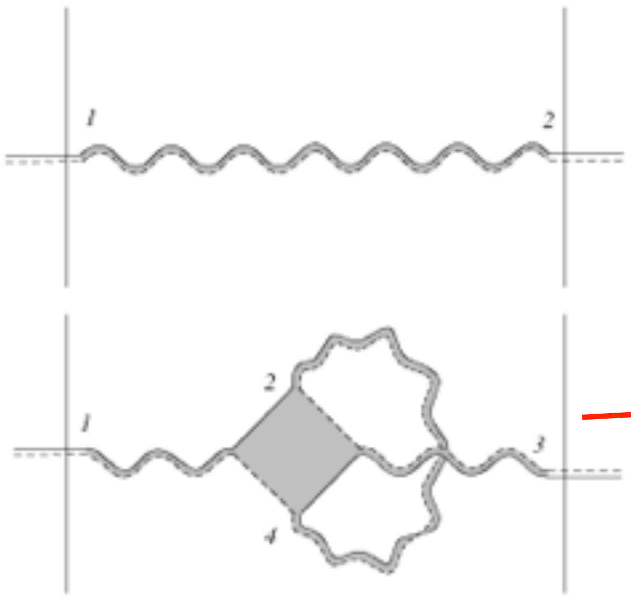


Weak localization corrections to the electrical conductance

$$\boxed{\frac{\Delta G}{G_{cl}} \propto -\frac{1}{g}\int_0^{\tau_D} Z(t)\frac{dt}{\tau_D}}$$

$$Z(t) = \int dr\, P_{cl}(r,r,t) = \left(\frac{\tau_D}{4\pi t}\right)^{d/2}$$

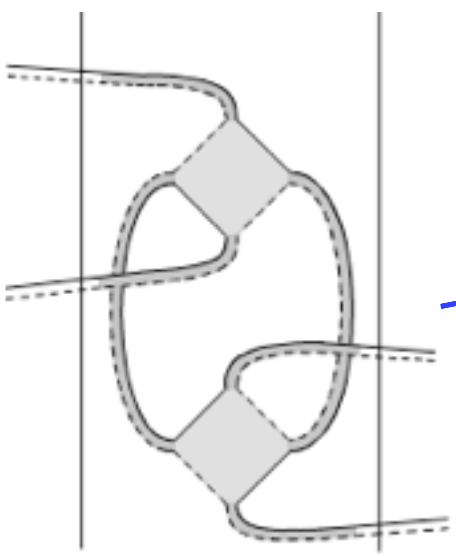
Summary ... and closed loops :



Weak localization corrections to the electrical conductance

$$\frac{\Delta G}{G_{cl}} \propto -\frac{1}{g}\int_0^{\tau_D} Z(t)\frac{dt}{\tau_D}$$

Conductance fluctuations



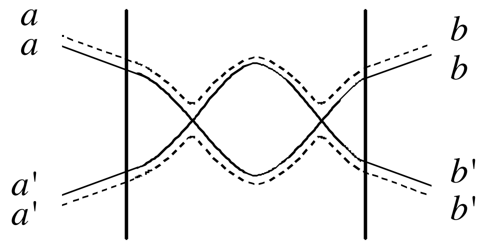
$$\frac{\overline{\delta G^2}}{G_{cl}^2} \propto \frac{1}{g^2}\int_0^{\tau_D} Z(t)\frac{t\,dt}{\tau_D^2}$$

