Topology of quasicrystals :
Measuring topological numbers with waves

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Department of Physics, UBC, Vancouver, August 2017

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Alberto Amo (LPN, Marcoussis)
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Program for today

- Fractal spectrum and quasicrystals
- Basics - Gap labelling theorem - the Fibonacci spectrum (experiments)
- Topological meaning of GLT : winding numbers
- Topological numbers from *structure* - Bragg (diffraction).

Fractal spectrum

Quasi-periodic structure (Fibonacci quasi-crystal)

S_0 B $\mathcal{F}_0=1$

S_1 A $\mathcal{F}_1=1$

S_2 A B $\mathcal{F}_2=2$

S_3 A B A $\mathcal{F}_3=3$

S_4 A B A A B $\mathcal{F}_4=5$

S_5 A B A A B A B A $\mathcal{F}_5=8$

S_6 A B A A B A B A A B A A B $\mathcal{F}_6=13$

⋮

Leonardo Pisano
(Fibonacci)

Concatenation rule

Fibonacci sequence: $S_{j\geq 2} = \left[S_{j-1} S_{j-2} \right], \quad S_0 = B, \quad S_1 = A$

A→AB→ABA→ABAAB→ABAABABA→…

Quasi-periodic structure (Fibonacci quasi-crystal)

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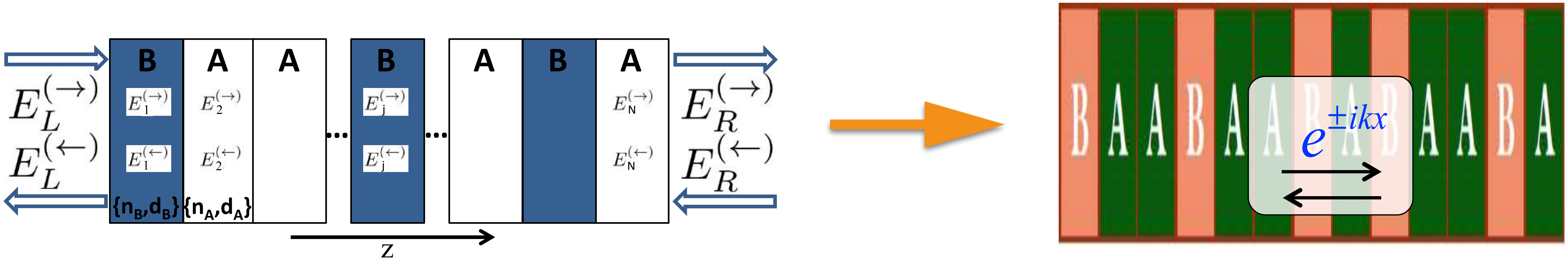
Concatenation rule

Quasi-periodic stack of dielectric layers of two types (n_A,n_B)



Spectrum of the Helmholtz equation on a quasi-periodic chain

$$-\frac{d^2\psi}{dx^2} - k_0^2 v(x)\psi(x) = k_0^2\psi(x)$$

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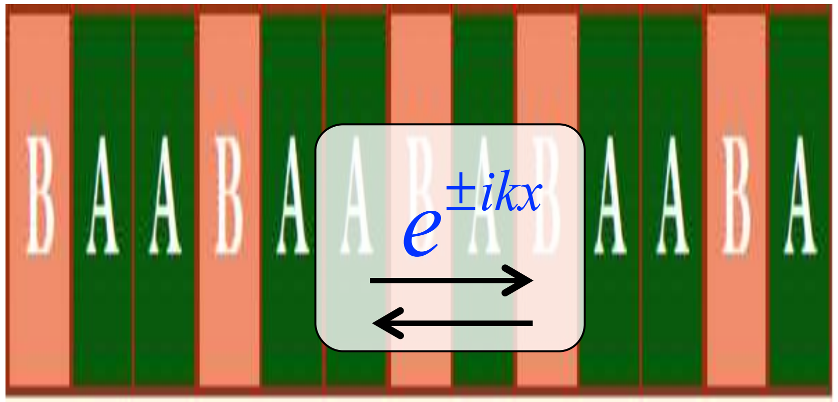
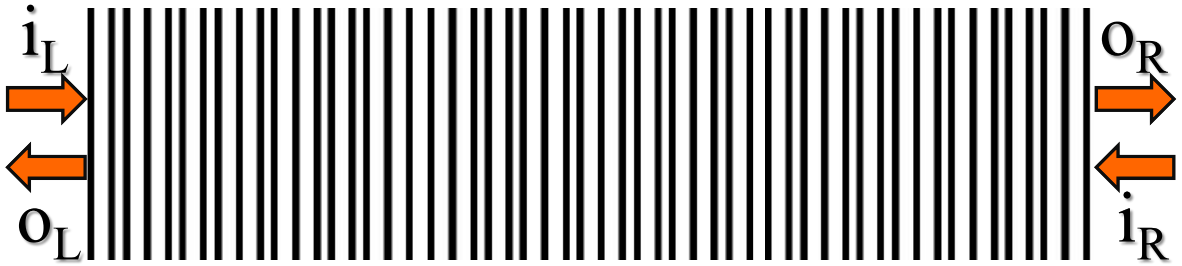
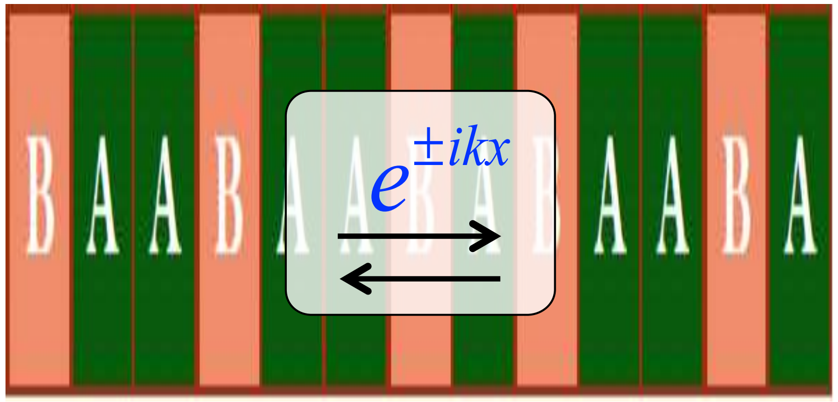
Scattering boundary conditions :



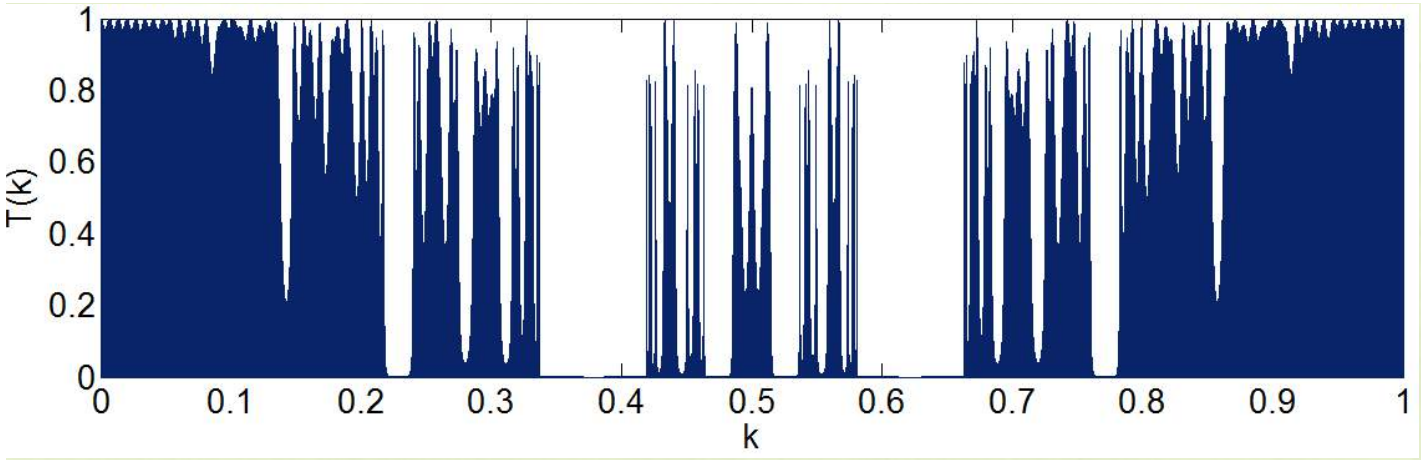
$\Leftrightarrow$



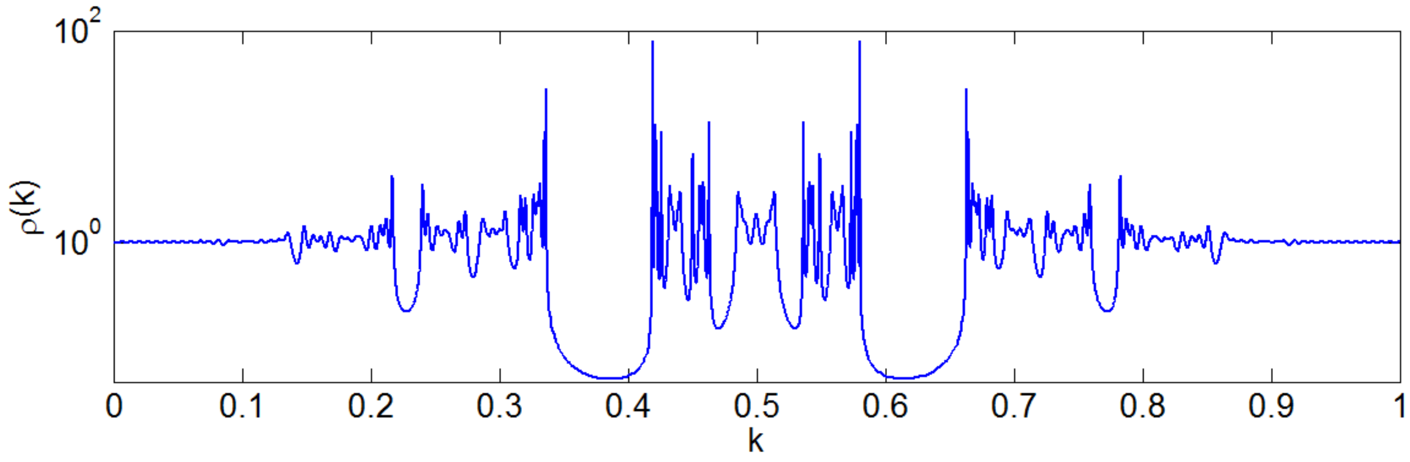
Quasi-periodic structure (Fibonacci quasi-crystal)



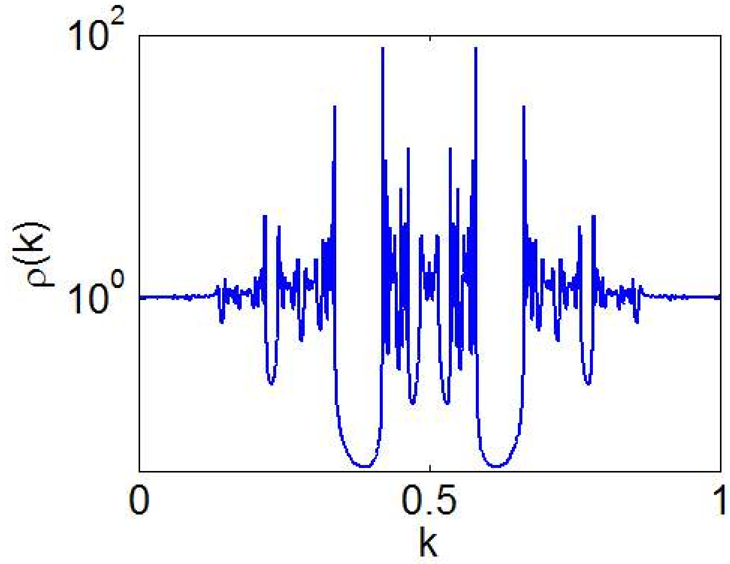
Spectrum of modes



Transmission spectrum



Density of modes



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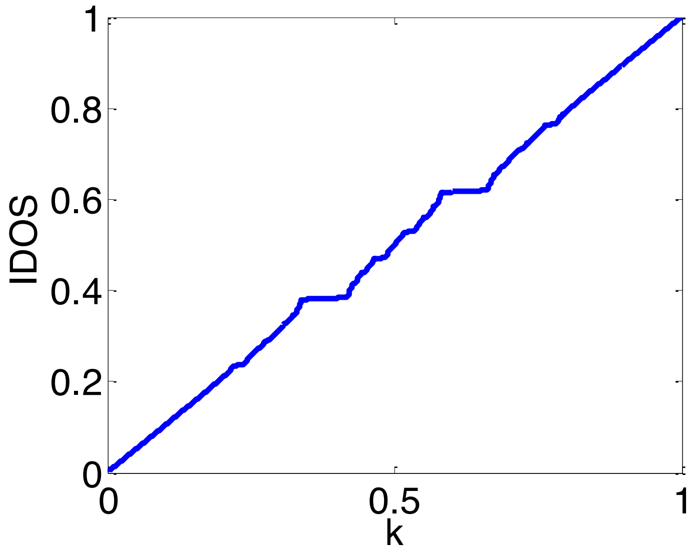
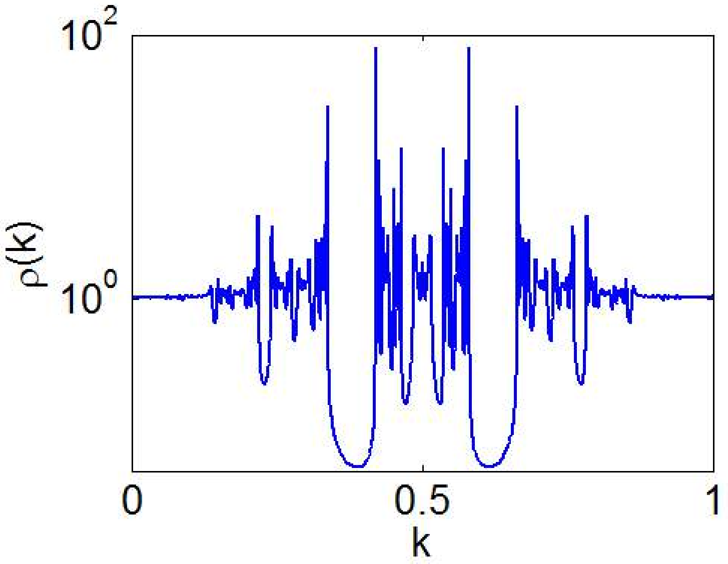


IDOM- counting function



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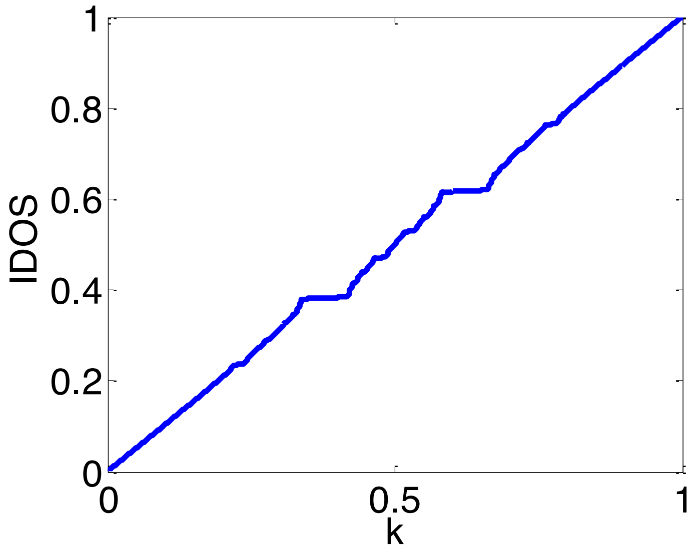
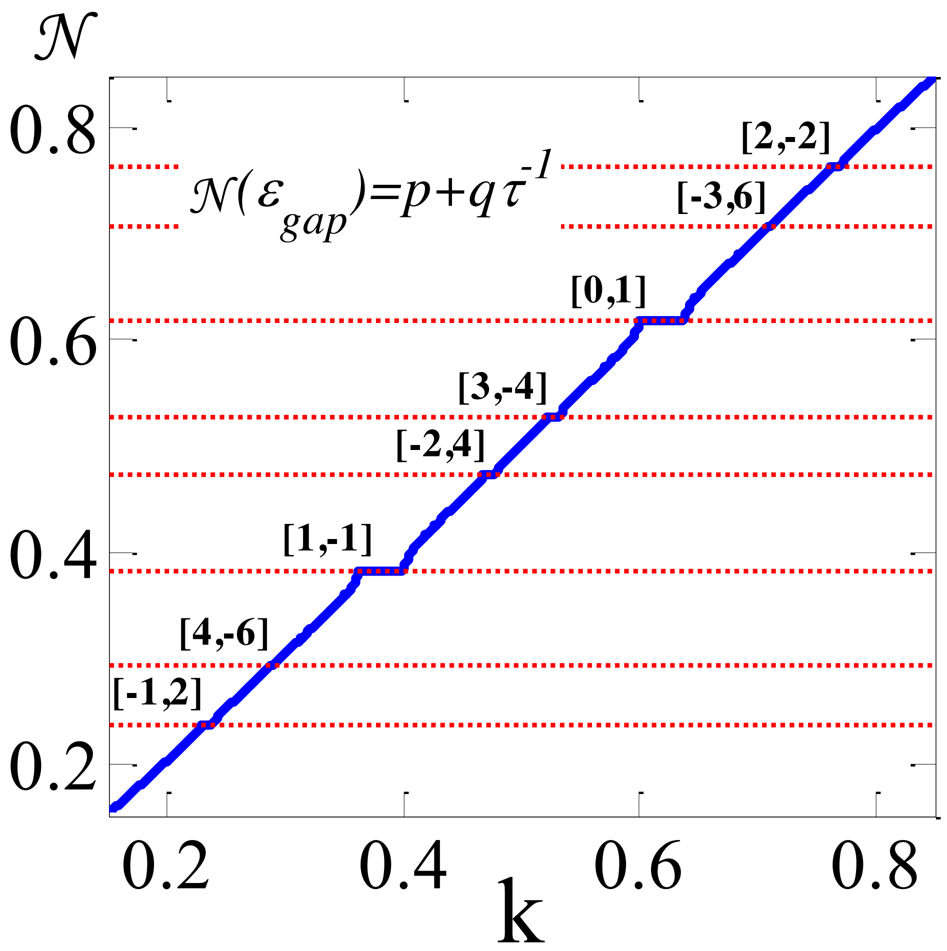
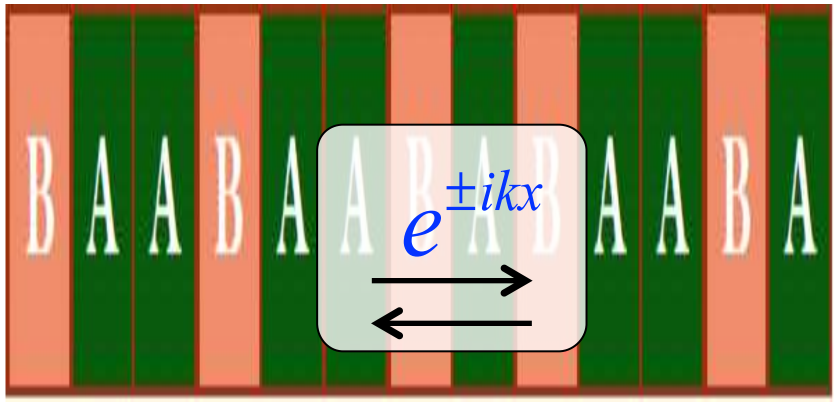


IDOS- counting function

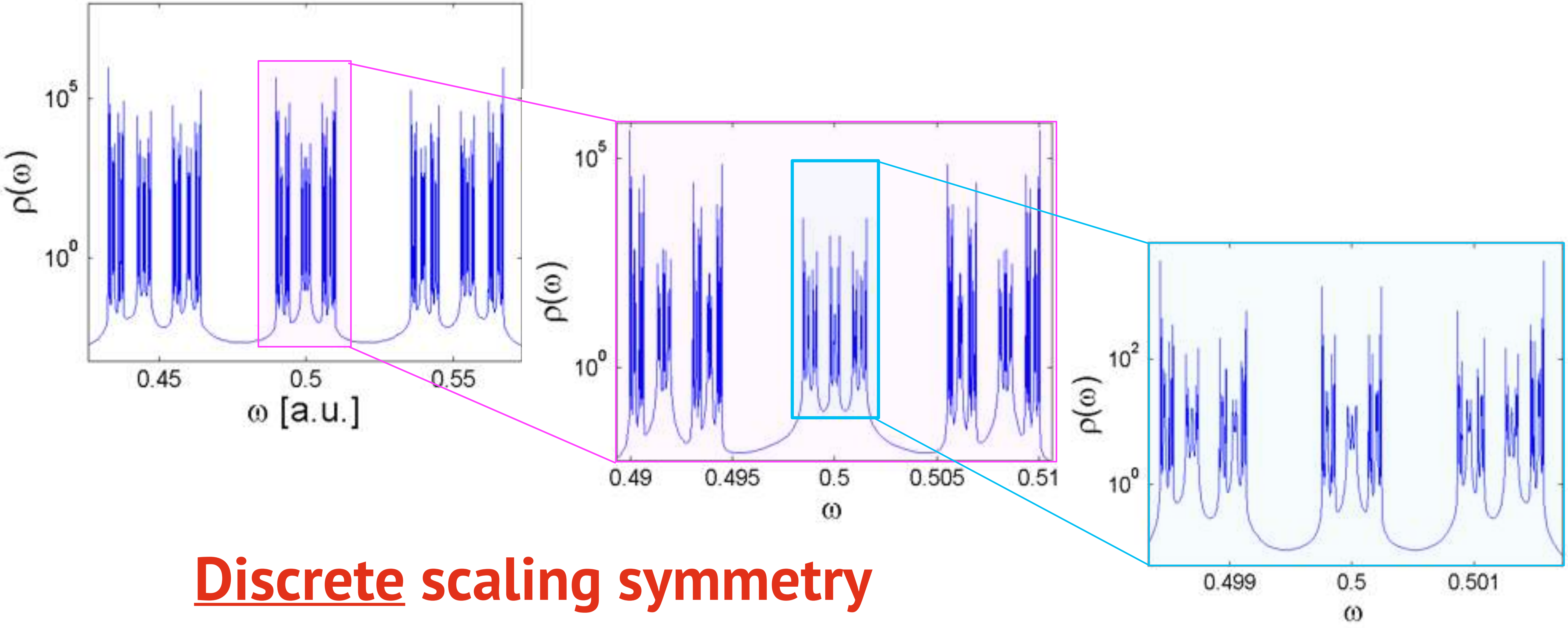


Gap Labeling Theorem (GLT)

Fractal spectrum of the Fibonacci quasi-periodic chain

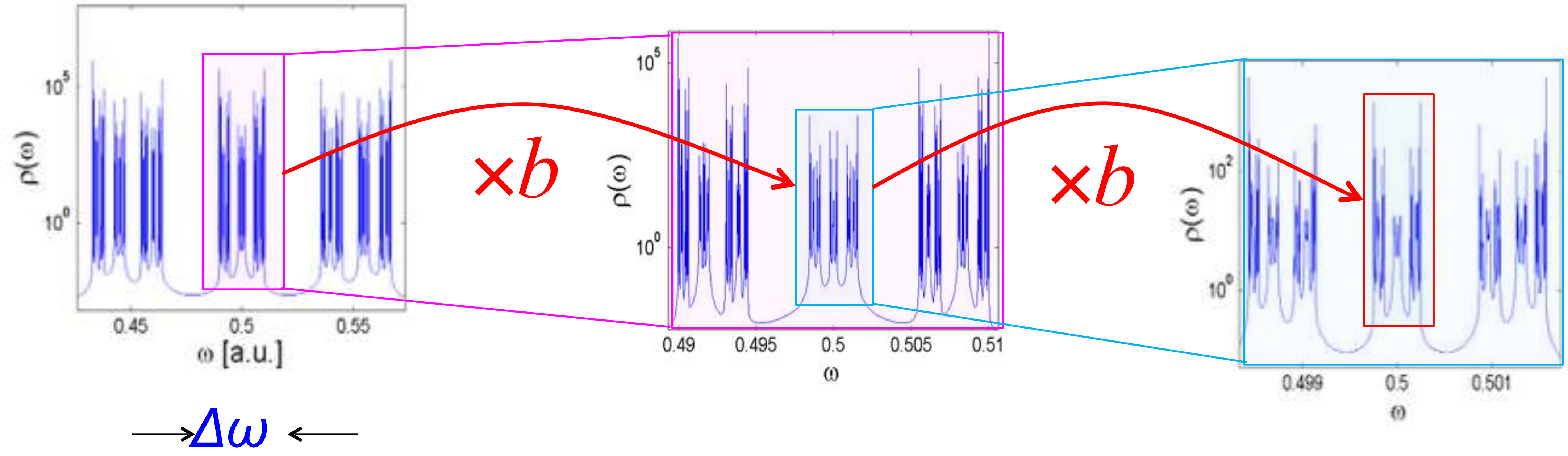


The density of modes $\rho(\omega)$:



Discrete scaling symmetry

Discrete scaling symmetry: formal description



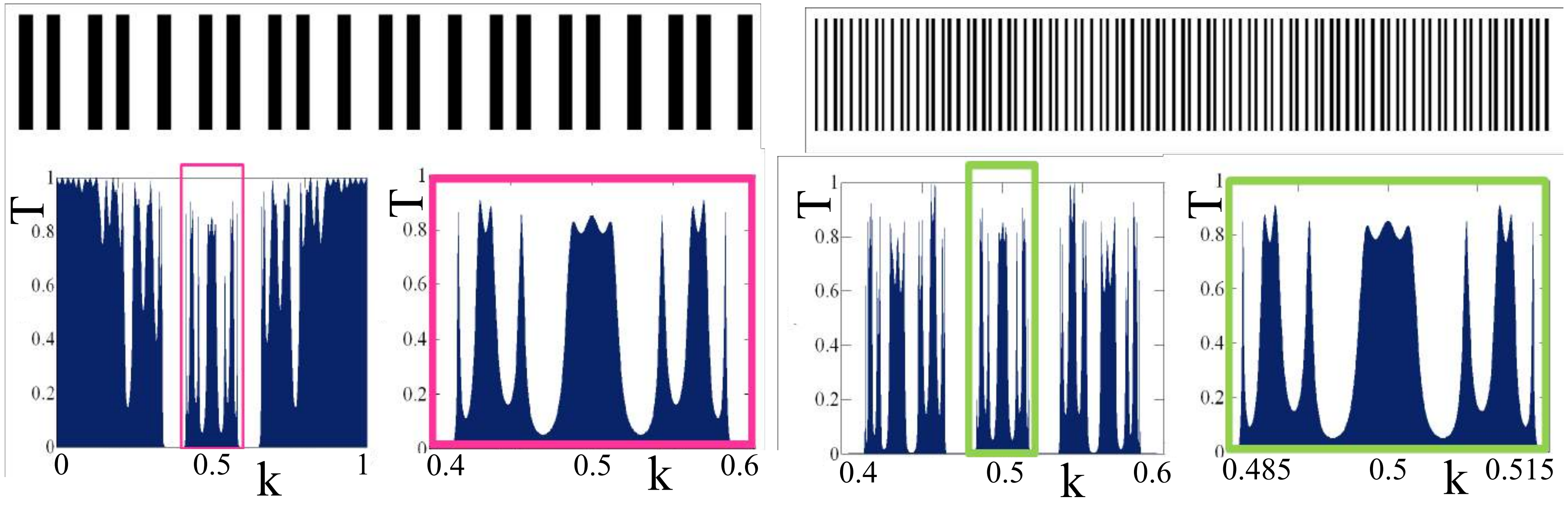
Counting function: $N_\omega(\Delta\omega) \equiv \int_{\omega}^{\omega+\Delta\omega} \rho(\omega')d\omega'$ = (# of states in $[\omega, \omega+\Delta\omega]$)

$$N_\omega(b^p\Delta\omega) = a^p N_\omega(\Delta\omega)\,, \quad p \in \mathbf{Z}$$

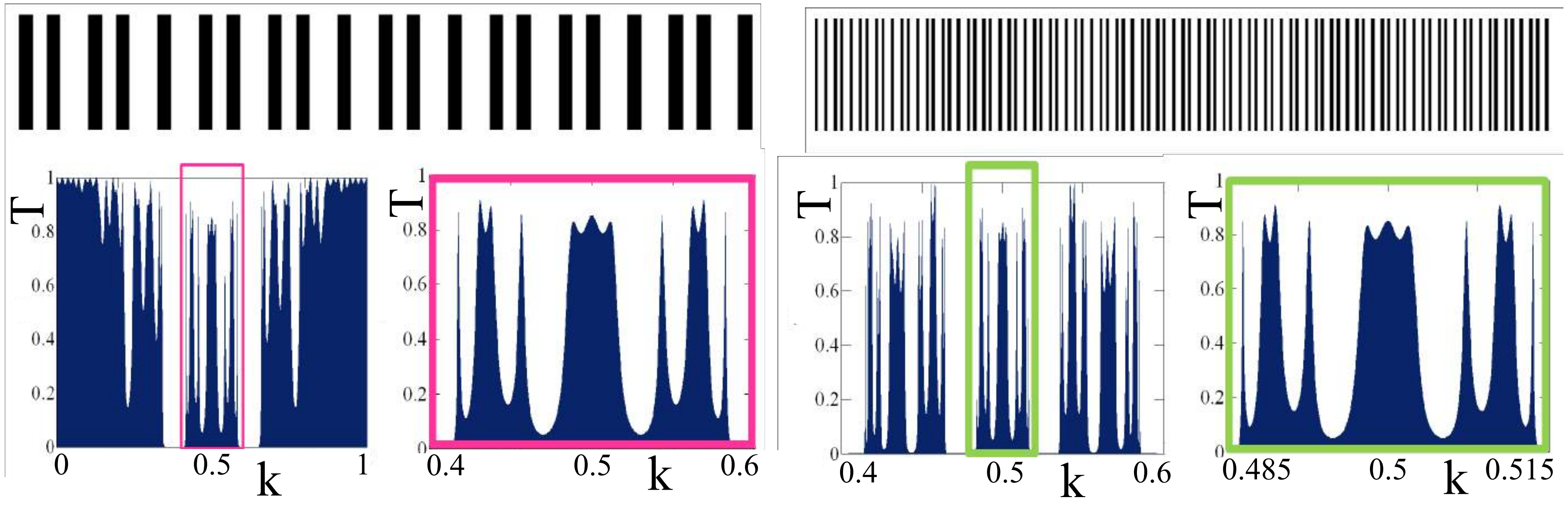
b, a - fixed scaling factors

⇒ **Discrete scaling symmetry**

Testing the discrete scaling symmetry



Testing the discrete scaling symmetry



$$N_k^{(g)}(\Delta k) \equiv \int g\left(\frac{k'-k}{\Delta k}\right)\rho(k')dk' = (\Delta k)^{\alpha} \times F_g\left(\frac{\ln|\Delta k|}{\ln b}\right)$$

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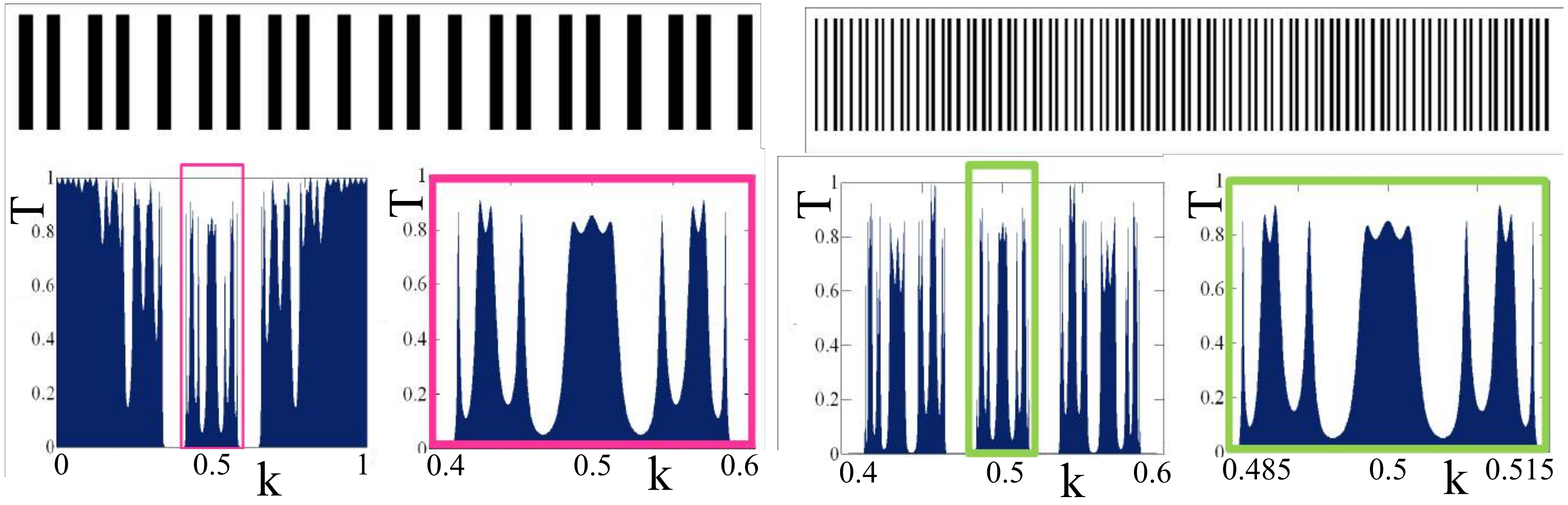

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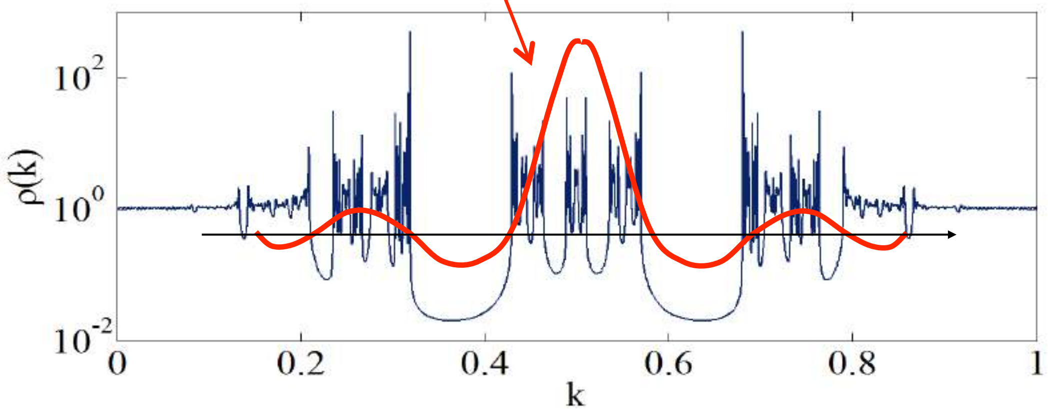
$$g(x) = \frac{\sin(x)}{\pi x}$$



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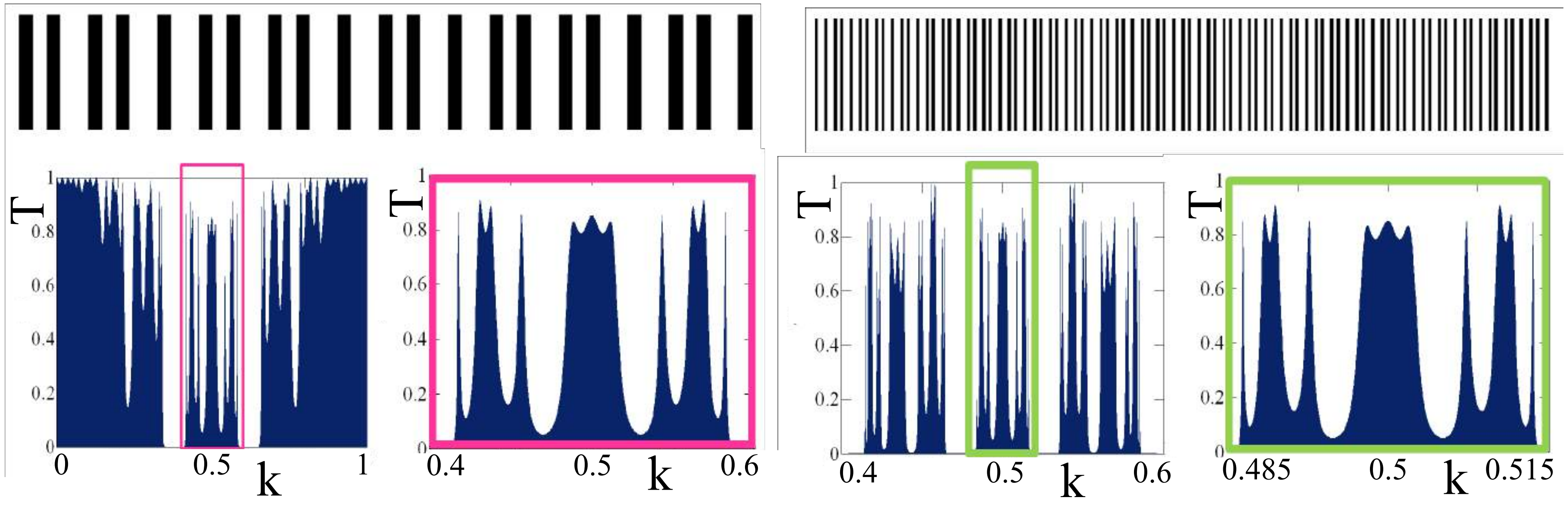

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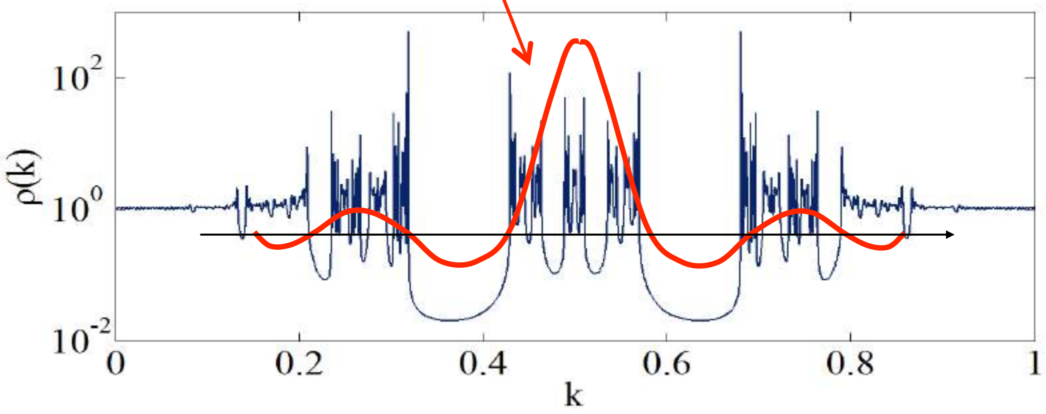
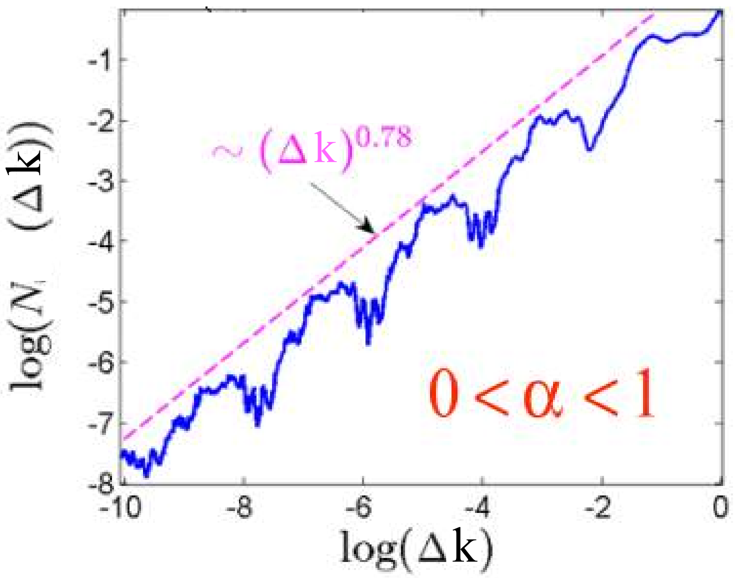


numerics



Summary

A quasi-periodic dielectric chain

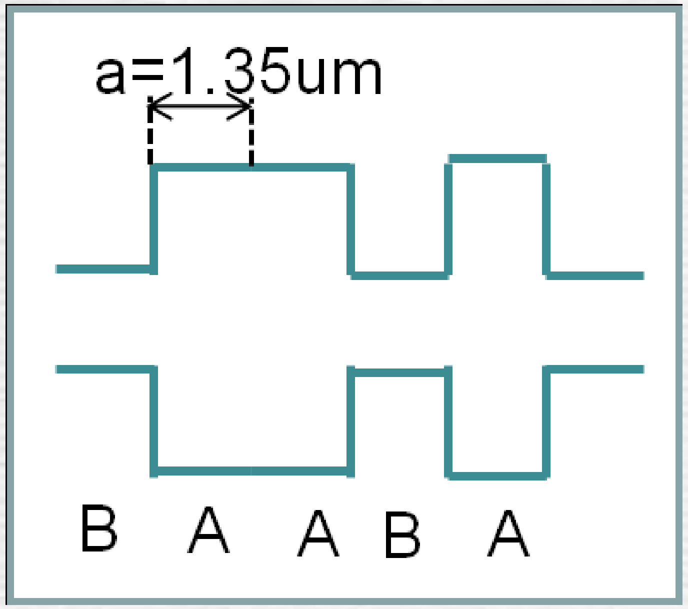
does not have a geometric fractal structure, but,

its spectrum has a fractal structure :

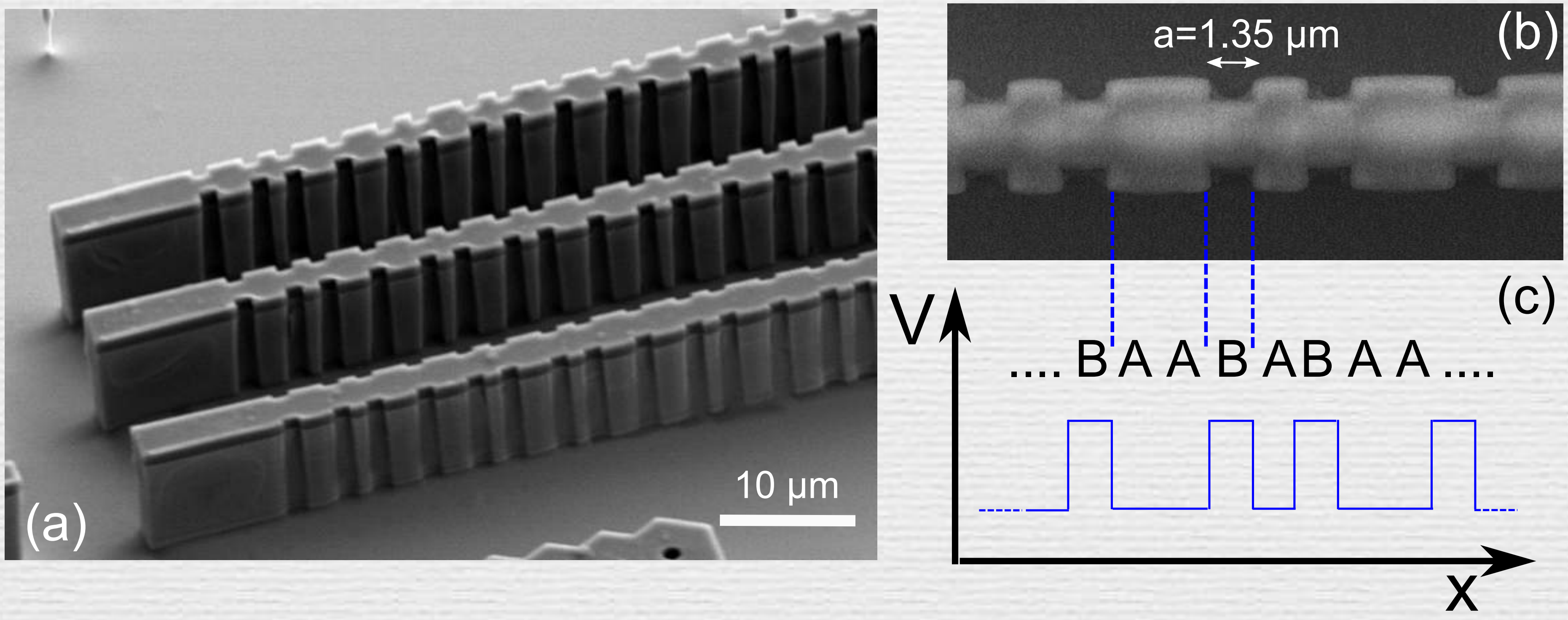
$$N_\omega(\Delta\omega) = (\Delta\omega)^{\alpha} \times F\left(\frac{\ln|\Delta\omega|}{\ln b}\right), \qquad \alpha = \frac{\ln a}{\ln b}, \quad F(x+1) = F(x)$$

Spectral fractal dimension

Experimental study of a fractal energy spectrum

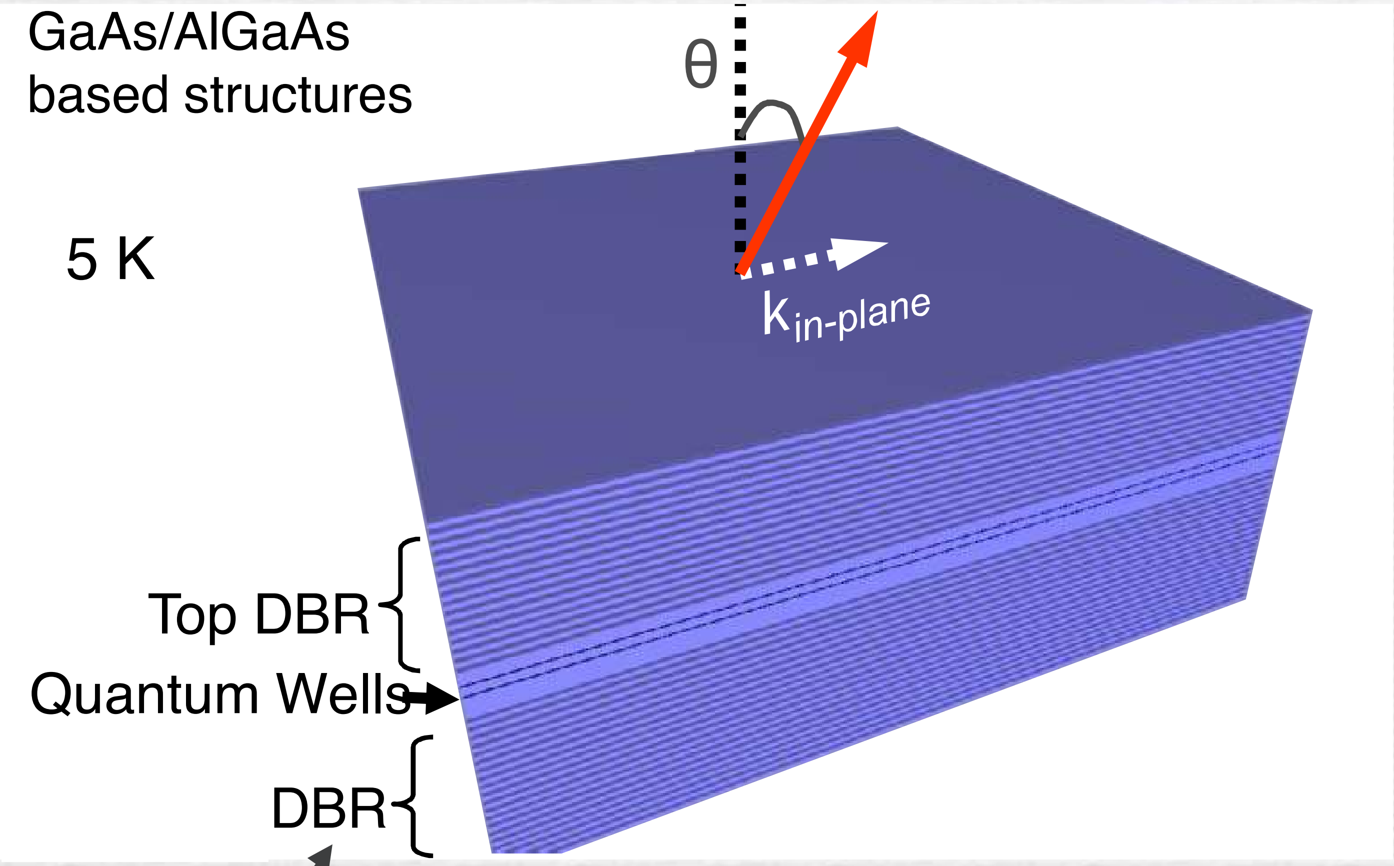


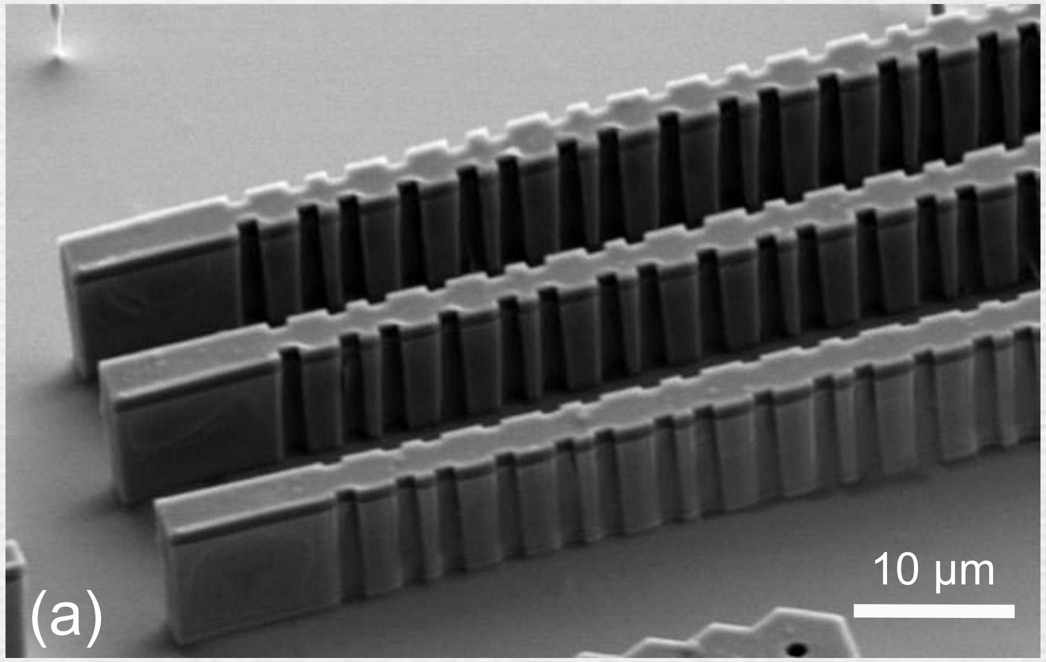
Number of letters of a sequence S_j is the Fibonacci number F_j so that $F_j = F_{j-1} + F_{j-2}$



(233 letters)

