Quantum symmetry breaking : Observation of a scale anomaly in graphene

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PAZY EXCELLING IN SCIENCE

ISRAEL SCIENCE FOUNDATION
הקרן הלאומית למדע

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Today's program

- Continuous vs. Discrete Scale Symmetry - a geometric tale.
- The $1/r^2$ - potential and Schrödinger : spectrum, universality and RG ideas.
- Dirac + Coulomb : do we know everything ? The graphene approach.
- An experimental surprise and a detour by Efimov physics.

Benefitted from discussions and collaborations with:

Technion:

Evgeni Gurevich (KLA-Tencor)
Dor Gittelman
Eli Levy (+ Rafael)
Ariane Soret (ENS Cachan)
Or Raz (HUJI, Maths)
Omrie Ovdat
Ohad Shpielberg

NRCN:
Ehoud Pazy

Elsewhere:

Gerald Dunne (UConn.)
Alexander Teplyaev (UConn.)
Jacqueline Bloch (LPN, Marcoussis)
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arXiv:1701.04121 (in press)

Continuous *vs*. discrete scale symmetry

Homogeneous string (uniform mass per unit length)

$d = 1$ —————————— $m(L)$ Expect : $m(L) \propto L$

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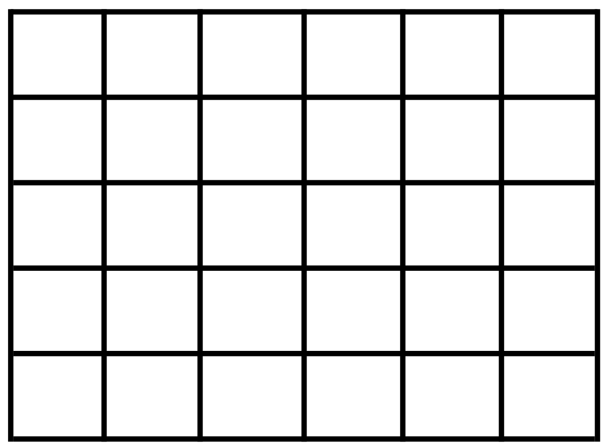
$d=2$



$m(L) \propto L^2$

spatial dimension

$$\Rightarrow \quad m(L) \propto L^d$$

Homogeneous string (uniform mass per unit length)

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How to obtain this result ?

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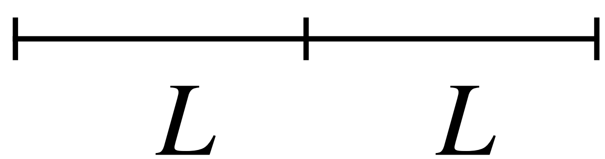

$$m(2L) = 2\ m(L)$$

or more generally, $\quad m(aL) = b\, m(L) \qquad \forall\, a \in \mathbb{R}$

Continuous scale invariance (CSI)

Scaling relation: $$f(ax) = b\,f(x)$$

If this relation is **satisfied for all** a and $b(a)$, the system has a continuous scale invariance (CSI).

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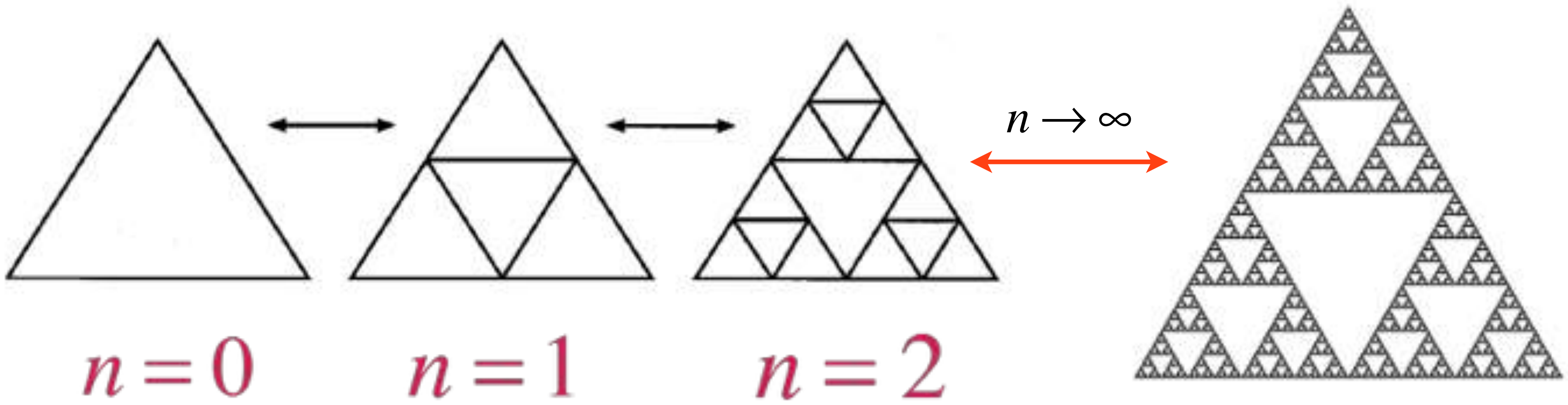
Discrete scale invariance (DSI)

discrete scale invariance is a weaker version of scale invariance, *i.e.*,

$$f(ax) = b\,f(x), \quad \textit{with fixed}\ (a,b)$$

Iterative lattice structures (fractals)



Sierpinski gasket

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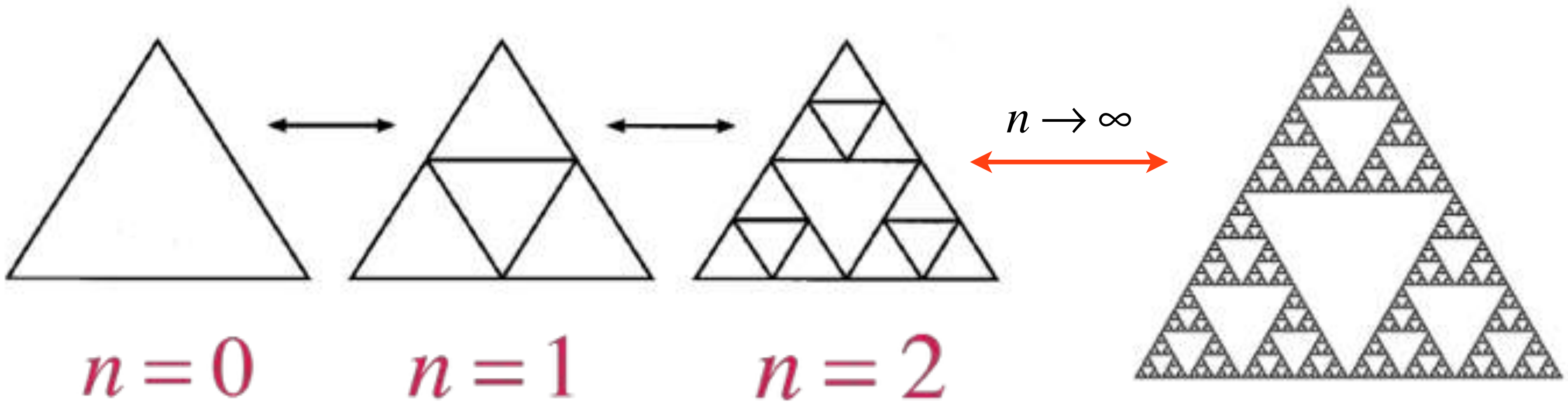

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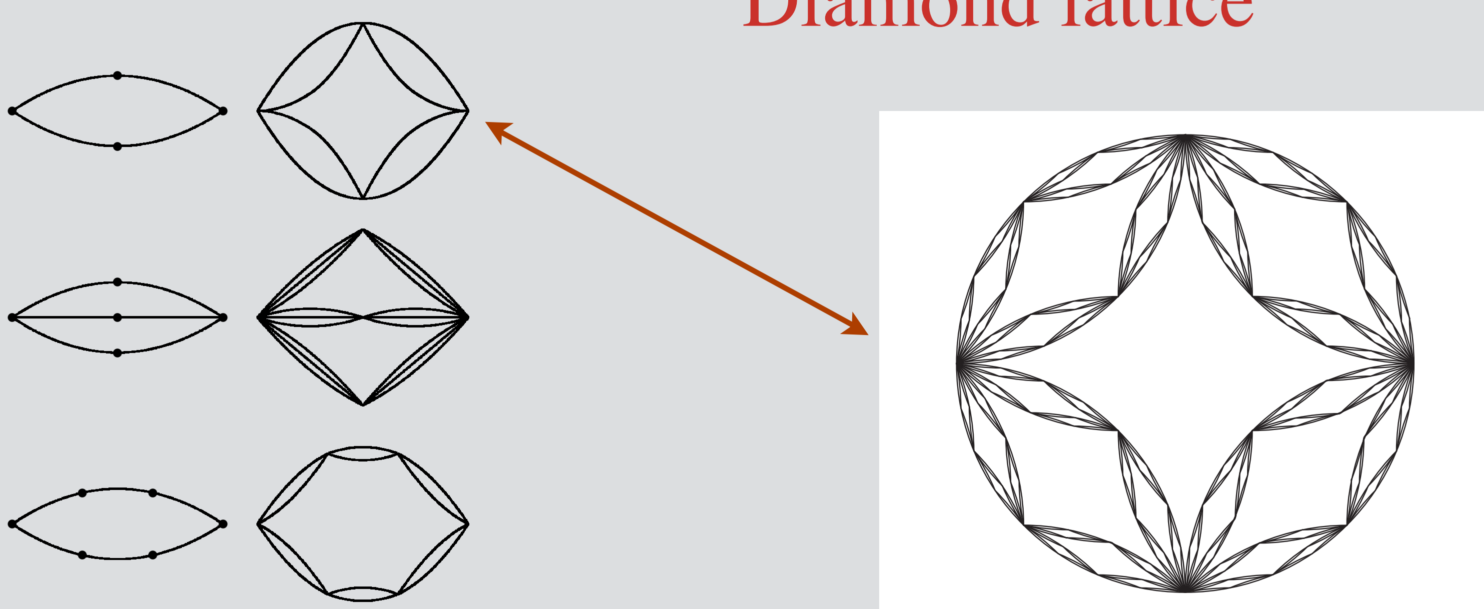
Diamond lattice



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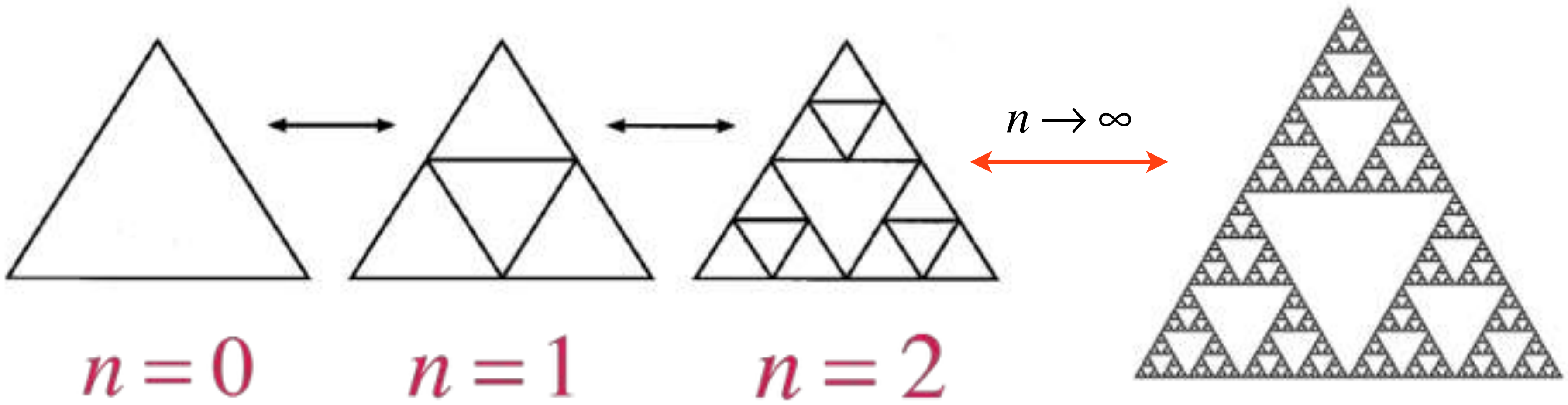

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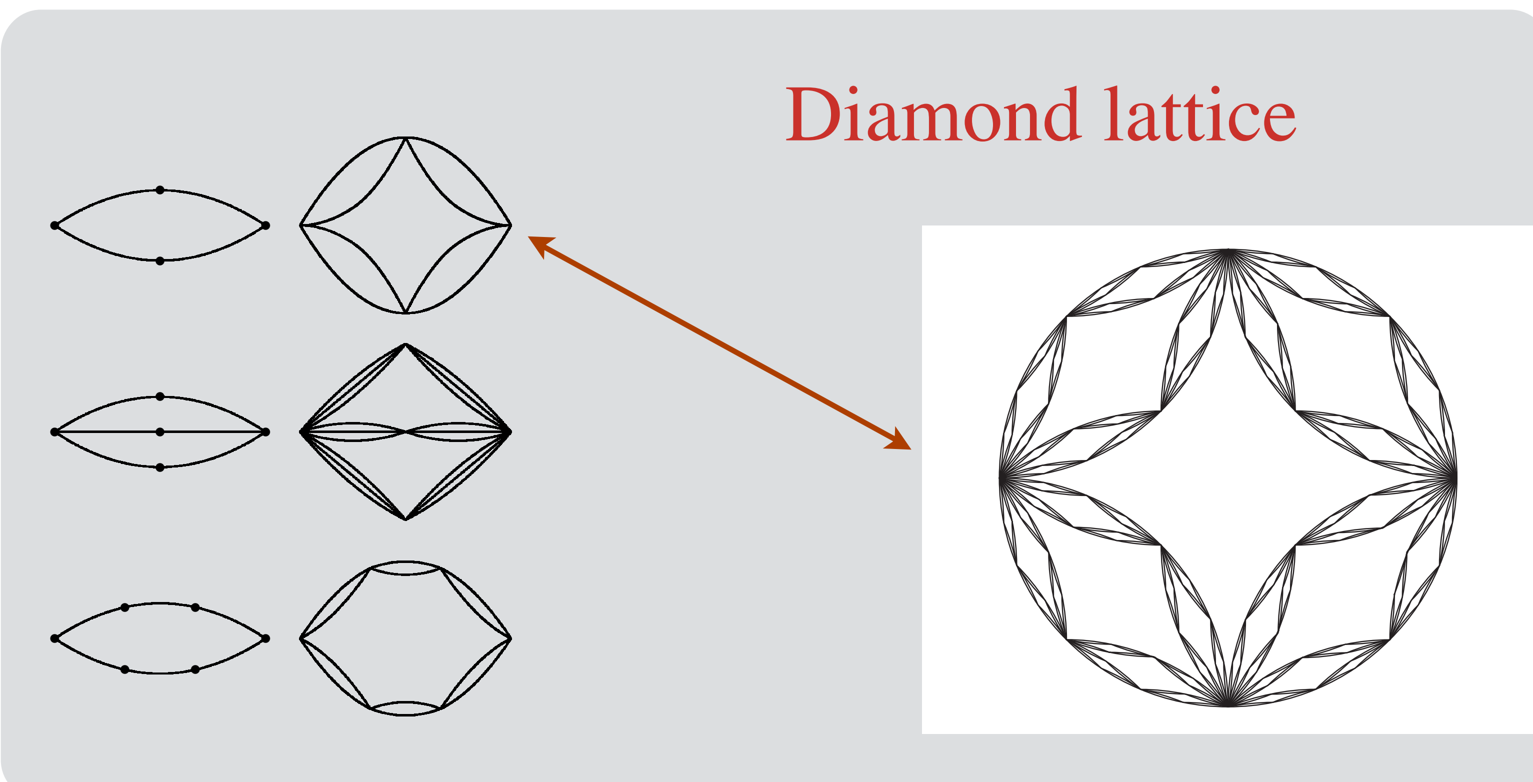
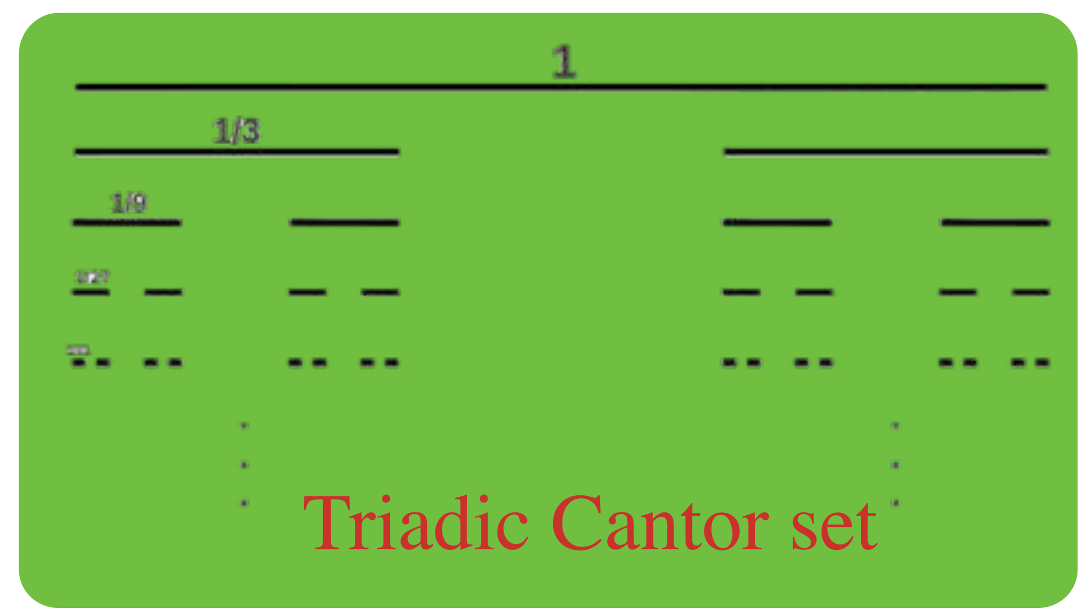
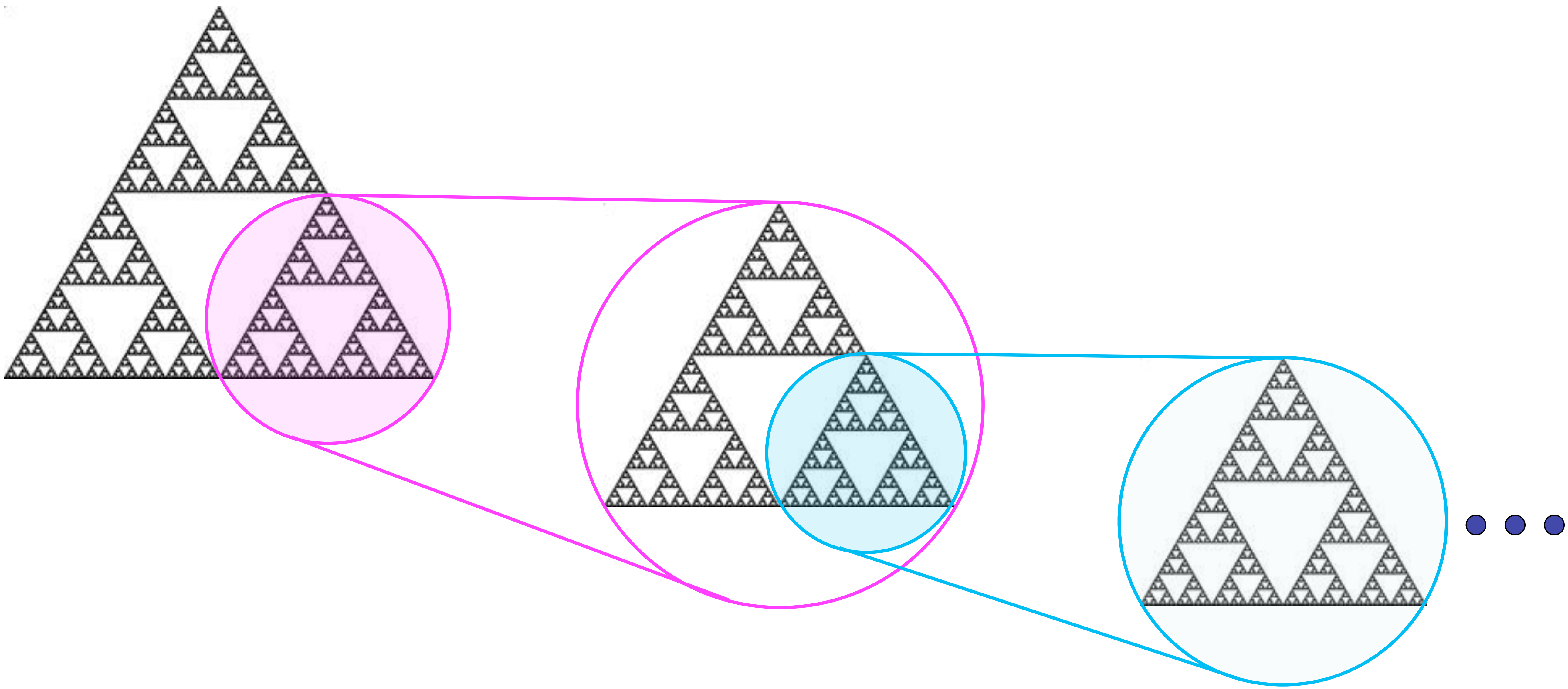




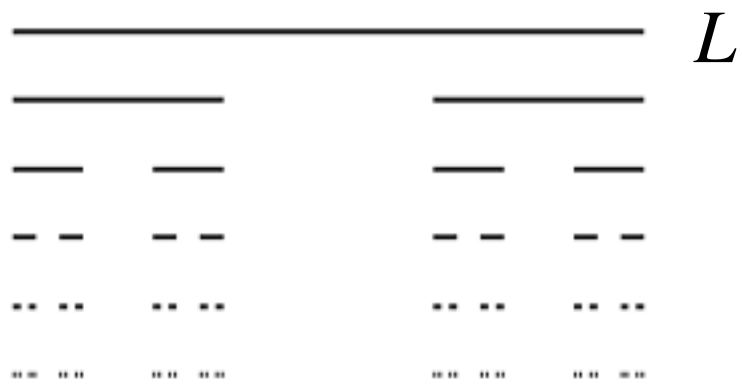
Fractals are self-similar objects

Fractal ↔ Self-similar



Discrete scaling symmetry

The Cantor set



$$M_n = 2^n M$$

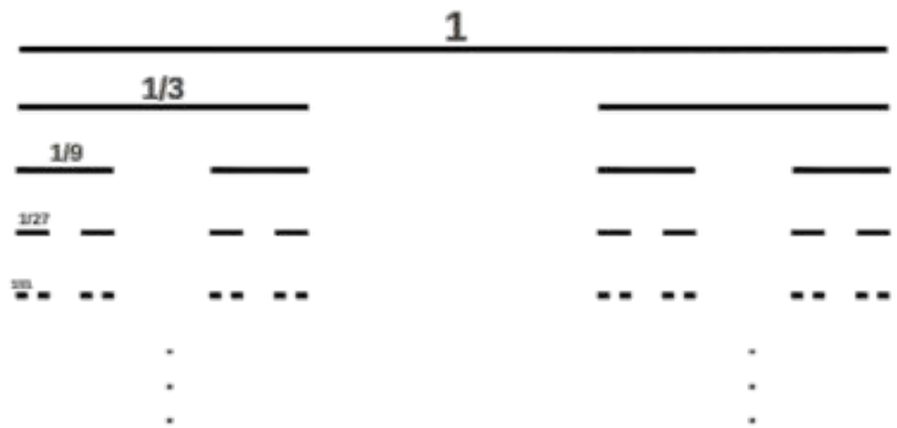
$$L_n = 3^n L$$

$$\frac{\ln M_n}{\ln L_n} \xrightarrow[n\to\infty]{} d_h = \frac{\ln 2}{\ln 3}$$

Alternatively, define the mass density $m(L)$ of the Cantor set

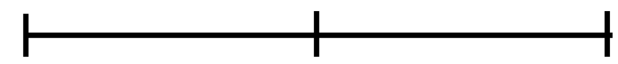
$$2\,m(L) = m(3L)$$

Relation between the different cases :



$$m(3L) = 2\,m(L) \qquad (a,b) = (3,2)$$

Cantor set

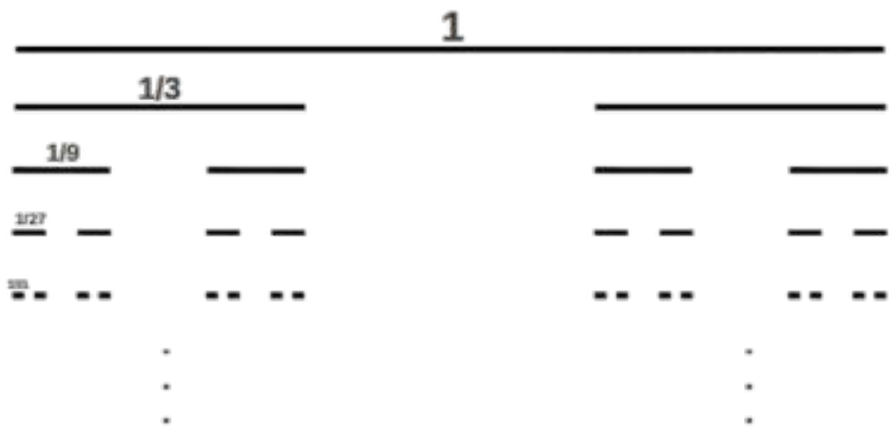


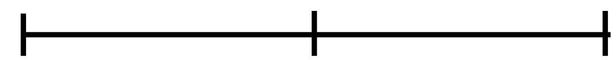
$$d = 1$$

$$m(2L) = 2\; m(L) \qquad \forall\, b(a) \in \mathbb{R}$$

Euclidean lattice

Relation between the two cases : discrete *vs*. continuous





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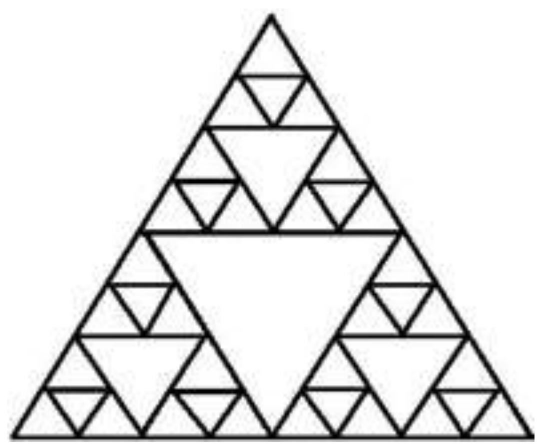

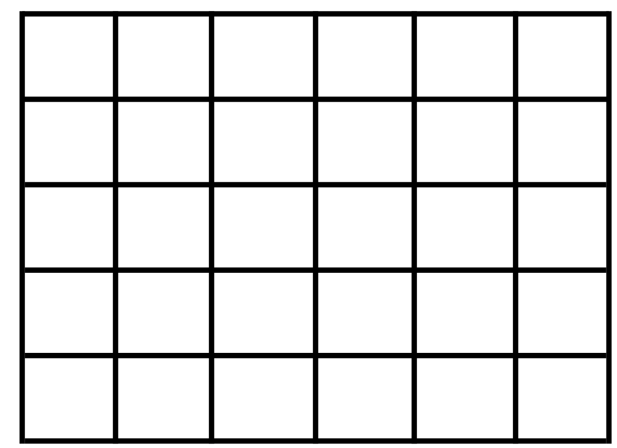

$d = 2$

$m(2L) = 3m(L)$ $\quad (a,b) = (2,3)$

Both satisfy $f(a\,x) = b\,f(x)$ but with fixed (a,b) for the fractals.