$$Z(t)=\int dr\,P_{cl}(r,r,t)=\left(\frac{\tau_D}{4\pi t}\right)^{d/2}$$

An exercise

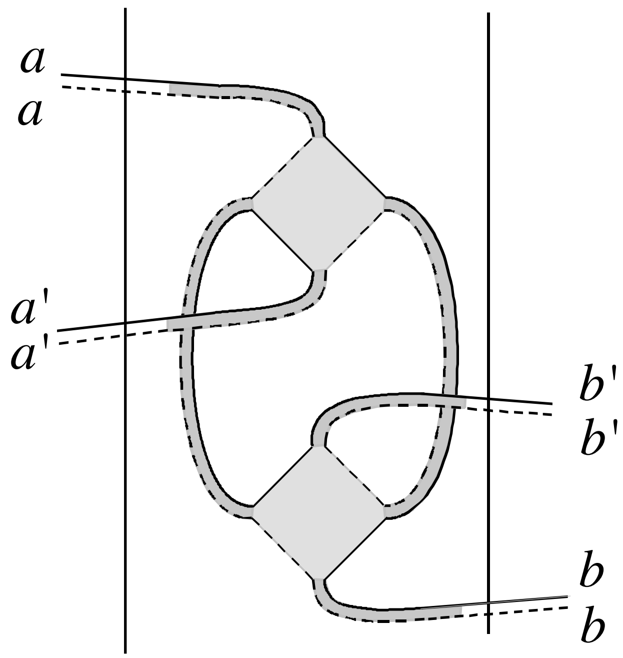
Dephasing and decoherence

Universal conductance fluctuations

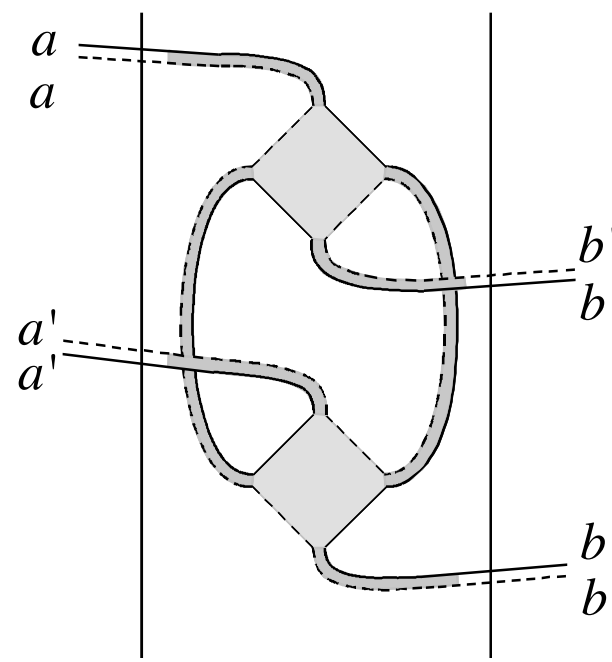


Dephasing and decoherence

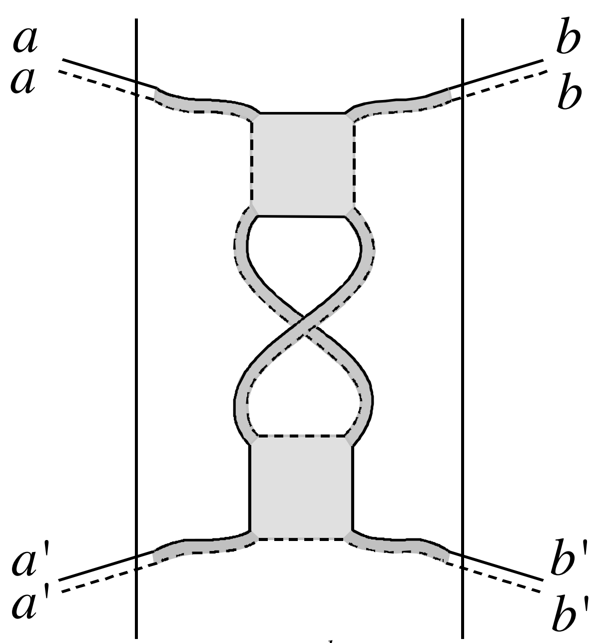