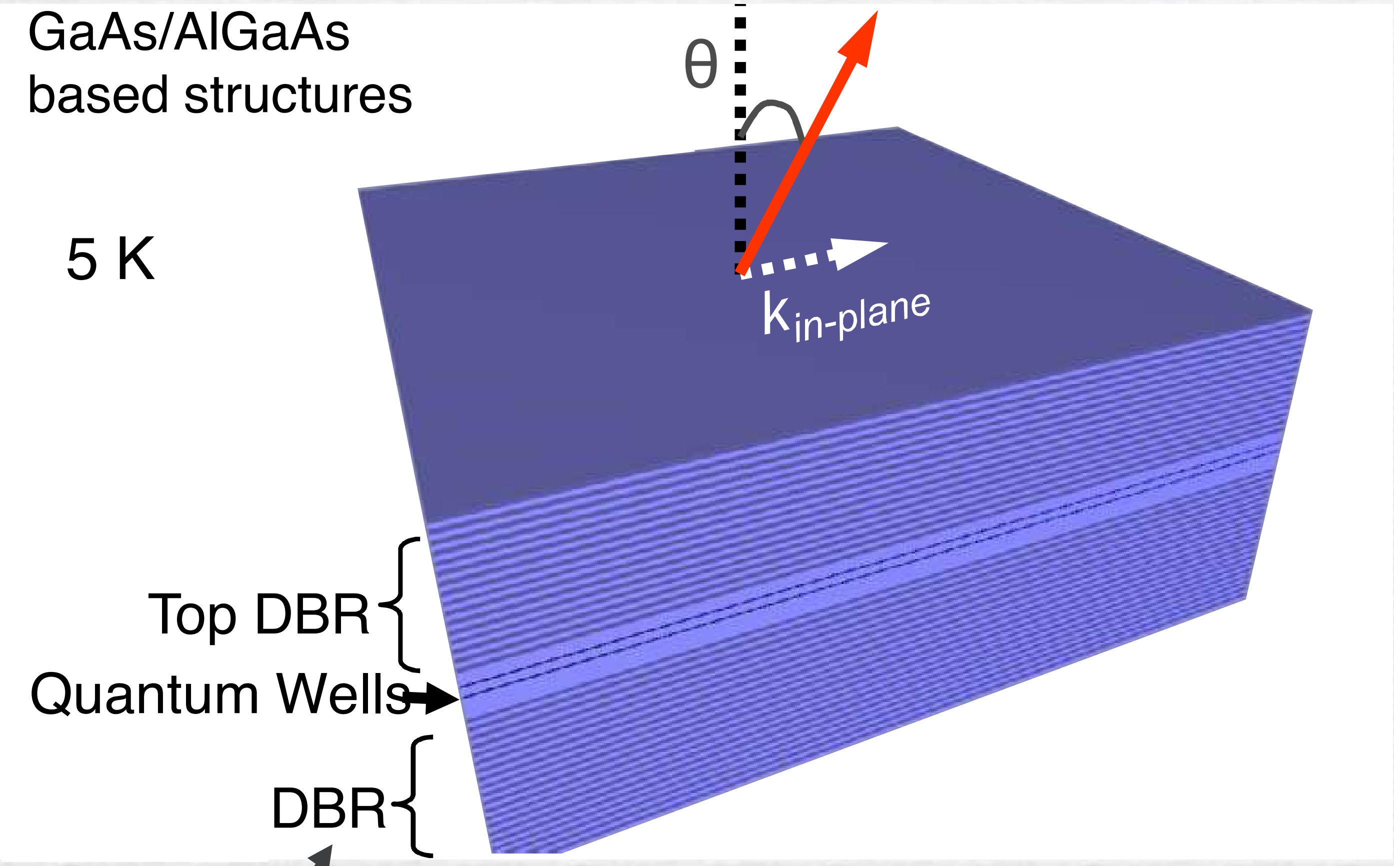
Cavity polaritons

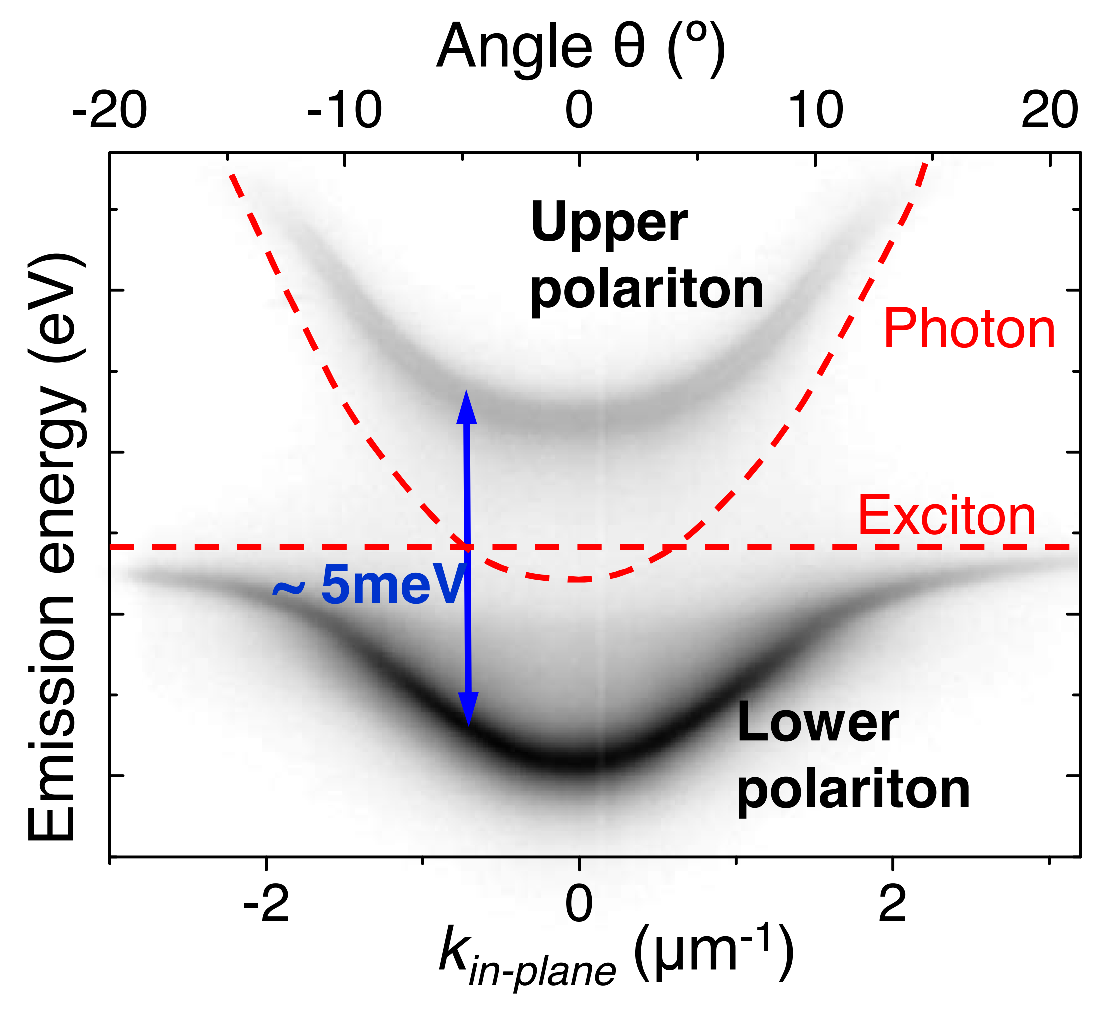




(Distributed) Bragg reflectors

Cavity polaritons





(Distributed) Bragg reflectors

Cavity polaritons :

between an optical cavity mode and confined excitons (quantum wells)

Massive photons

Cavity polaritons obey a d=2 *Schrödinger* equation

$$E\psi(x,y) = -\frac{\hbar^2}{2m_{ph}}\Delta_\perp \psi(x,y)$$

Effective photon mass $m_{ph} = {n^2 E_c}/{c^2}$

n = effective refraction index, $\Delta_\perp \equiv \partial_x^2 + \partial_y^2$

$E_c = \frac{\hbar c}{n} k_z$ = energy of the fundamental mode of the cavity

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Eigenmodes of the d=2 problem ⟶ numerics

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Eigenmodes of the d=2 problem ⟶ numerics

Well controlled d=1 effective model is preferable !

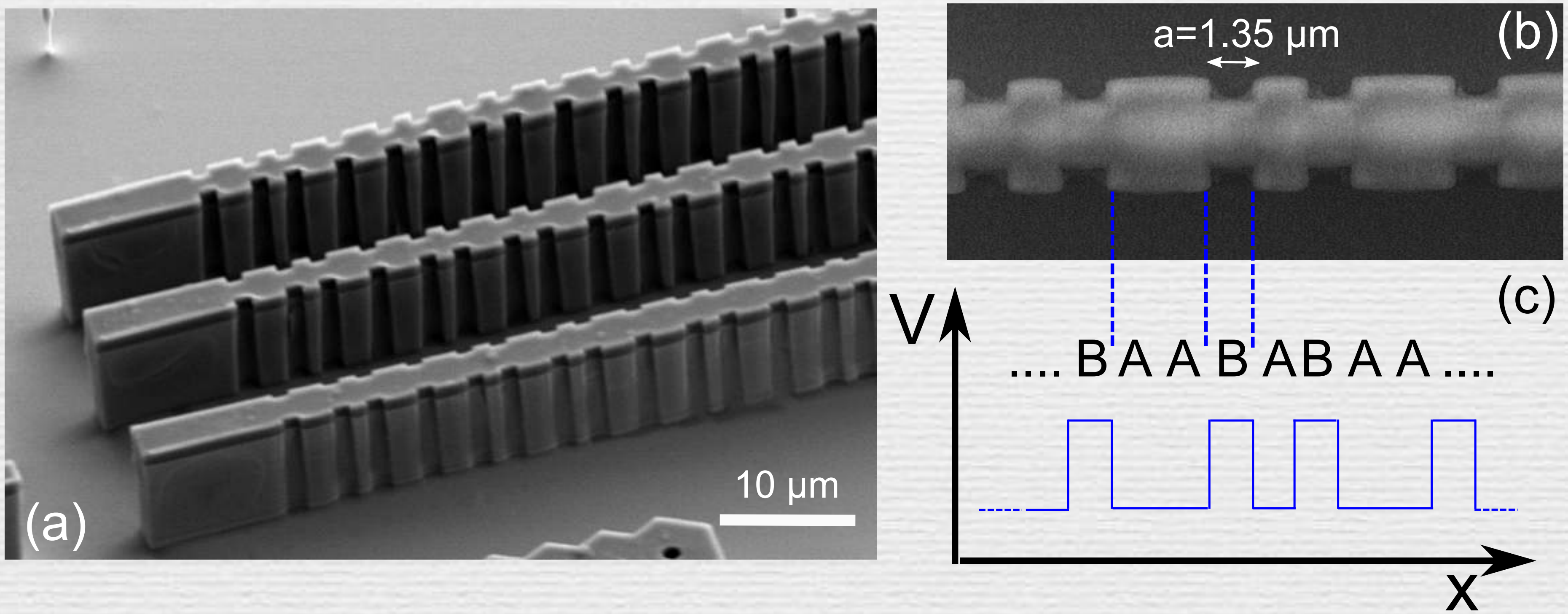
$$E\varphi(x)=\frac{\hbar^2}{2m_{ph}}\left[-\frac{d^2}{dx^2}+V(x)\right]\varphi(x)$$

Effective 1D model

$$\left[-\frac{\hbar^2}{2M}\frac{d^2}{dx^2}+V(x)\right]\psi(x)=E\psi(x)$$

with

$$V(x)=\sum_n \chi\left(\tau^{-1}n\right)u_b\left(x-an\right)$$



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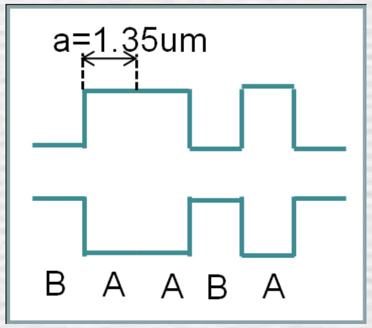
$$\chi(x)=\begin{cases}1, & 0<x<2-\tau\\ 0, & 2-\tau<x<1\end{cases}$$

Characteristic function

Shape of each letter



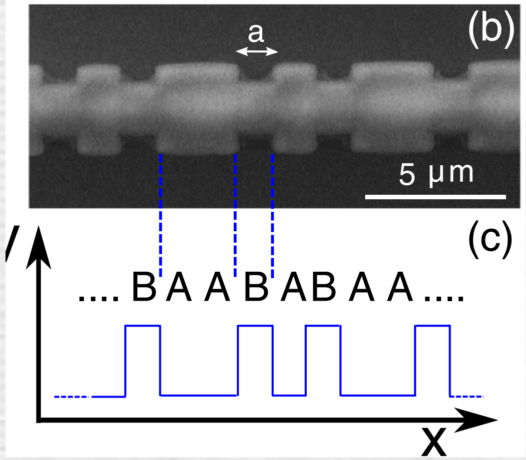
$\tau = \sqrt{5}+1 \big/ 2 \approx 1.62$ is the golden mean

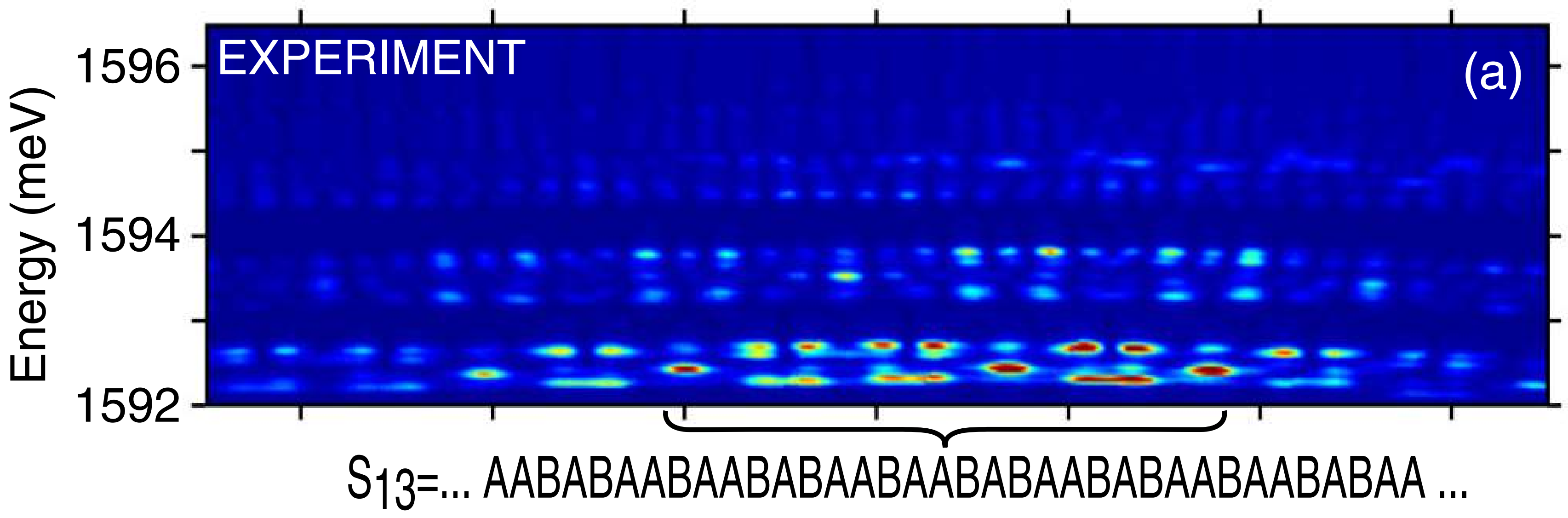
Advantages of cavity polaritons :

allow for a

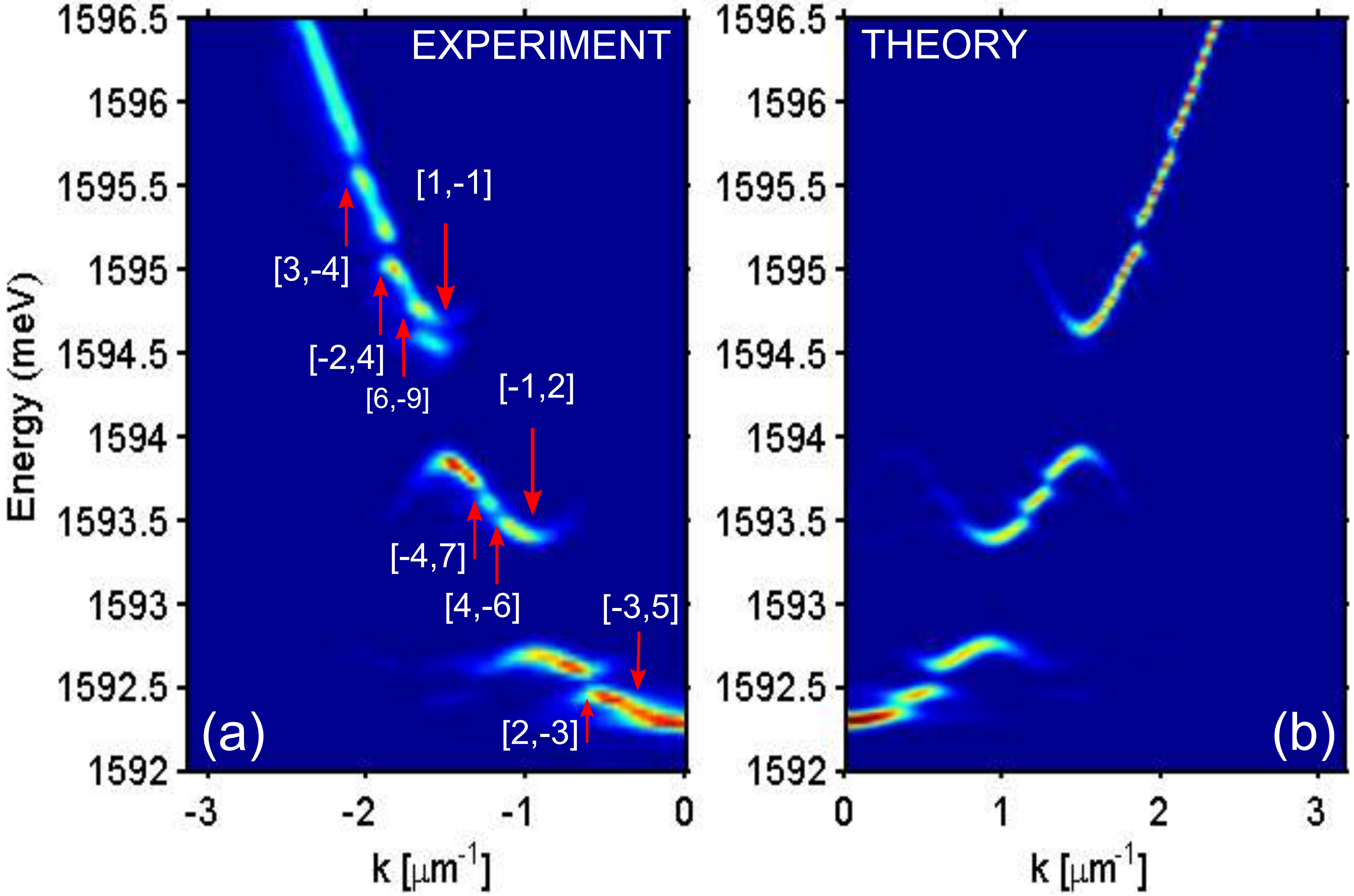
excitations in real and momentum spaces.

⟶ Visualisation/imaging of individual eigenmodes





Labeling the gaps...



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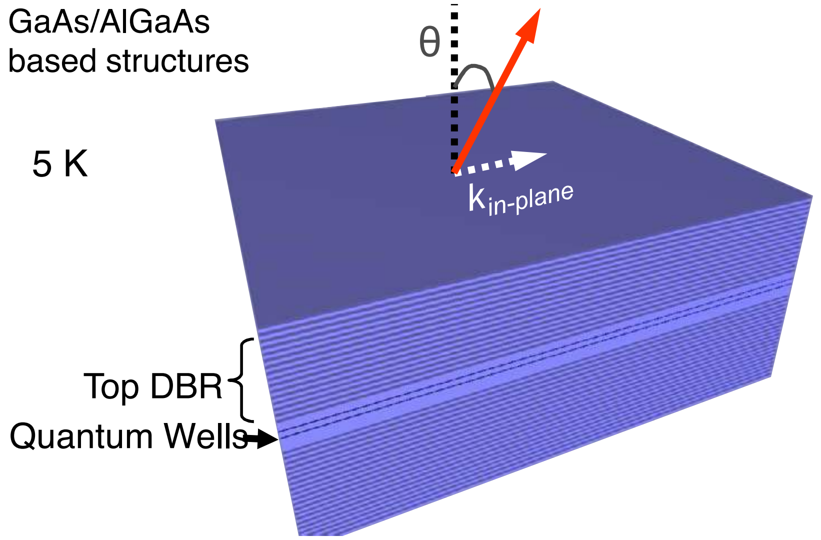
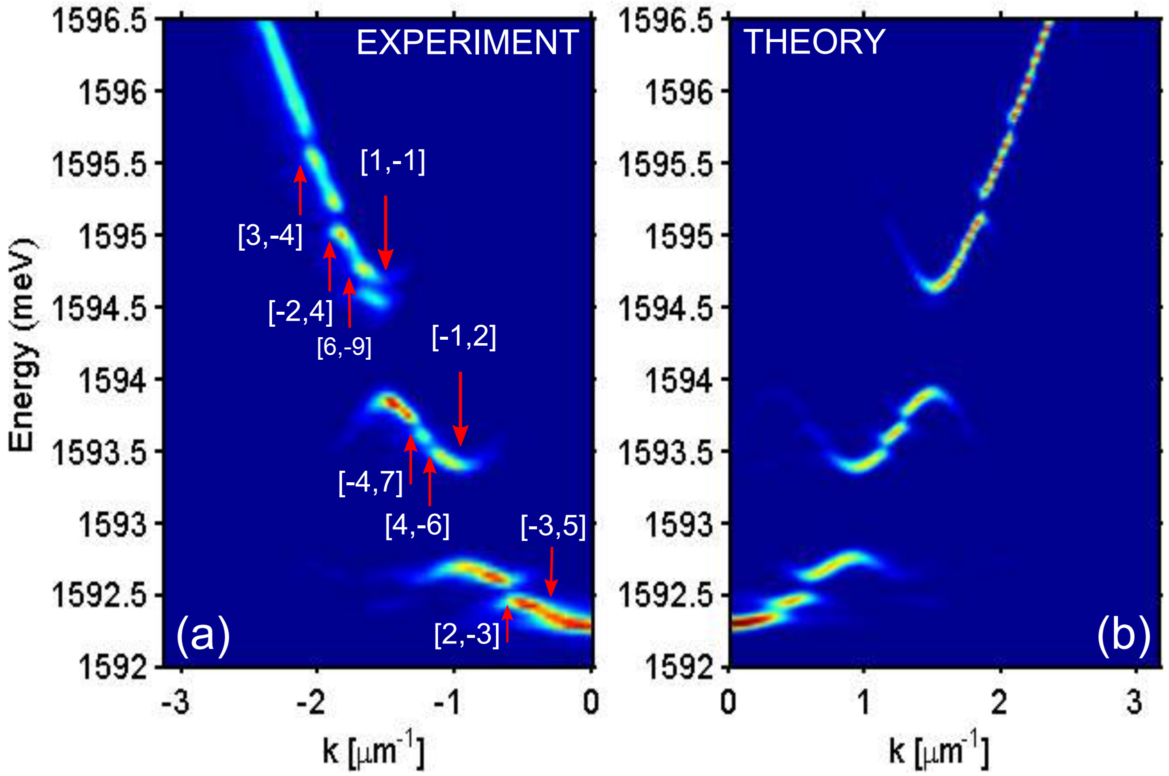
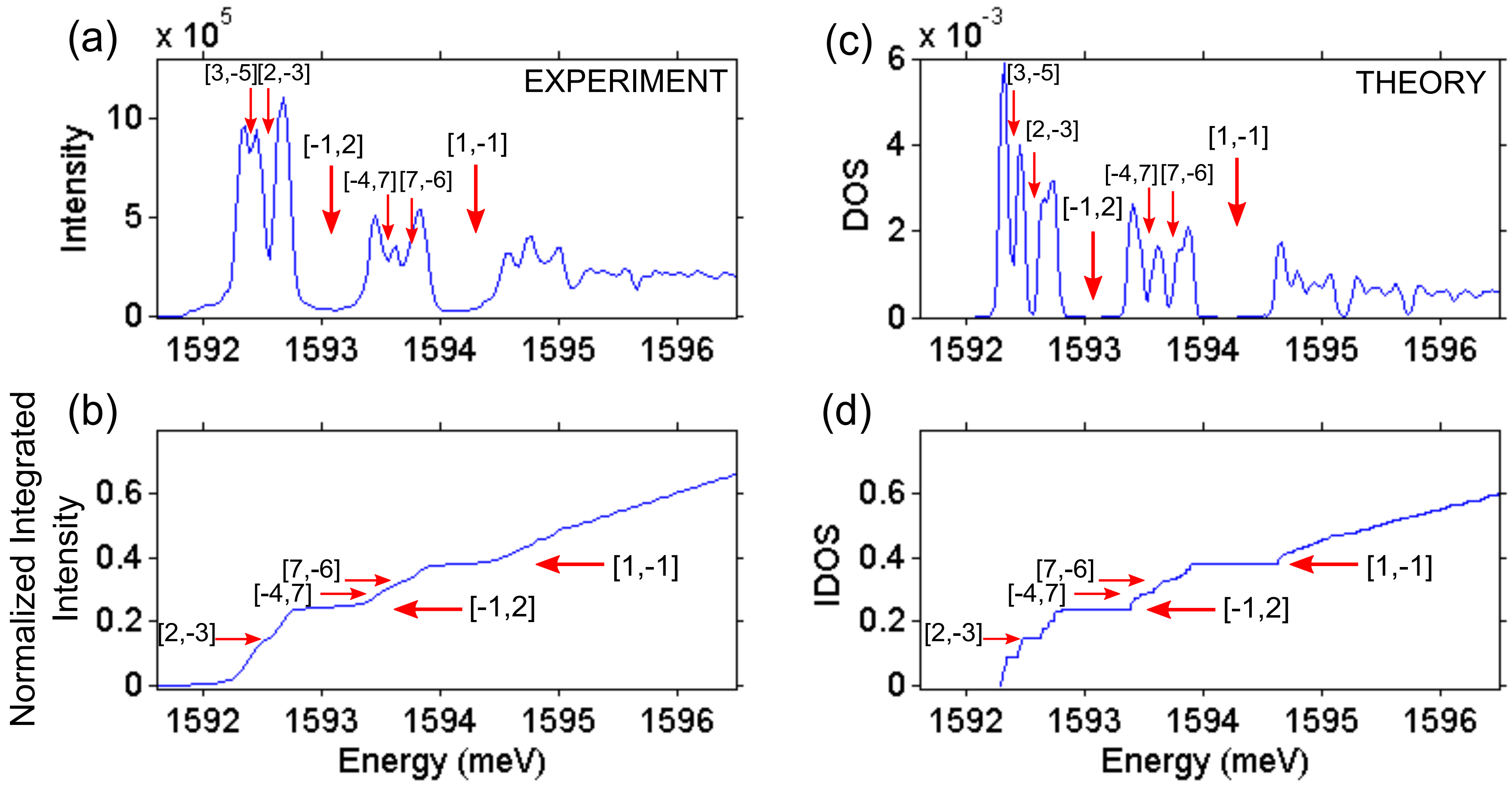
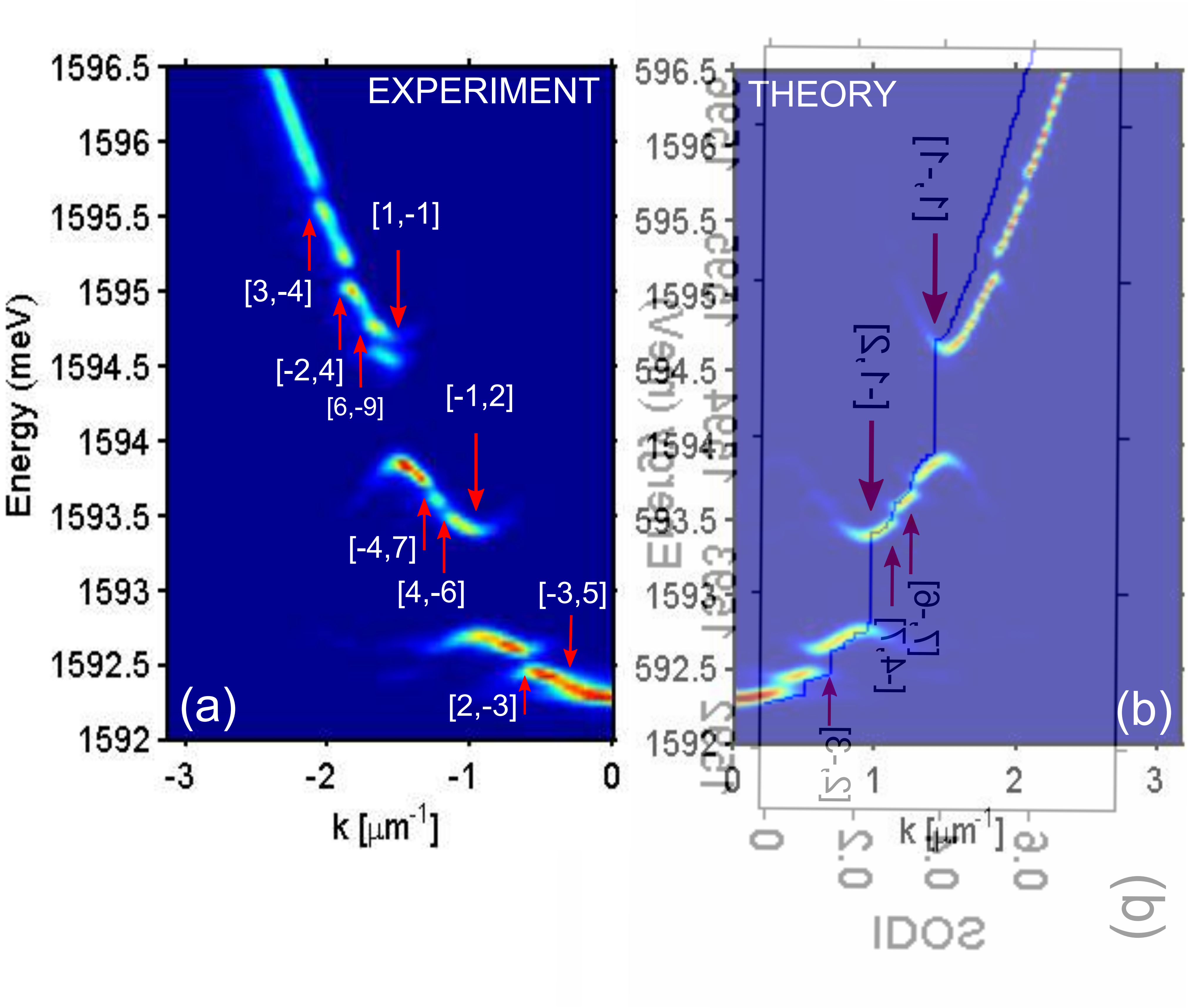