Continuous *vs*. discrete scale invariance (CSI *vs*. DSI)

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General solution :

$$f(x) = C\,x^{\alpha}$$

with $\alpha = \frac{\ln b}{\ln a}$

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General solution (by direct inspection)

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General solution is

$$f(x) = x^{\alpha}\, G\left(\frac{\ln x}{\ln a}\right)$$

Break CSI into DSI ?

Claim : breaking of CSI into DSI occurs at the quantum level : quantum phase transition (scale anomaly)

Part 2

A simple example of continuous scale invariance in quantum physics

An illustration of continuous scale invariance in (simple) quantum mechanics

Schr$\ddot{o}$dinger equation for a particle of mass μ in d-dimensions in an attractive potential :

$$V(r) = -\frac{\xi}{r^2}$$

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Schrödinger equation for a particle of mass μ in d-dimensions in an attractive potential :

$$V(r) = -\frac{\xi}{r^2}$$

$$\hat{H} = -\frac{\hbar^2}{2\mu}\Delta - \frac{\xi}{r^2}$$

Redefining $k^2 = -2\mu E$

$$\zeta = 2\mu\xi - l(l+d-2)$$

$$\psi''(r) + \frac{d-1}{r}\psi'(r) + \frac{\zeta}{r^2}\psi(r) = k^2\psi(r)$$

l: orbital angular momentum

$$\psi''(r)+\frac{d-1}{r}\psi'(r)+\frac{\zeta}{r^2}\psi(r)=k^2\psi(r)$$

The only parameter ζ in the problem is dimensionless : no characteristic length (energy) scale, $e.g.$ Bohr radius $a_0 = \hbar^2 / \mu e^2$ for the Coulomb potential.

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Consequence: Schrödinger eq. displays continuous scale invariance :
it is invariant under the transformation:

$$\begin{cases} r \to \lambda r \\ k \to \frac{1}{\lambda} k \end{cases} \qquad \forall \lambda \in \mathbb{R}$$

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The existence of one bound state implies those of a continuum of related bound states. No ground state. Problem !

It is a problem, but a well known (textbook) one.

It results essentially from :

- the ill-defined behaviour of the potential $V(r) = -\frac{\xi}{r^2}$ for $r \rightarrow 0$

- the absence of characteristic length/energy.

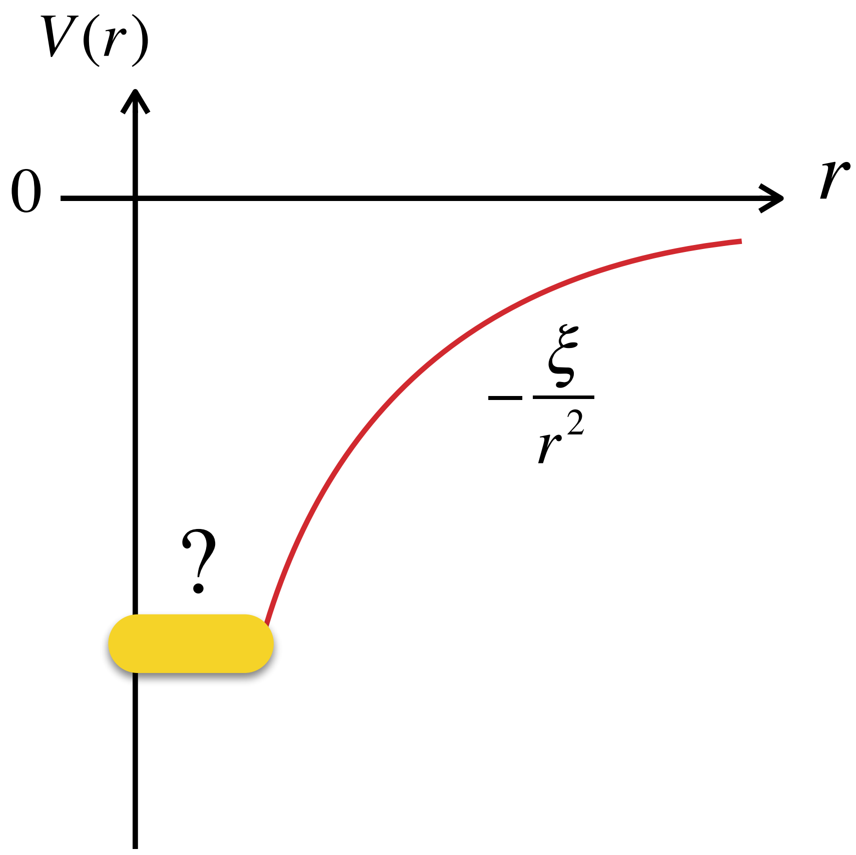
Technically : non hermitian (self-adjoint) Hamiltonian.
To cure it : need to properly define boundary conditions (somewhere)

Outline of the main results

$$\hat{H} = -\frac{\hbar^2}{2\mu}\Delta - \frac{\xi}{r^2}$$ is scale invariant (CSI) : $$r \rightarrow \lambda r \Rightarrow \hat{H} \rightarrow \frac{1}{\lambda^2}\hat{H}$$

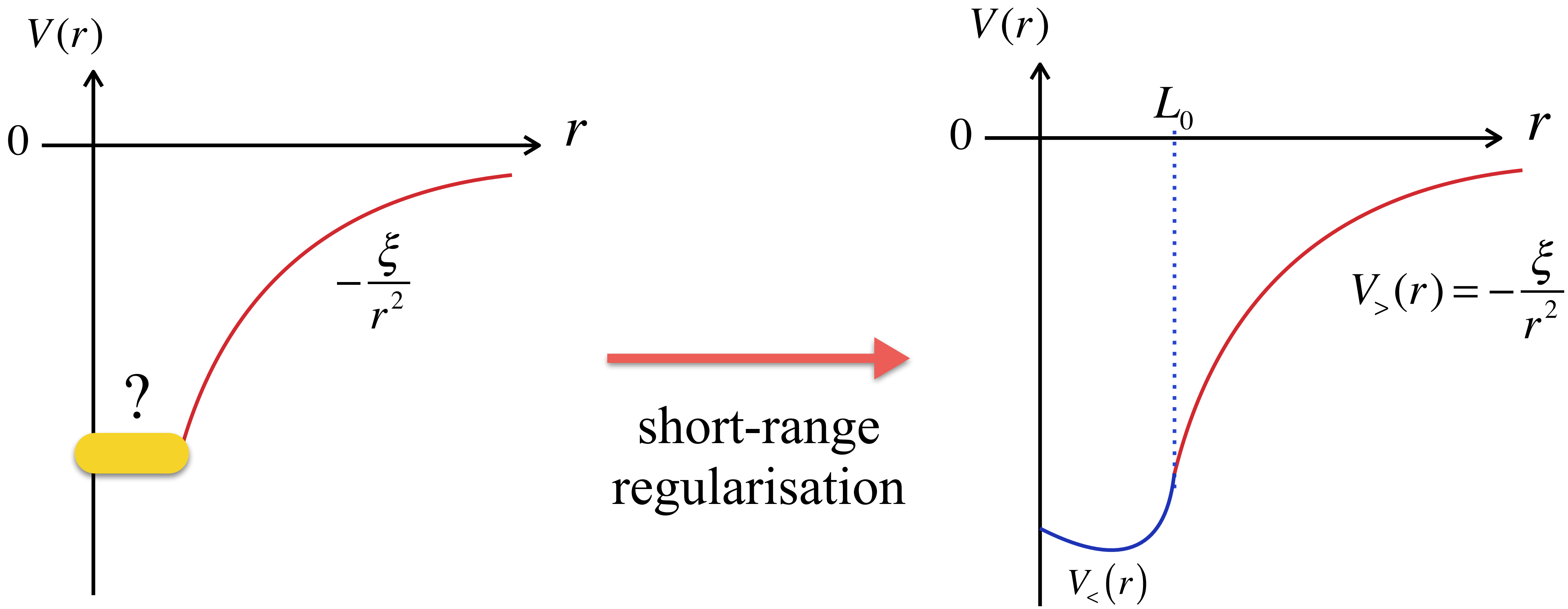
Any type of boundary conditions needed to find a well defined hermitian Hamiltonian break CSI.

Outline of the main results



No characteristic scale

Outline of the main results



No characteristic scale

Some potential $V_<(r)$: accounts for “real” short-range physics.

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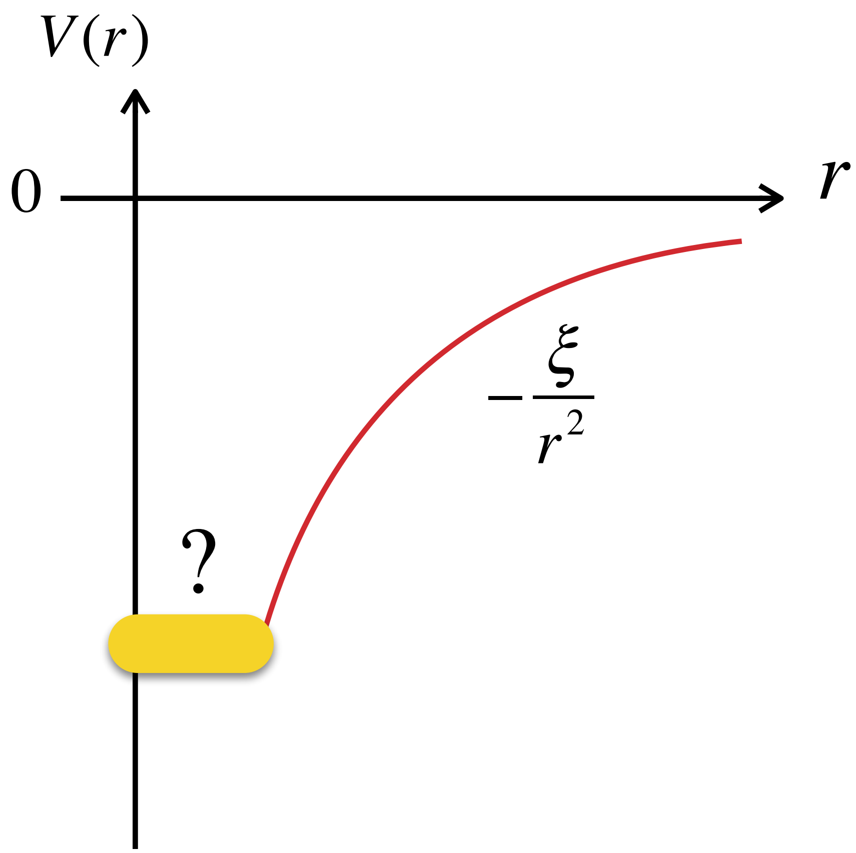
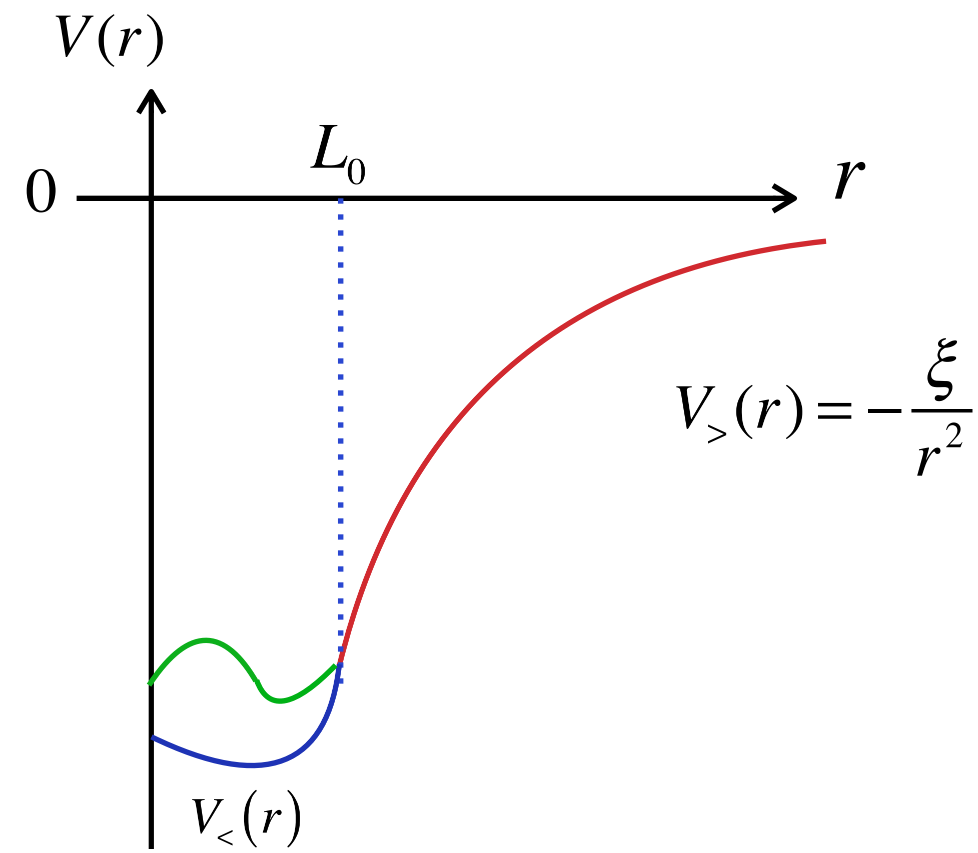


short-range regularisation



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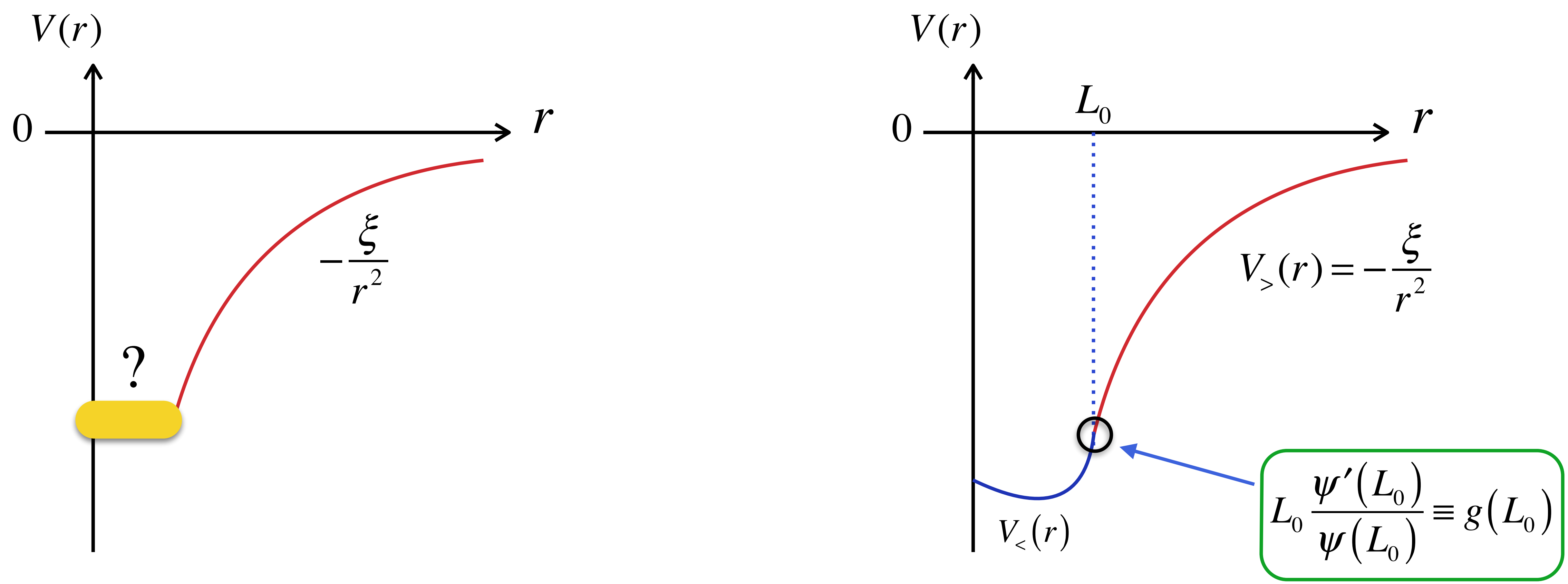
Exact expression is not important .

Outline of the main results



Problem becomes well-defined :

- characteristic length L_0
- continuity of ψ and ψ' at L_0 (boundary condition)

$\Longrightarrow$ energy spectrum

How the energy spectrum looks like ?

At low enough energies $\left(E \simeq 0\right)$,the spectrum has a "universal" behaviour.

- It depends on the parameter $\zeta = 2\mu\xi - l\left(l+d-2\right)$
- It exists a singular value $\zeta_{cr} = \frac{\left(d-2\right)^2}{4}$

Universal part of the energy spectrum

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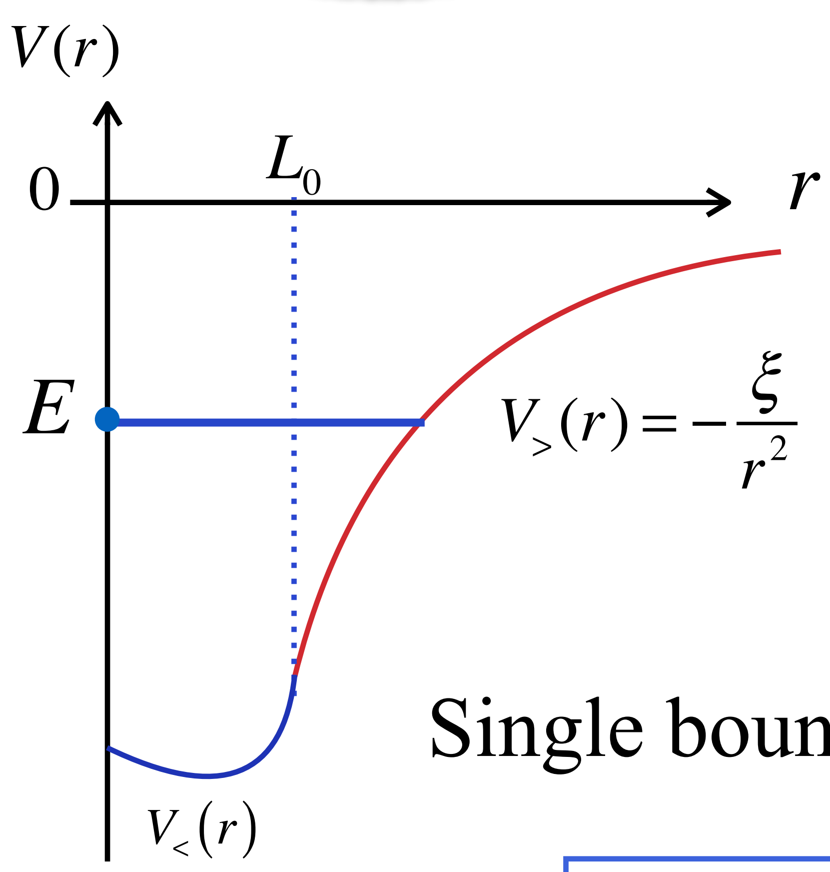

Single bound state

$$E = -\frac{1}{L_0^2} f(g)$$

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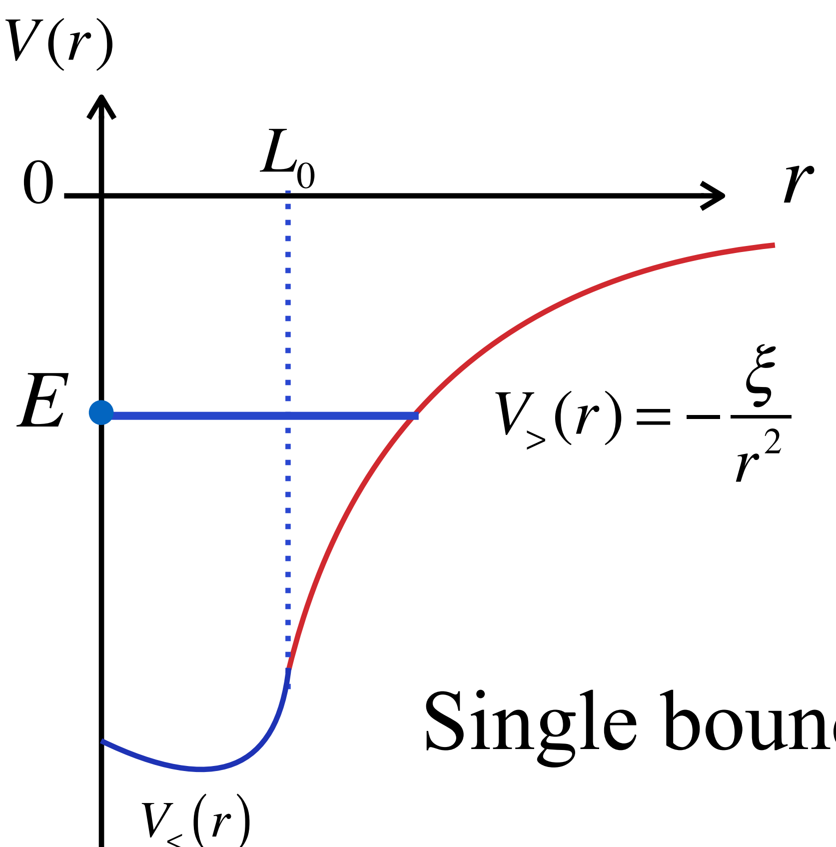