Universal conductance fluctuations



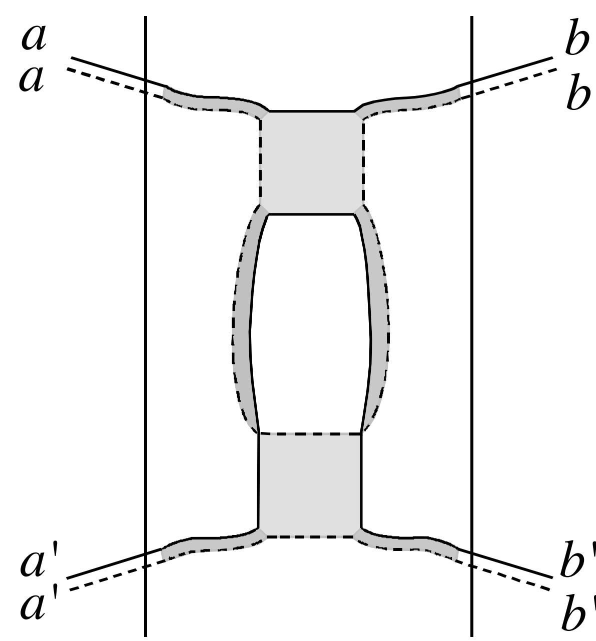
(K_1^d)



(K_2^c)



(K_3^d)



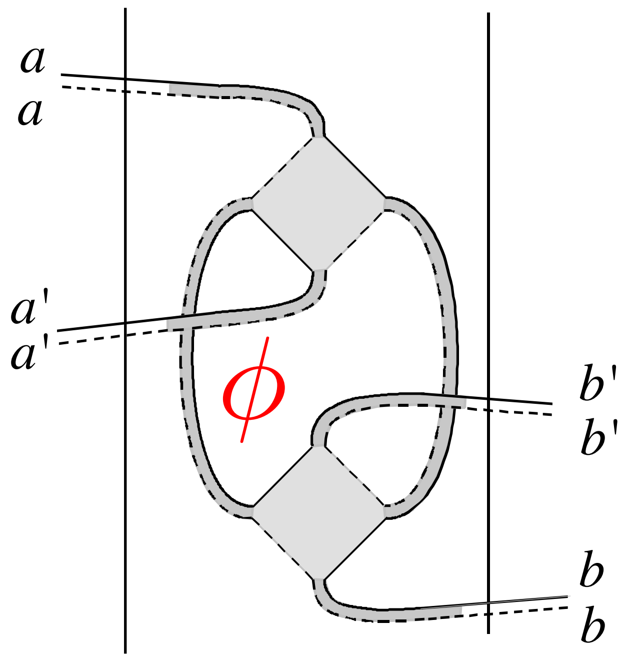
(K_3^c)

There are 4 diagrams : 2 involve diffusons and 2 cooperons.

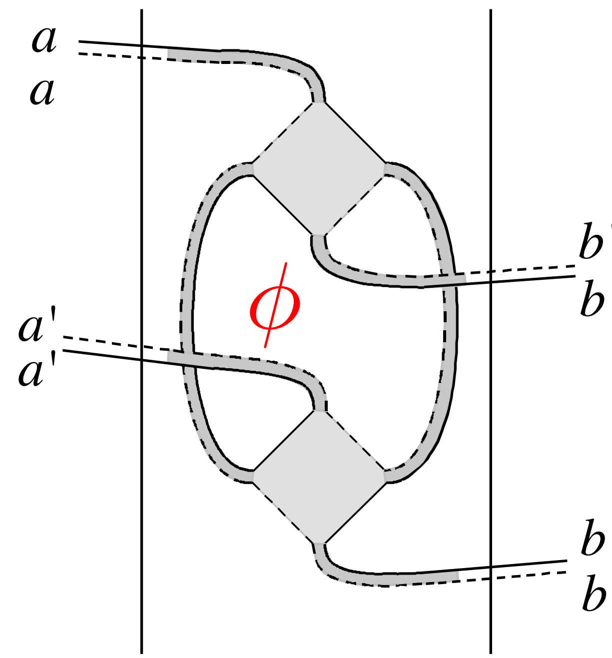
How to differentiate them ?