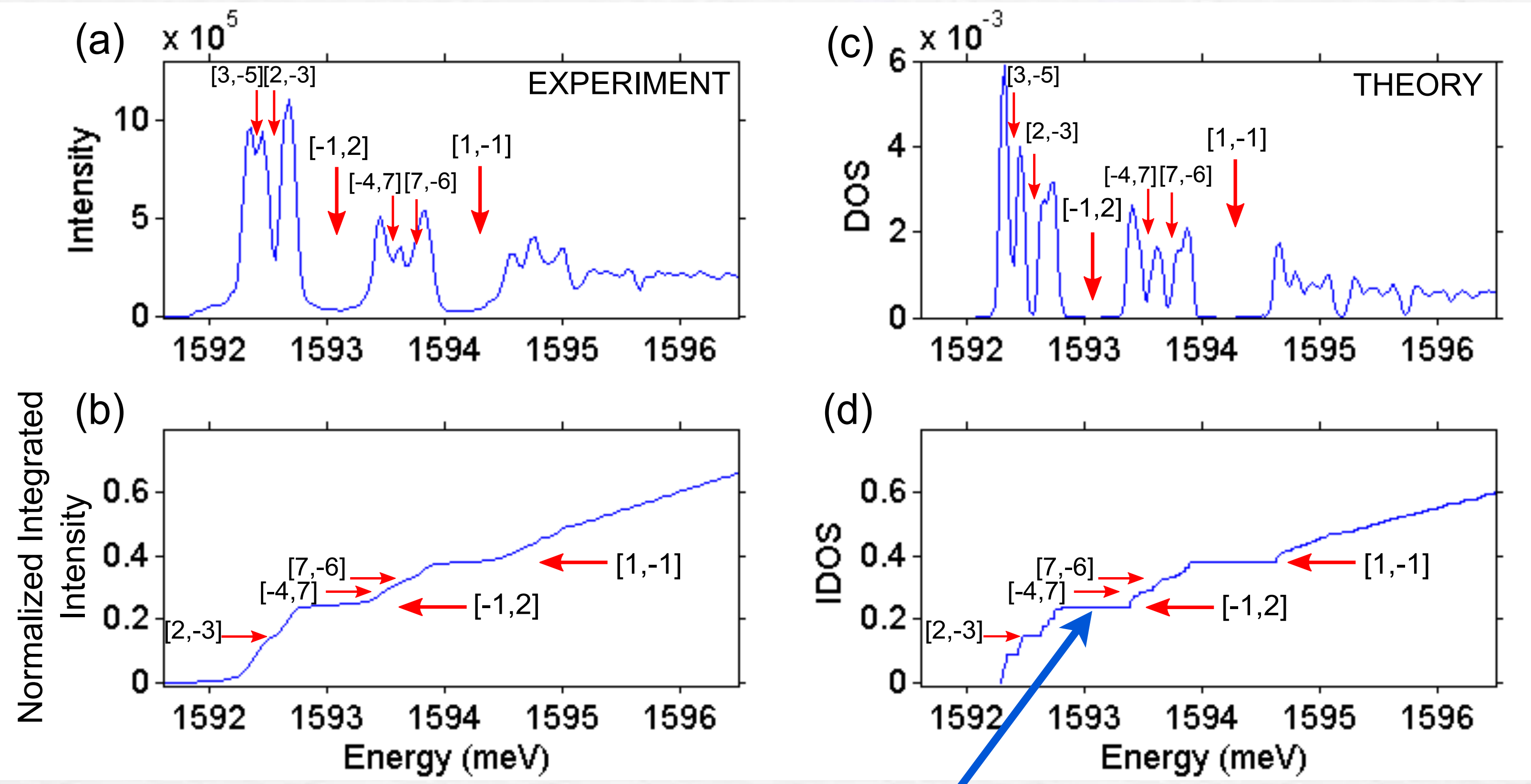


Calculating the integrated density of states (IDOS)




EXPERIMENT
(a)
Energy (meV)
1596.5
1596
1595.5
1595
1594.5
1594
1593.5
1593
1592.5
1592
k [μm^{-1}]
-3
-2
-1
0
[1,-1]
[3,-4]
[-2,4]
[6,-9]
[-1,2]
[-4,7]
[4,-6]
[-3,5]
[2,-3]
THEORY
(b)
596.5
1596
595.5
1595
594.5
1594
593.5
1593
592.5
1592
0
1
2
3
k [μm^{-1}]

Integrated Density of States-Gap Labeling



$$N\left(\varepsilon = E_{Q_{p,q}/2}\right) = p + q\tau^{-1}$$

within a $\{p,q\}$ gap

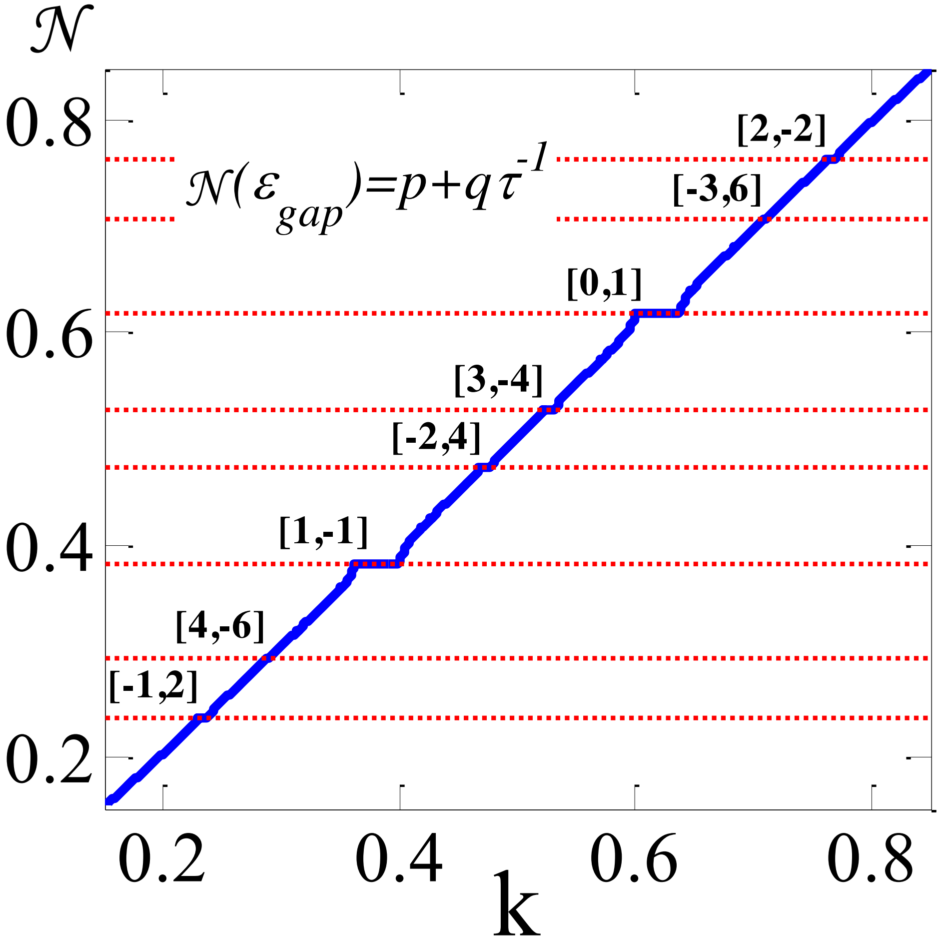
$p,q \in \mathbb{Z}$

Labeling the gaps...

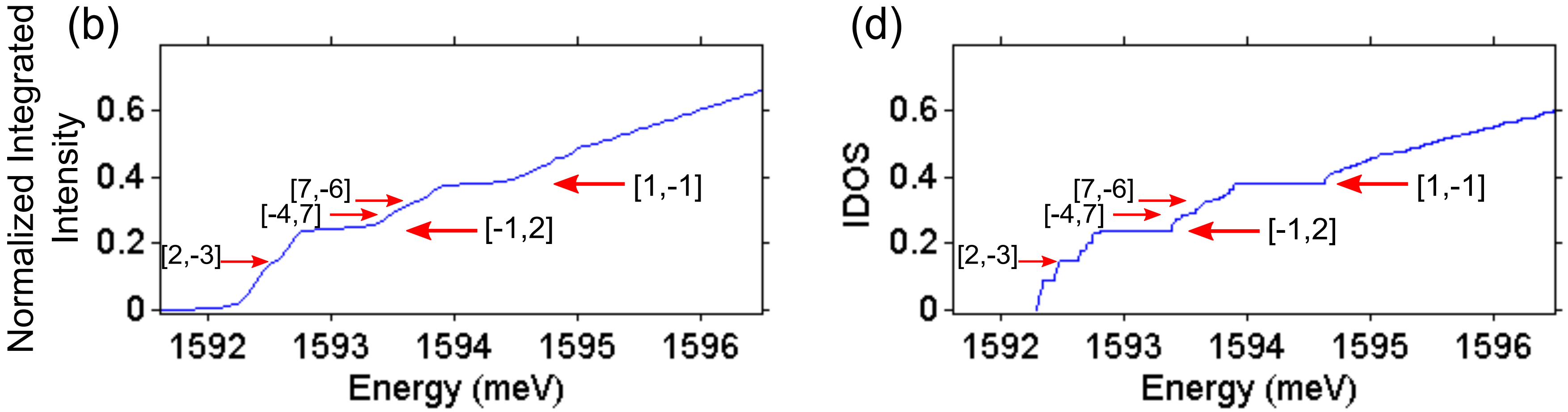
$$N\left(\varepsilon_{gap}\right) = p + q\tau^{-1}$$

$$p,q \in \mathbb{Z}$$

$$\tau = \sqrt{5}+1\big/2 \approx 1.62$$



Integrated density of states (IDOS)



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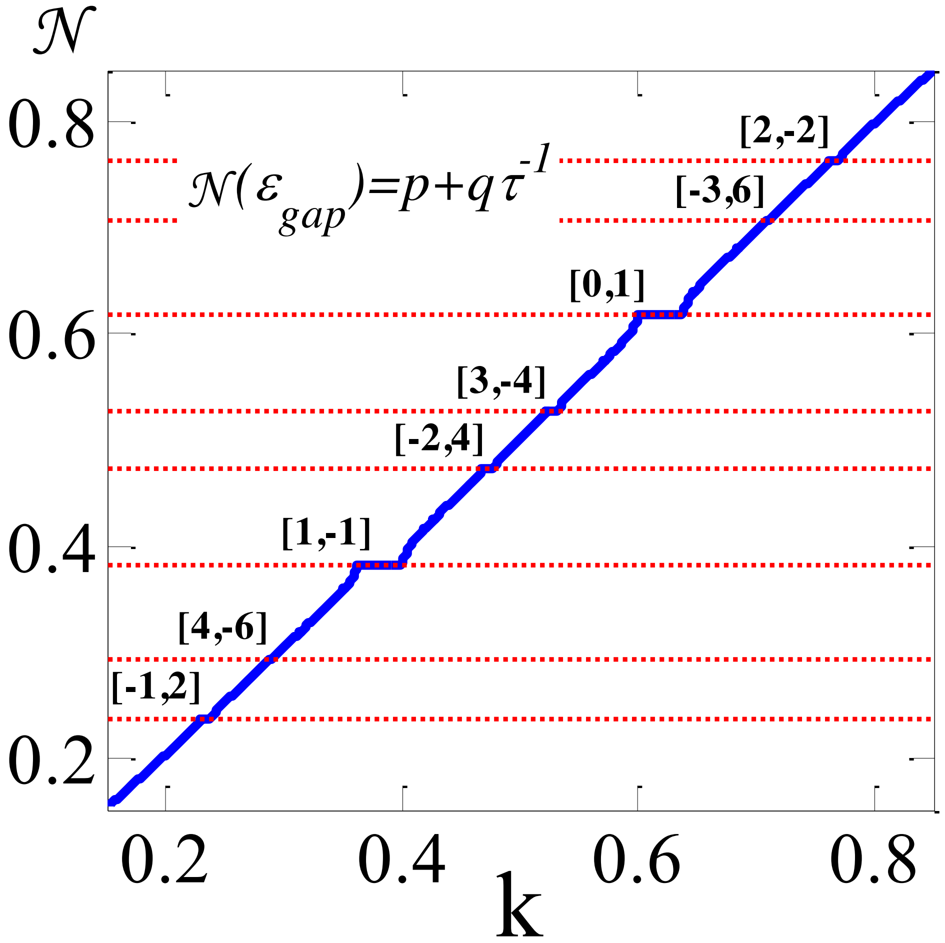

Integrated density of states (IDOS)

This result has a broader range of validity : Gap labeling theorem (Bellissard, 1982)

Labeling the gaps...

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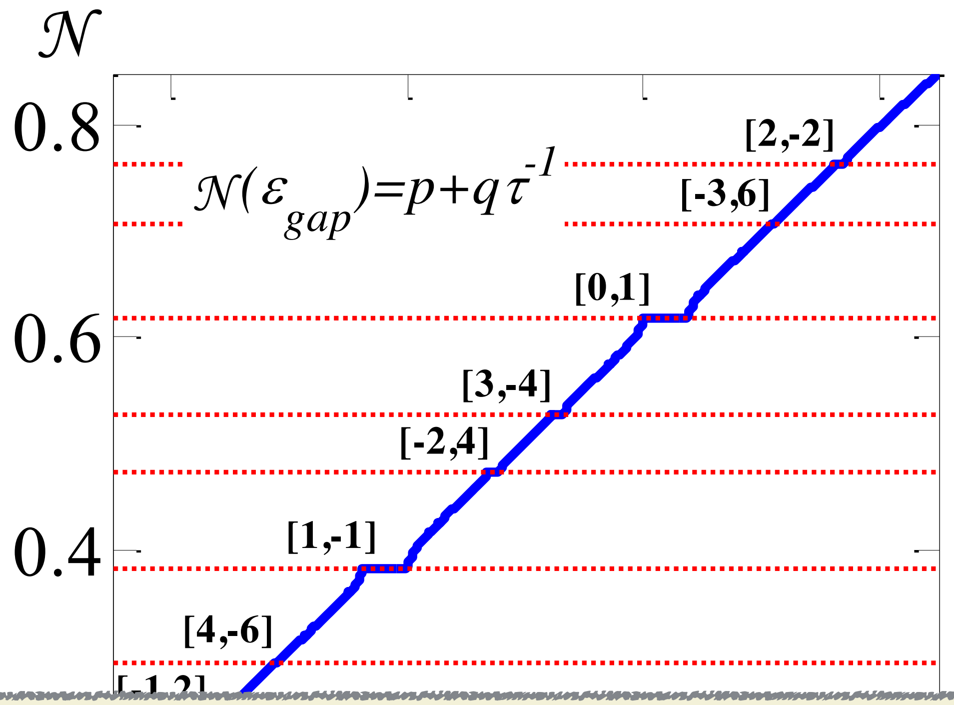

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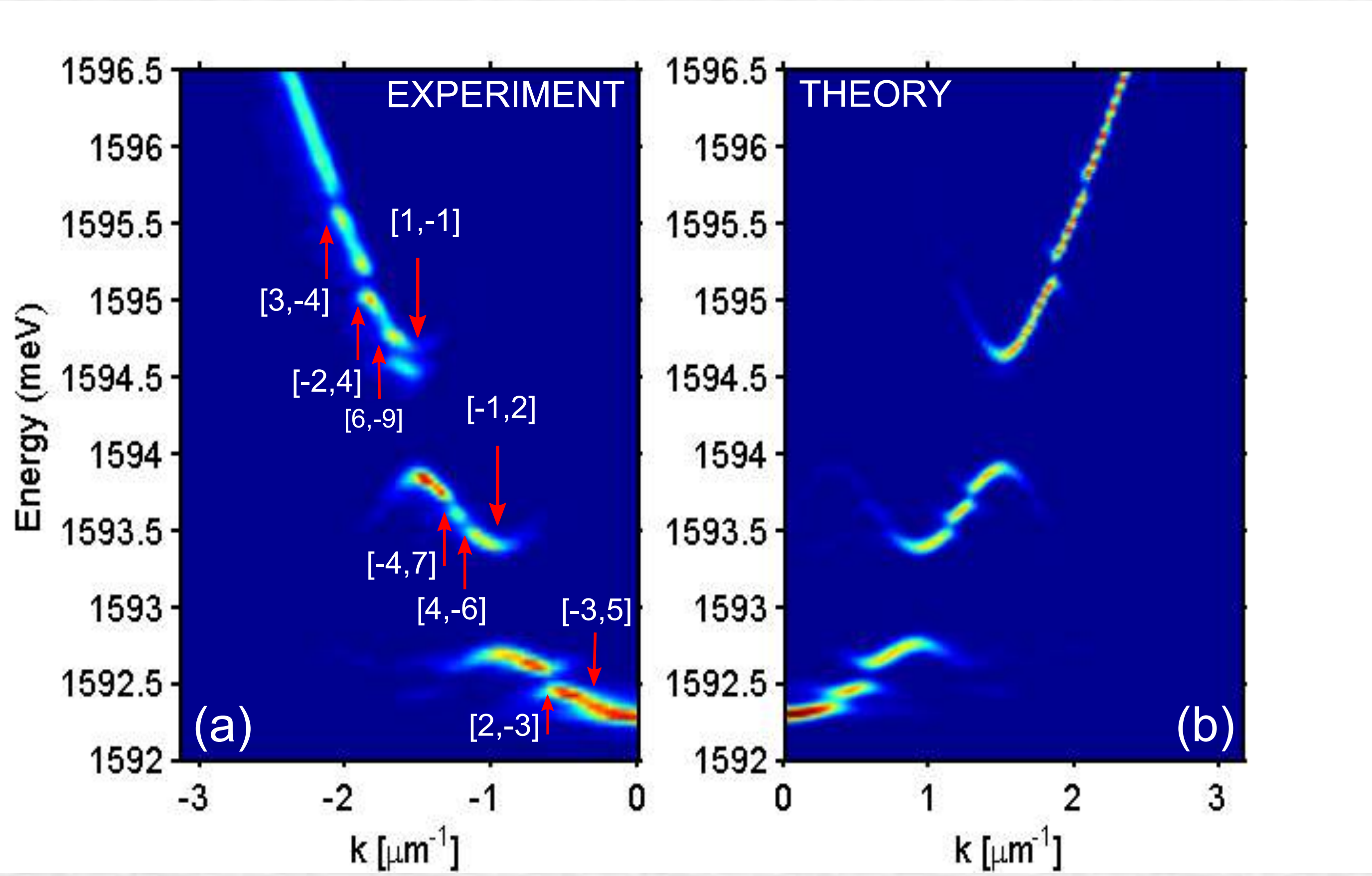
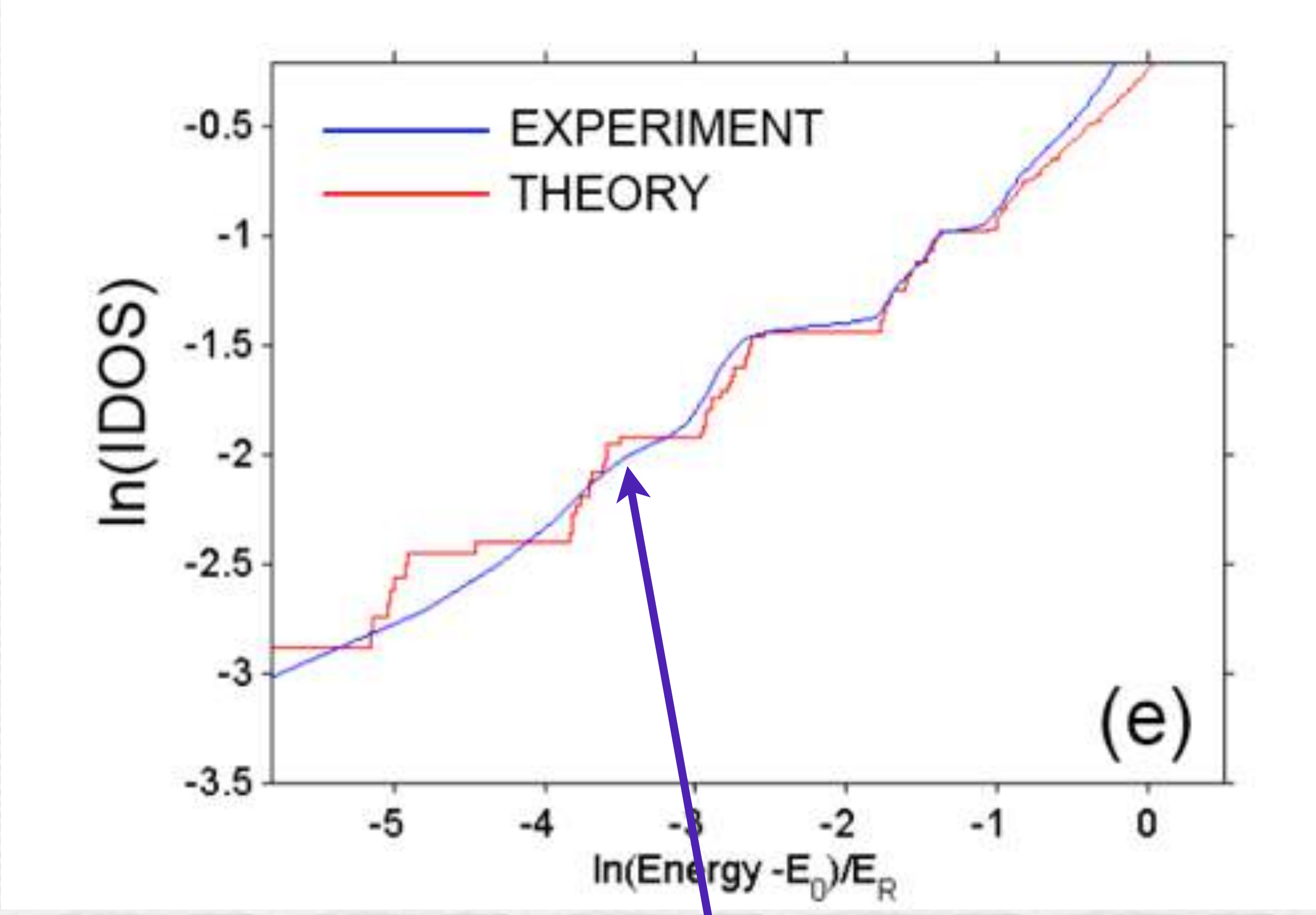
Topological invariants

independent of potential strength,
inhomogeneity, ...