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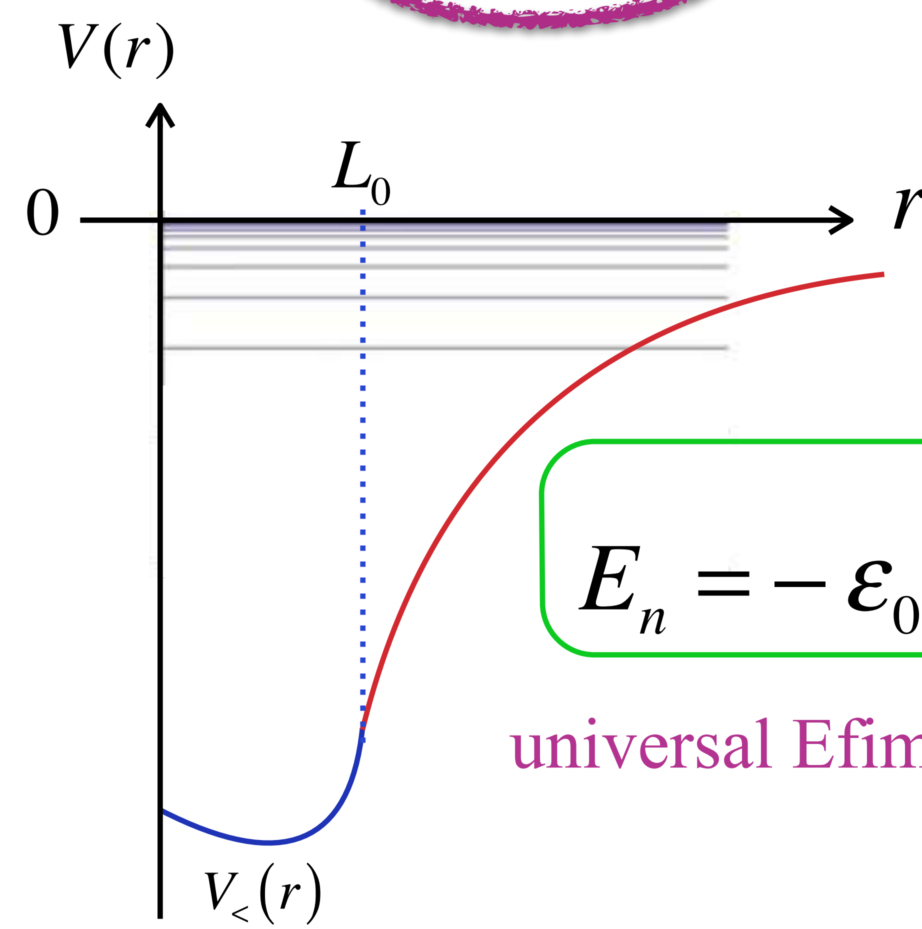

$$E_n = -\varepsilon_0 \, e^{-\frac{\pi n}{\sqrt{\zeta - \zeta_{cr}}}}$$

universal Efimov spectrum

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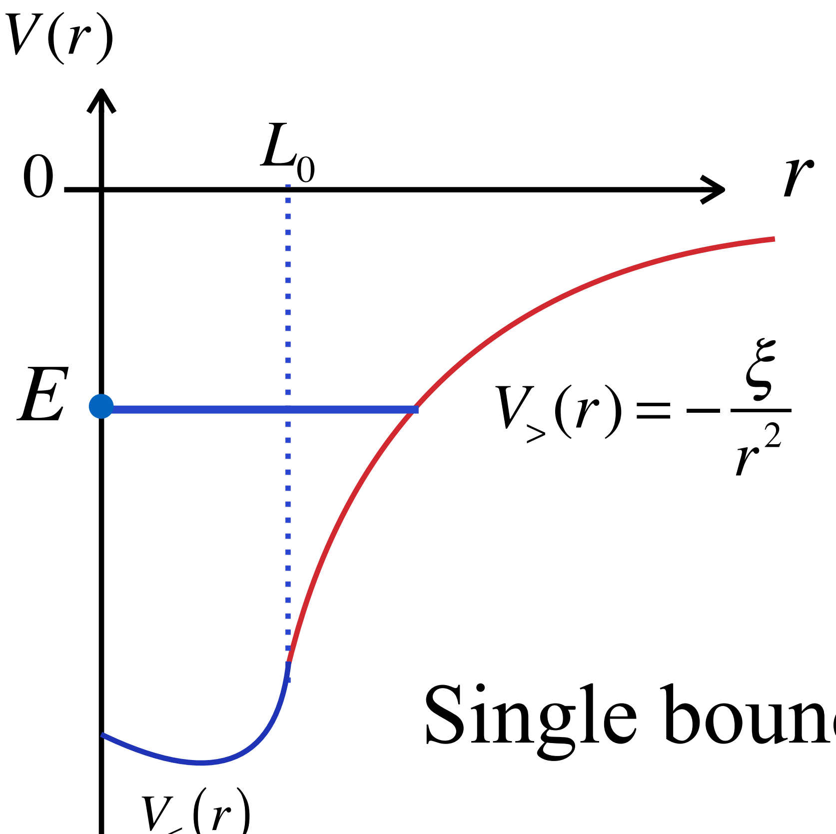

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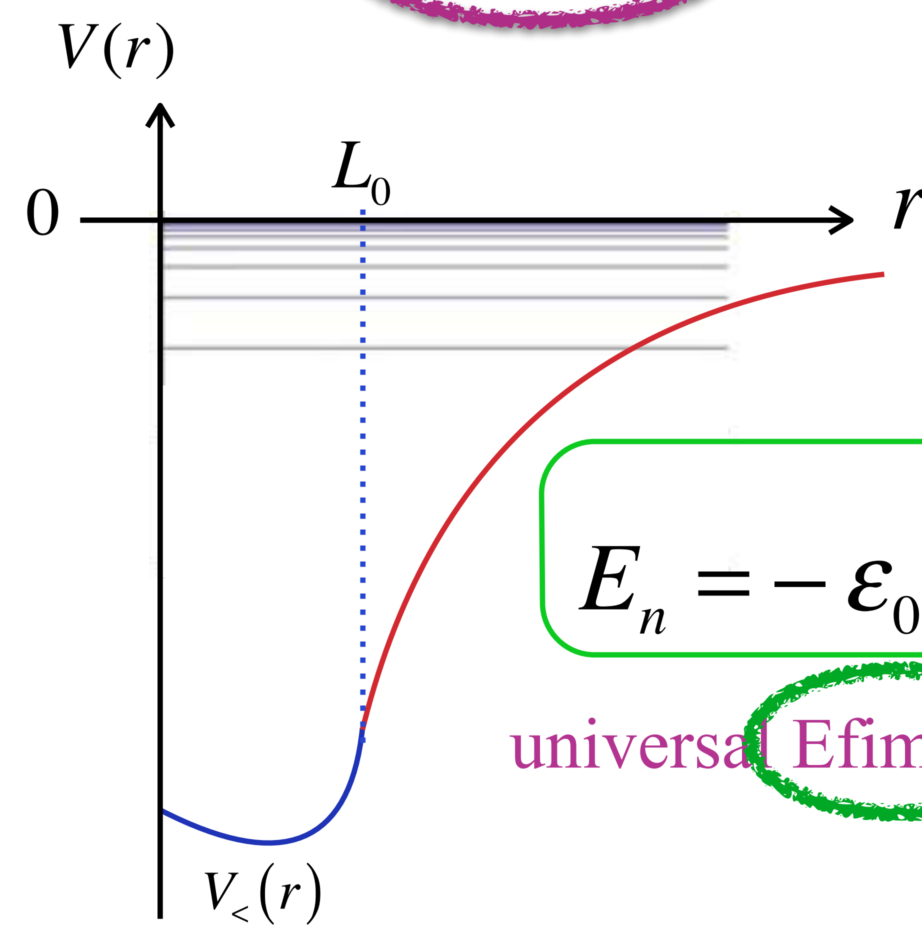

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Just a name for the moment

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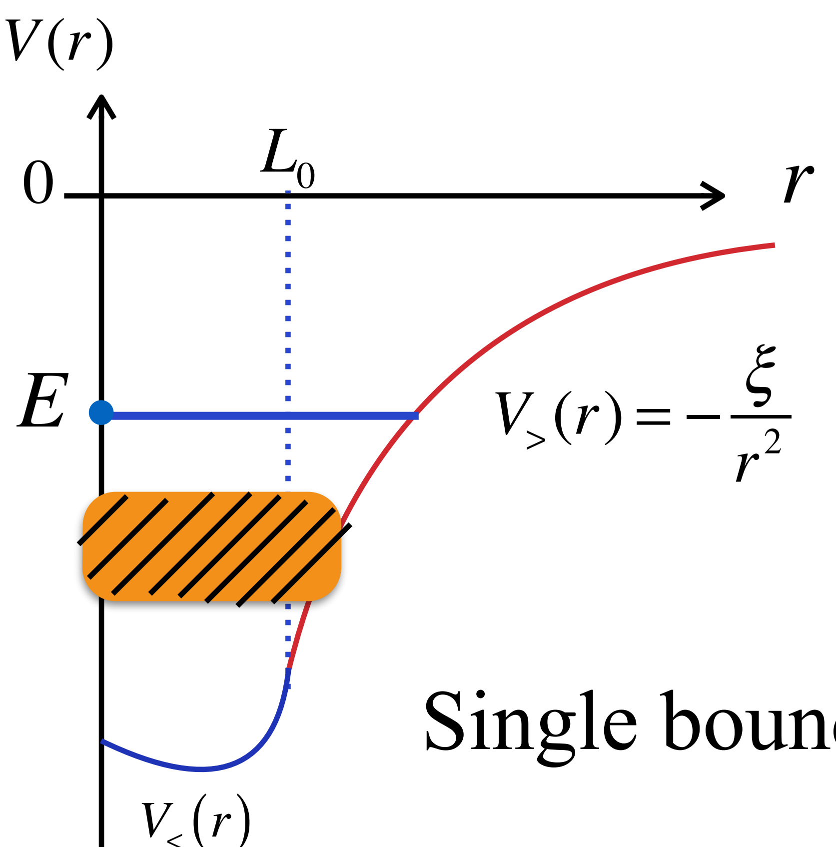

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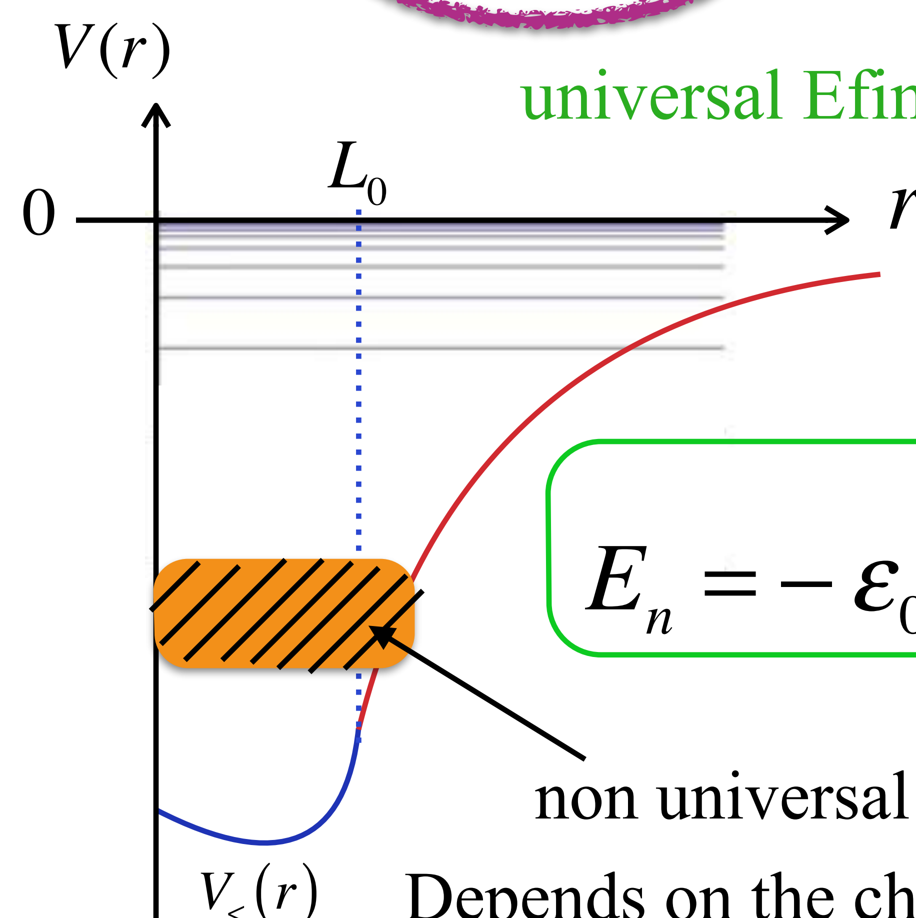

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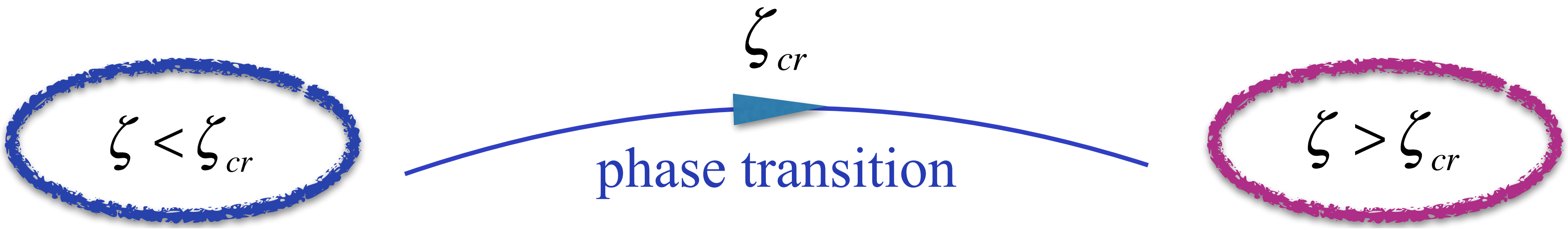
non universal part

Depends on the choice of $V_<(r)$

A quantum phase transition

It exists a singular value $$\zeta_{cr} = \frac{(d-2)^2}{4}$$

Take the limit $L_0 \to \infty$ with $E L_0^2$ fixed



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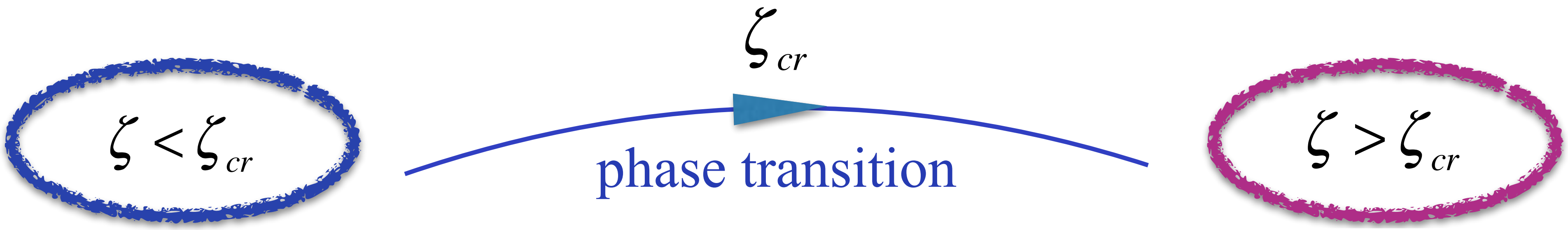

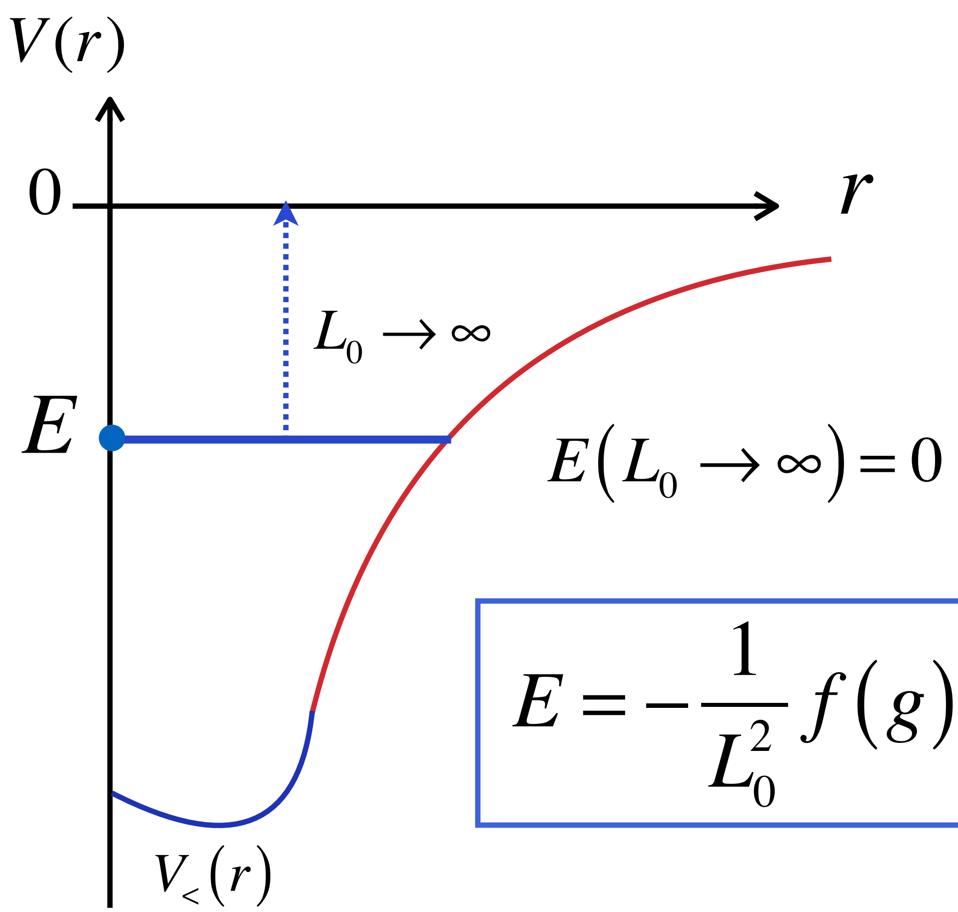

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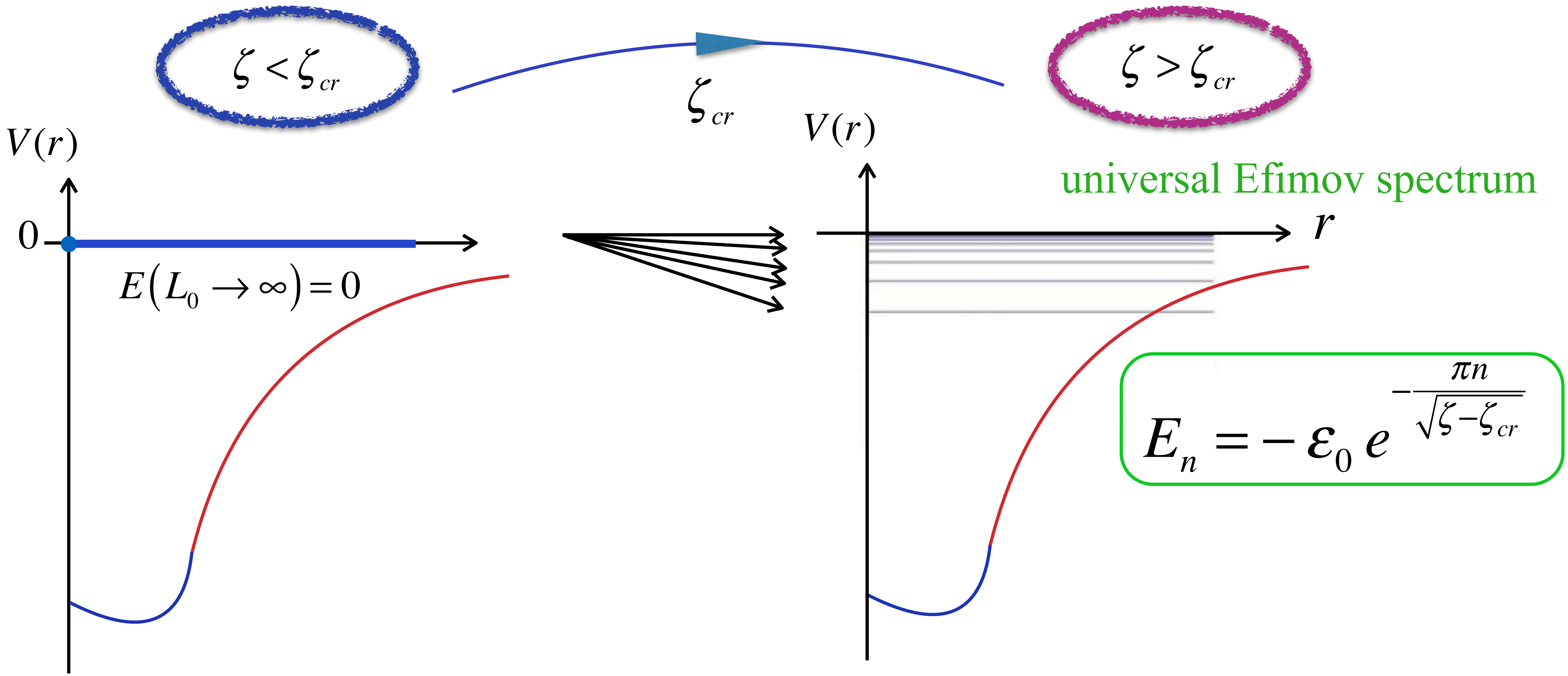

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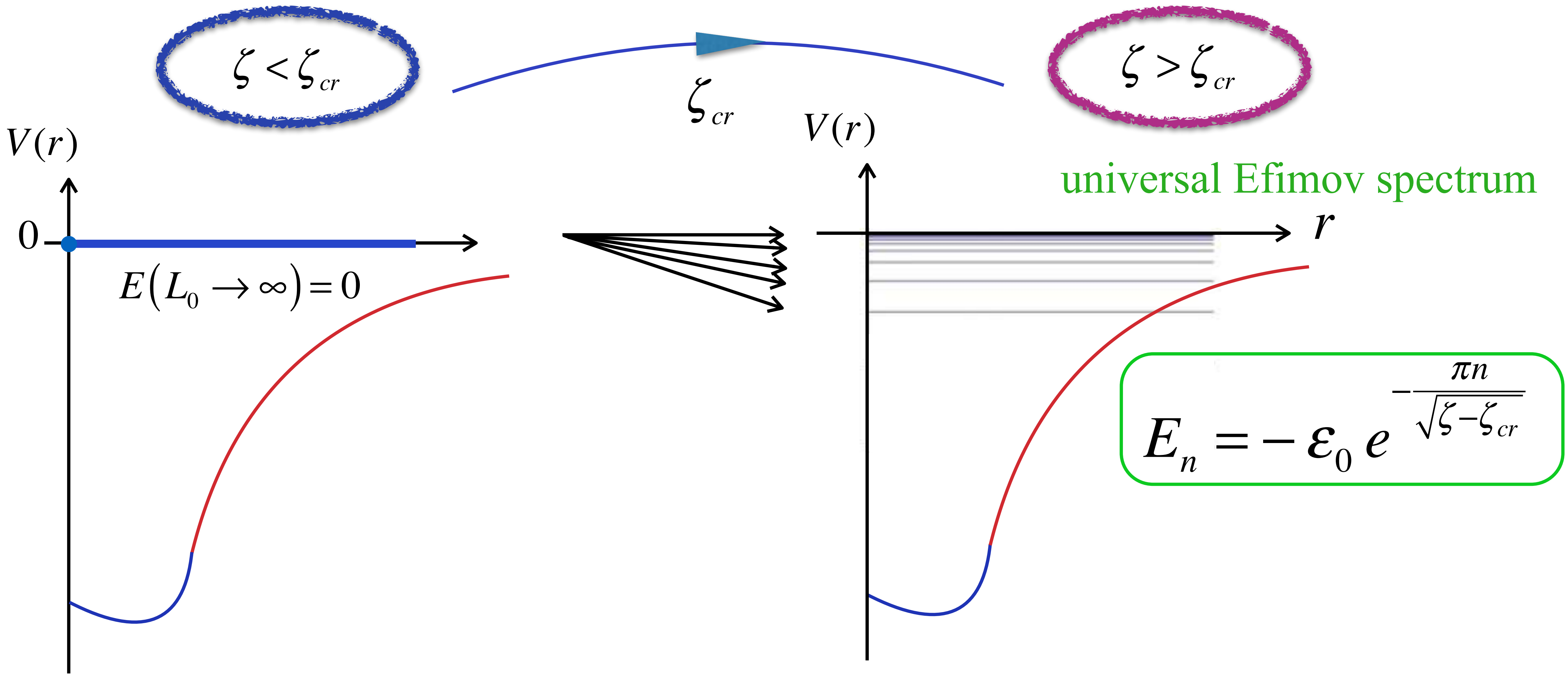