Dephasing and decoherence

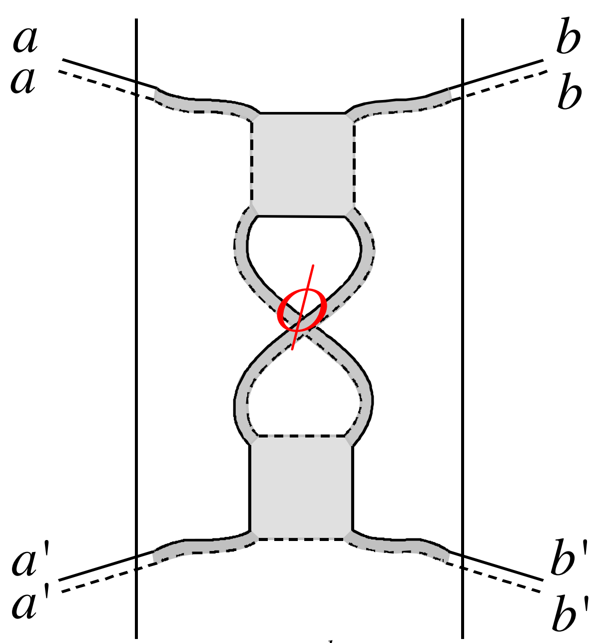
Universal conductance fluctuations



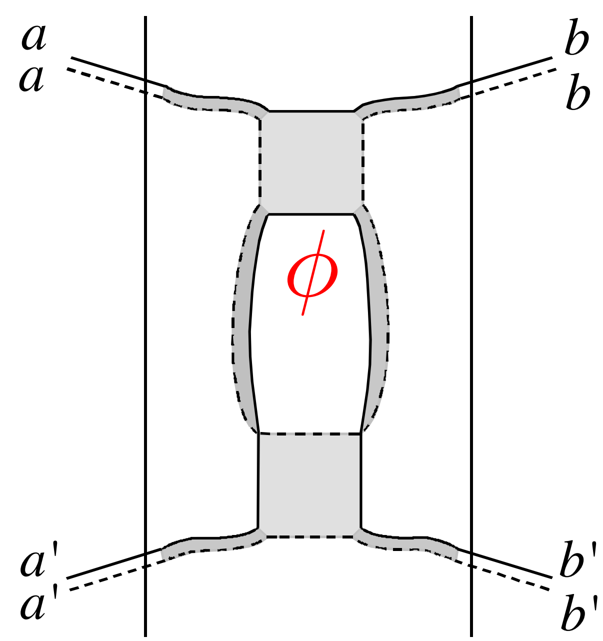
(K_1^d)



(K_2^c)



(K_3^d)