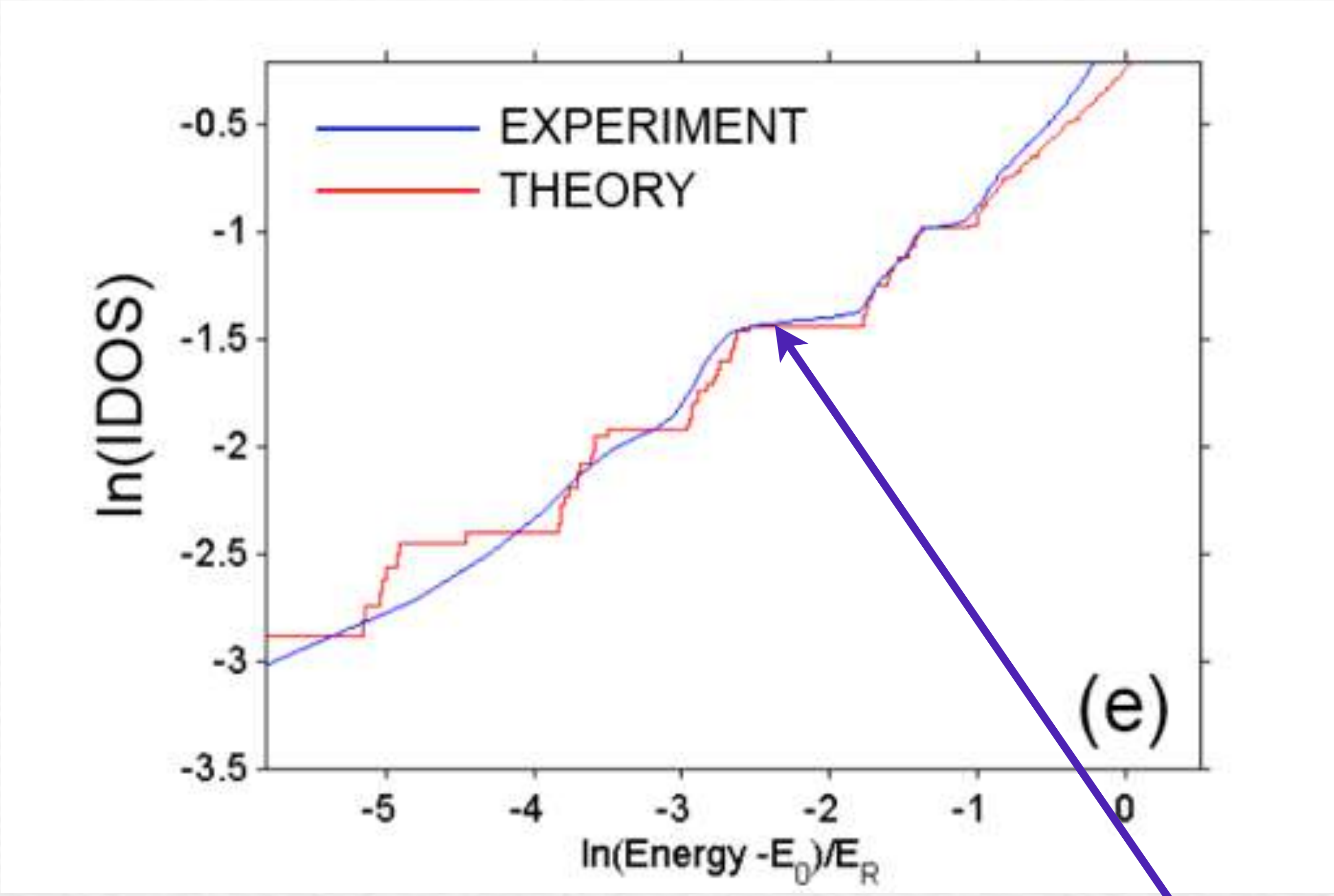
IDOS -





$$N_{\omega}(\Delta\omega) = (\Delta\omega)^{\alpha} \times F\left(\frac{\ln|\Delta\omega|}{\ln b}\right), \qquad \alpha = \frac{\ln a}{\ln b}, \quad F(x+1) = F(x)$$

Log-periodic oscillations : fingerprint of the fractal spectrum



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Part 3

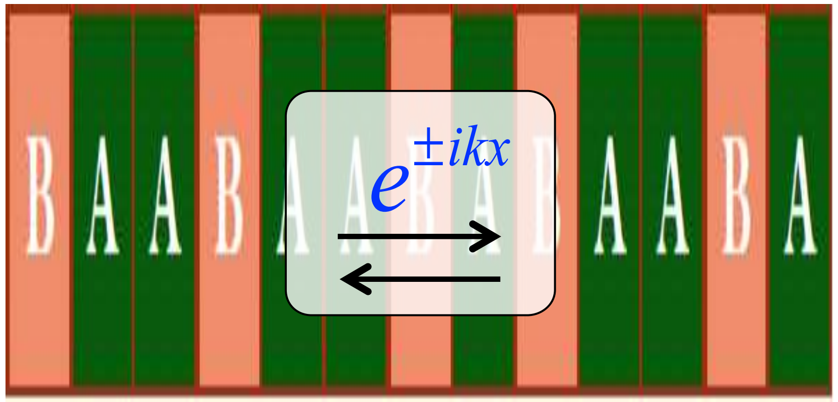
Topological content of the gap labelling theorem (GLT) - winding (Chern) numbers

Chern numbers and gap labeling.

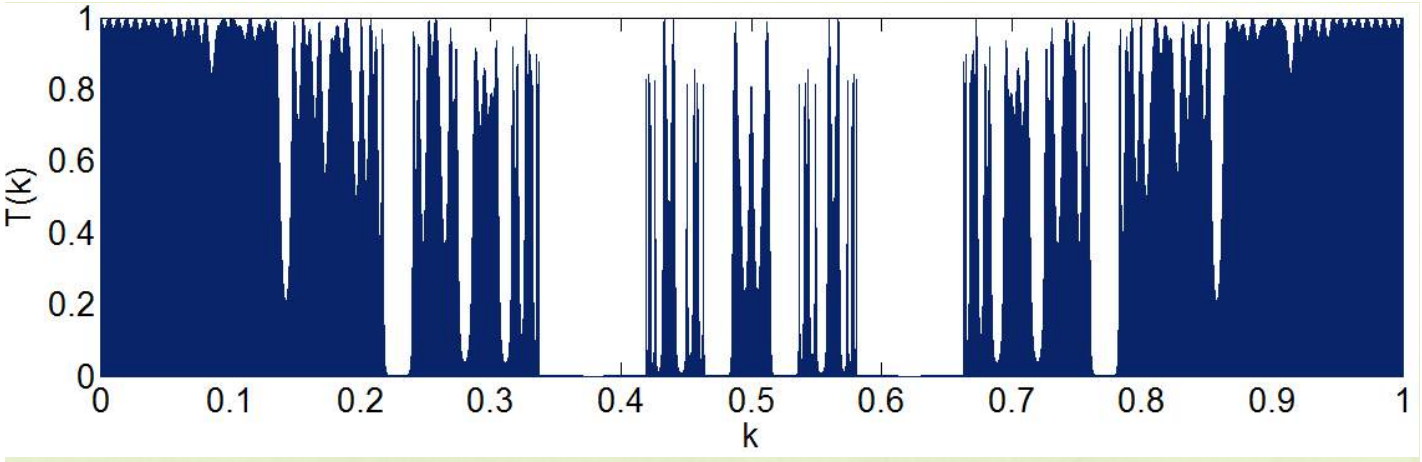
- Is there a relation with other occurrences of Chern numbers (e.g. quantum Hall effect, topological insulators, graphene, Weyl semi-metals…) ?

- Not so obvious : in the previous cases, topology and associated Berry curvature result from the existence of underlying magnetic fields, Aharonov-Bohm fluxes, Dirac structure…

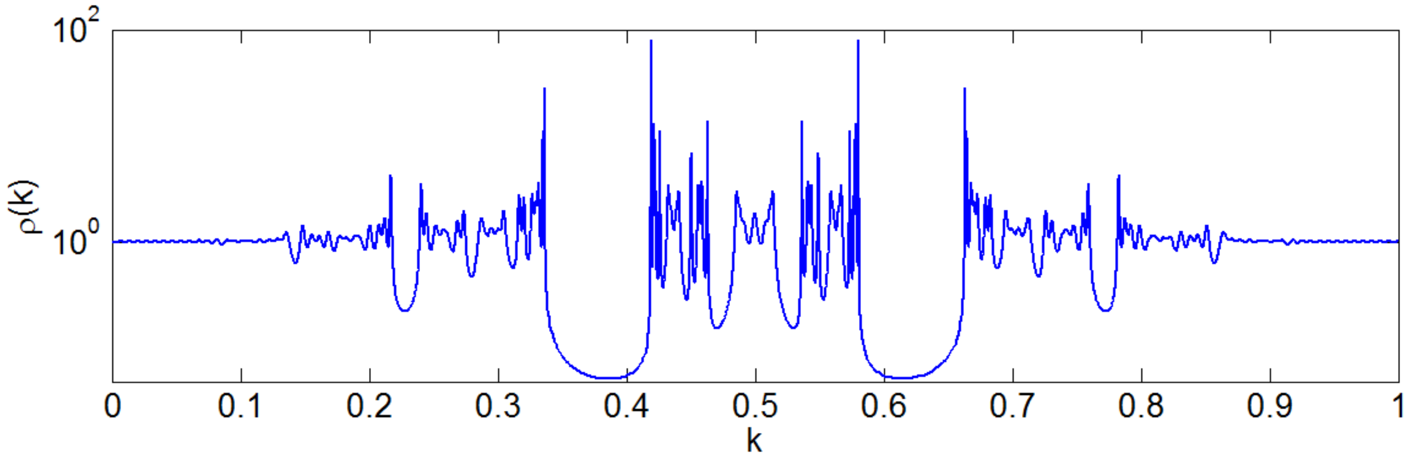
Quasi-periodic structure (Fibonacci quasi-crystal)



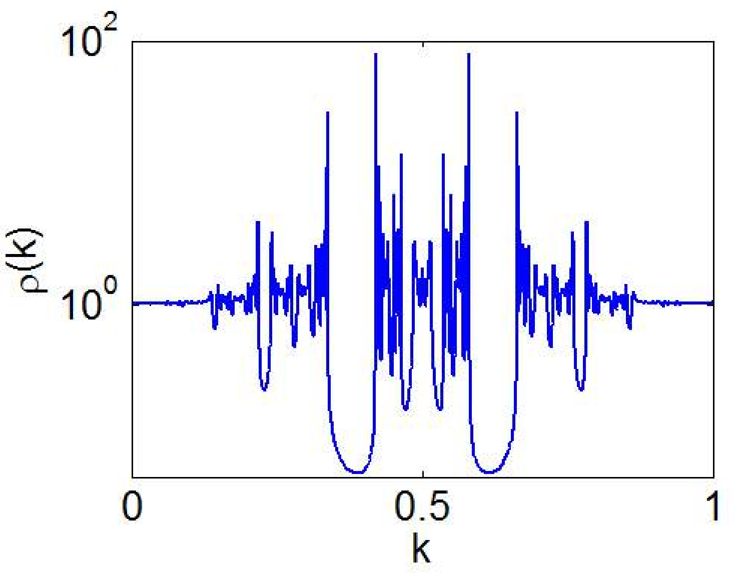
Spectrum of the cavity modes



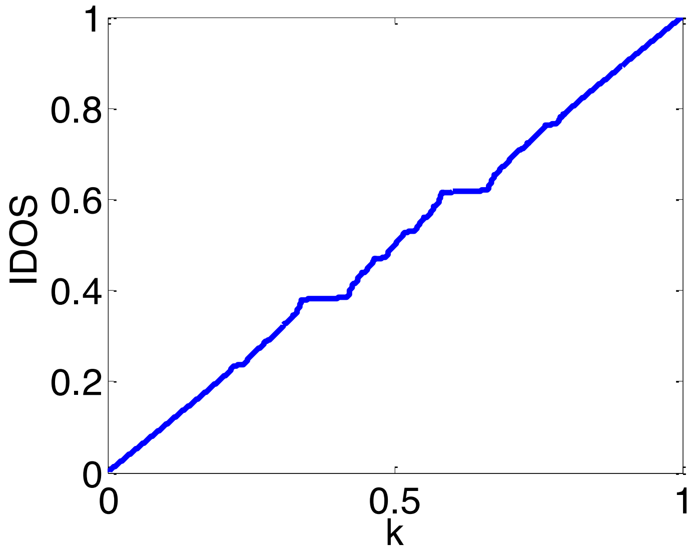
Transmission spectrum



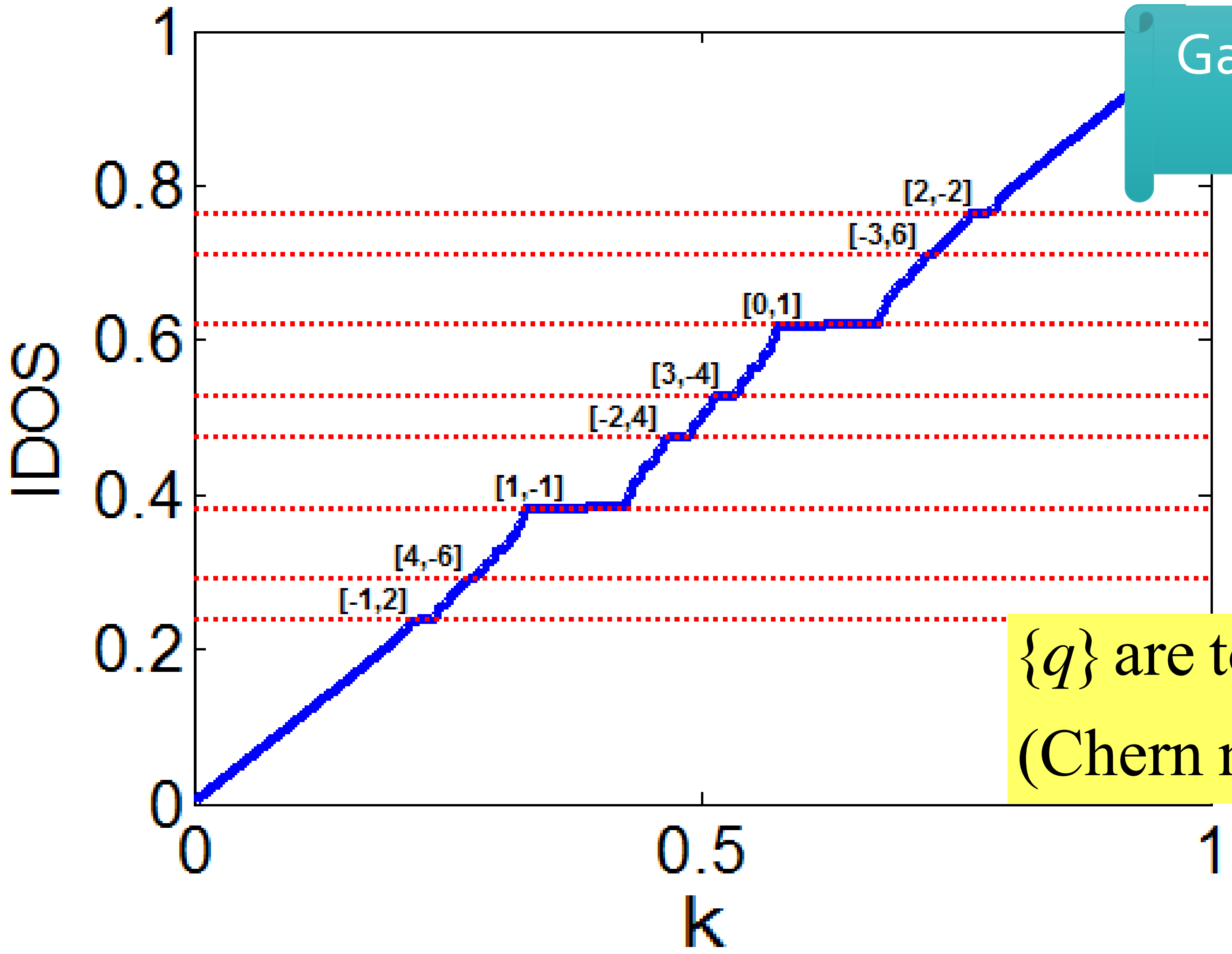
Density of modes



Density of modes



IDOS- counting function



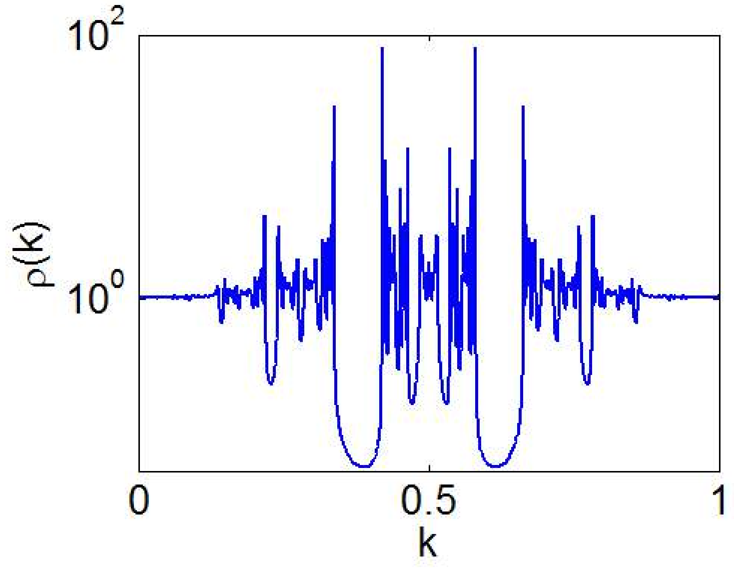
Gap labeling theorem, Bellissard, 1982

$$IDOS_{gap} = p + q\tau^{-1}$$

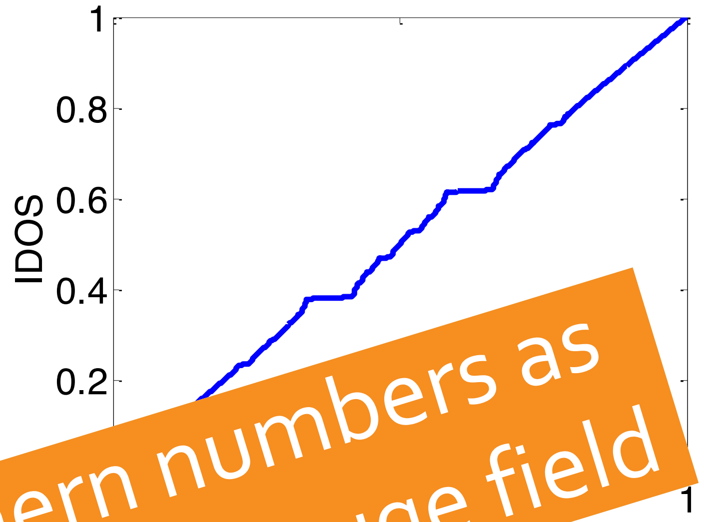
$$\tau = \tfrac{1}{2}(1+\sqrt{5})$$

$$\{p,q\} \in \mathbb{Z}$$

$\{q\}$ are topological numbers (Chern numbers)

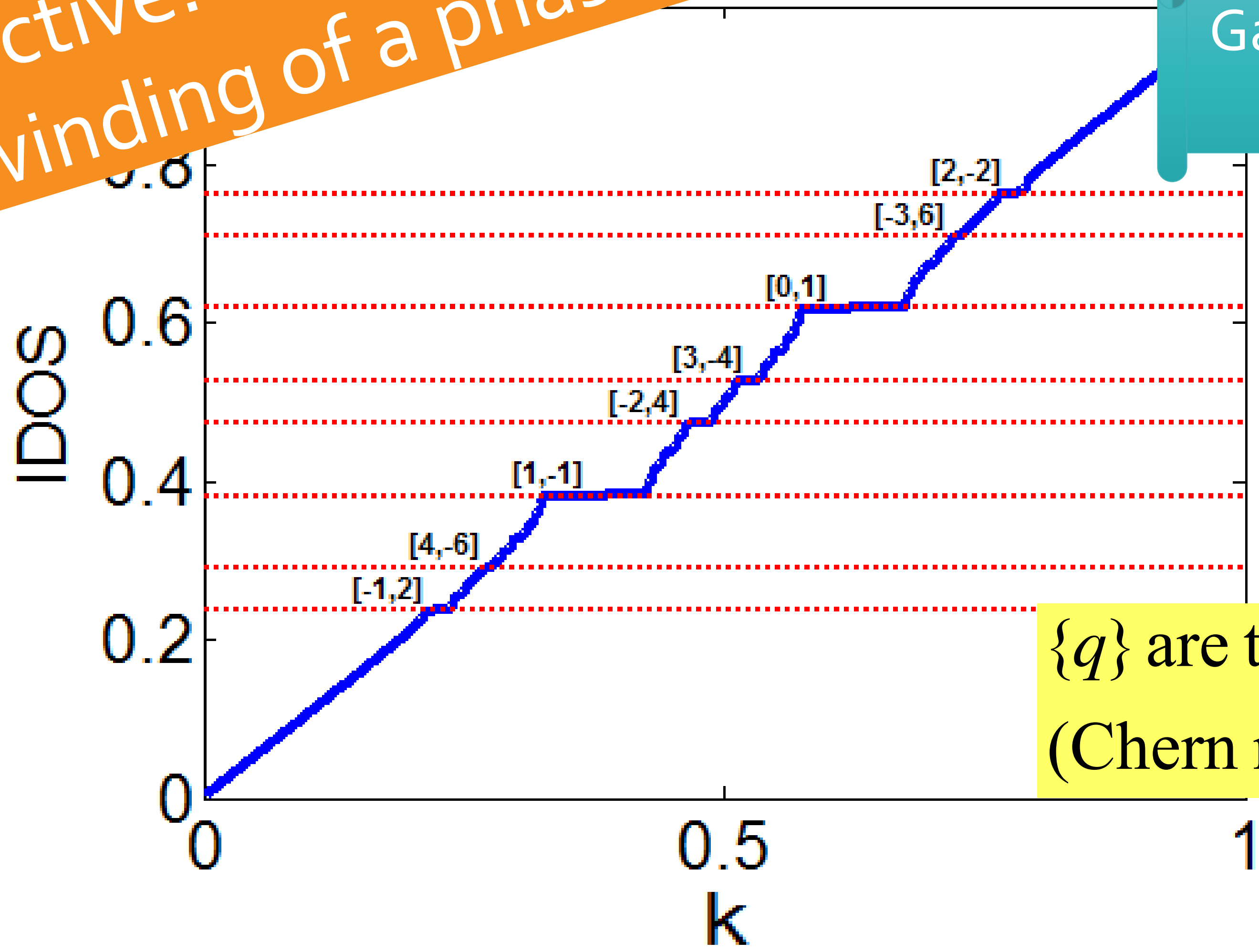


Density of modes



DOS- counting function

Objective: describe the gap labeling Chern numbers as the winding of a phase with respect to some gauge field



Gap labeling theorem, Bellissard, 1982

$$IDOS_{gap} = p + q\tau^{-1}$$

$$\tau = \tfrac{1}{2}(1+\sqrt{5})$$

$$\{p,q\} \in \mathbb{Z}$$

$\{q\}$ are topological numbers (Chern numbers)