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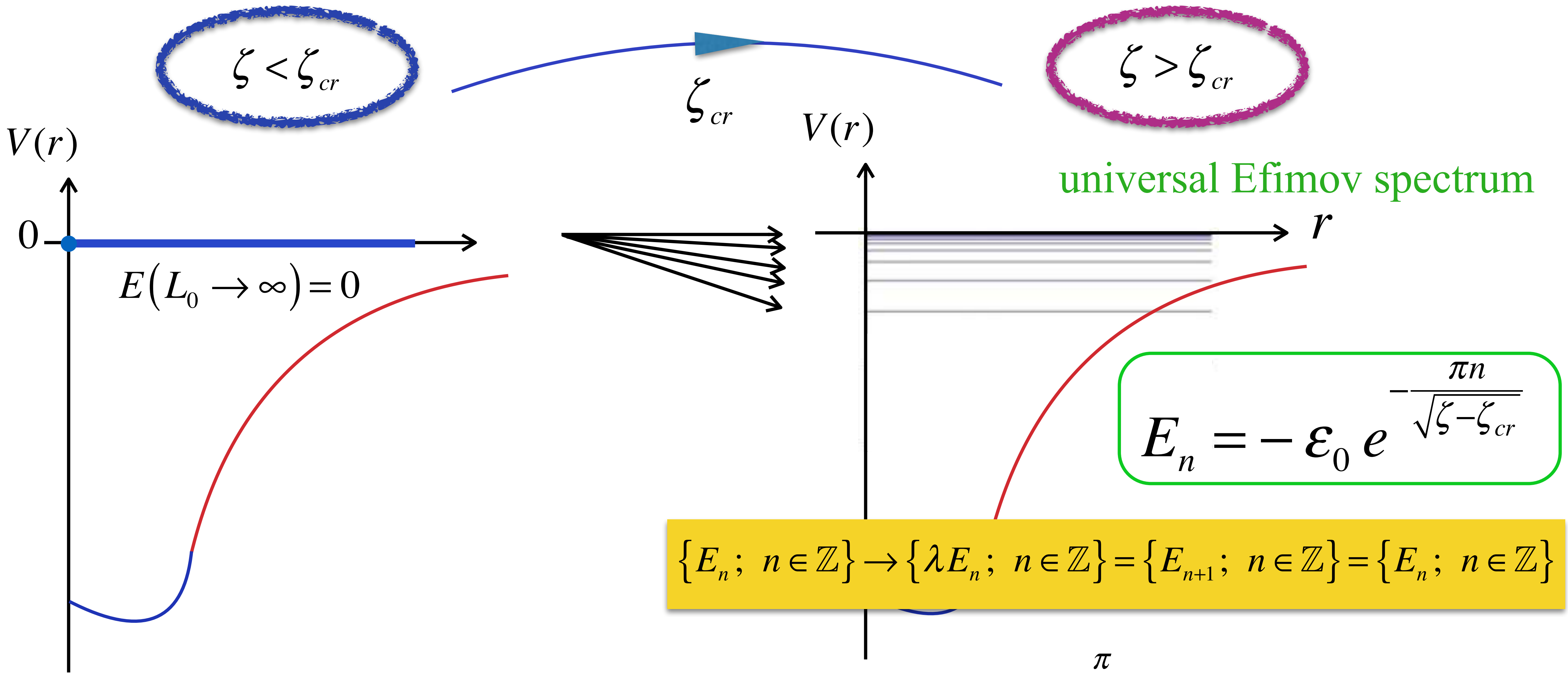

continuous scale invariance (CSI)

but trivial : $\lambda E = 0 \quad \forall \lambda$

$\lambda \equiv e^{-\frac{\pi}{\sqrt{\zeta - \zeta_{cr}}}}$ is fixed :

discrete scale invariance (DSI)

<u>Universal</u> Efimov energy spectrum

$$E_n = -\varepsilon_0 e^{-\frac{\pi n}{\sqrt{\zeta - \zeta_{cr}}}} \equiv -\varepsilon_0 \lambda^n$$

<u>Non universal</u> energy parameter

Universal Efimov energy spectrum

$$E_n = -\varepsilon_0 e^{-\frac{\pi n}{\sqrt{\zeta - \zeta_{cr}}}} \equiv -\varepsilon_0 \lambda^n$$

- The Efimov spectrum is invariant under a discrete scaling *w.r.t.* the parameter :

$$\lambda \equiv e^{-\frac{\pi}{\sqrt{\zeta - \zeta_{cr}}}}$$

where $\zeta = 2\mu\xi - l(l+d-2)$

$$\zeta_{cr} = \frac{(d-2)^2}{4}$$

Universal Efimov energy spectrum

$$E_n = -\varepsilon_0 \, e^{-\frac{\pi n}{\Lambda}} \equiv -\varepsilon_0 \, \lambda^n$$

- The Efimov spectrum is invariant under a discrete scaling *w.r.t.* the parameter :

$$\lambda \equiv e^{-\frac{\pi}{\sqrt{\zeta-\zeta_{cr}}}}$$

where $\zeta = 2\mu\xi - l(l+d-2)$

- Density of states $\rho(E) = \sum_{n\in\mathbb{Z}} \delta(E - E_n)$

$$\rho(\lambda^2 E) = 2\mu \sum_{n\in\mathbb{Z}} \delta(\lambda^2 k^2 - k_n^2) = \cdots = \lambda^{-2} \rho(E)$$

so that $\rho(E) = \frac{1}{E} G\left(\frac{\ln E}{\ln \lambda}\right)$ where $G(u+1) = G(u)$

Universal Efimov energy spectrum

$$E_n = -\varepsilon_0\, e^{-}$$

- The Efimov spectrum is invaria
  parameter :

$$\lambda \equiv e^{-\frac{\pi}{\sqrt{\zeta-\zeta_{cr}}}}$$

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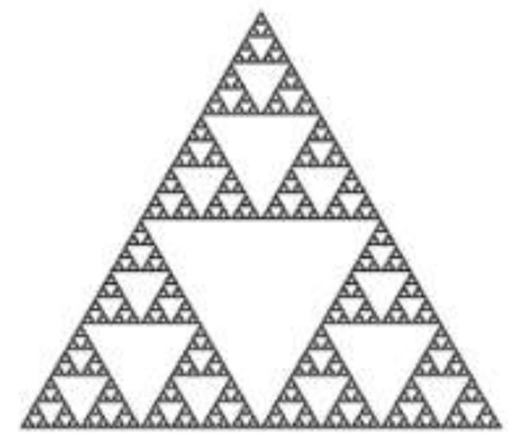
w



$$f(a\,x) = b\,f(x)$$

If satisfied *with fixed* (a,b) (DSI),

General solution is

$$f(x) = x^{\alpha}\, G\left(\frac{\ln x}{\ln a}\right)$$

where $G(u+1) = G(u)$ is a periodic function of period unity

The same problem from another point of view

Renormalisation group (RG) and limit cycles

It is interesting to re-phrase the previous problem using the language of RG transformations.

Why ?

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- Insert that problem in a broader perspective.
- Connects to other physical problems.

As we saw, the problem of the potential $V(r) = -\frac{\xi}{r^2}$ results from

- its behaviour for $r \to 0$
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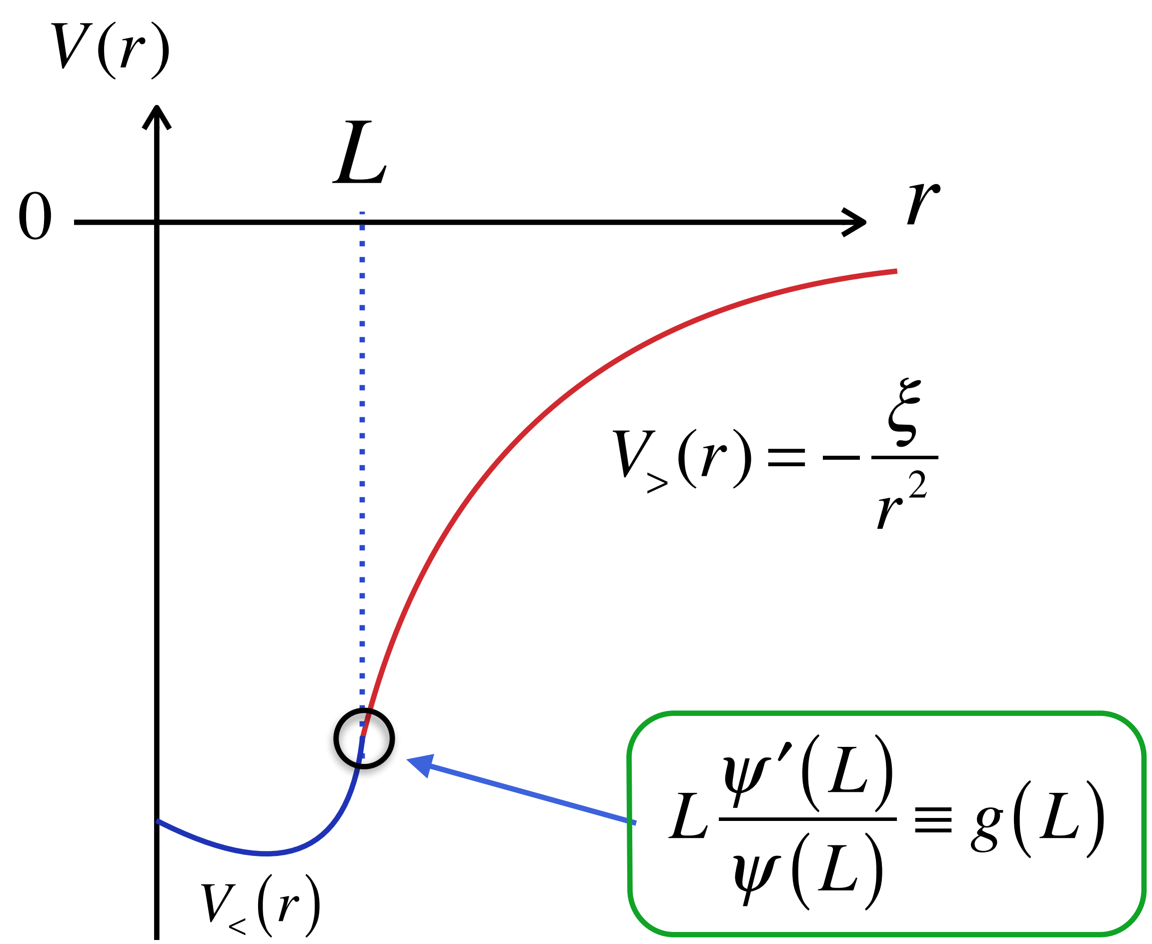
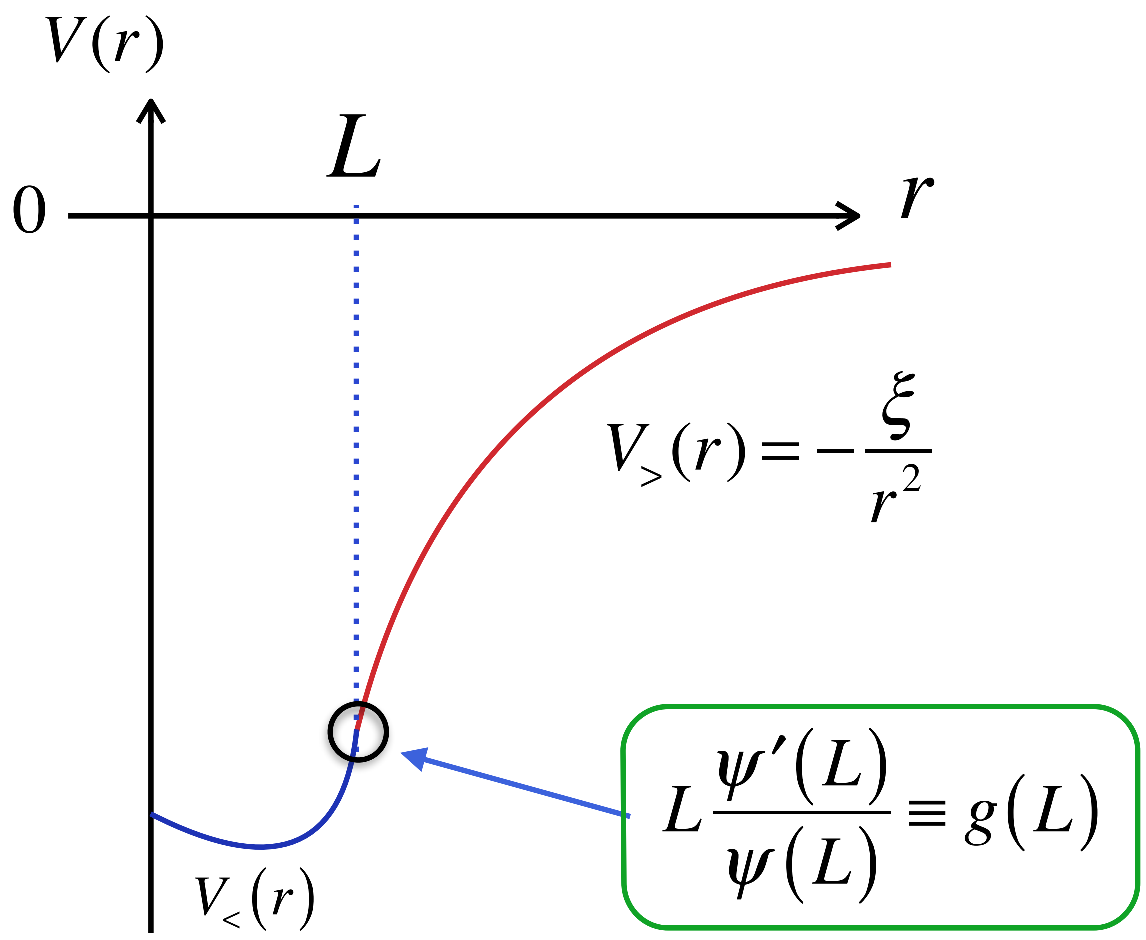


Problem becomes well-defined :

- characteristic length L
- continuity of ψ and ψ' at L

$\Rightarrow$ energy spectrum

Is it possible to consistently change (L, ξ, g) so that the energy spectrum remains unchanged ?



Problem becomes well-defined :

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- continuity of ψ and ψ' at L

$\Rightarrow$ energy spectrum

ξ is a dimensionless number. To make it change with L we take

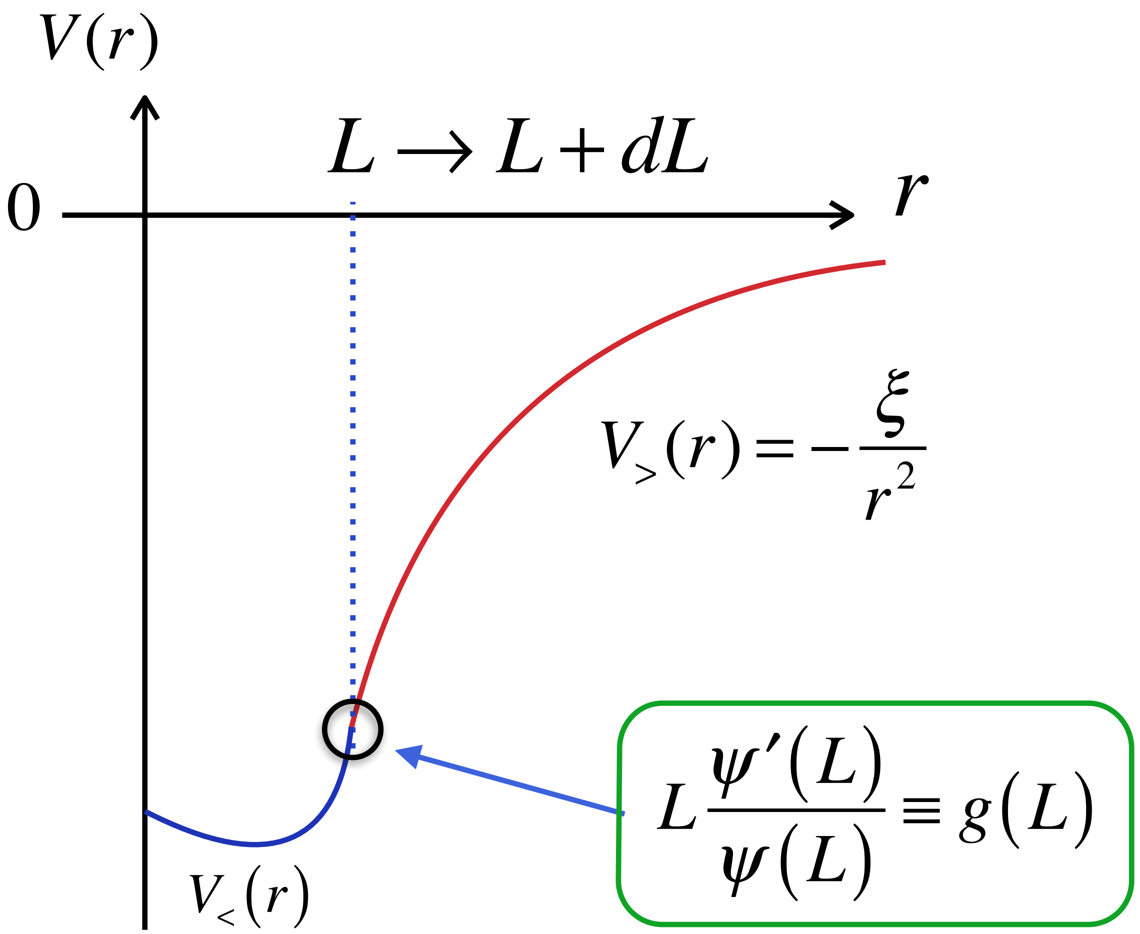
Note

$$\begin{cases} V_>(r) = -\xi / r^s & for\ r > L \\ V_<(r) & for\ \ r < L \end{cases}$$

eventually, $s \to 2$

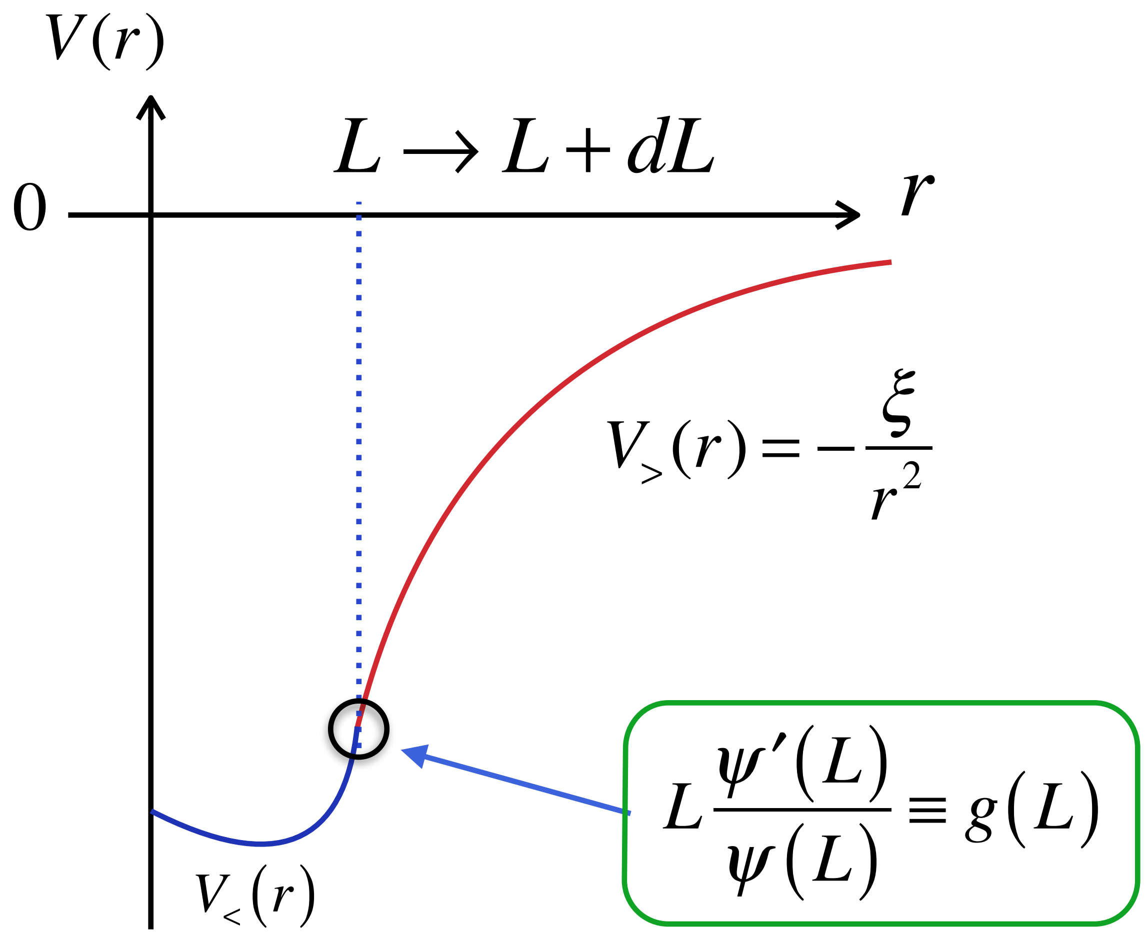
so that now, $\left(L, \xi(L), g(L)\right)$

Perform a RG transformation : change the cutoff distance $L \to L + dL$



leaves the energy spectrum unchanged provided :

Perform a RG transformation : change the cutoff distance $L \to L + dL$



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- coupling strength changes as

$$L\frac{d\xi}{dL} = (2 - s)\xi$$

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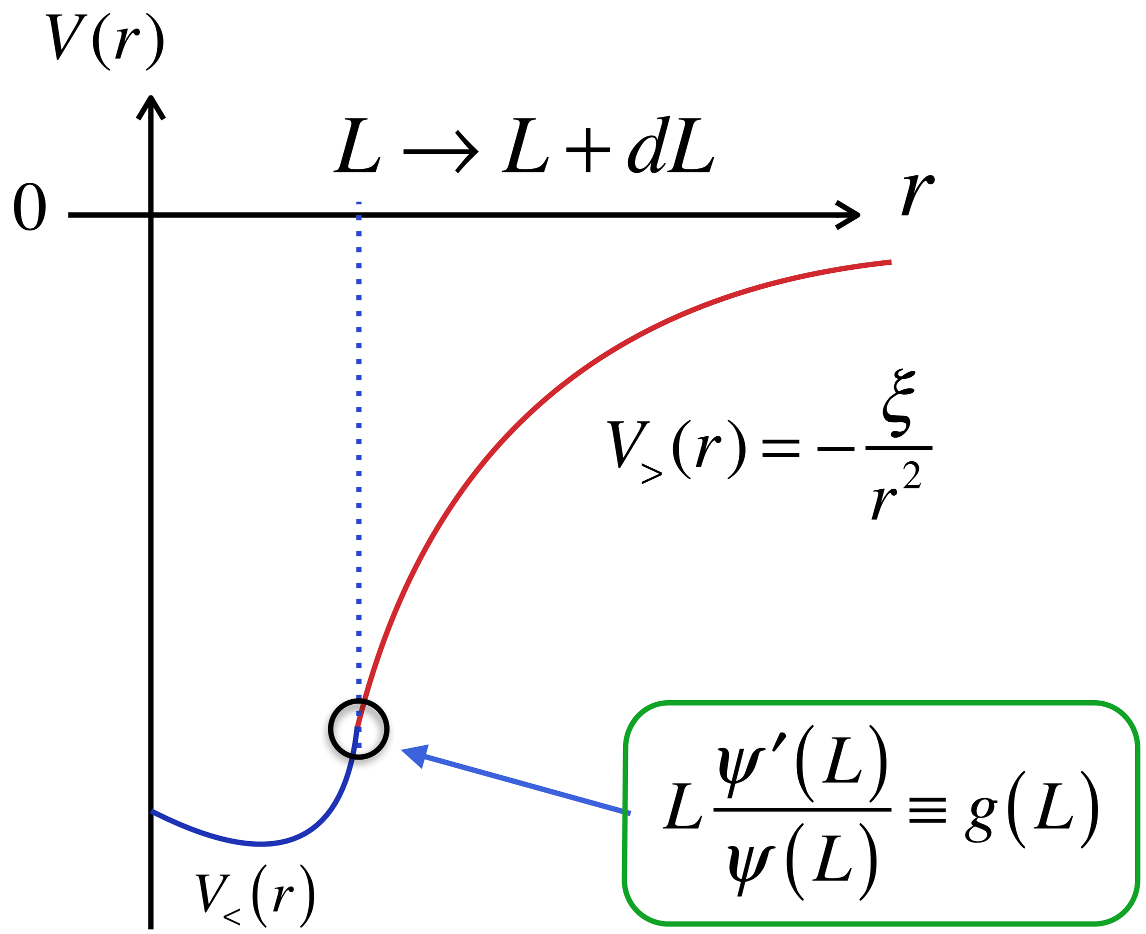