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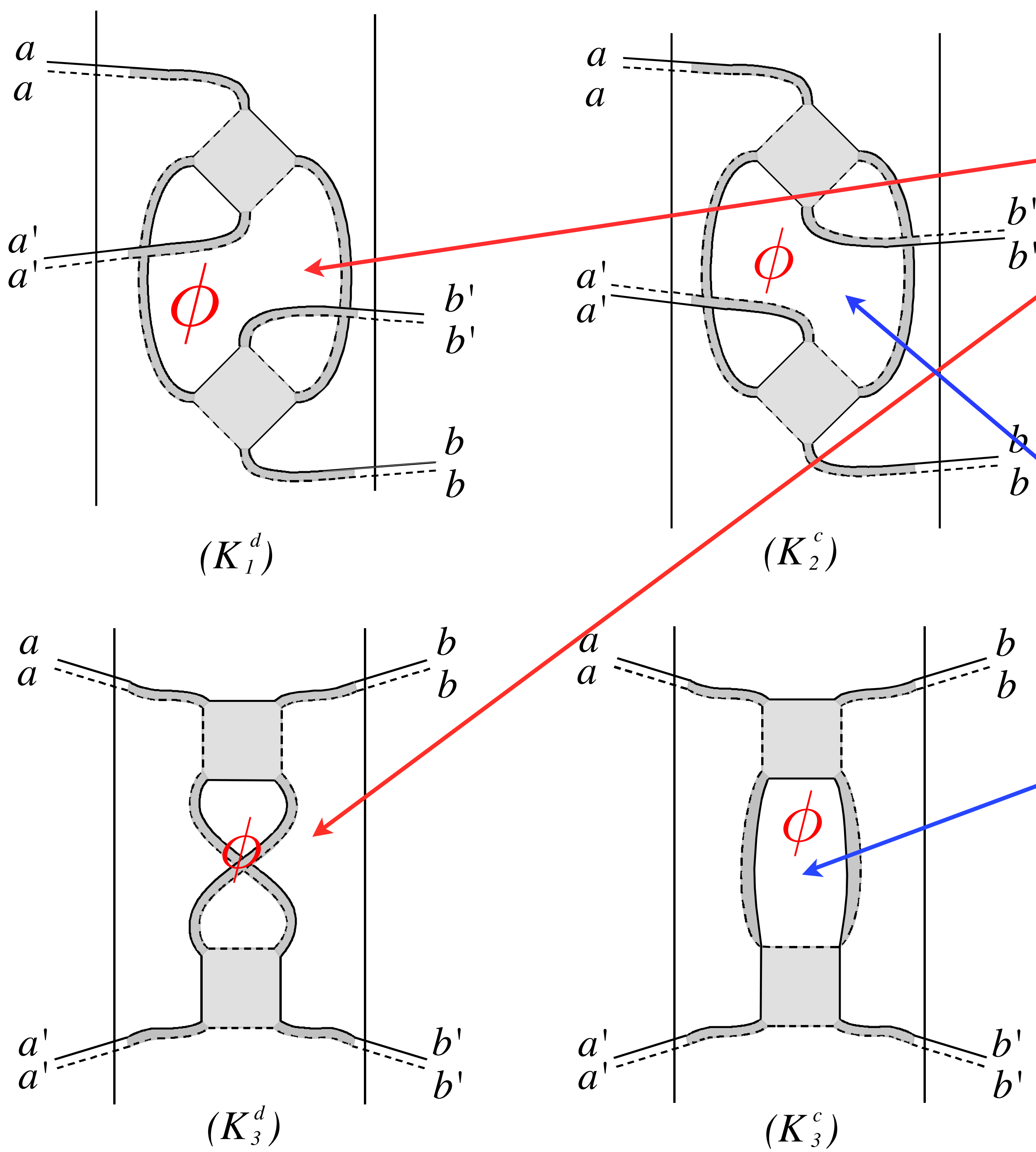