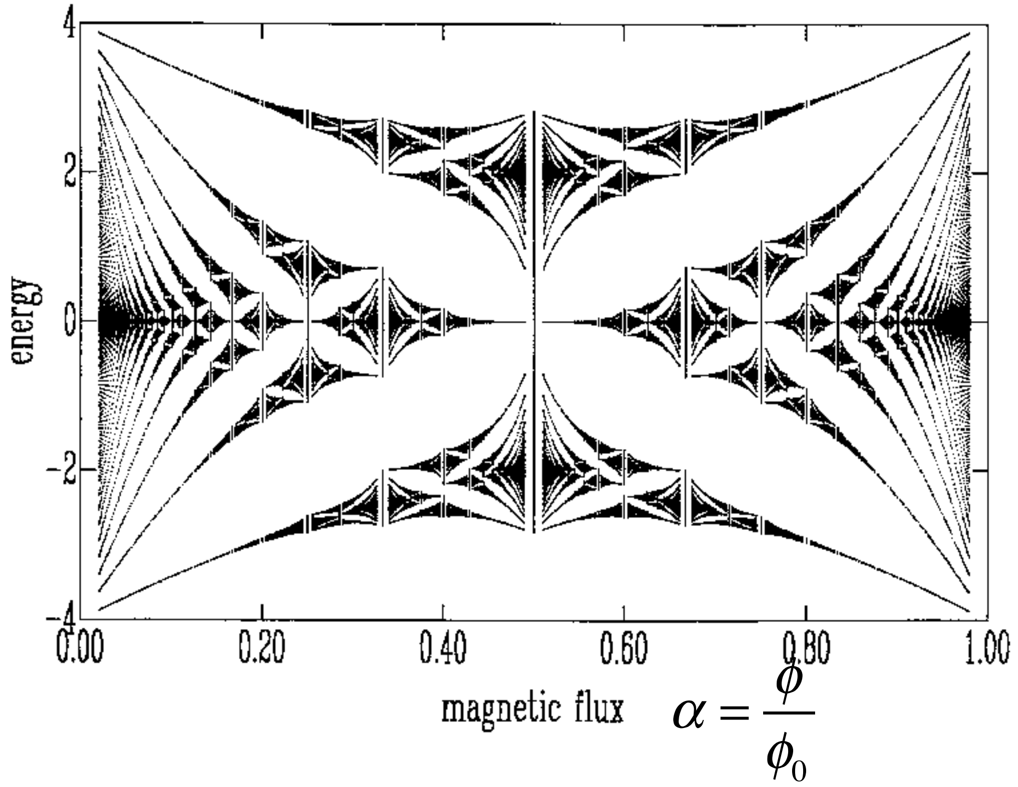
Example : Free electrons in a 2D crystal + magnetic field

(Harper problem)

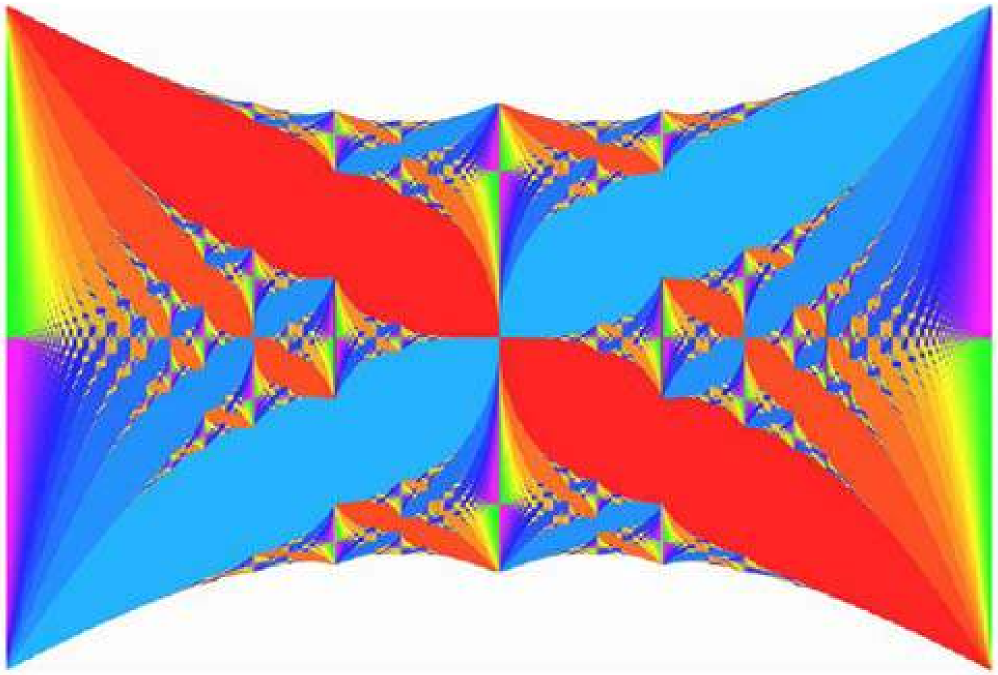
Non trivial group of magnetic translations: $\begin{cases} U_1 = e^{iK_x} \\ U_2 = e^{iK_y} \end{cases}$

$$\boxed{U_1 U_2 = e^{2i\pi\alpha} U_2 U_1}$$

$$\alpha = \frac{\phi}{\phi_0} \qquad \phi_0 = \frac{h}{e}$$



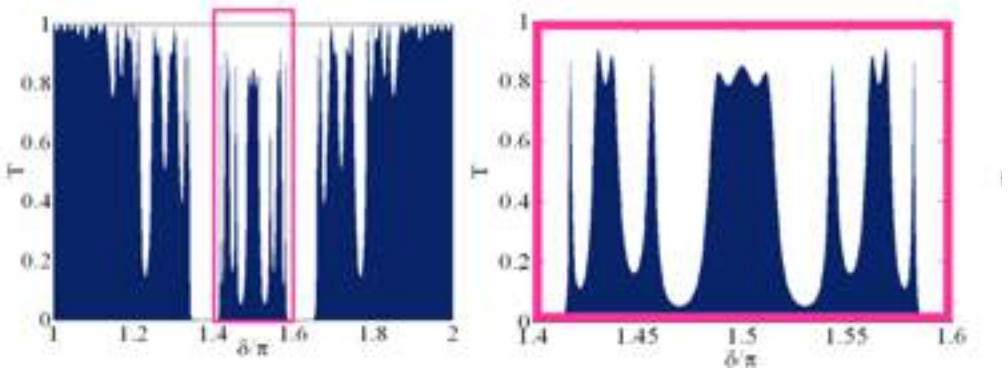
Hofstadter butterfly



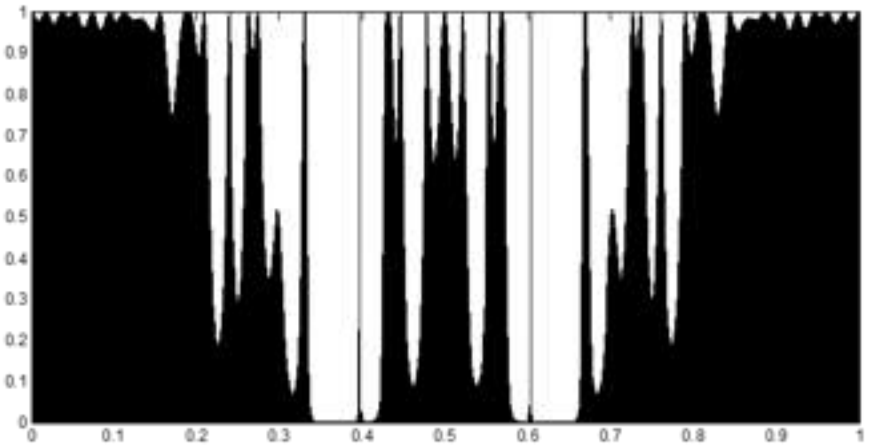
Osadchy, Avron, (2001)

Topological features manifest through specific properties of edge (gap) modes in the presence of boundaries

Under certain (boundary) conditions, instead of observing

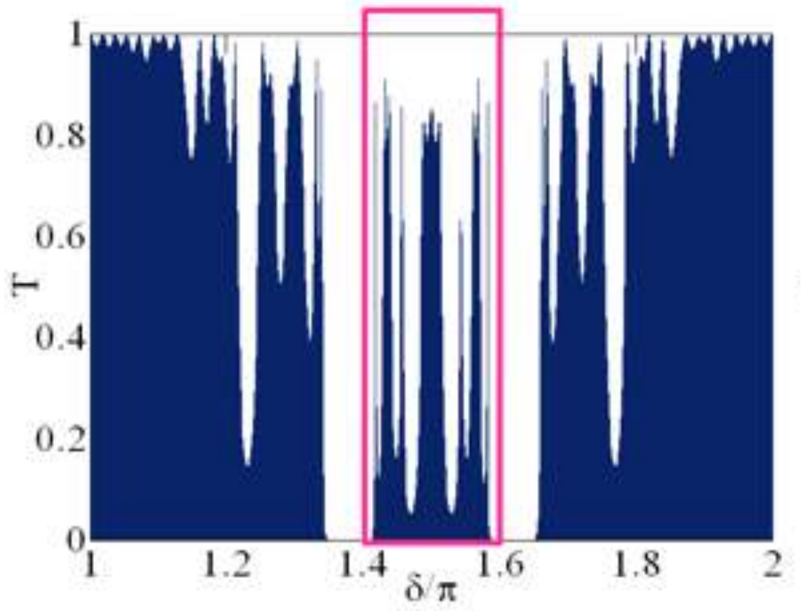


we observe



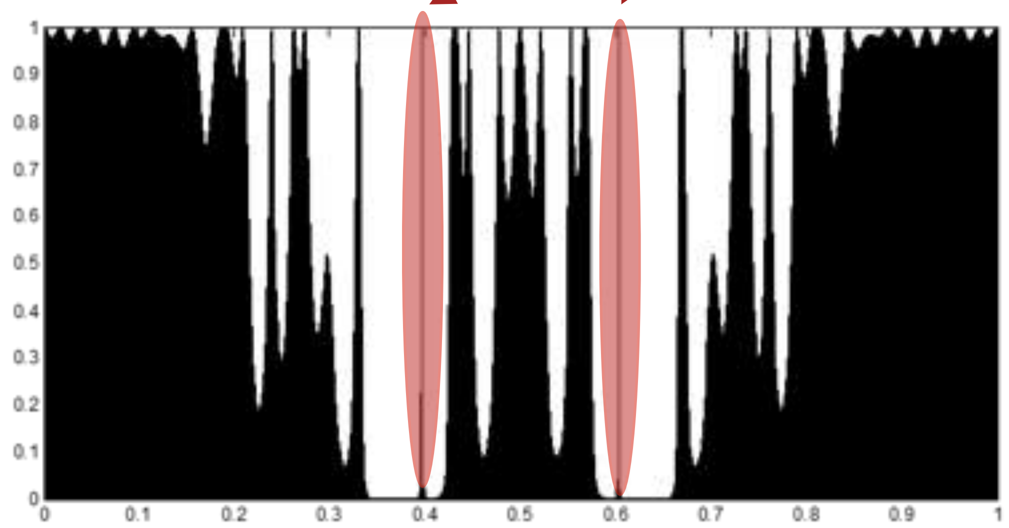
Topological features manifest through the behaviour of gap modes

Under certain (boundary) conditions, instead of observing

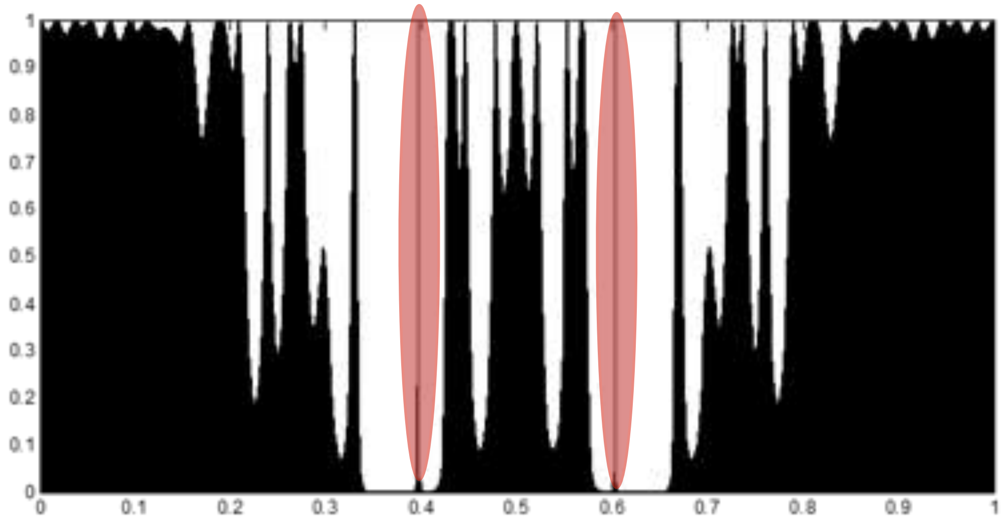


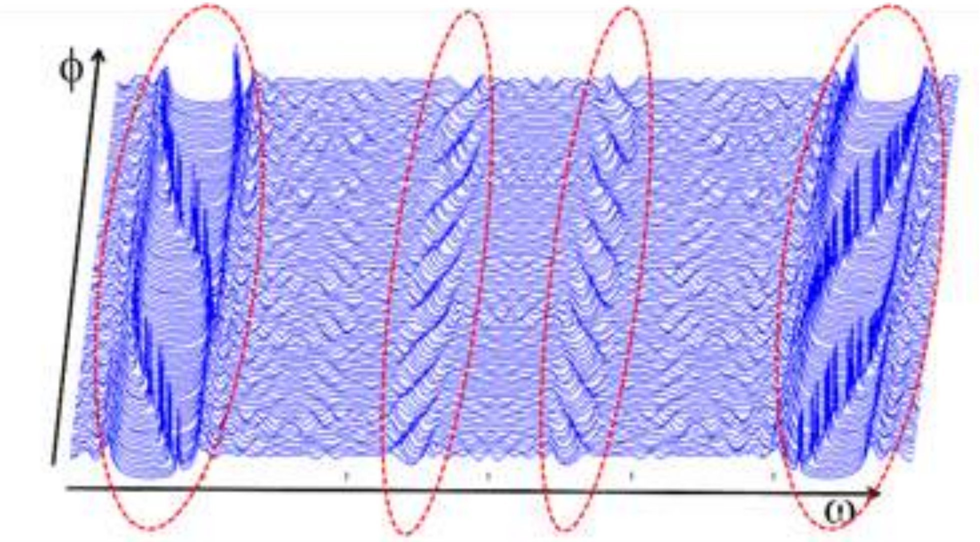
we observe

Additional gap modes

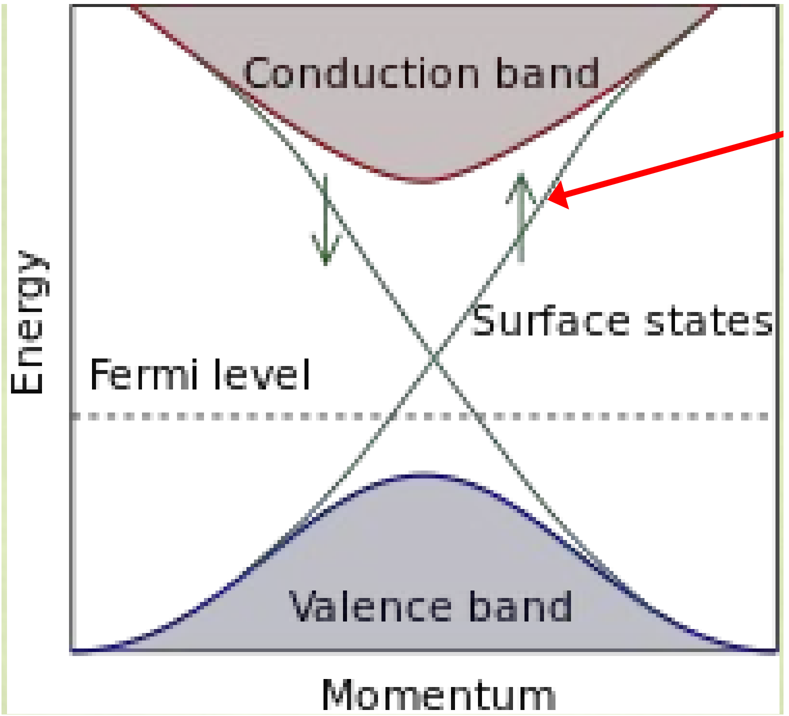


Gap modes cross over the gaps while varying a parameter ϕ yet to be determined





analogous to
topological insulators



Building quasi-periodic chains -

Winding (Chern) numbers

Equivalent ways to build quasi-periodic chains

- Characteristic function
- Cut & Project

Characteristic function

$$\chi_n = sign\left[\cos\left(2\pi n\tau^{-1}+\phi\right)-\cos\left(\pi\tau^{-1}\right)\right]$$

$-1 =$ B
$+1 =$ A

$$F_N(\phi) = \left[\chi_1\,\chi_2\cdots\chi_n\cdots\chi_N\right] \Leftrightarrow$$ A B A A B A B A A B A A B ◆◆◆

The angle ϕ is a (legitimate) degree of freedom.

ϕ is known as a phason

$\tau = \sqrt{5}+1 \big/ 2 \approx 1.62$

Meaning ?

Characteristic function

$$\chi_n = sign\left[\cos\left(2\pi n\tau^{-1}+\phi\right)-\cos\left(\pi\tau^{-1}\right)\right]$$

$-1 =$ B
1 = A

$$F_N(\phi) = [\chi_1\,\chi_2 \dots \chi$$

A B A A B A A B ◆ ◆ ◆

It is our gauge field candidate !

The angle ϕ timate) degree of freedom.

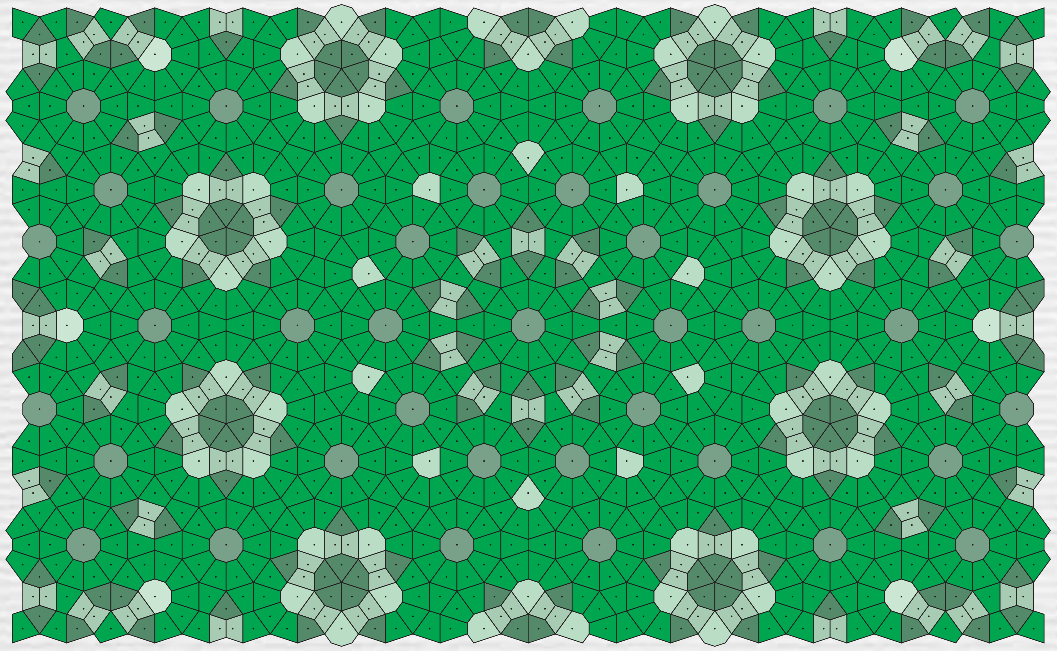
ϕ is known as a phason

$$\tau = {\sqrt{5}+1}\big/_{2} \approx 1.62$$

Meaning ?

Cut & Project method

Very active branch in maths of tiling, dynamical systems

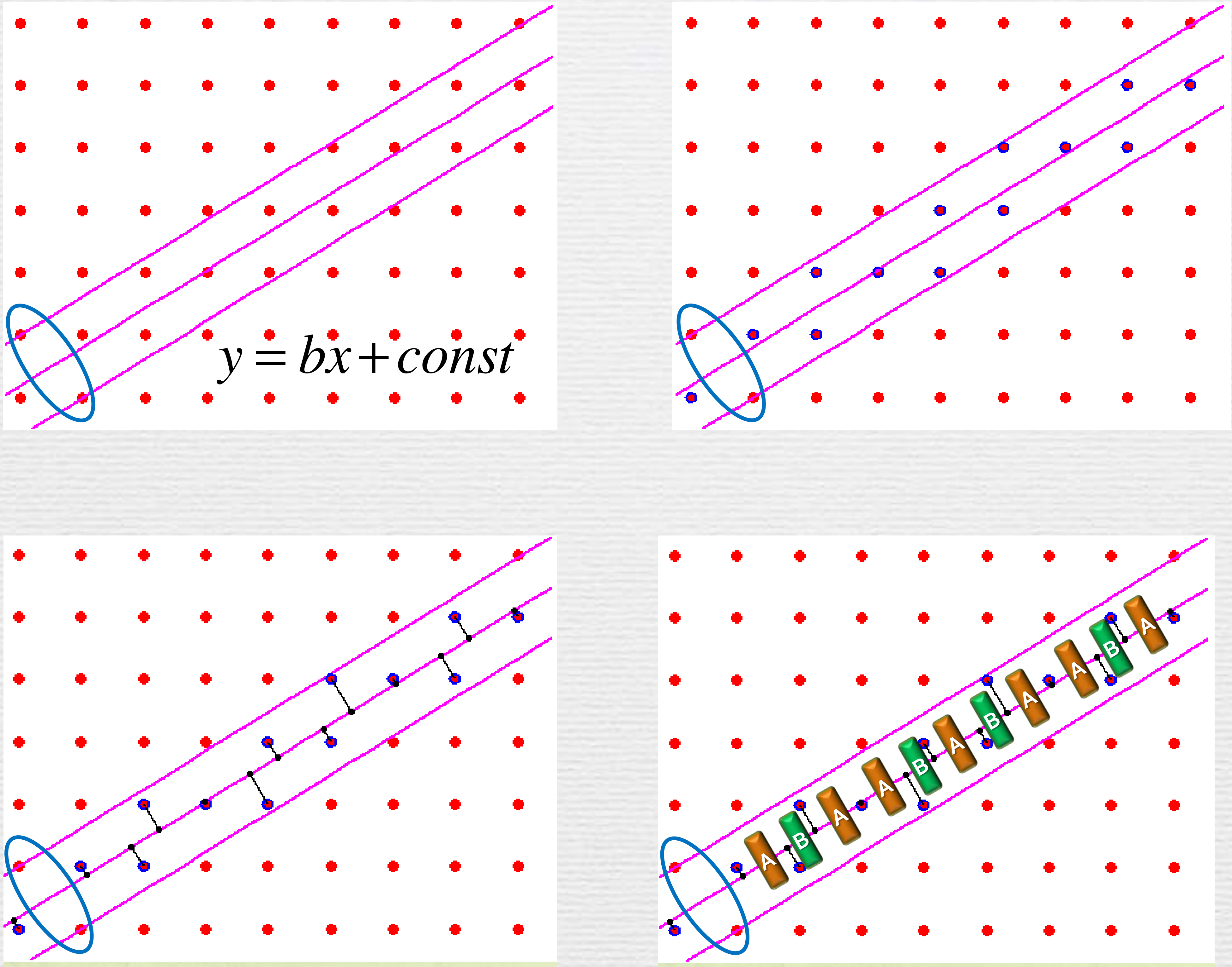


Duneau & Katz
Moody, Meyer
Pinsner, Voiculescu
Mendes-France, Allouche
Bombieri, Taylor,
Kellendonk, Grimm,
Queffelec, Bellissard, ……

Generate both periodic and quasi-periodic (quasicrystals) structures.

A brief tutorial for practical implementation.