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$$L\frac{dg}{dL} = (2-d)g - g^2 - \zeta$$

(for low enough energies, i.e. for $L \to \infty$)

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Scale invariant coupling ξ

We define the β -function :

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Scale invariant coupling ξ

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The coupling ξ is scale (L) independent

$\Rightarrow \zeta$ is also scale independent

Evolution of the coupling $g(L)$ - quantum phase transition

$$\beta(g) = \frac{\partial g}{\partial \ln L} = (2-d)g - g^2 - \zeta$$

Evolution of the coupling $g(L)$ - quantum phase transition

$$\beta(g)=\frac{\partial g}{\partial \ln L}=(2-d)g-g^2-\zeta=-(g-g_+)(g-g_-)$$

$$g_\pm=\frac{2-d}{2}\pm\sqrt{\zeta_{cr}-\zeta}$$

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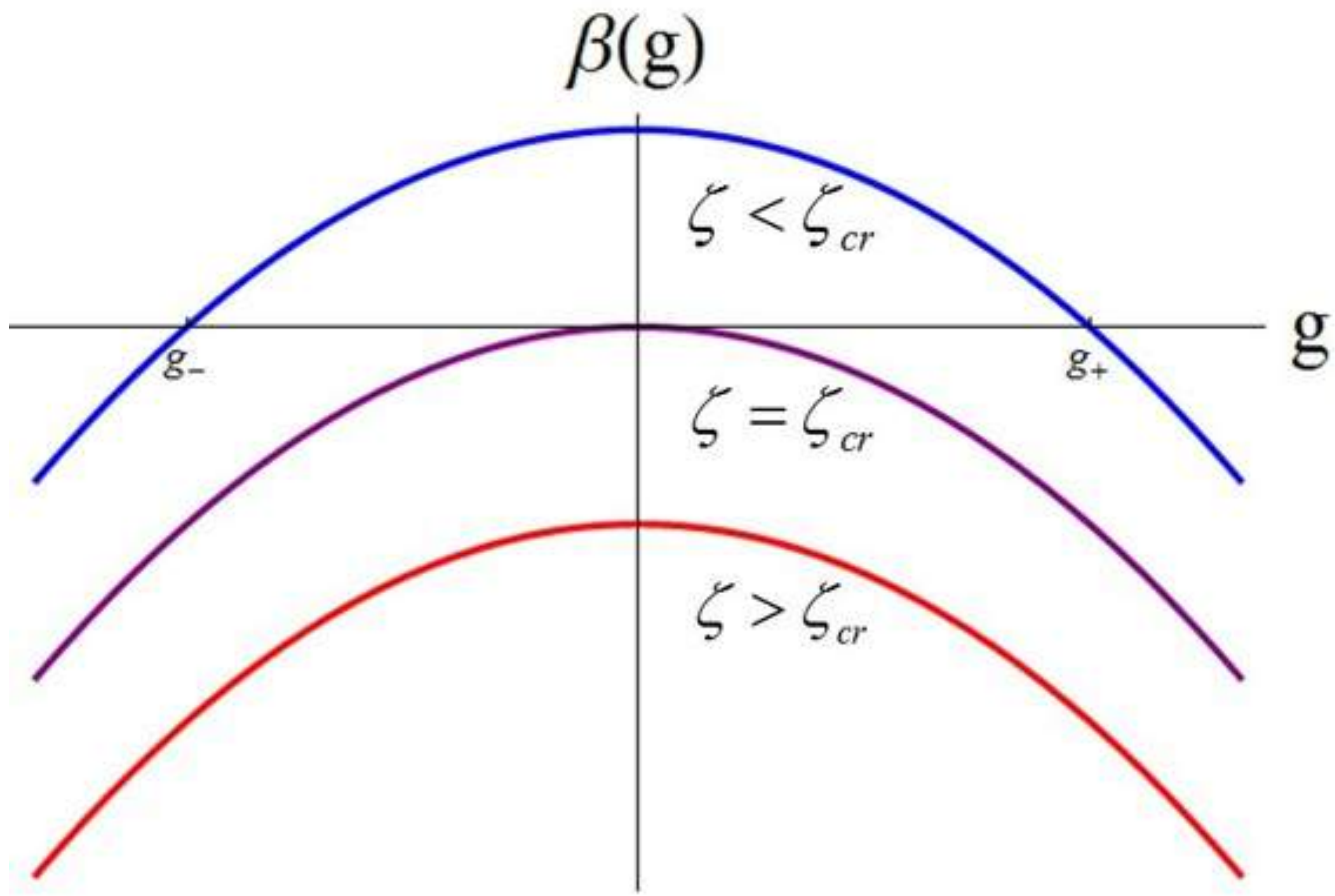

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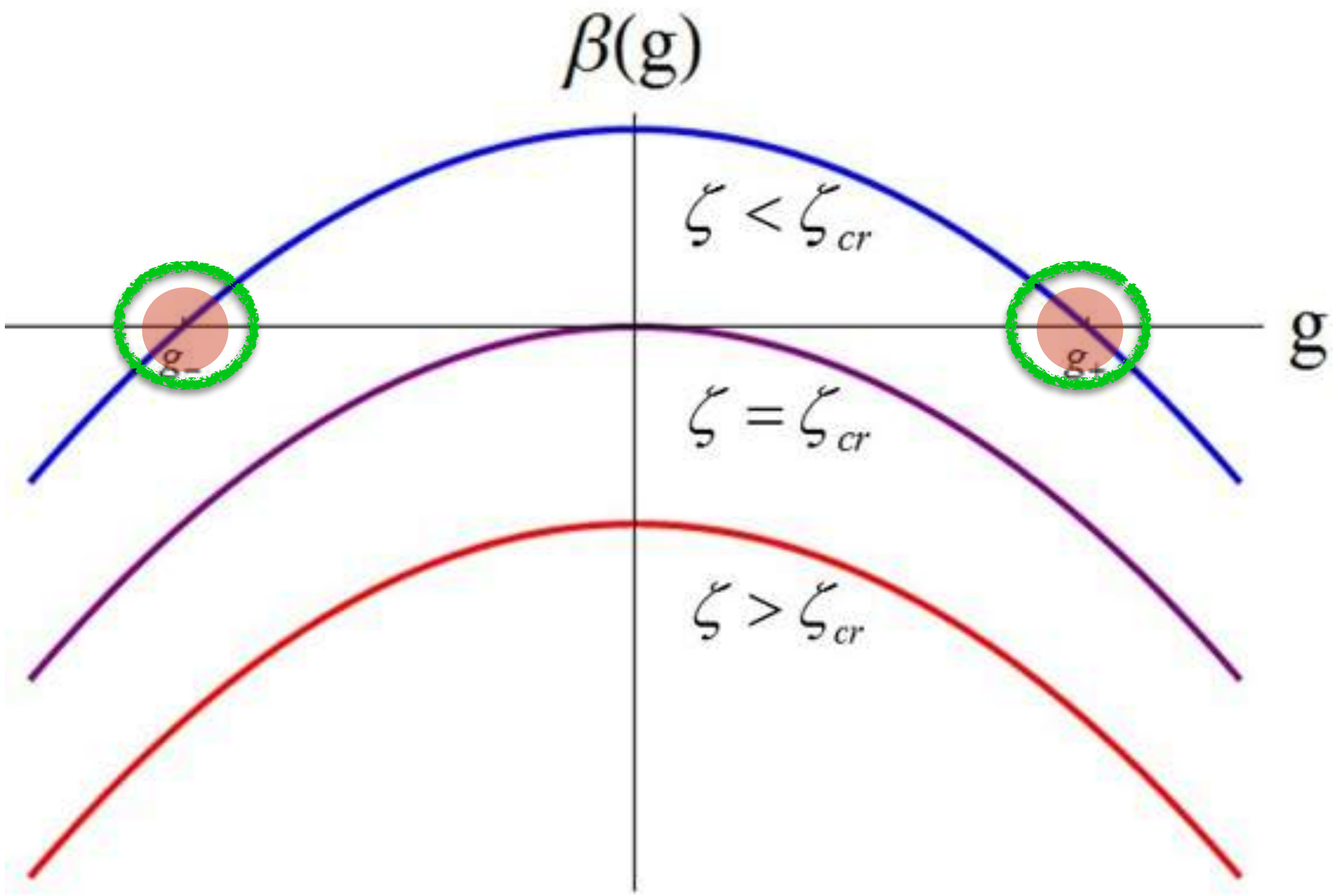
Evolution of the coupling $g(L)$ - quantum phase transition

For $\zeta < \zeta_{cr}$ two fixed points (g_+, g_-)



$$\beta(g_\pm) = 0$$

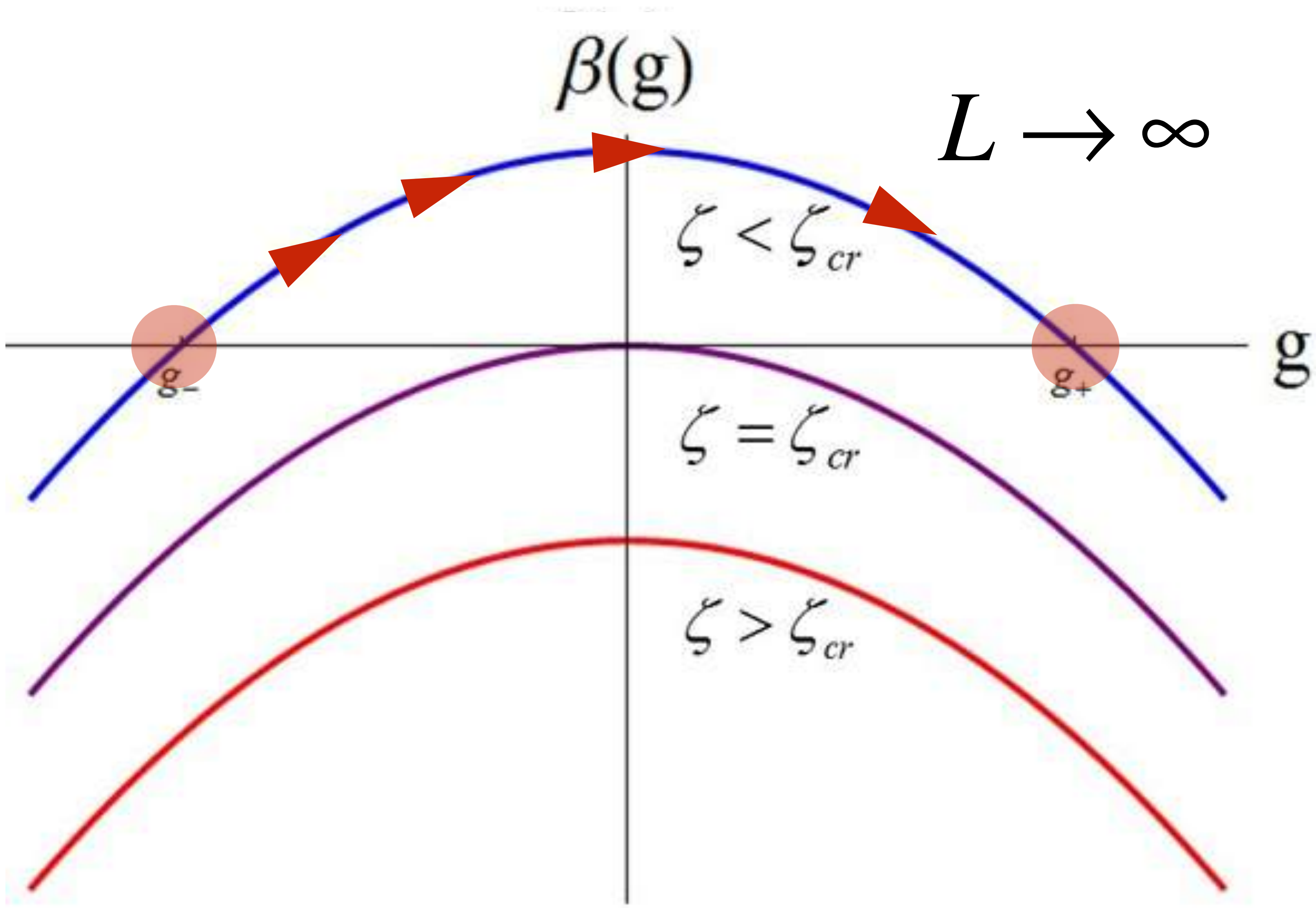
$$\Downarrow$$

$g_\pm$ are L-independent

Evolution of the coupling $g(L)$ - quantum phase transition

For $\zeta < \zeta_{cr}$ two real fixed points (g_+ , g_-)

- g_+ → stable
- g_- → unstable



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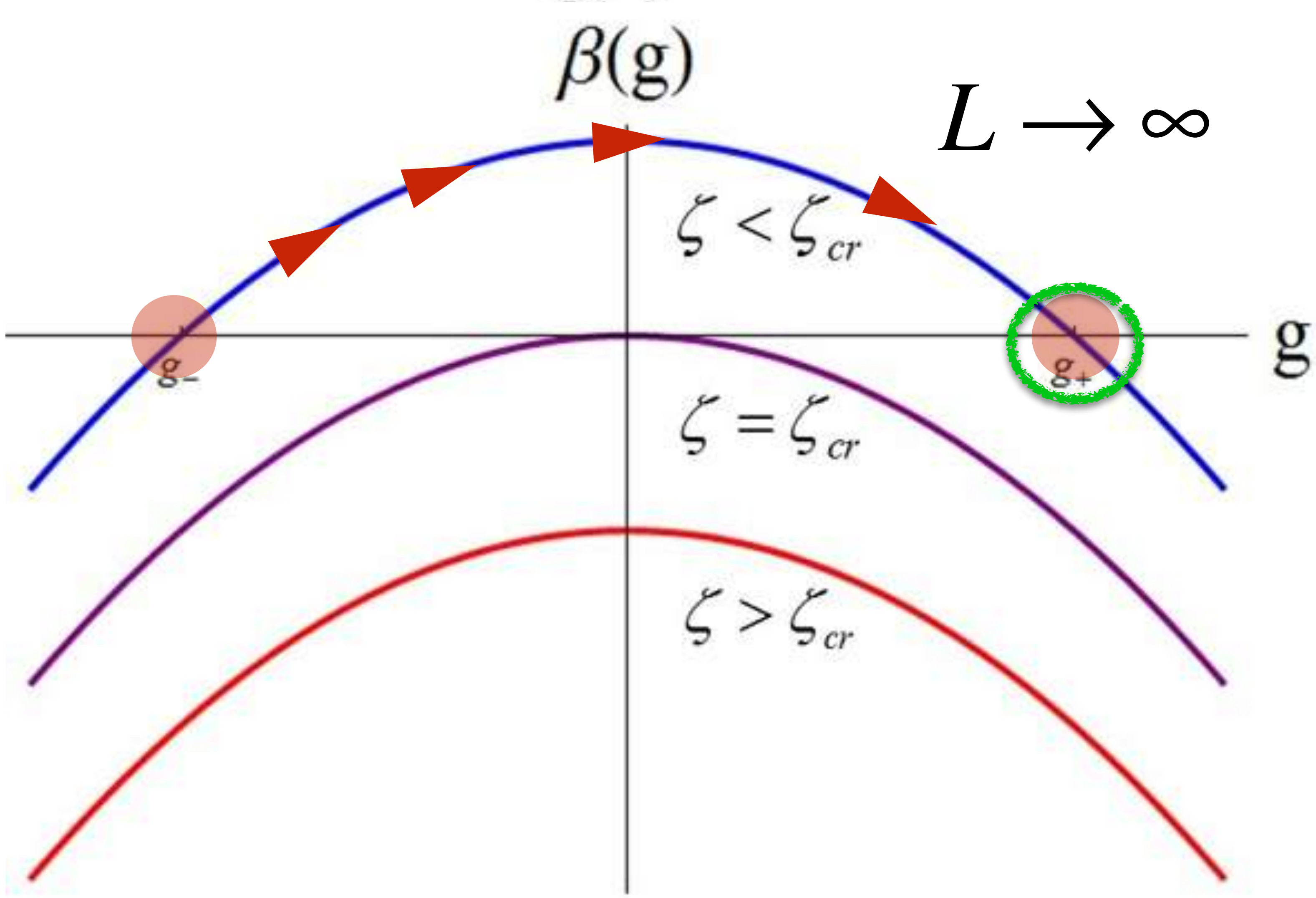

Universal behaviour of the energy spectrum

Evolution of the coupling $g(L)$ - quantum phase transition

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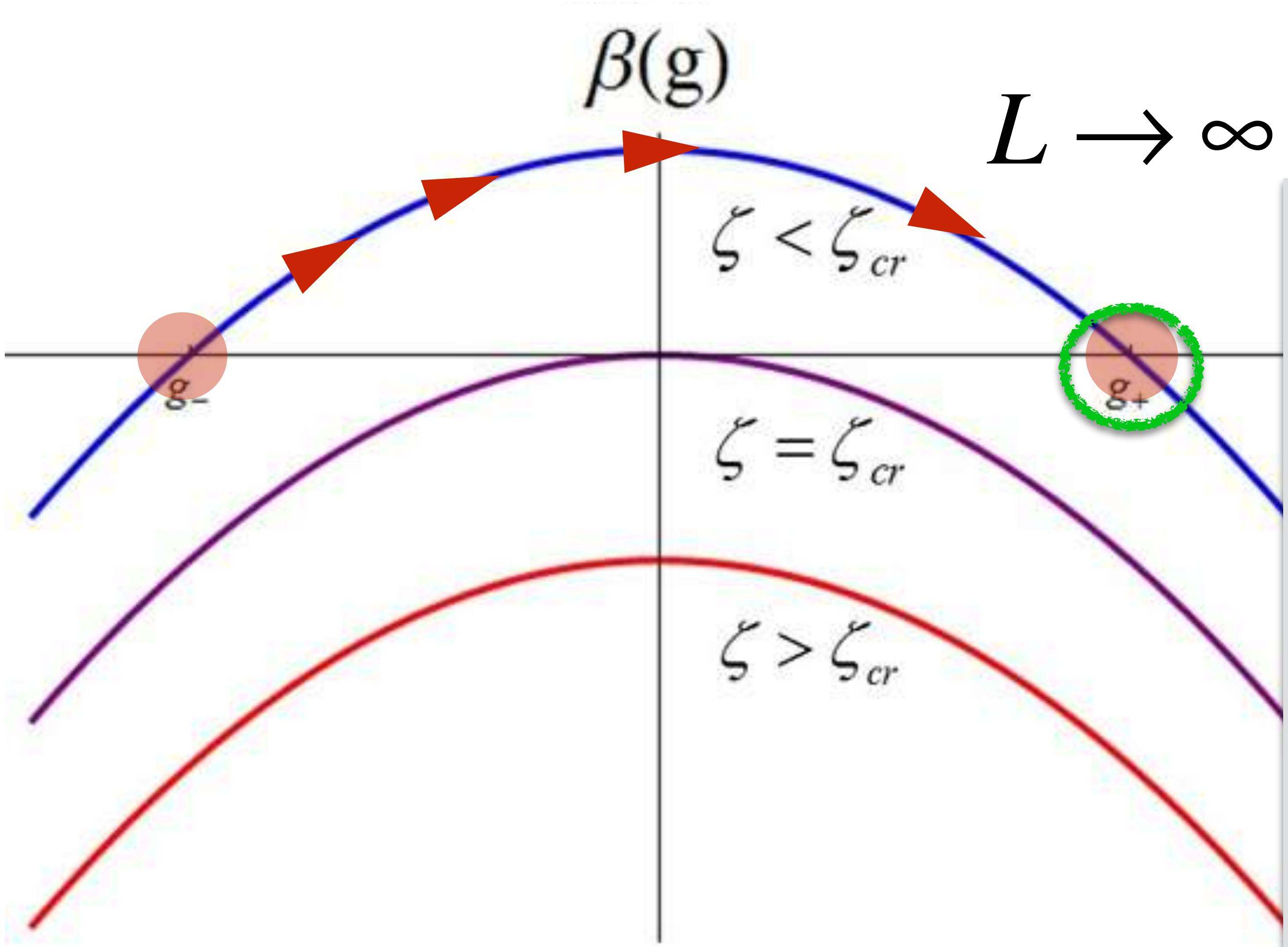
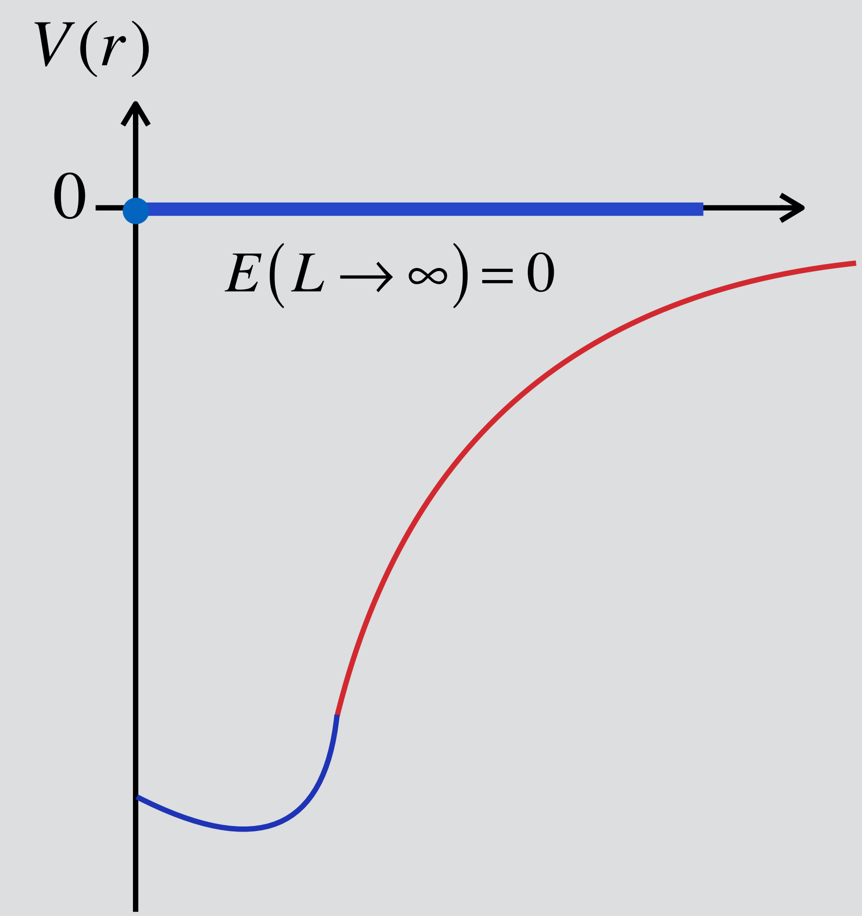


$L \to \infty$



continuous scale invariance (CSI)