2 Diffusons

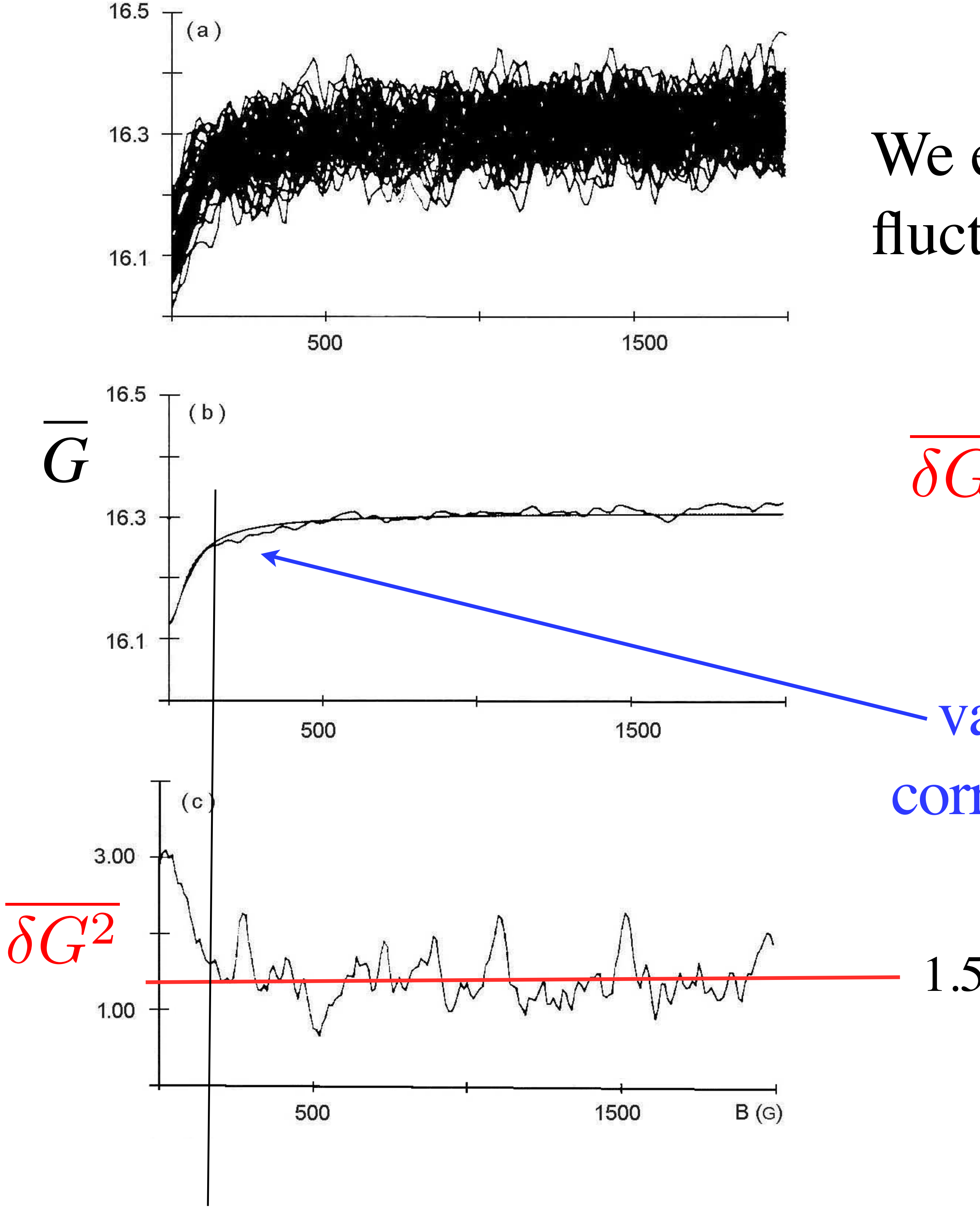
2 Cooperons

sensitive to an applied Aharonov-Bohm magnetic flux

ϕ

46 *Si*-doped *GaAs* samples at 45 mK

(Mailly-Sanquer)



We expect the conductance fluctuations to be reduced by a factor 2

$$\overline{\delta G^2} \xrightarrow{\phi} \frac{\overline{\delta G^2}}{2}$$

vanishing of the weak localization correction for the same magnetic field

In the presence of incoherent processes $L > L_\phi$:

$$\overline{\delta G^2} \rightarrow 0$$

Beyond weak disorder - a glimpse of Anderson localization phase transition

Weak disorder physics

Weak disorder limit: $\lambda << l \Rightarrow \mathrm{g} >> 1$

Probability of a crossing $\left(\propto 1/g\right)$ is small: phase coherent corrections to the classical limit are small.

Quantum crossings modify the classical probability (*i.e.* the Diffuson) but it remains normalized.

Due to its long range behavior, the Diffuson propagates (localized) coherent effects over large distances.

Quantum crossings are independently distributed :

We can generate higher order corrections to the Diffuson as an expansion in powers of $1/g$

A quantum phase transition: Anderson localization

Expansion in powers of quantum crossings $1/g$ allows to calculate quantum corrections to physical quantities.