Start from a 2D lattice $L = \mathbb{Z}^2$



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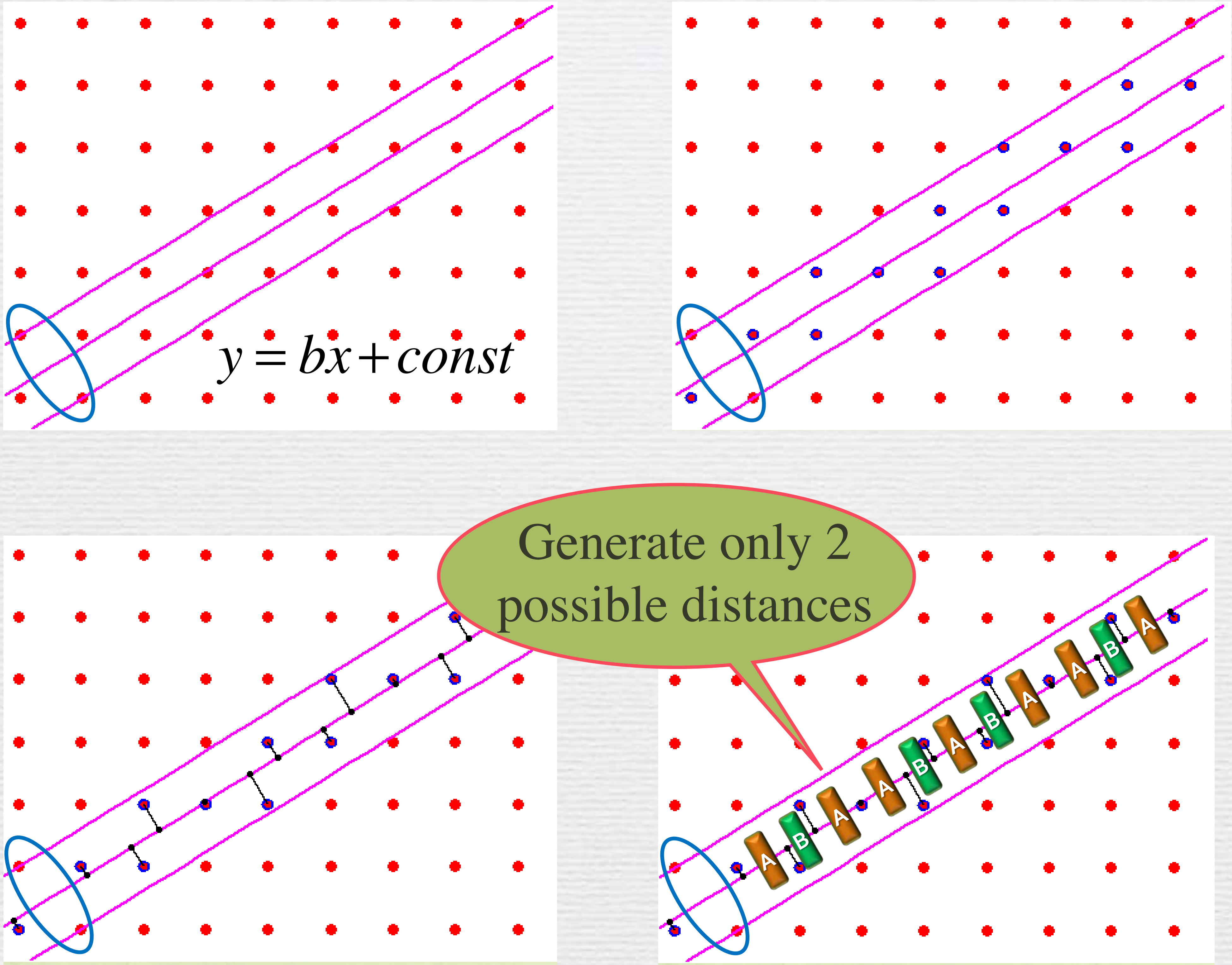
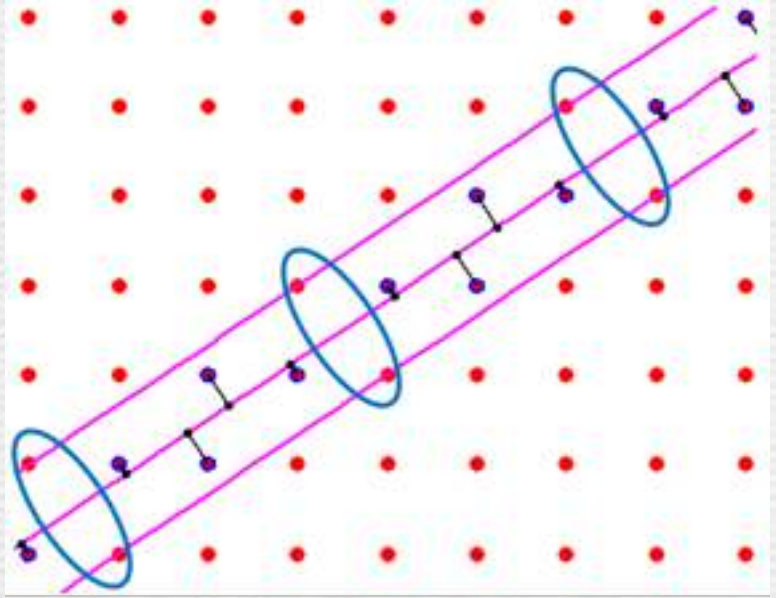


Generate only 2 possible distances

A B A A B A B A A B A ⋯

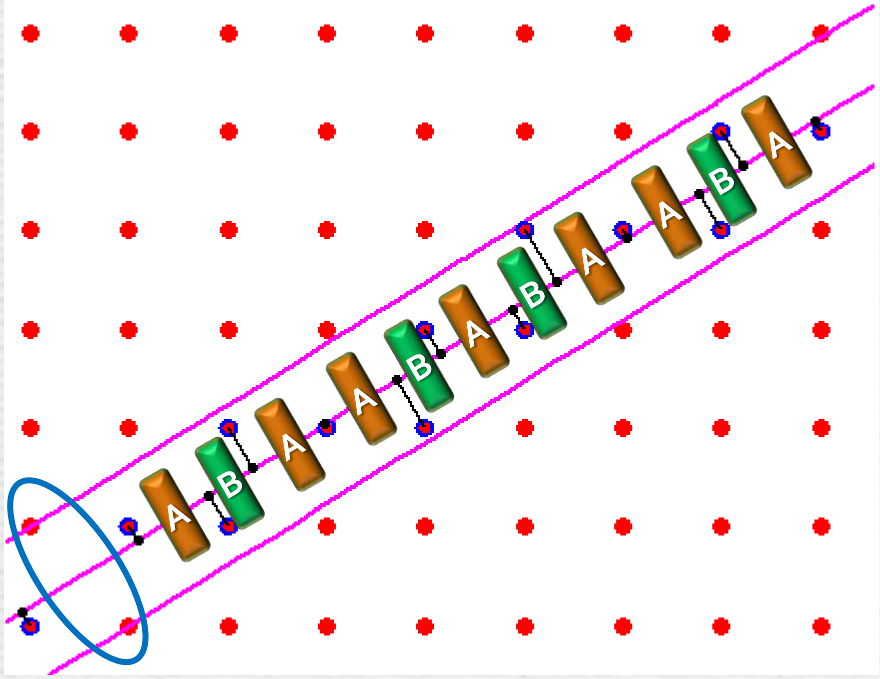
For a rational slope : periodic superlattice



$$y = \tfrac{2}{3}x + const$$

A B A A B A A B A ···

For an irrational slope : quasi-periodic structure



$$y = \tau^{-1} x + const$$

A B A A B A B A A B A A B ···

$$\tau = {\left(1+\sqrt{5}\right)}\Big/_{2}$$

golden mean

Characteristic function

C&P method

$$\chi_n = sign\left[\cos\left(2\pi n\,\tau^{-1}+\phi\right)-\cos\left(\pi\,\tau^{-1}\right)\right]$$

ϕ is an innocuous and thus ignored modulation phase.

For an infinite Fibonacci chain :

$$\phi_\infty = 3\pi\sigma = 3\pi\,\tau^{-1}$$

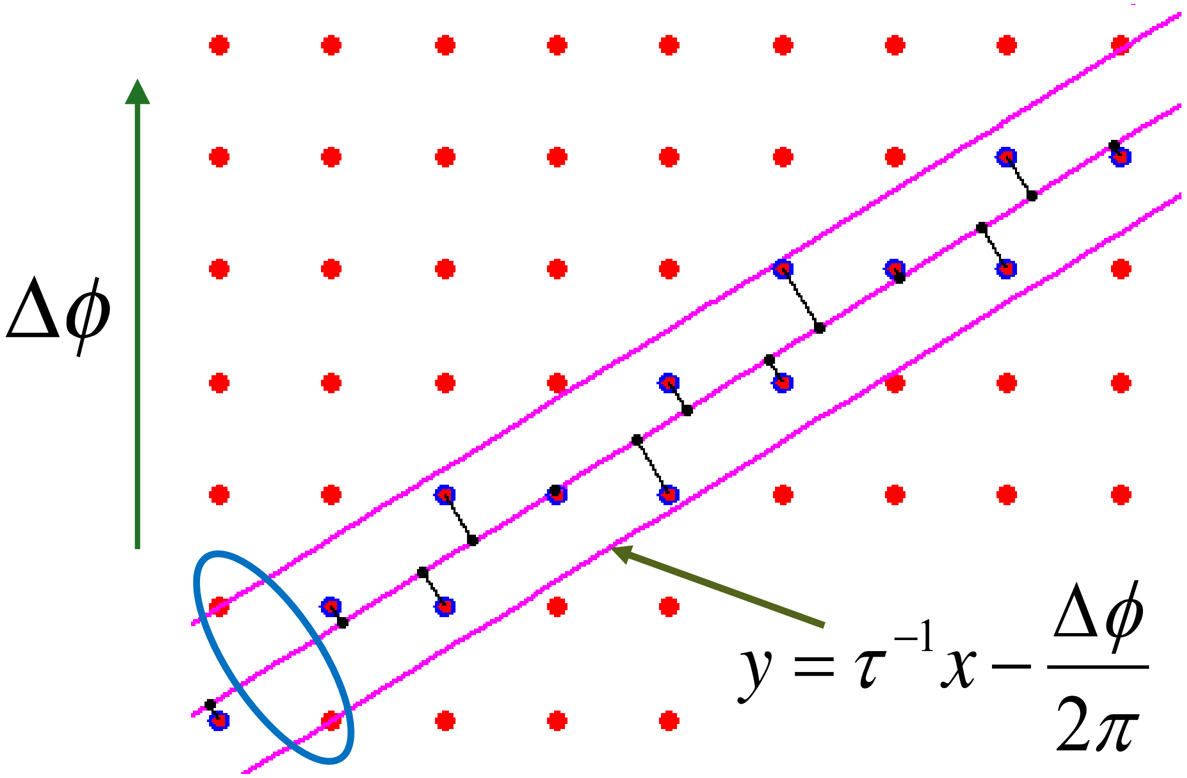
Define instead

$$\chi_n = sign\left[\cos\left(2\pi n\tau^{-1}+\phi_\infty+\Delta\phi\right)-\cos\left(\pi\tau^{-1}\right)\right]$$

$$\tau = {\left(1+\sqrt{5}\right)}\big/{2}$$

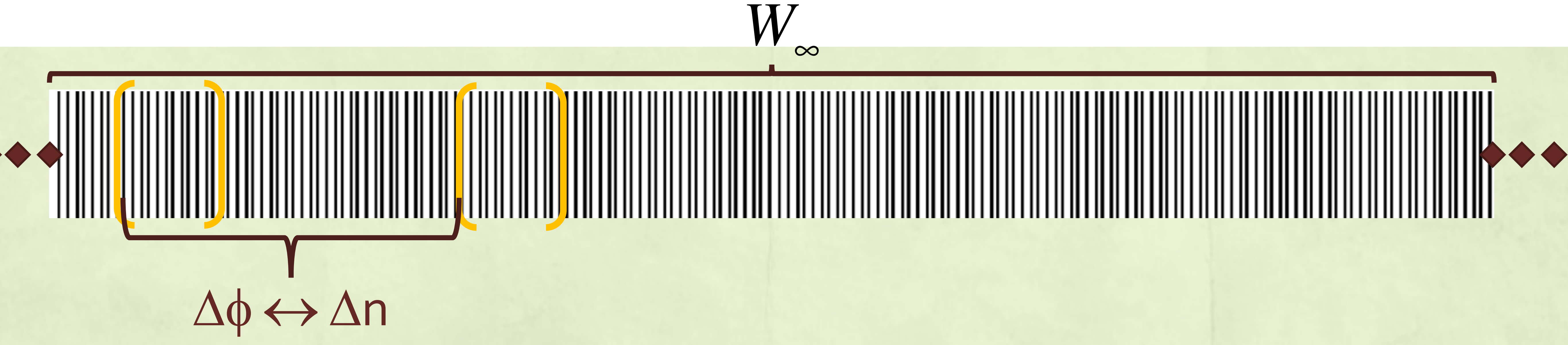
golden mean

Is it possible to give a meaning to $\Delta\phi$ using the C&P method ?



We understand the meaning of $\Delta\phi$

Consider the infinite chain W_∞

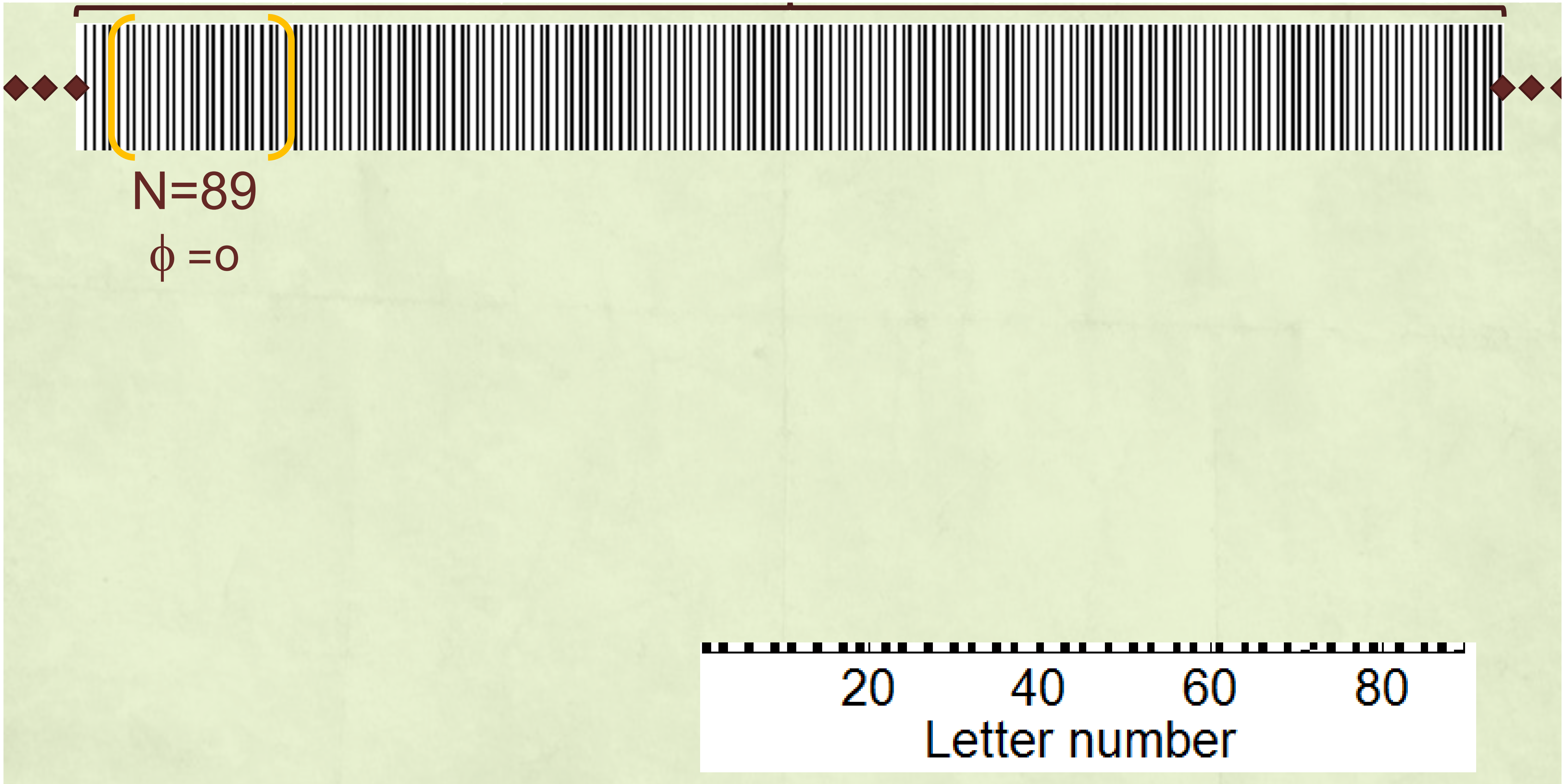


Changing ϕ is equivalent to moving along W_∞

There are finitely many different finite segments

$$F_N(\phi)=[\chi_1\,\chi_2\cdots\chi_n\cdots\chi_N]$$

which are unitary related.



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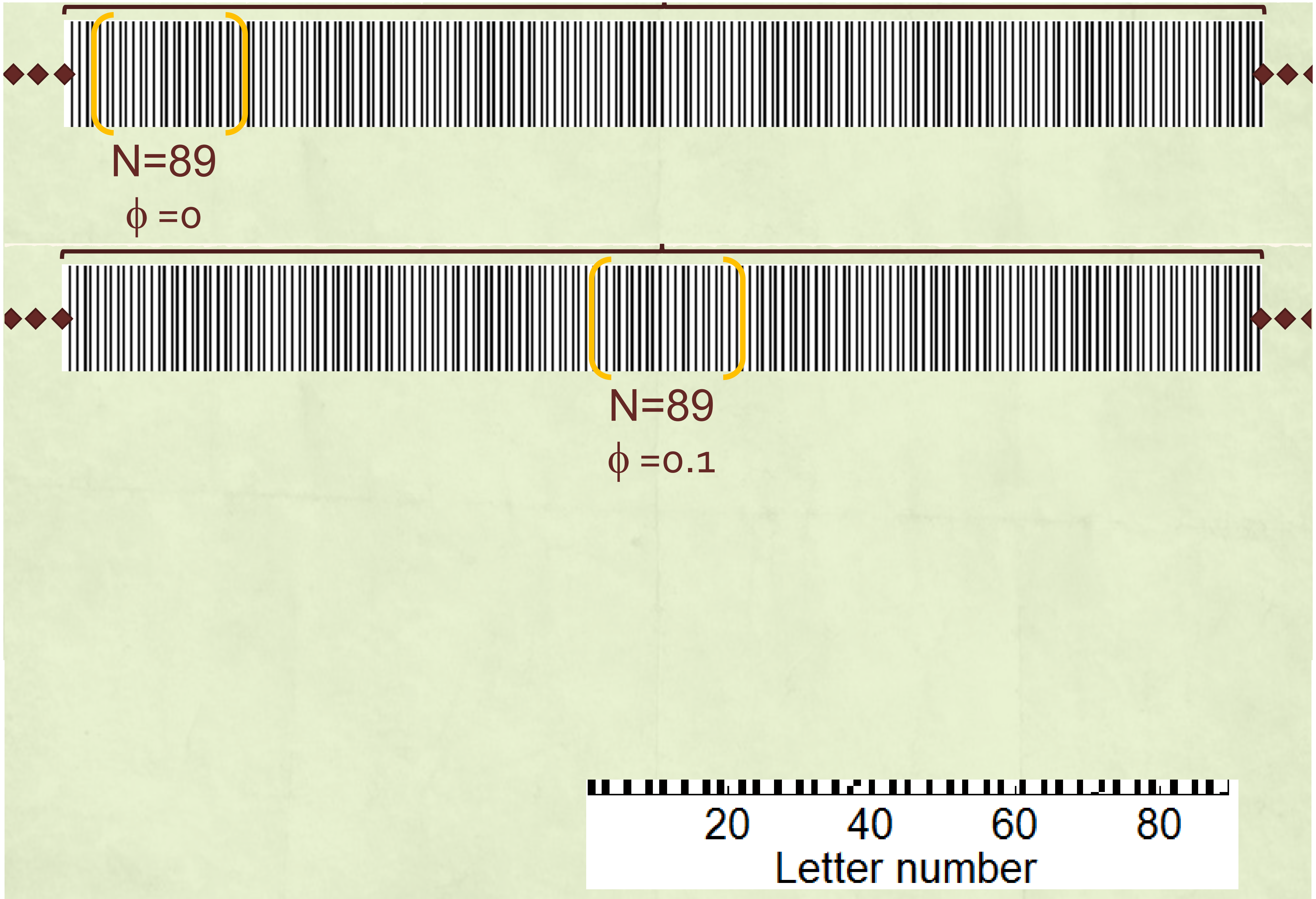
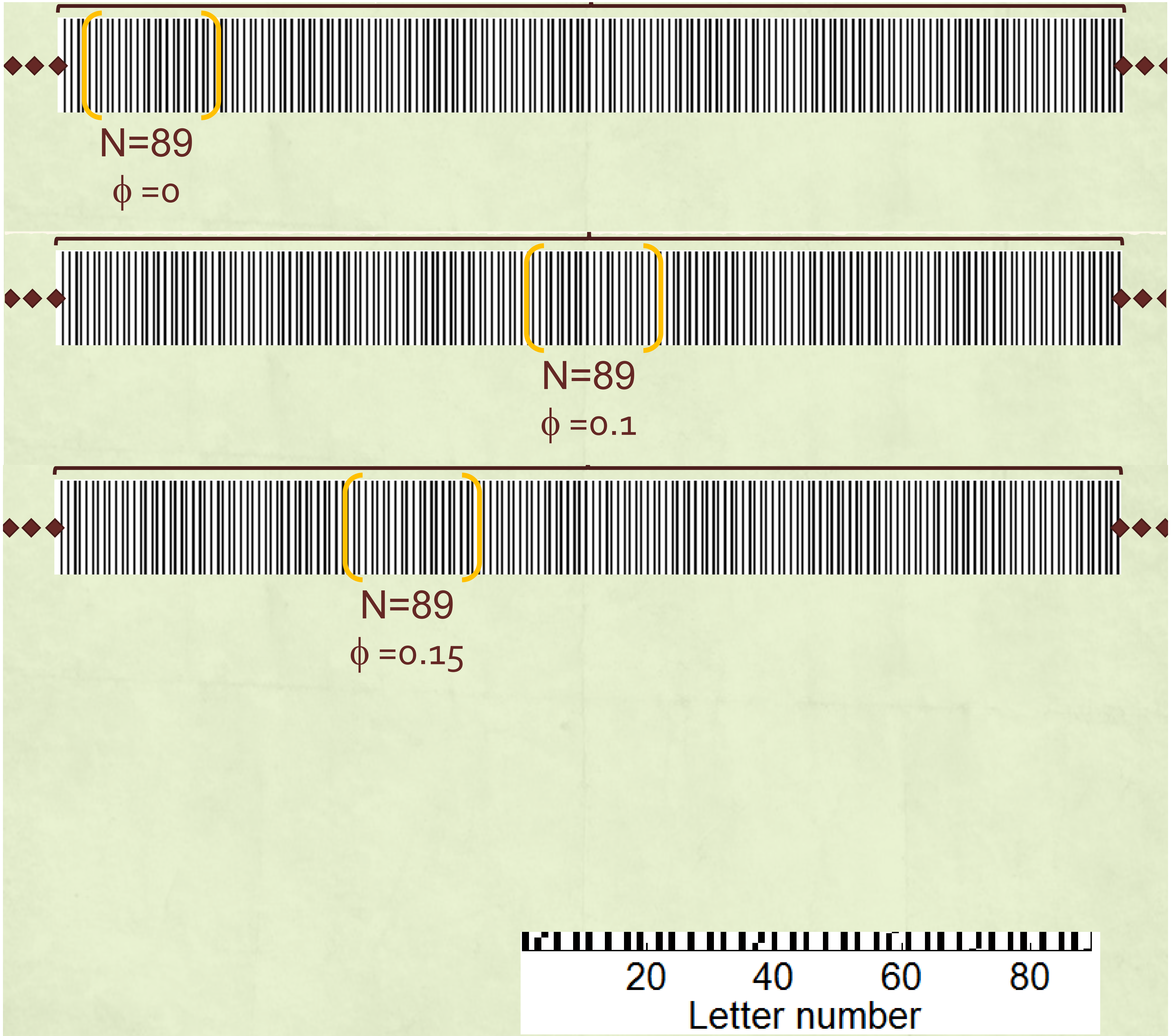
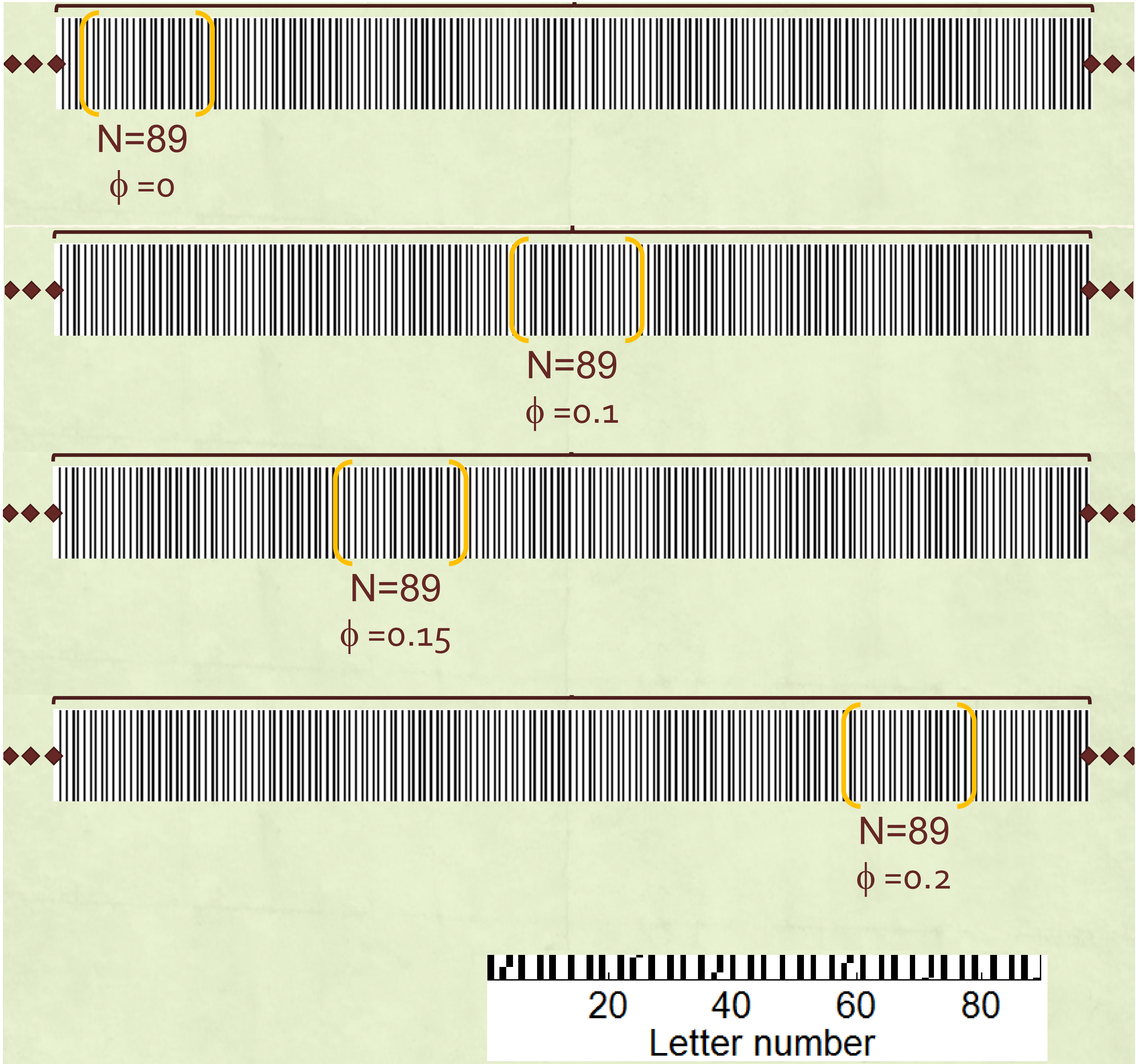
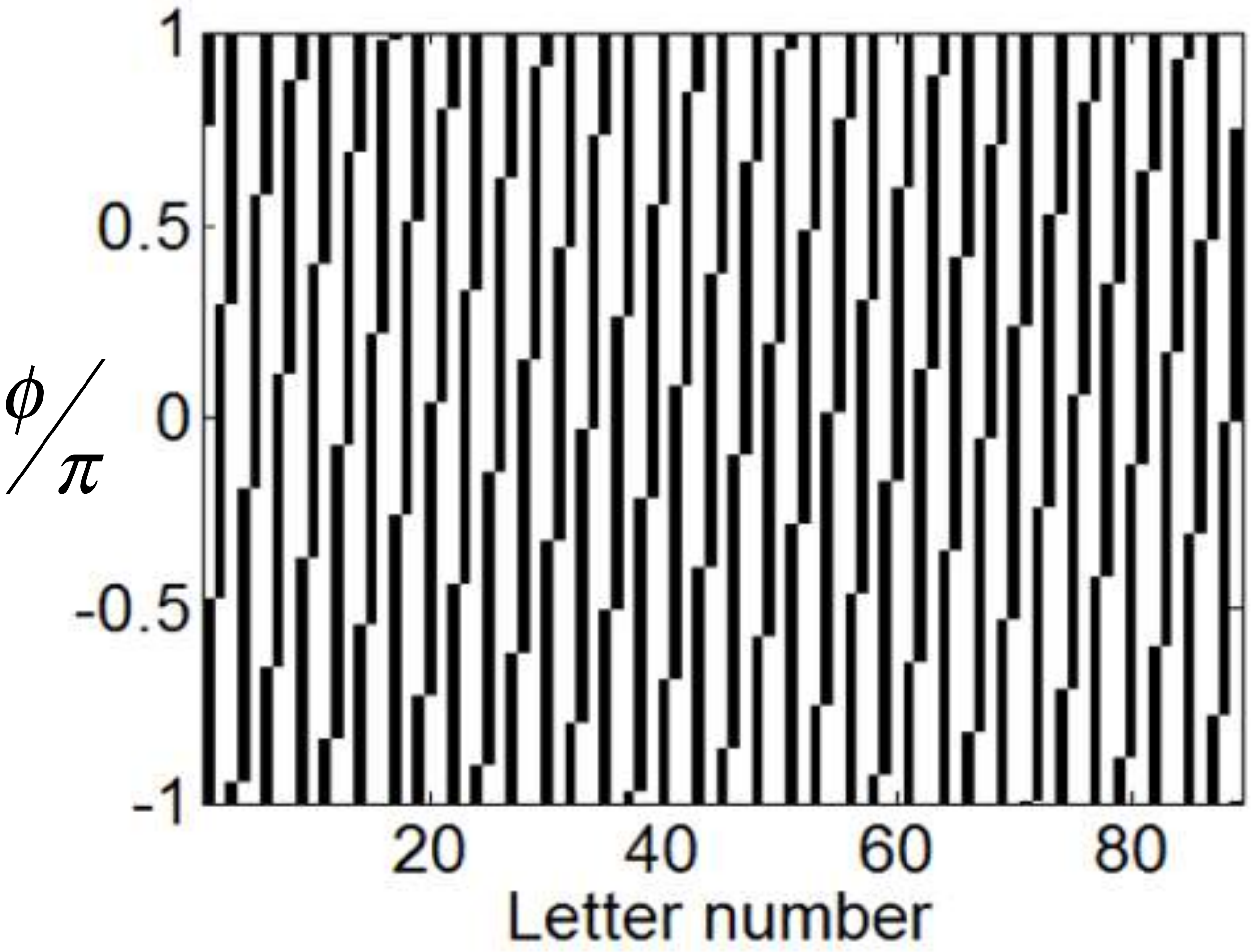