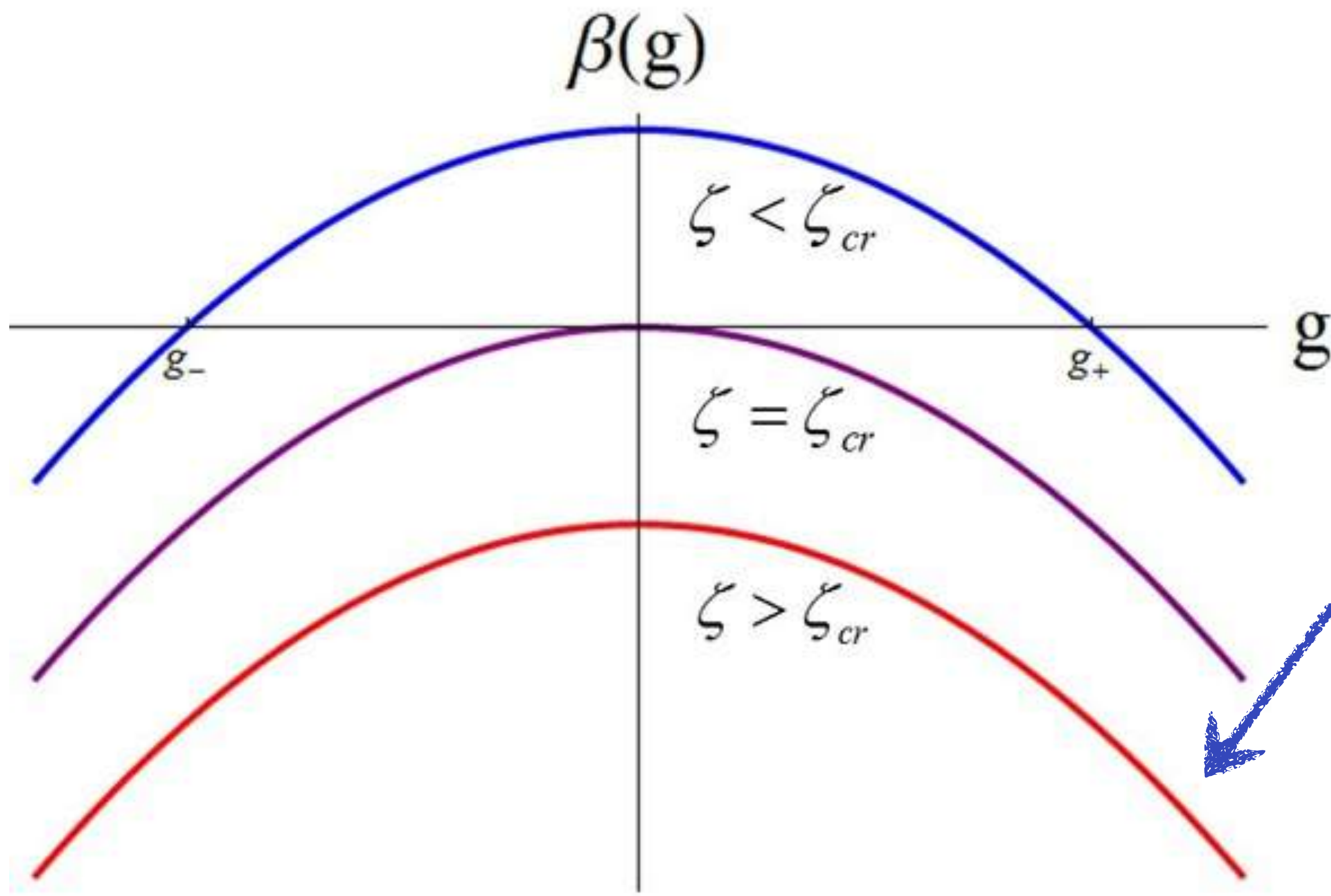
but trivial : $\lambda E = 0 \quad \forall \lambda$

Universal behaviour of the e

Evolution of the coupling $g(L)$ - quantum phase transition

For $\zeta > \zeta_{cr}$

$$\beta(g) = \frac{\partial g}{\partial \ln L} = -(g - g_+)(g - g_-)$$



$$g_{\pm} = \frac{2-d}{2} \pm \sqrt{\zeta_{cr} - \zeta}$$

Two complex valued solutions

No fixed point

The solution for $g(L)$ is a limit cycle.

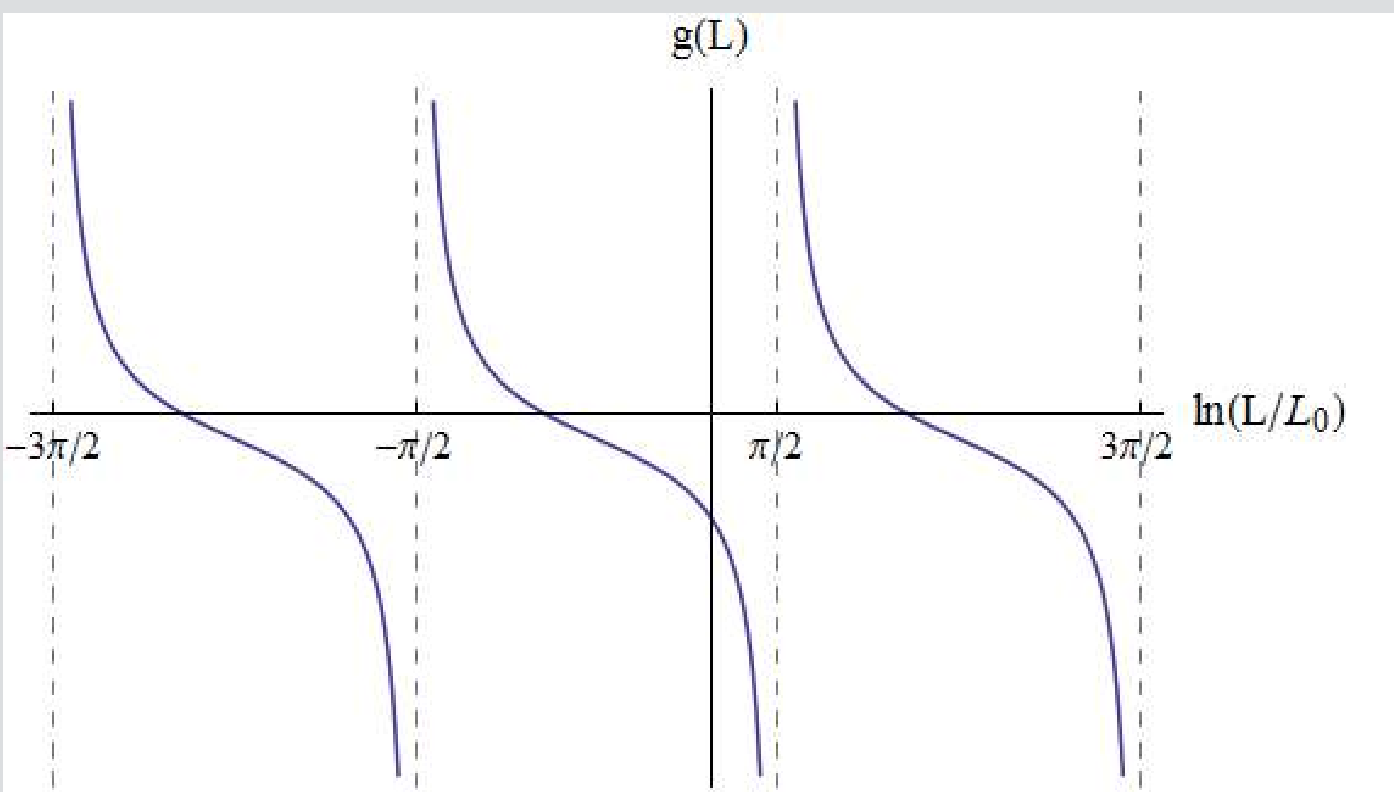
Evolution of the coupling $g(L)$ - quantum phase transition

For $\zeta > \zeta_{cr}$

$$\beta(g) = \frac{\partial g}{\partial \ln L} = -(g - g_+)(g - g_-)$$

The solution for $g(L)$ is a limit cycle.

The cycle completes a period for every $L \to e^{\frac{-\pi}{\sqrt{\zeta - \zeta_{cr}}}} L$



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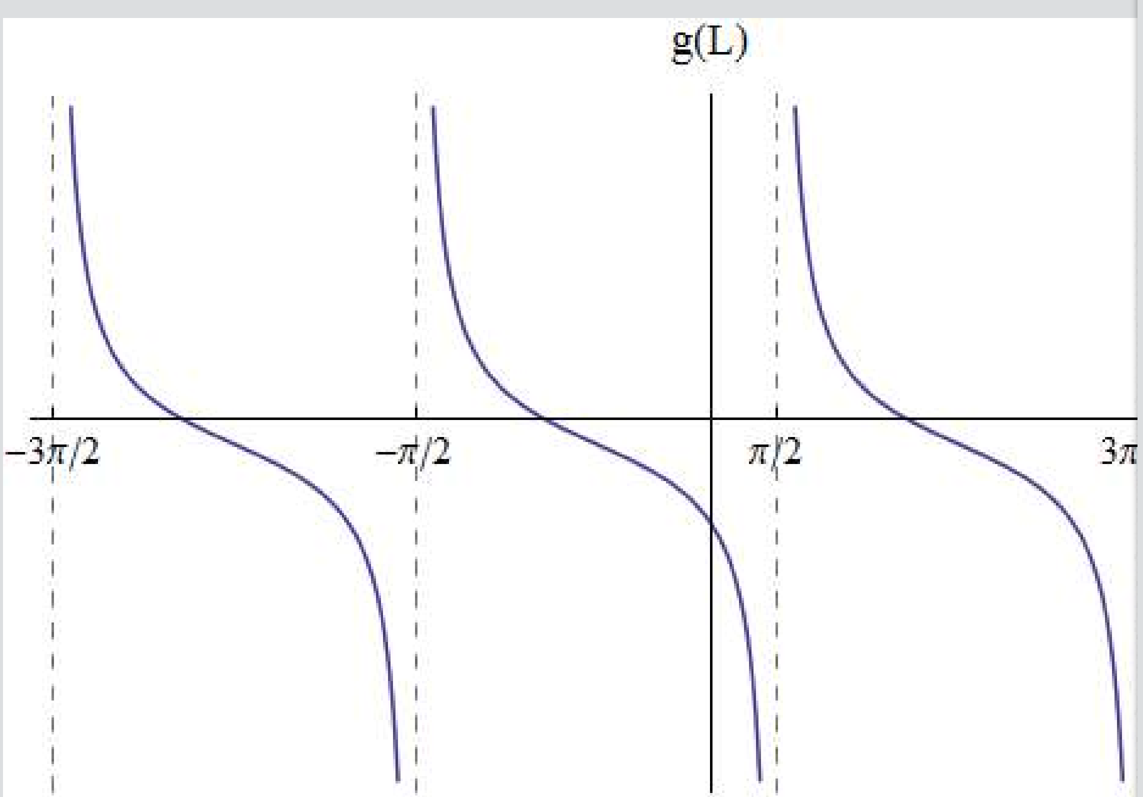
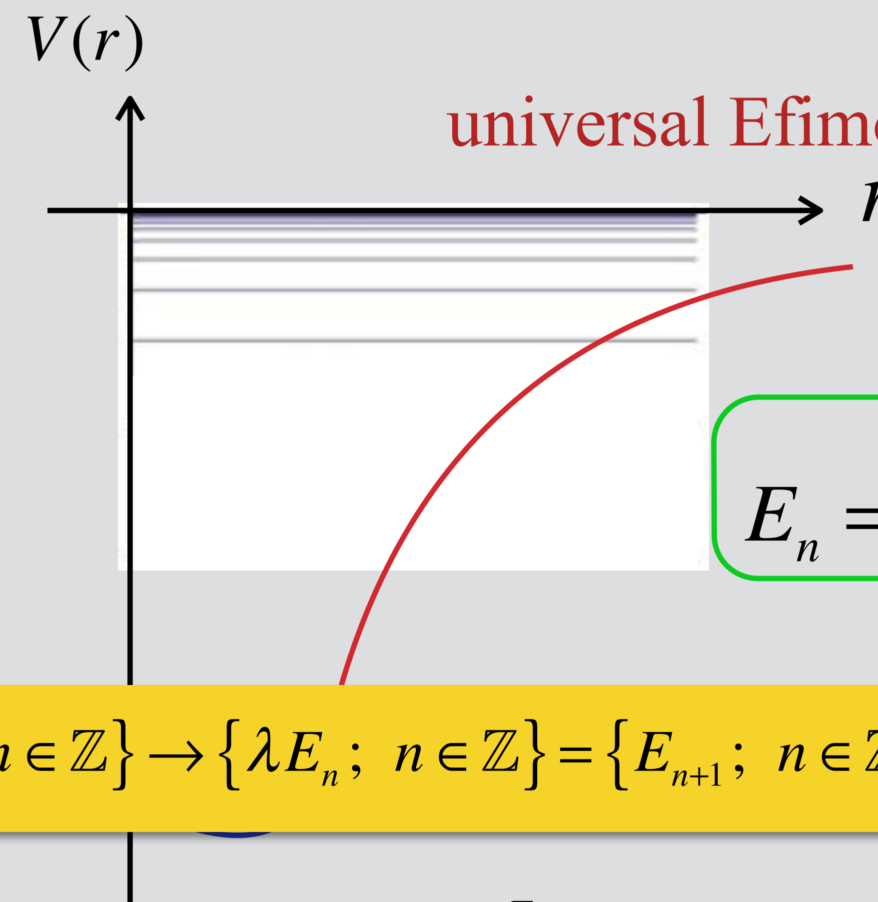




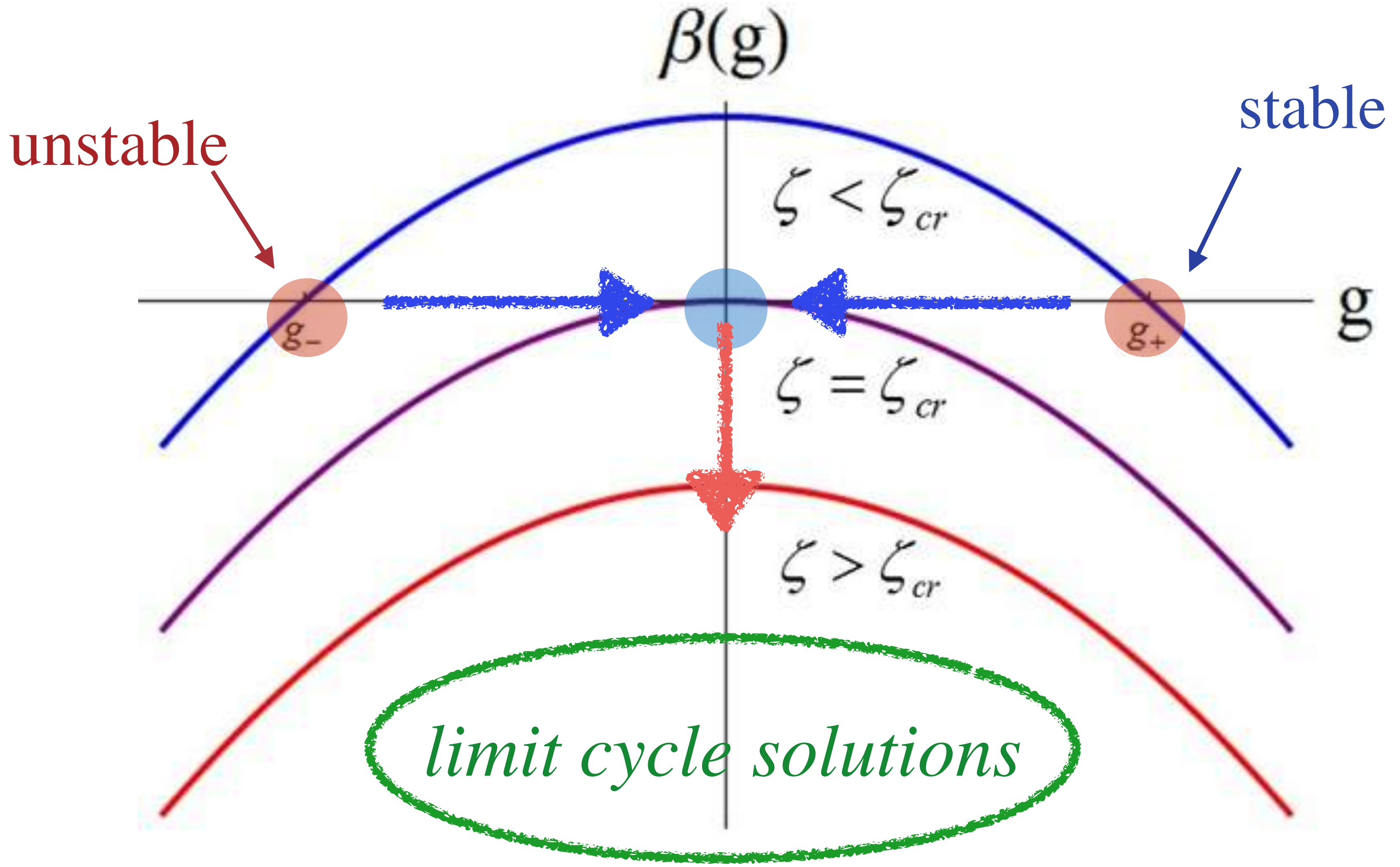
$$E_n = -\varepsilon_0 \, e^{-\frac{\pi n}{\sqrt{\zeta - \zeta_{cr}}}}$$

$$\{E_n;\ n \in \mathbb{Z}\} \to \{\lambda E_n;\ n \in \mathbb{Z}\} = \{E_{n+1};\ n \in \mathbb{Z}\} = \{E_n;\ n \in \mathbb{Z}\}$$

$\lambda \equiv e^{-\frac{\pi}{\sqrt{\zeta - \zeta_{cr}}}}$ is fixed :

discrete scale invariance (DSI)

Breaking of CSI into DSI is now interpreted as
a transition of the RG flow from a stable fixed point
into the emergence of limit cycle solutions.



Part 4

Dirac equation + Coulomb :
Do we know everything ?
The graphene approach

Dirac equation + Coulomb potential

Continuous scale invariance (CSI) of the Hamiltonian :

$$\hat{H} = -\frac{\hbar^2}{2\mu}\Delta - \frac{\xi}{r^2}$$

A immediate question : What about the Dirac eq. with a Coulomb potential ?

Dirac equation + Coulomb potential

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Dirac eq. $$i\sum_{\mu=0}^{d}\gamma^{\mu}\left(\partial_{\mu} + ieA_{\mu}\right)\Psi\left(x^{\nu}\right) = 0$$ is linear with momentum and

Coulomb potential

$$eA_0 = V(r) = -\frac{\xi}{r}, \quad \xi \equiv Z\alpha$$
$$A_i = 0, \quad i = 1,\ldots,d$$

fine structure constant

These two problems share the same continuous scale invariance (CSI).

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Dirac equation + Coulomb potential

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Don't we know everything about the Dirac Hydrogen atom ?

A immediate question : What abo… …ulomb potential ?

Dirac e… is linear with momentum and

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Old problem (Pomeranchuk, 1945) of a relativistic electron in a super critical Coulomb potential.

Success of QED lies in the domain of weak fields and perturbation theory in the small dimensionless parameter :

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Perturbation theory fails for $Z\alpha \geq 1$

In that case, we expect instability of the vacuum (ground state) against creation of electron-positron pairs.

How to understand this instability ?

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The Dirac-Kepler problem

Heuristic argument : classical expression for the energy of an electron of mass m , momentum p in the field of a charge Ze

$$\varepsilon = c\sqrt{p^2 + m^2c^2} - \frac{Ze^2}{r}$$

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<u>Estimate of the ground state energy :</u>

electron position cannot be determined to better than $\hbar / p$

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Minimising w.r.t p :

$$\varepsilon_0 = mc^2\sqrt{1 - (Z\alpha)^2}$$

which reproduces well known features of the Hydrogen ground state in the non relativistic $(Z\alpha \ll 1)$ and relativistic limits.

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For $Z\alpha > 1$ the ground state energy becomes imaginary.

Problem : to observe this instability, we need $Z \geq {}^{1}\!/_{\alpha} \simeq 137$

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<u>Idea</u>: consider analogous condensed matter systems with a "much larger effective fine structure constant".

Graphene : Effective massless Dirac excitations with a Fermi velocity $v_F \simeq 10^6\, m/s$ so that

$$\alpha_G = e^2/\hbar v_F \approx 2.5$$

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- Charged impurities in graphene (Coulomb potential)

This instability in the Dirac + Coulomb problem is an example of the breaking of CSI into DSI.

Is there an Efimov like spectrum for the massless Dirac problem ?

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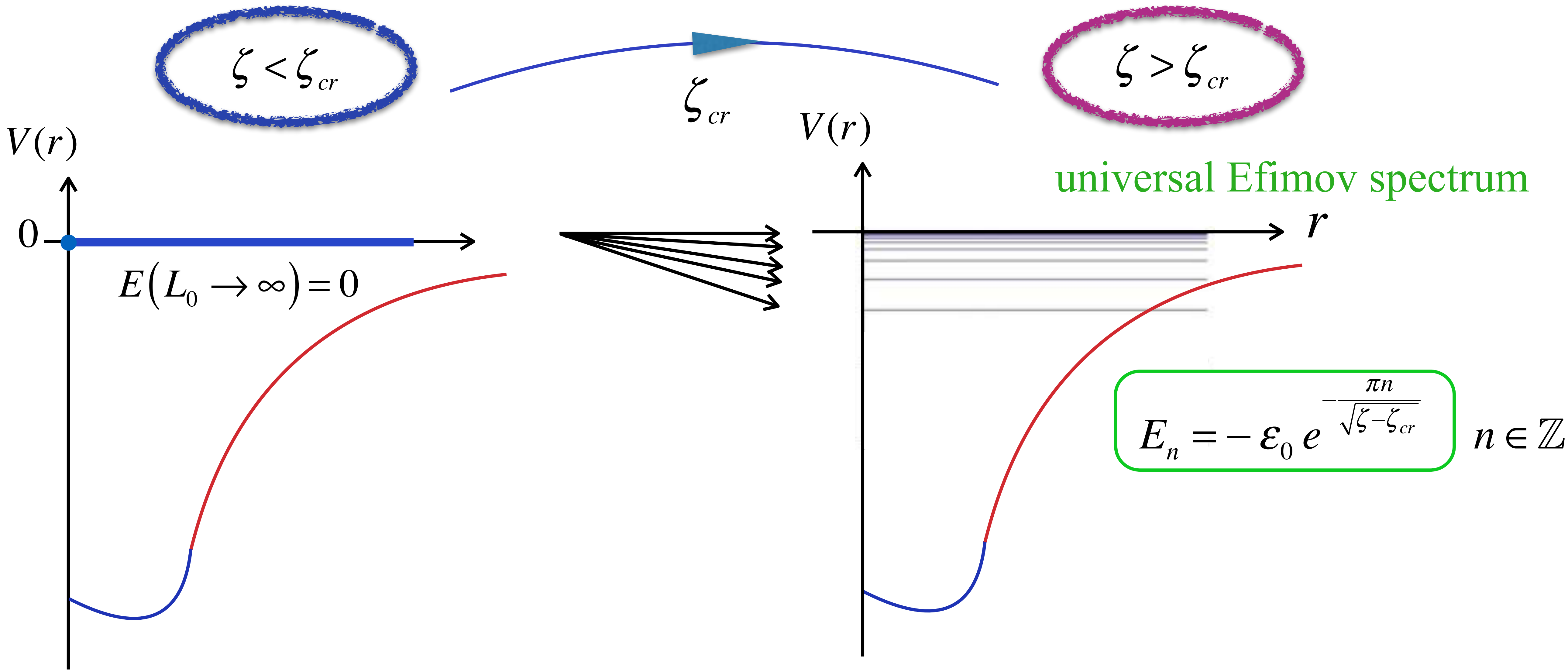
- Charged impurities in graphene (Coulomb potential)

This instability in the Dirac + Coulomb problem is an example of the breaking of CSI into DSI.