A quantum phase transition: Anderson localization

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The diffusion coefficient D is reduced (weak localization) and becomes size dependent :

$$D(L) = D\left(1 - \frac{1}{\pi g}\ln\left(L/l\right) + \cancel{\left(\frac{1}{\pi g}\ln\left(L/l\right)\right)^2} + \ldots\right) \qquad (d = 2)$$

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A renormalization of *D(L)* changes also *g(L)*:

$$g(L) = \frac{D(L)}{c\,\lambda^{d-1}} L^{d-2}$$

Scaling and its meaning : (P.W. Anderson *et al*.,1979)

If we know $g(L)$, we know it at any scale :

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Scaling behavior :

$$\boxed{g(L,W) = f\left(L/\xi(W)\right)}$$

$\xi(W)$ is the localization length

Numerical calculations on the (universal) Anderson Hamiltonian

$d = 2$ $\qquad$ $d = 3$

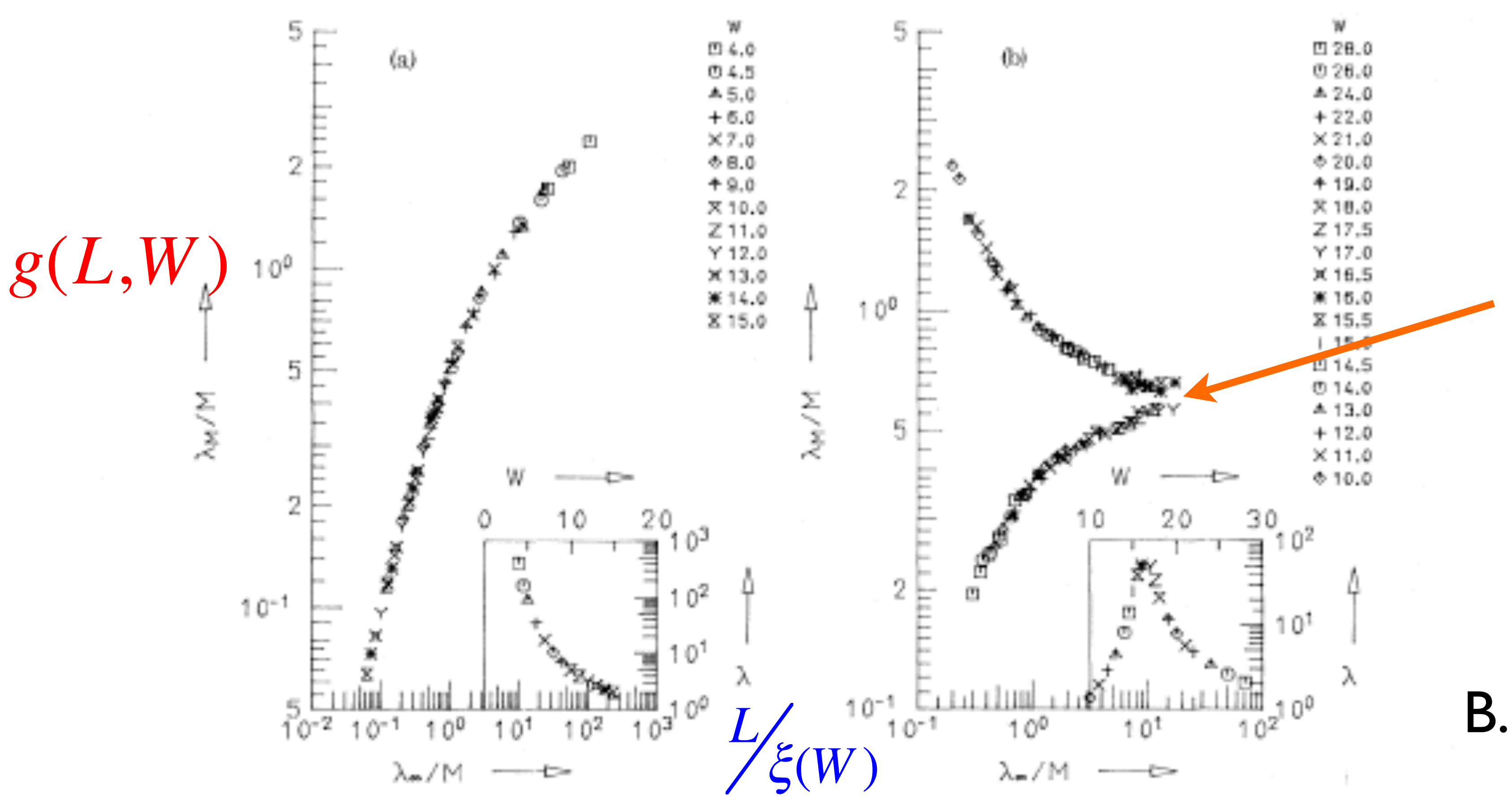

FIG. 1. Scaling function λ_M/M vs λ_∞/M for the localization length λ_M of a system of thickness M for (a) $d=2$ ($M \geq 4$) and (b) $d=3$ ($M \geq 3$). Insets show the scaling parameter λ_∞ as a function of the disorder W.

B.Kramer, A. McKinnon, 1981

Anderson localization phase transition occurs in $d > 2$

It has been observed experimentally with electromagnetic waves

(Aegarter, Maret *et al.*, 2006)

Intermezzo : heat and waves

From classical diffusion to wave propagation

There is a very important relation between classical diffusion and wave propagation on a manifold.

It expresses this profound idea that it is possible to measure and characterize a manifold using waves, more precisely with the eigenvalue spectrum of the Laplacian operator.

Can you hear the shape of a drum ?
(Kac, 1966)
Blackbody radiation (Lorentz,1910)

How does it work ?

$$\frac{\partial \phi}{\partial t} = D \frac{\partial^2 \phi}{\partial x^2}$$ Diffusion (heat) equation in d=1

whose solutions are such that $$Z(x,y,t) = \frac{e^{-\frac{(x-y)^2}{4Dt}}}{(4\pi Dt)^{1/2}}$$

Probability of diffusing from x to y in a time t.

In d space dimensions: $$Z_d(x,y,t) = \frac{e^{-\frac{(x-y)^2}{4Dt}}}{(4\pi Dt)^{d/2}}$$

We can characterize the "space geometry" by watching how the heat flows. The heat kernel Z(t) is

$$Z_d(t) = \int_{Vol.} d^d x \, Z_d(x,x,t) = \frac{Volume}{(4\pi Dt)^{d/2}}$$

Find the volume of the manifold

Boundary terms- Hearing the shape of a drum

Mark Kac (1966)

0 L

$$Dirichlet: \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1,2,\ldots$$

$$Neumann: \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 0,1,2,\ldots$$

$$Z_N(t) = \sum_{n=0}^{\infty} e^{-\left(\frac{n\pi}{L}\right)^2 t} = 1 + Z_D(t)$$

Poisson formula $\Longrightarrow$

$$Z_{\left\{\frac{D}{N}\right\}}(t) = \frac{L}{\sqrt{4\pi t}} \mp \frac{1}{2} + \ldots$$

<u>**Weyl expansion**</u> (2d)
:

$$Z(t) \sim \frac{Vol.}{4\pi t} - \frac{L}{4}\frac{1}{4\pi t} + \frac{1}{6} + \ldots$$

bulk

sensitive to boundary

integral of bound. curvature

The heat kernel is related to the density of states of the Laplacian

There are Laplace transform of each other:

$$Z(t) = \int_0^\infty d\omega\, \rho(\omega)\, e^{-t\omega}$$

From the Weyl expansion, it is thus possible to obtain the density of states.

And now the miracle !

We do <u>**not**</u> need to know the spectrum and/or the local form of the Laplacian (differential eq.) to calculate the Heat Kernel

Very convenient on a fractal since both are not available.

End of the Intermezzo !