The RG picture is rather simple here and it gives the expected Efimov spectrum

A quantum phase transition

It exists a singular value $\zeta_{cr} = \frac{(d-2)^2}{4}$ Take the limit $L_0 \to \infty$



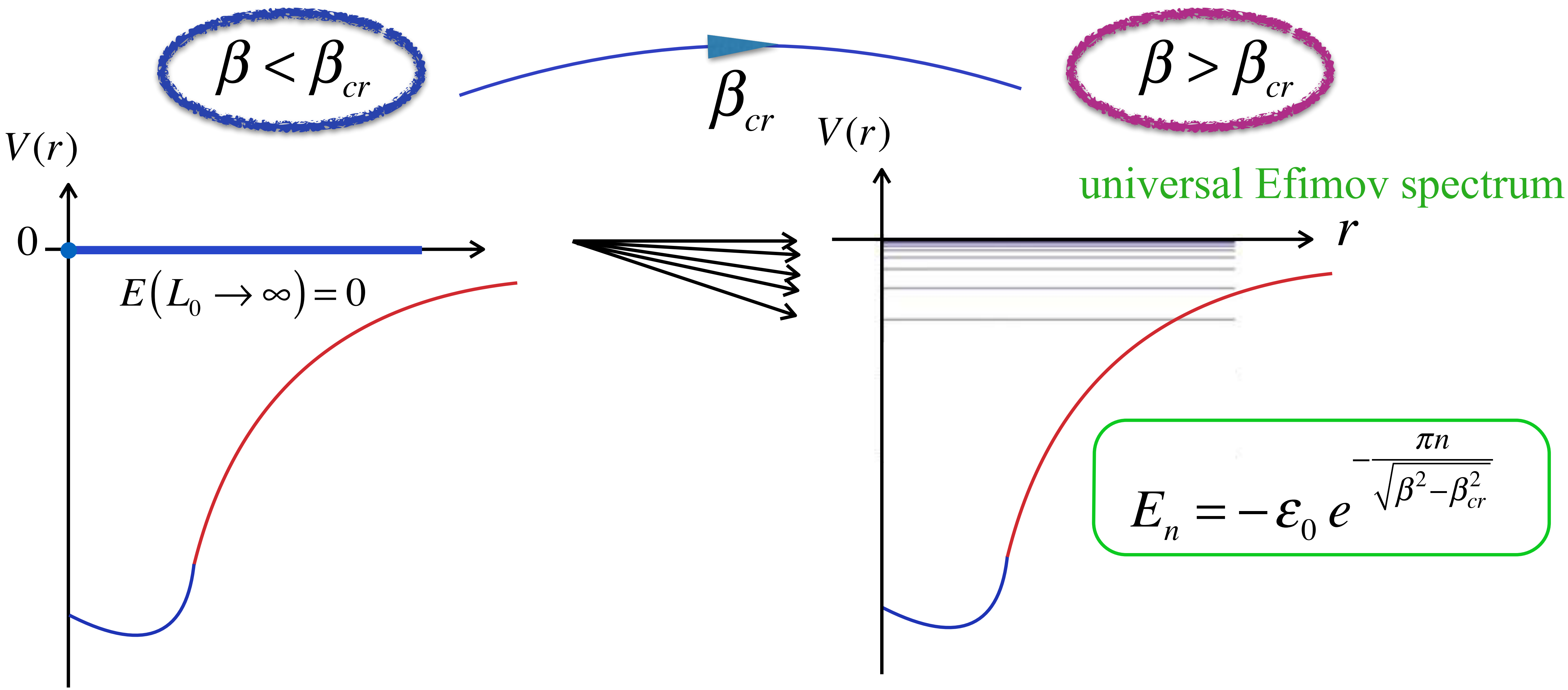
CSI to DSI quantum phase transition for the

$V(r) = -\frac{\xi}{r^2}$ potential

Dirac quantum phase transition

Dimensionless coupling $\beta \equiv Z\alpha$

Singular value $\beta_{cr} = \frac{d-1}{2} = \frac{1}{2}$

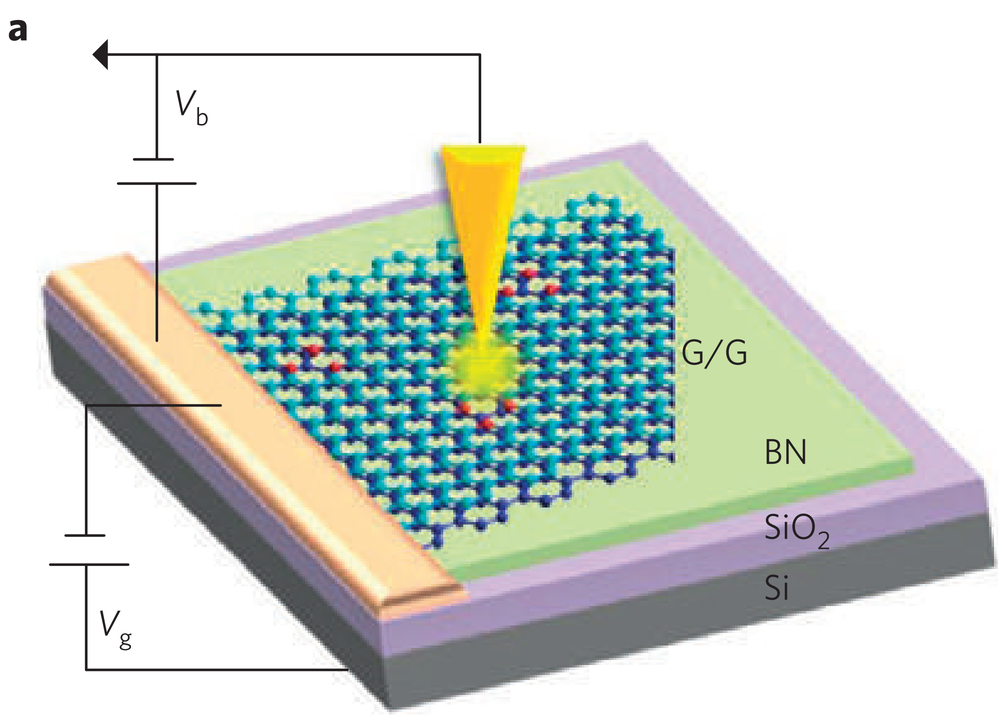


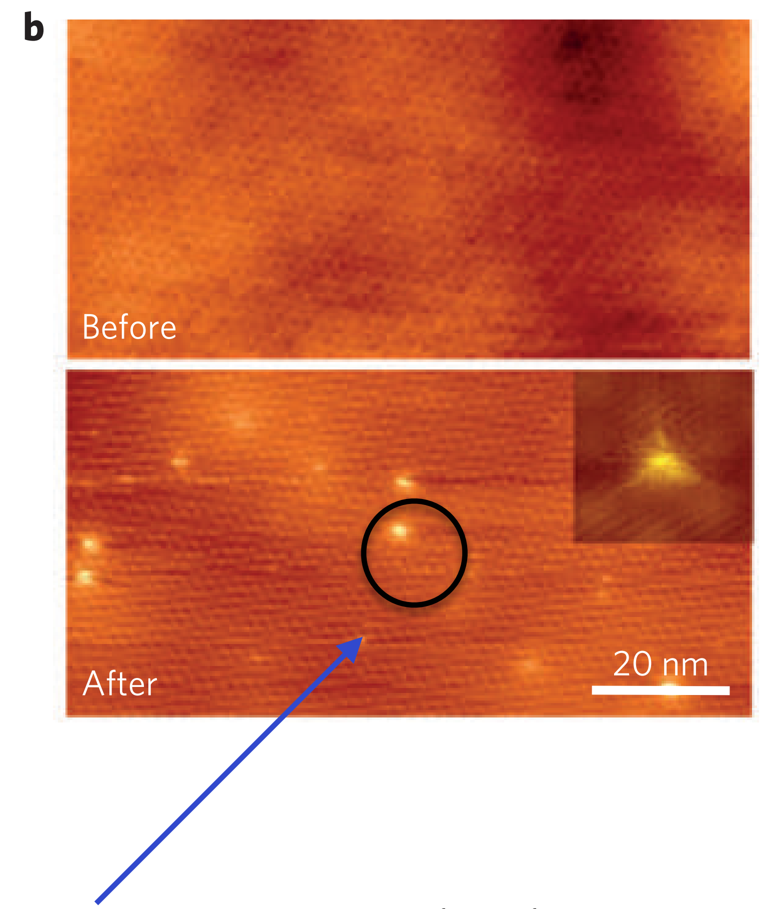
Continuous scale invariance (CSI)

Discrete scale invariance (DSI)

Building an artificial atom in graphene

Jinhai Mao, Eva Andrei et al. (2016)



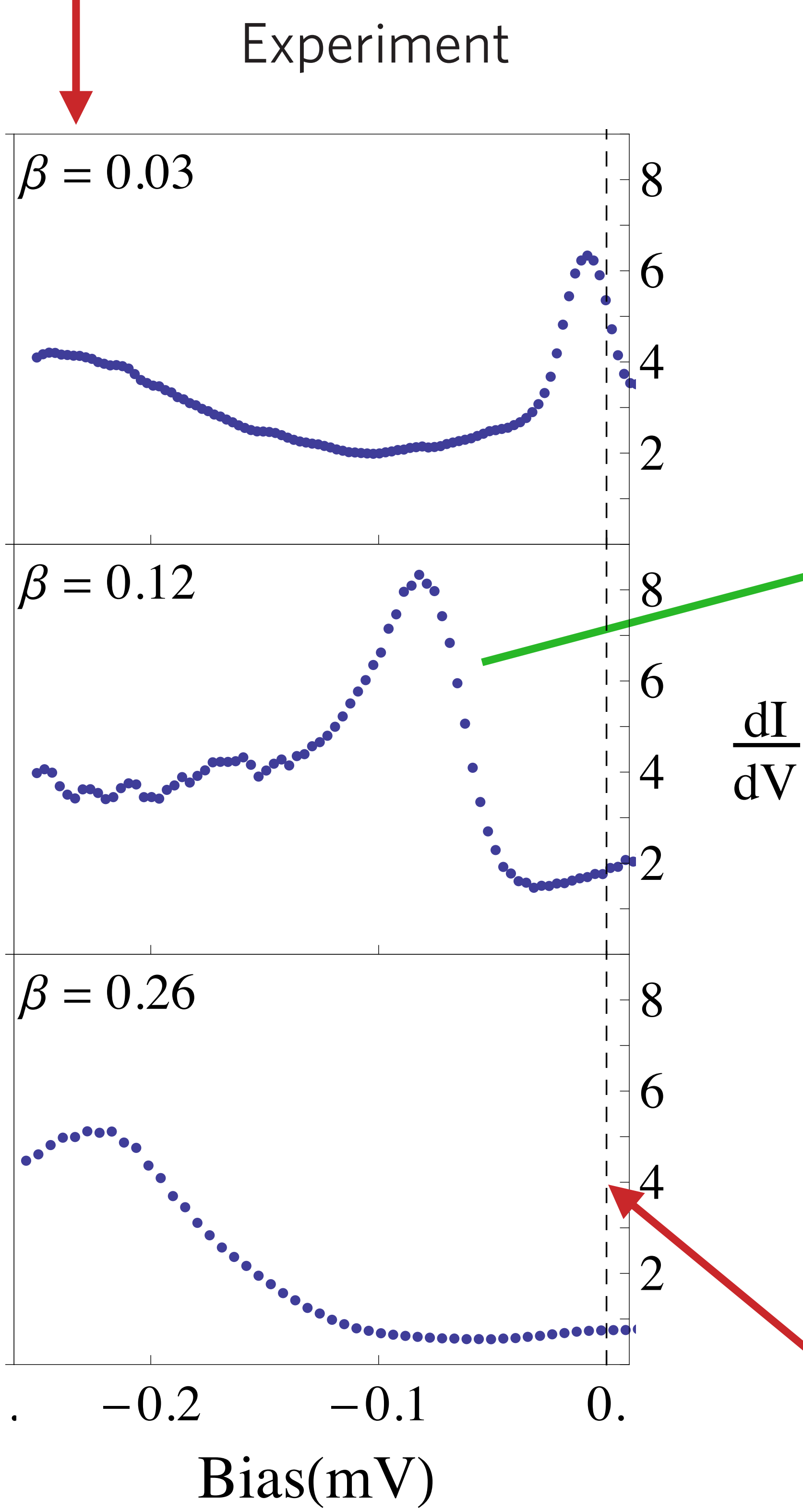


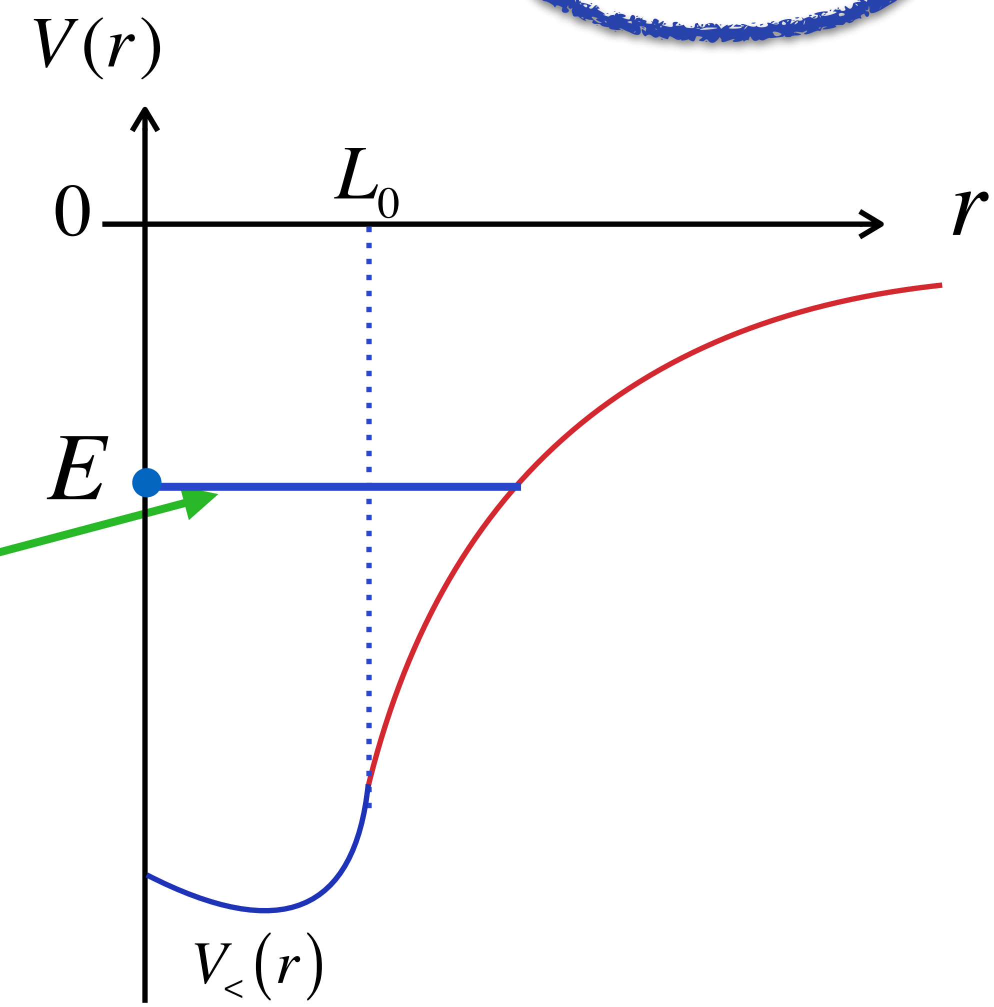
Local vacancy. Local charge is changed by applying voltage pulses with the tip of an STM

$$\beta < \beta_{cr} \equiv \frac{1}{2}$$

Increasing the charge, quasi-bound states are trapped.

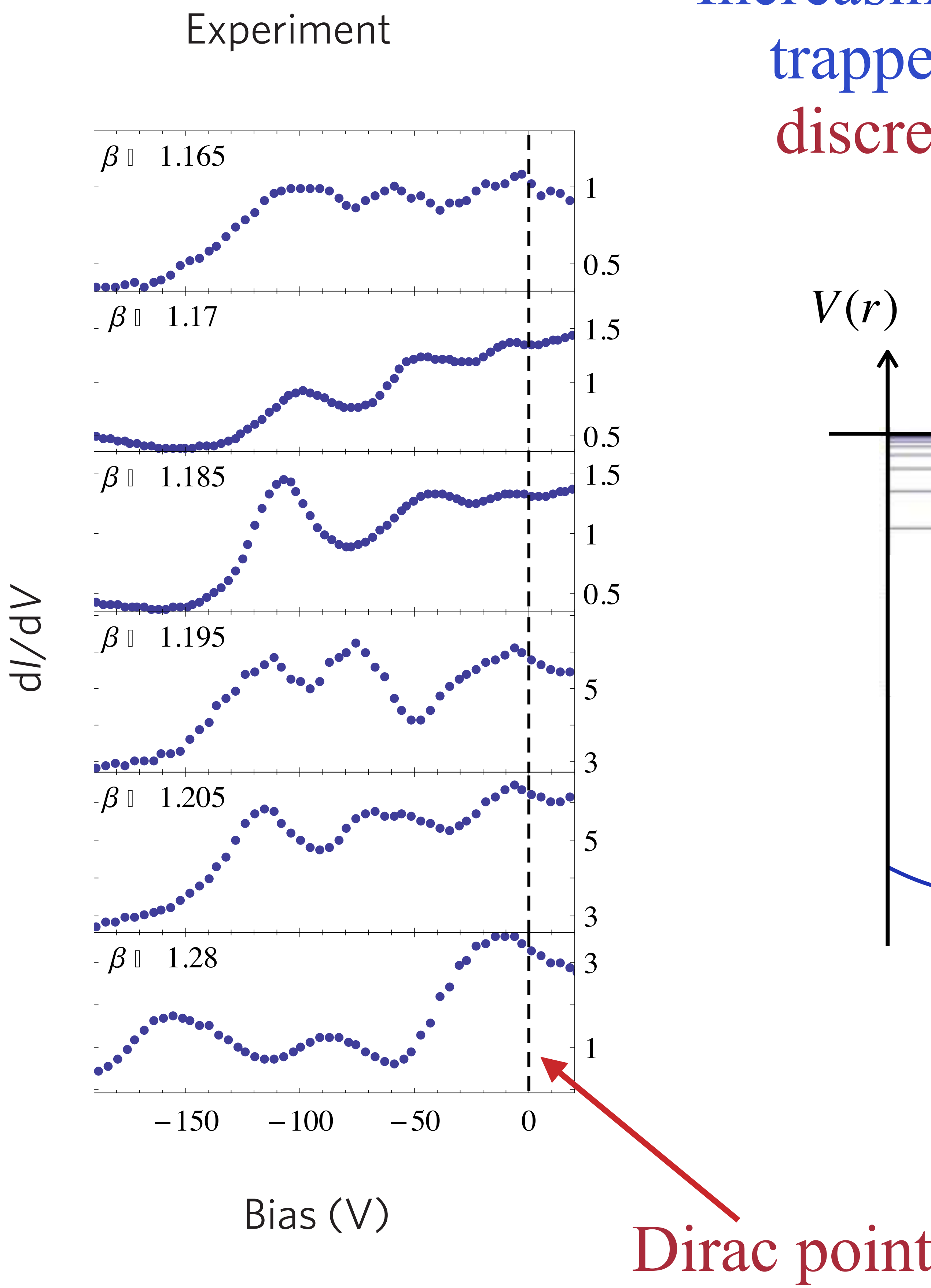
$$\beta < \beta_{cr}$$

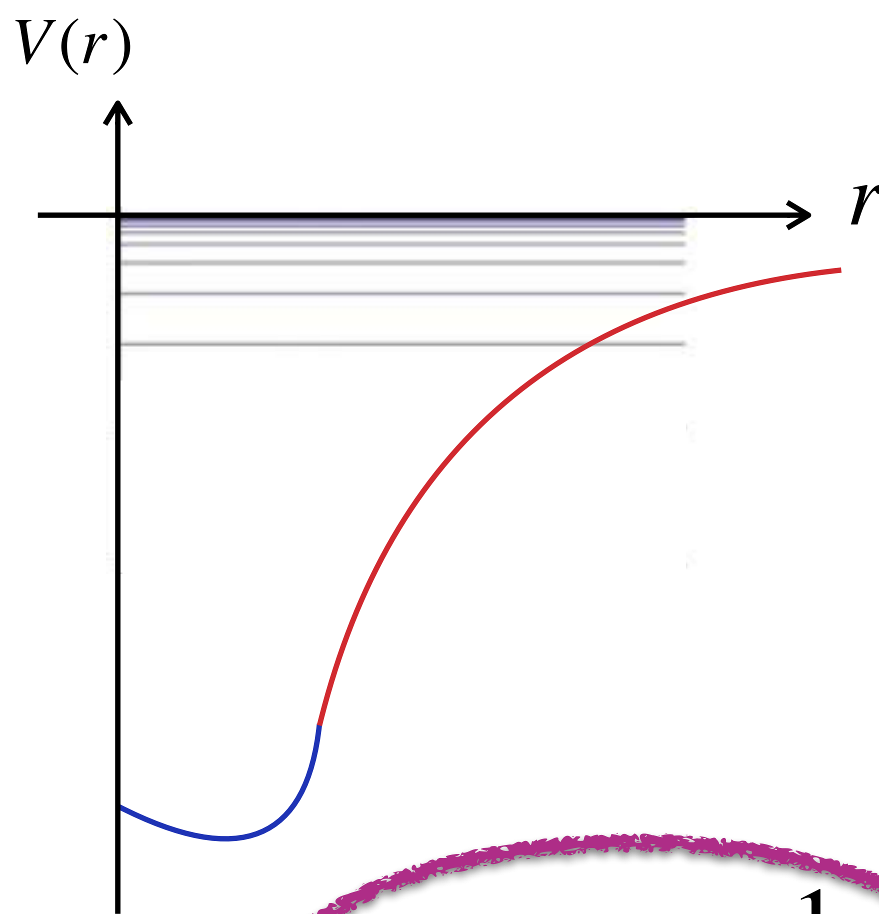




Measure the local density of states from the tunnelling conductance of the STM

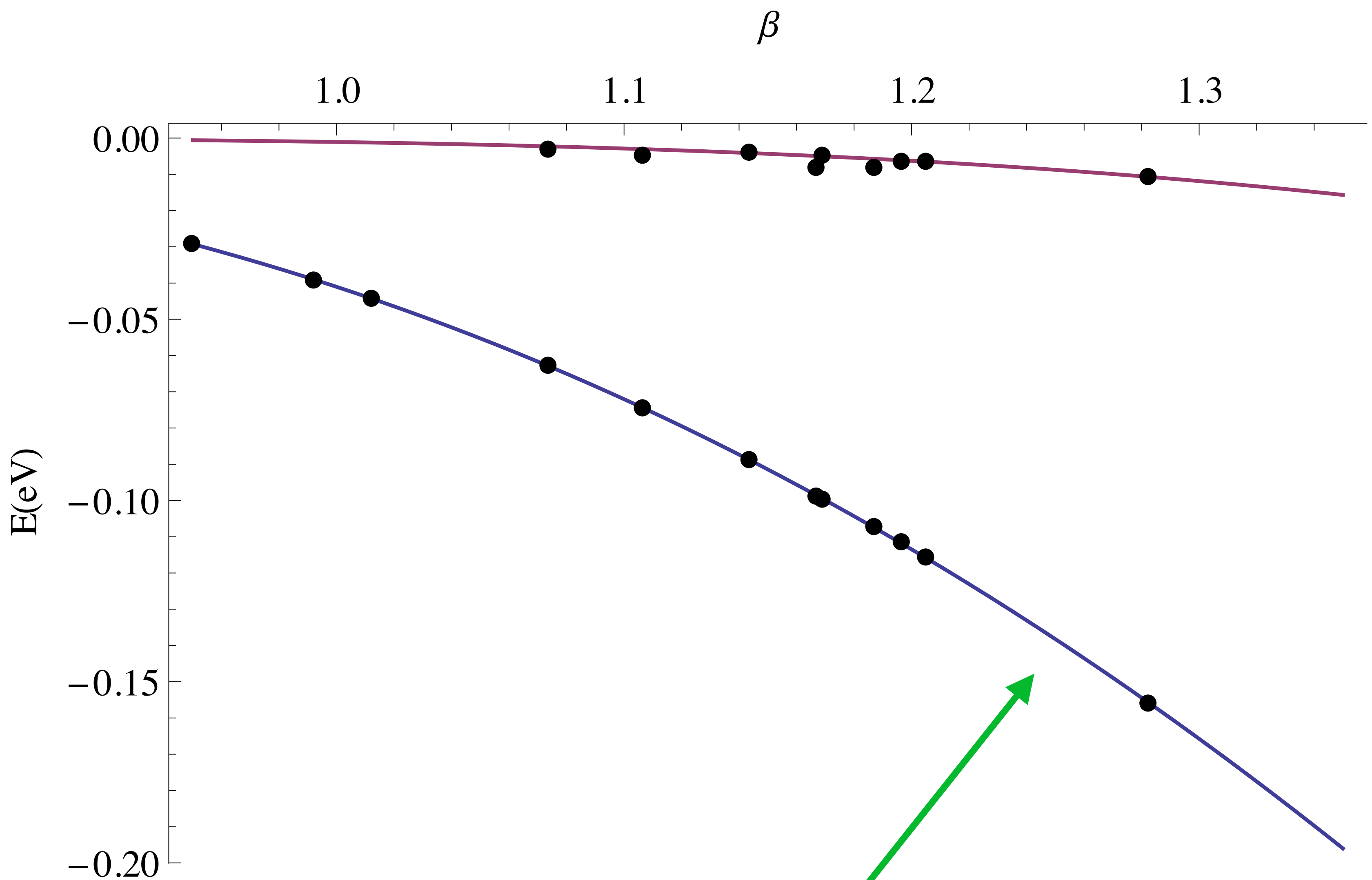
Increasing the charge, quasi-bound states are trapped. For a large enough coupling, a discrete set of Efimov states shows up.





$$\beta > \beta_{cr} \equiv \frac{1}{2}$$

The Efimov universal spectrum



$$\beta > \beta_{cr} \equiv \frac{1}{2}$$

$n = 2$

$n = 1$

$$E_n = -\varepsilon_0\, e^{-\frac{\pi n}{\sqrt{\beta^2 - \beta_{cr}^2}}}$$

The Efimov universal spectrum

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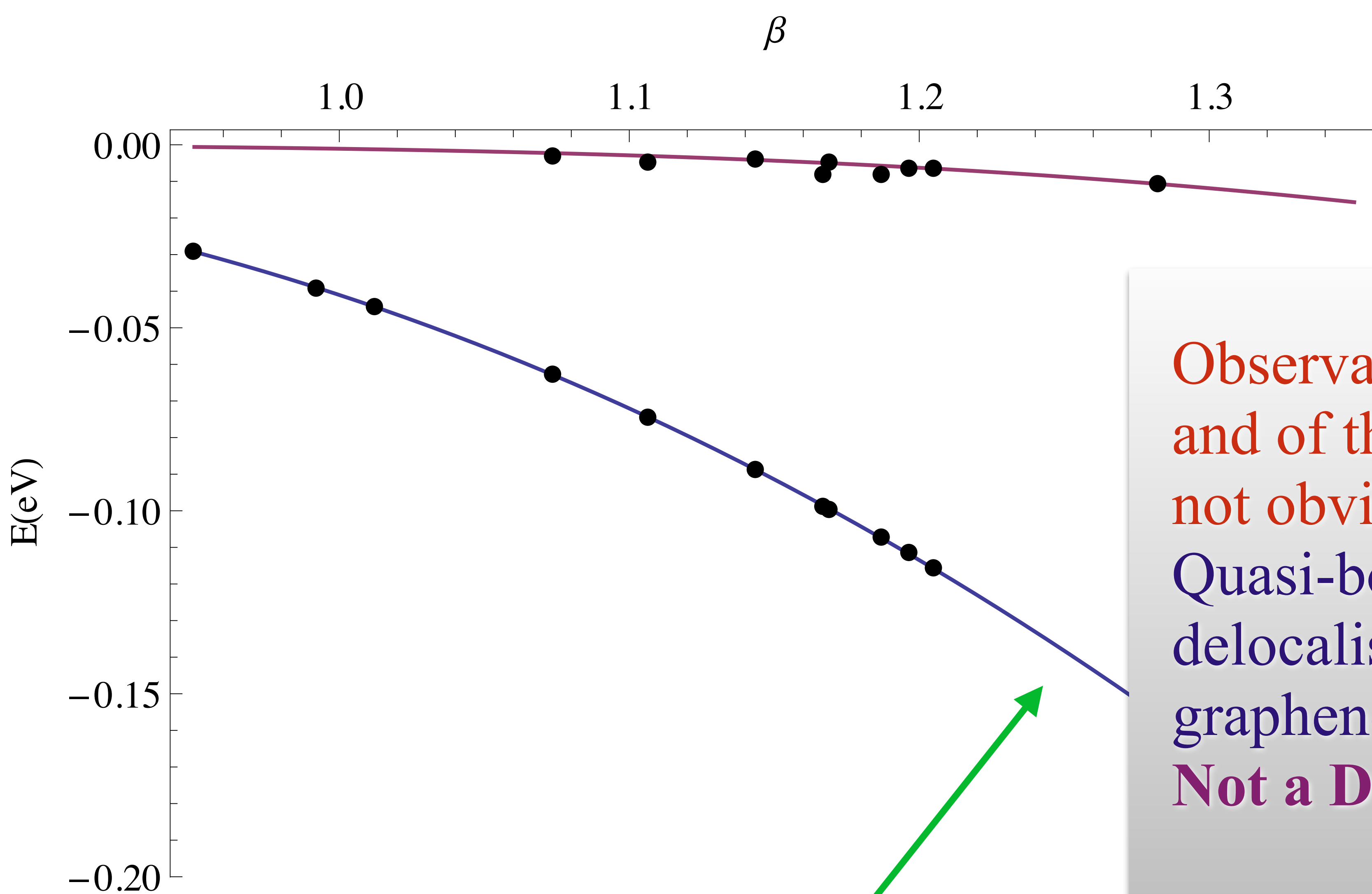

Observation of the universal spectrum and of the quantum phase transition is not obvious :

Quasi-bound states are the response of delocalised conduction electrons in graphene to the local charge.

Not a Dirac hydrogen atom !

$$E_n = -\varepsilon_0\, e^{-\frac{\pi n}{\sqrt{\beta^2-\beta_{cr}^2}}}$$

What is Efimov physics ?

Universality in cold atomic gases : Efimov physics

DSI in the non relativistic quantum 3-body problem

Universality in cold atomic gases : Efimov physics

non relativistic quantum 3-body problem

Efimov (1970) analysed the 3-nucleon system interacting through zero-range interactions (r_0). He pointed out the existence of universal physics at low energies, $E \ll \hbar^2 / m r_0^2$

Universality in cold atomic gases : Efimov physics

non relativistic quantum 3-body problem

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When the scattering length a of the 2-body interaction becomes $a \gg r_0$ there is a sequence of 3-body bound states whose binding energies are spaced geometrically in the interval between $\hbar^2 / m a^2$ and $\hbar^2 / m r_0^2$

As $|a|$ increases, new bound states appear according to

$$E_n = -\varepsilon_0 \, e^{-\frac{2\pi n}{s_0}}$$

Efimov spectrum

where $s_0 \approx 1.00624$ is a universal number

Efimov showed that the corresponding 3-body problem reduces to an effective Schrödinger equation with the attractive potential :

$$V(r) = -\frac{s_0^2 + 1/4}{r^2}$$

Efimov physics is always super-critical :

Schrodinger equation with an effective attractive potential $(d=3)$:

$$V(r) = -\frac{s_0^2 + 1/4}{r^2} \qquad s_0 \approx 1.00624$$

$$\zeta_{cr} = \frac{(d-2)^2}{4} = \frac{1}{4} \quad \Longrightarrow \quad \text{Efimov physics occurs at :}$$

$$\zeta_E = s_0^2 + 1/4 = 1.26251 > \zeta_{cr}$$

ζ_E is <u>fixed</u> in Efimov physics. It cannot be changed !

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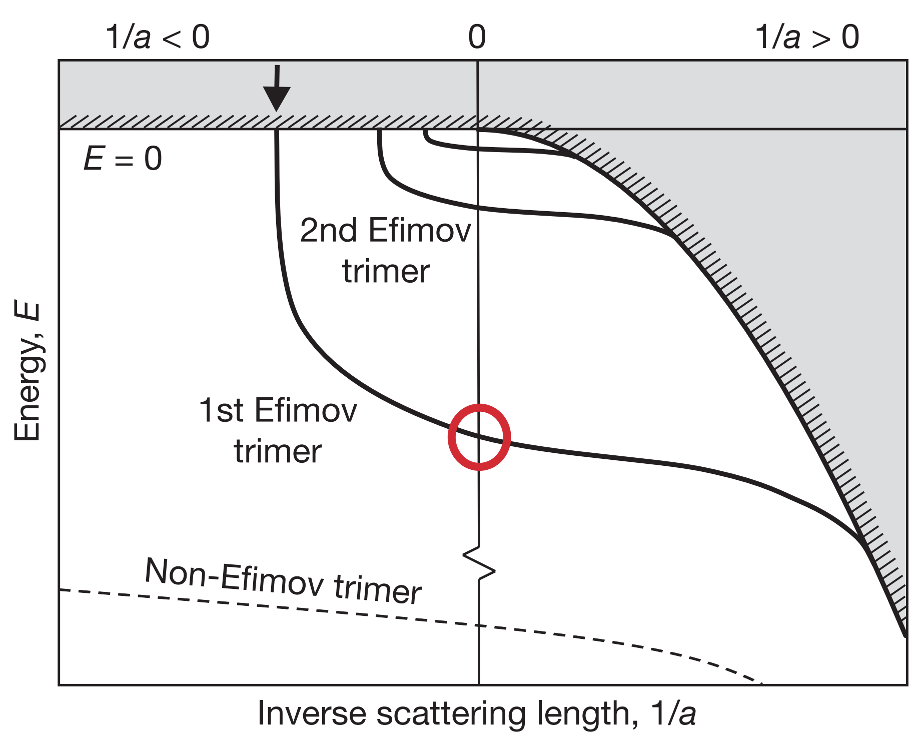
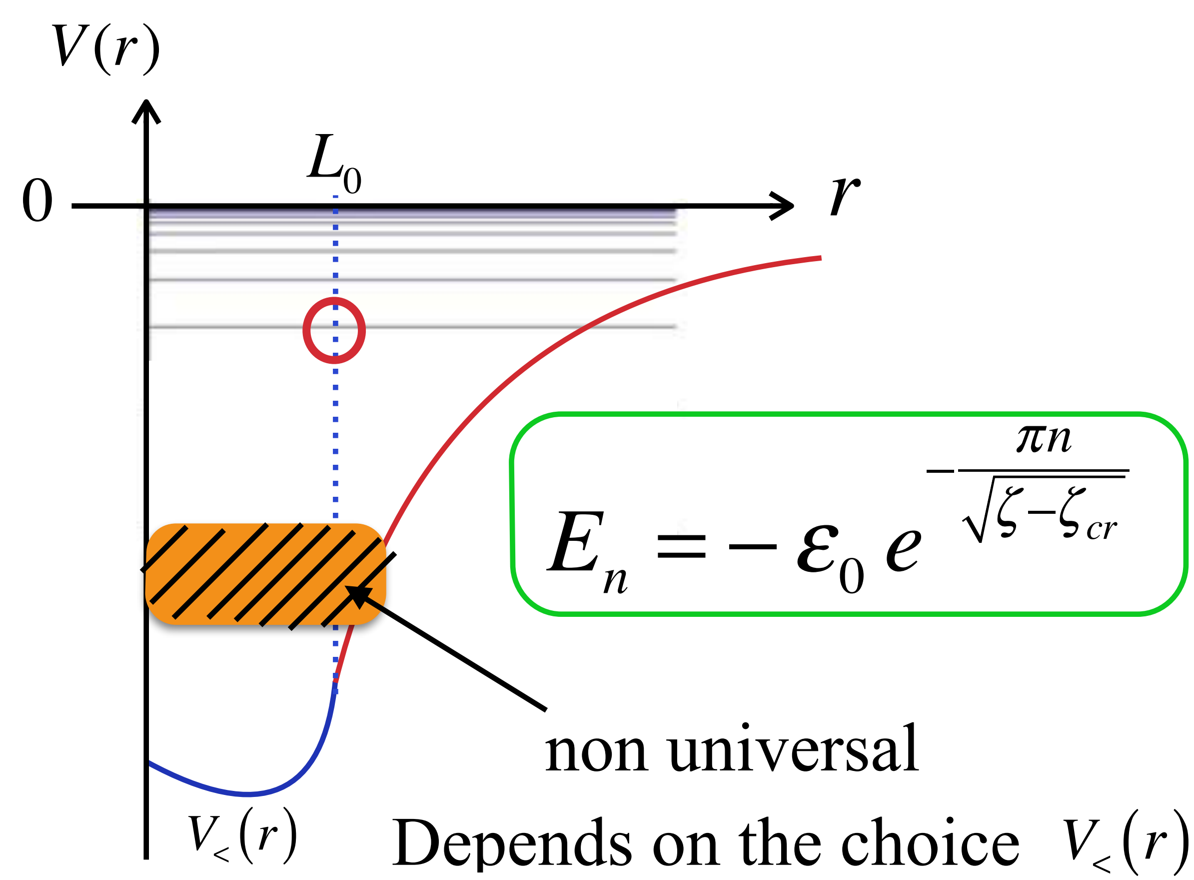
Measurement of a single Efimov state : n=1





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Selected for a Viewpoint in *Physics*

PRL **112,** 190401 (2014) PHYSICAL REVIEW LETTERS week ending 16 MAY 2014

Observation of the Second Triatomic Resonance in Efimov's Scenario

Bo Huang (黄博),[1] Leonid A. Sidorenkov,[1,2] and Rudolf Grimm[1,2]

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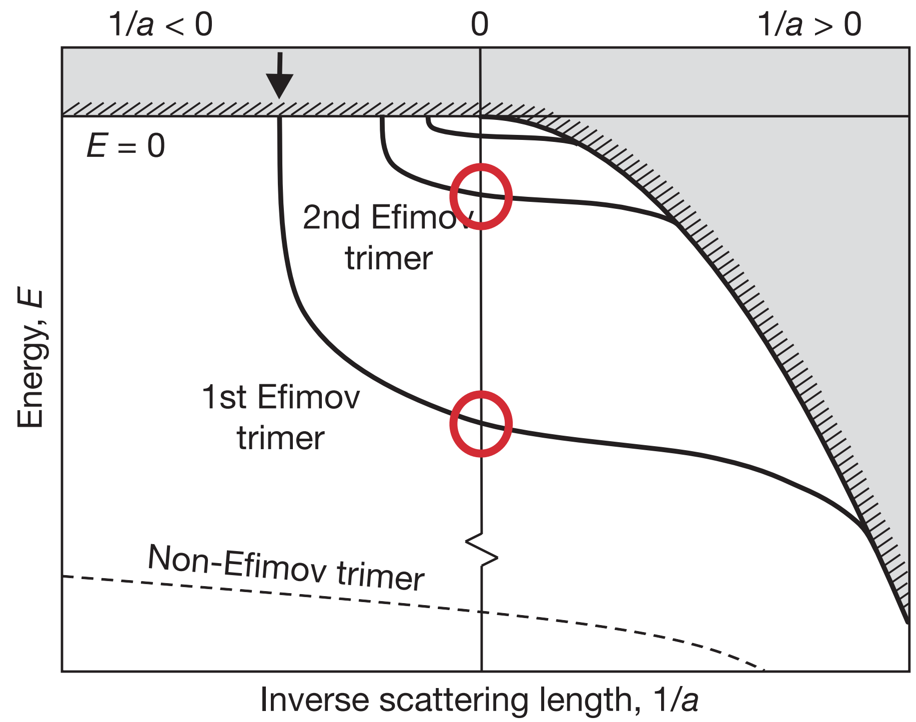
(Received 26 February 2014; published 12 May 2014)

Measurement of a second Efimov state : n=2



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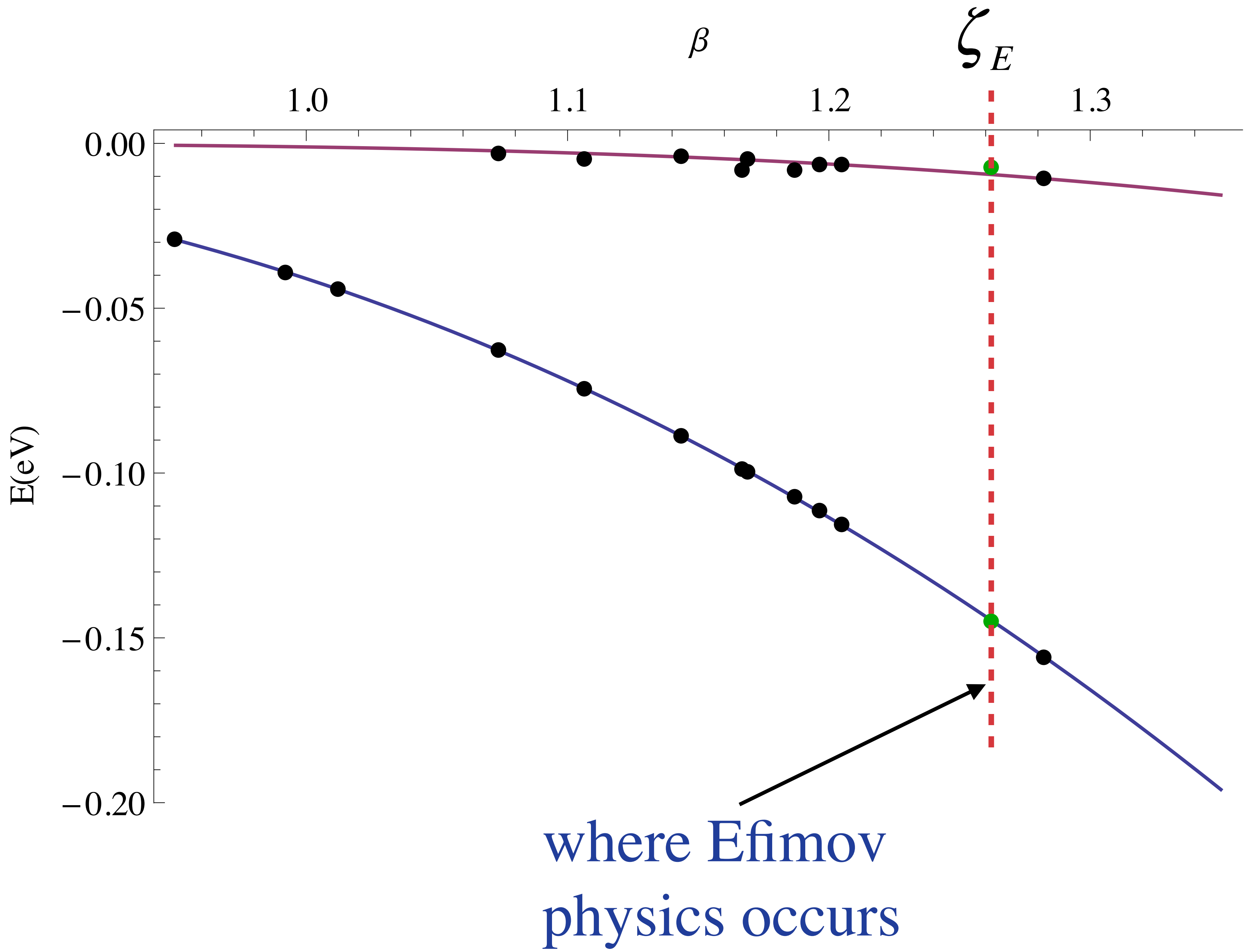
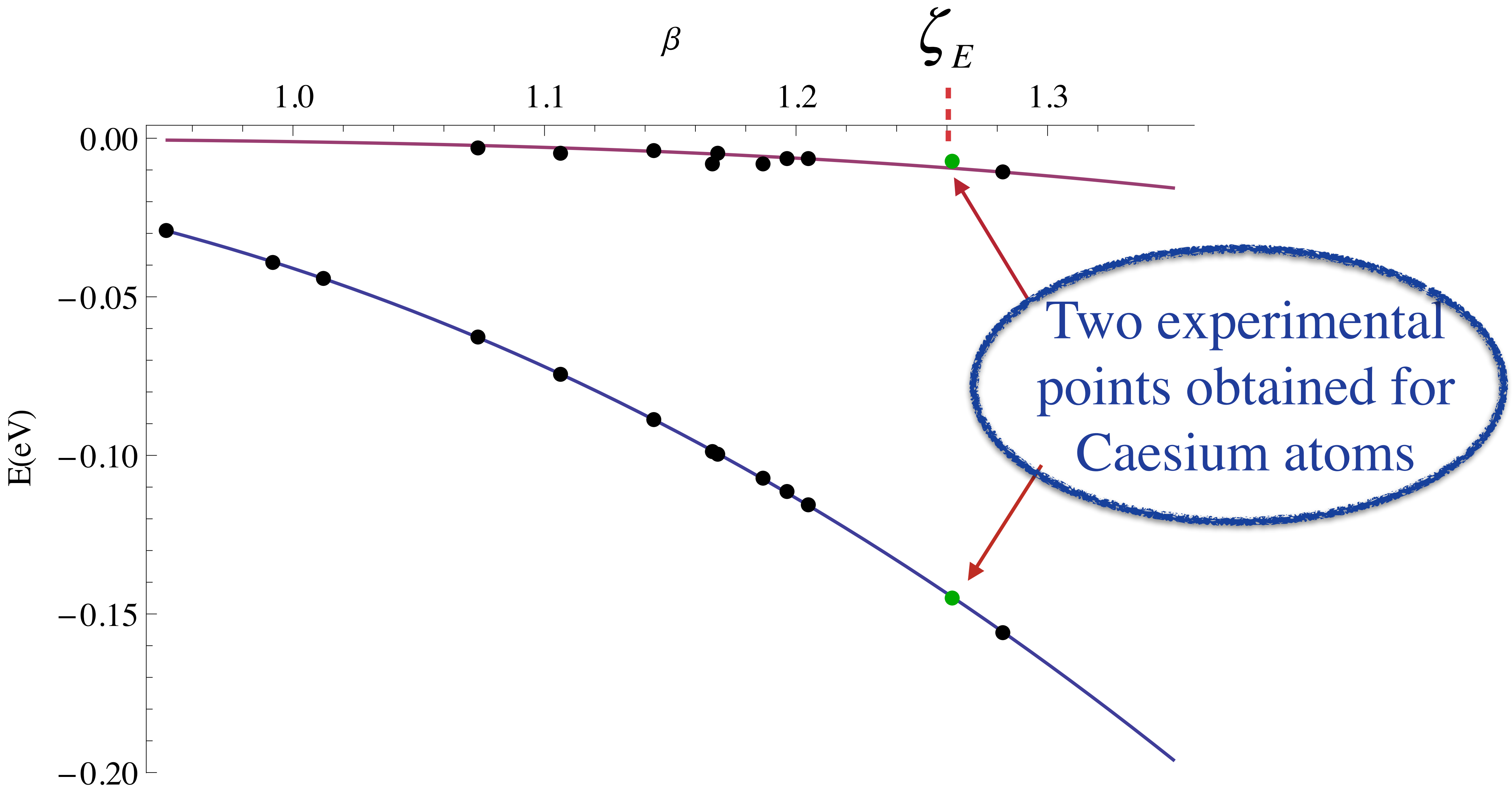
Universality



Universality



Not obvious at all ! Two very different physical phenomena share the same universal energy spectrum.

Summary-Further directions

- Breaking of continuous scale invariance (CSI) into discrete scale invariance (DSI) on two examples.
- Observed this quantum phase transition on graphene. It raises more questions than it solved.
- Efimov physics belongs to this universality class. It does not allow observing the transition.
- Breaking of the CSI is interpreted using the Renormalisation Group picture : stable fixed points evolve into limit cycles. Proposal of K.G. Wilson: RG and strong interactions, 1971.

- Other problems can be described similarly as “conformality lost” (Kaplan et al., 2009) and emergence of limit cycles:

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  - Kosterlitz-Thouless transition (deconfinement of vortices in the XY-model at a critical temp. above which the theory is conformal): mapping between the XY-model and the T=0 sine-Gordon in 1+1 dim.

$$L = \frac{T}{2}\left(\partial_\mu \phi\right)^2 - 2z\cos\phi$$

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  - Breitenlohner-Freedman bound for free massive scalar field on AdS_{d+1} space.

Thank you for your attention.

arXiv:1701.04121 (in press)