N=89
ϕ =0
N=89
ϕ =0.1
N=89
ϕ =0.15
20
40
60
80
Letter number

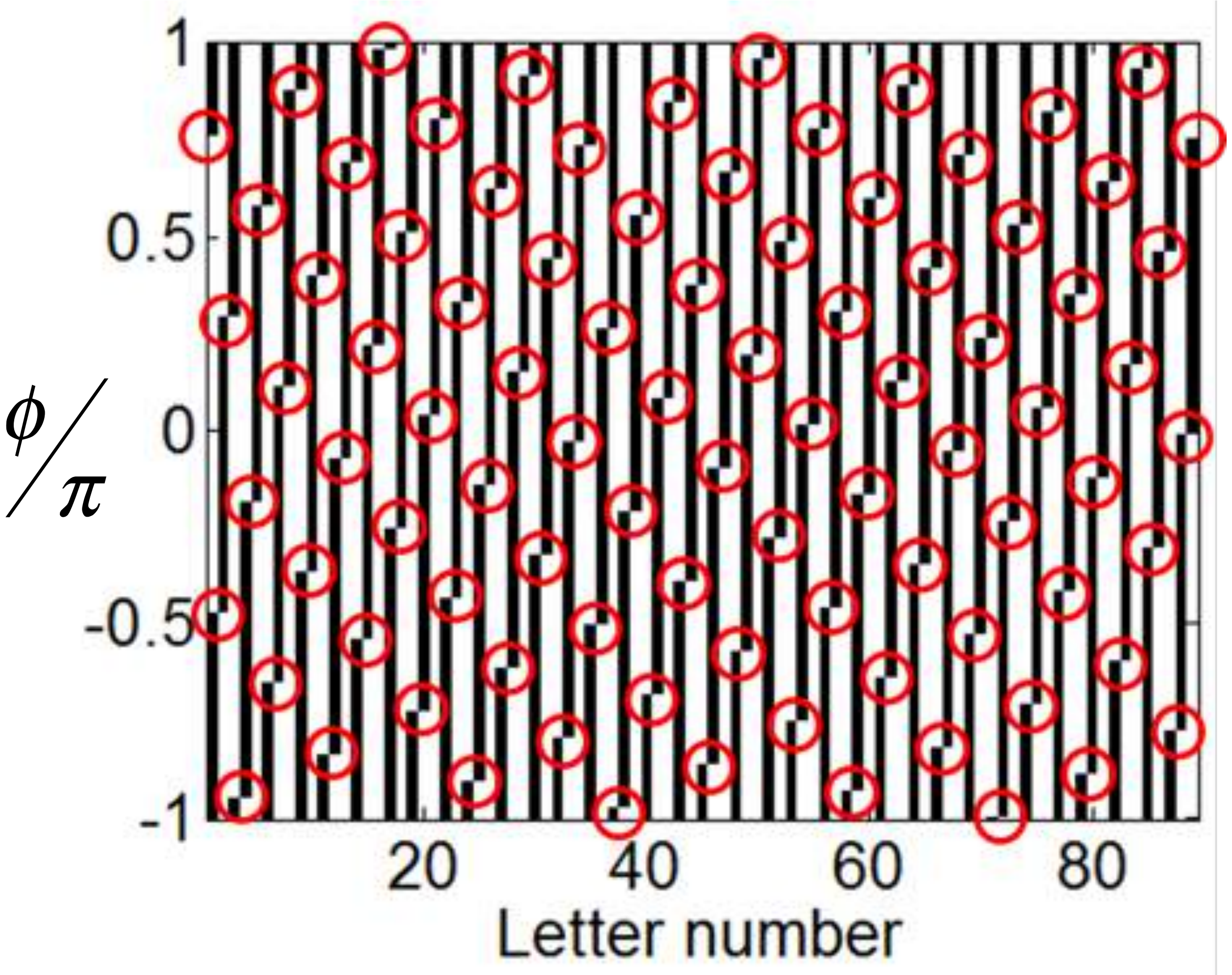

N=89
ϕ =0
N=89
ϕ =0.1
N=89
ϕ =0.15
N=89
ϕ =0.2
20
40
60
80
Letter number

Varying ϕ over a period 2π



$$\chi_n = sign\left[\cos\left(2\pi n\tau^{-1}+\phi\right)-\cos\left(\pi\tau^{-1}\right)\right]$$

Varying ϕ over a period 2π

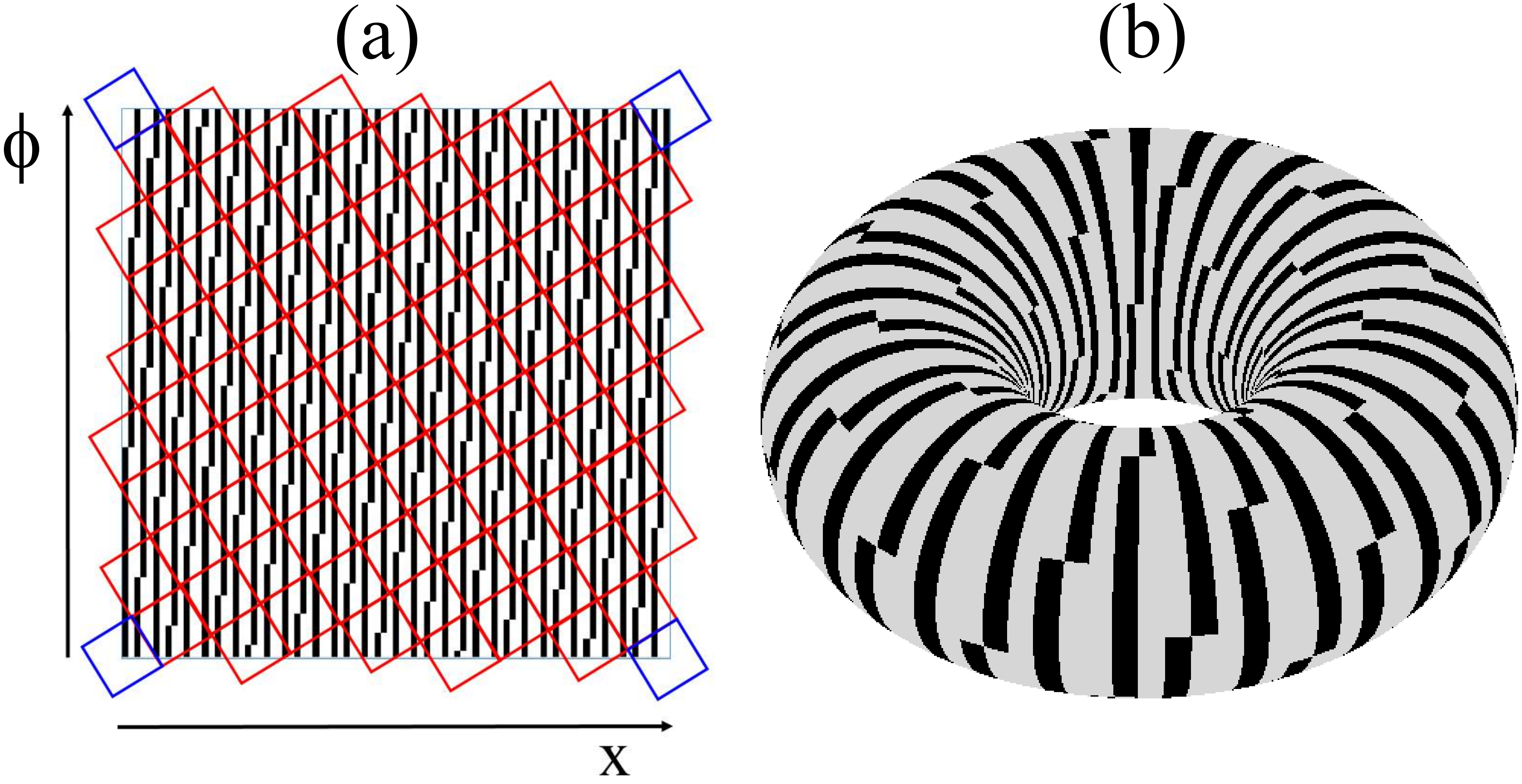


$$\chi_n = sign\left[\cos\left(2\pi n\tau^{-1}+\phi\right)-\cos\left(\pi\tau^{-1}\right)\right]$$

Scanning ϕ generates **local** structural changes.

A structural degree of freedom

A torus



Are there spectral consequences
of these structural properties ?

Are there spectral consequences
of these structural properties ?

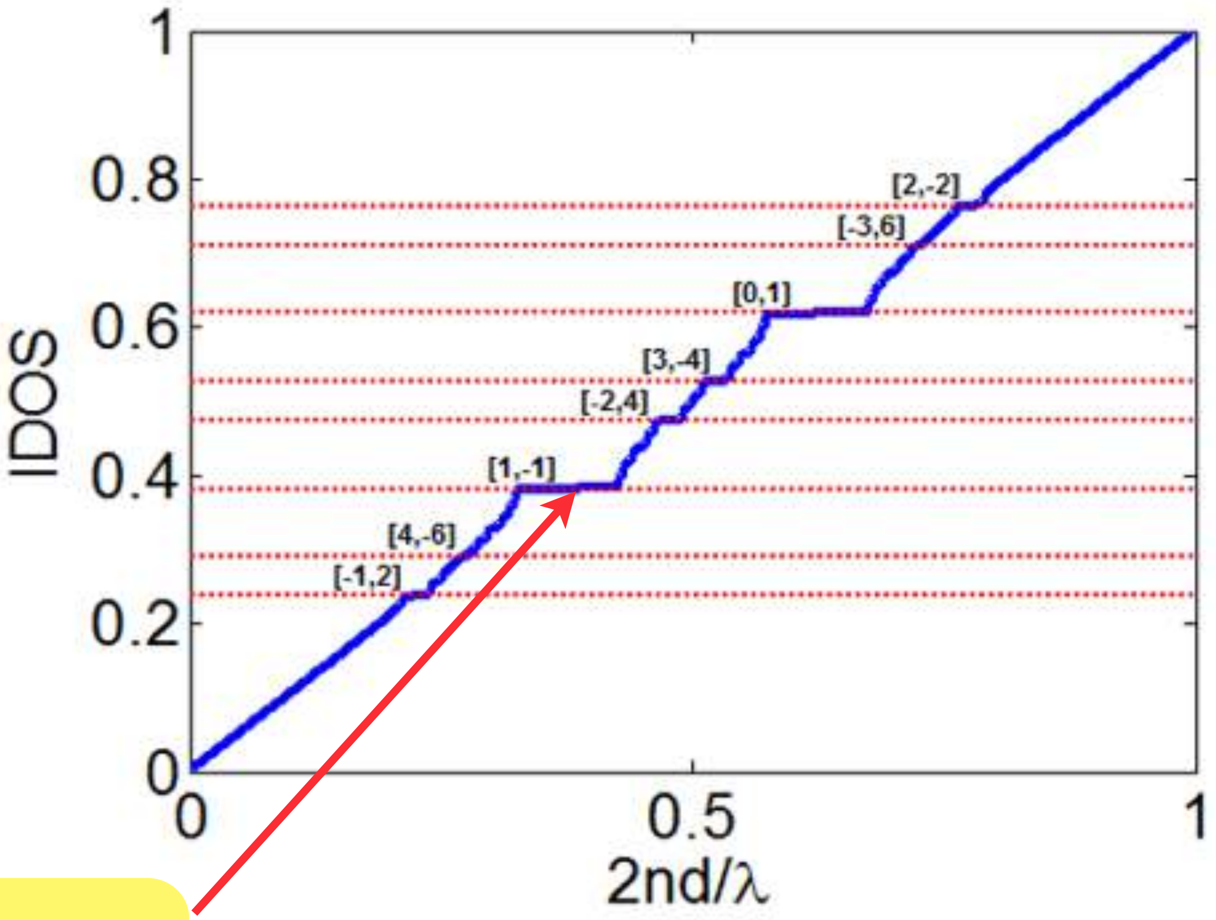
Almost No…

We have already calculated and measure the spectrum in details

Integrated Density of States-Gap Labeling

$$\tau = \left(1+\sqrt{5}\right)\Big/2$$

golden mean



$$N\left(\omega_{gap}\right) = p + q\tau^{-1}$$

within a $\left[p,q\right]$ gap

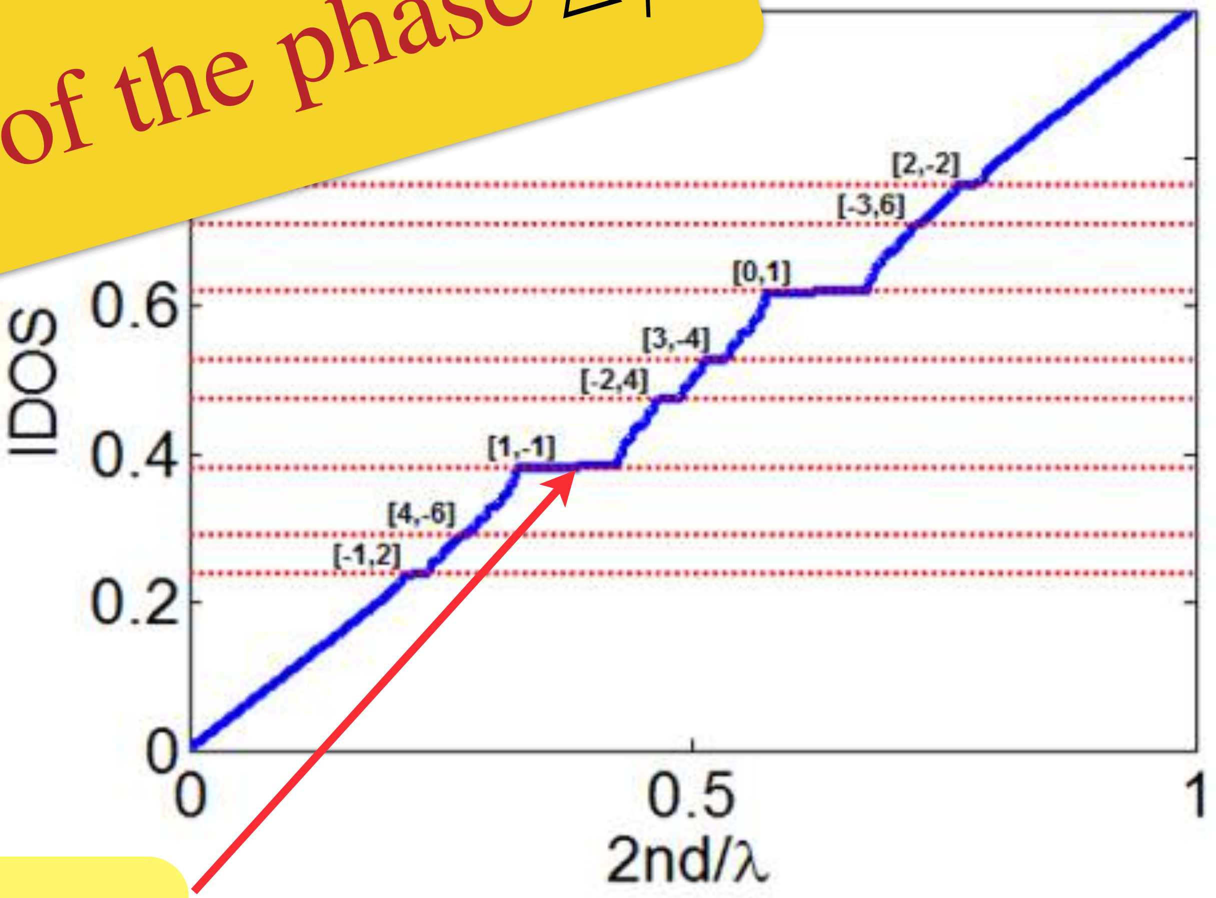
$(p,q) \in \mathbb{Z}$ are topological invariants (Chern numbers).

Independent of the potential strength, inhomogeneity, ...

Integrated Density of States-Gap Labeling

Independent of the phase $\Delta\phi$

golden mean



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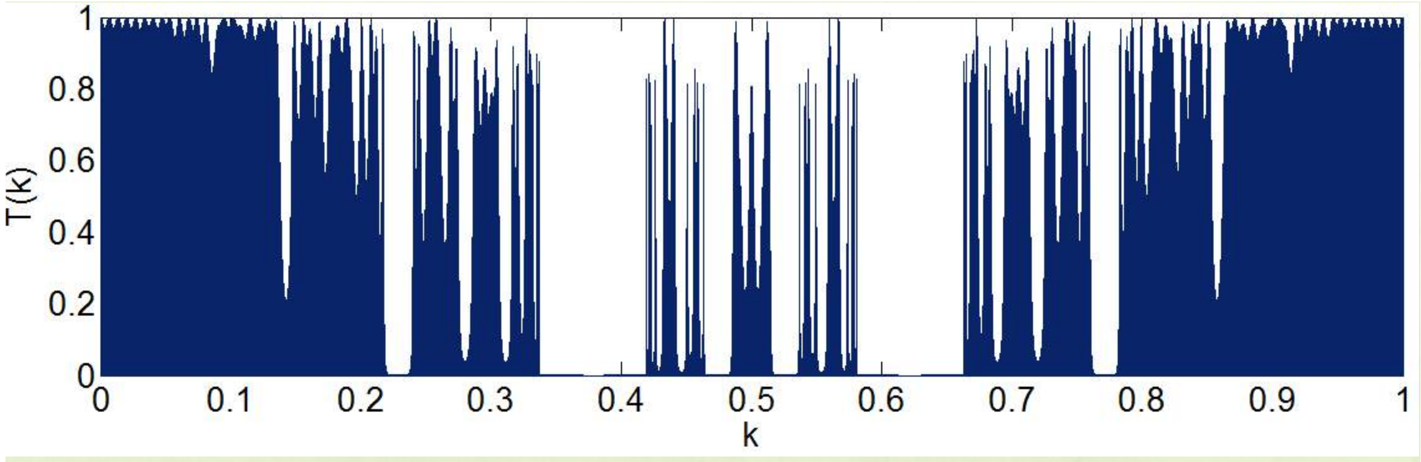
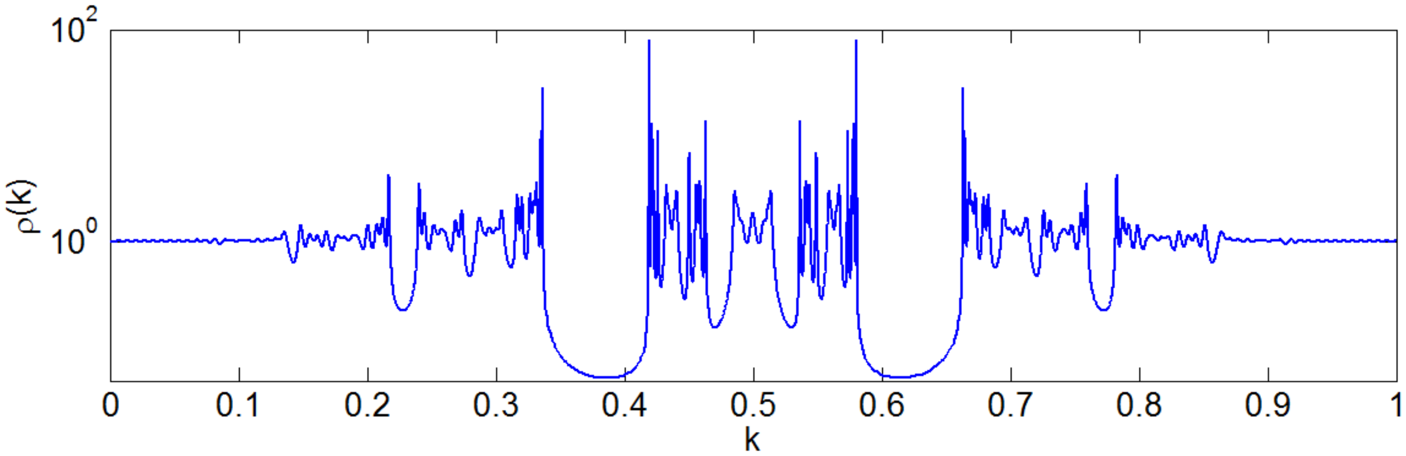
Independent of the potential strength, inhomogeneity, ...

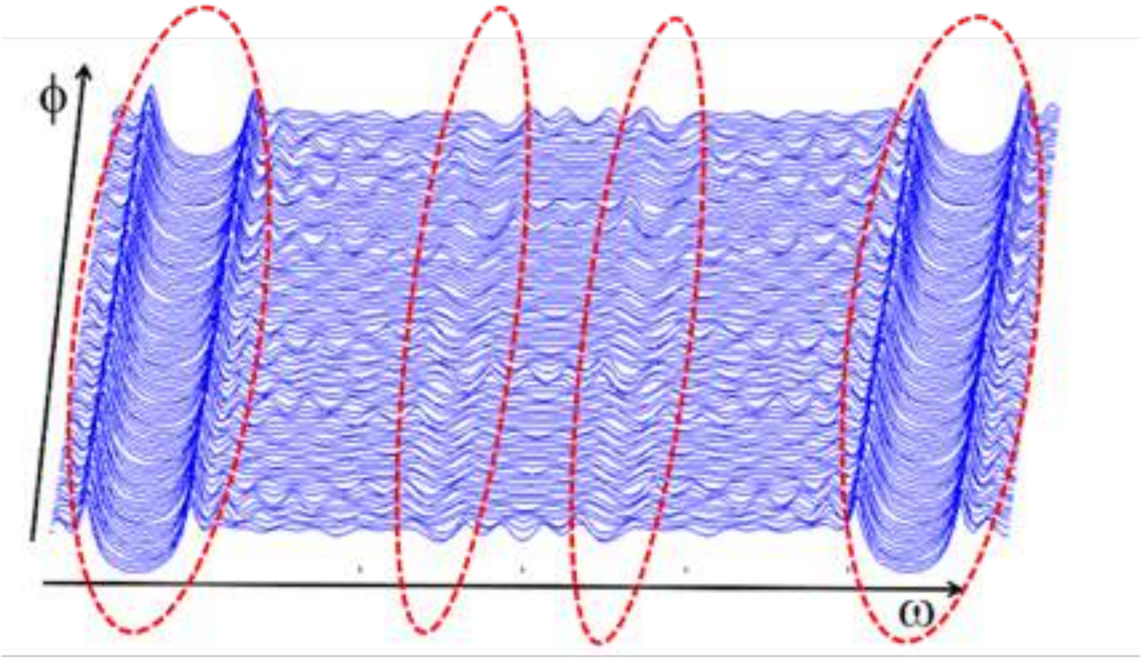
Spectral characteristics are independent of the phase $\Delta\phi$



Transmission spectrum



Density of modes



Spectral characteristics are independent of the phase $\Delta\phi$

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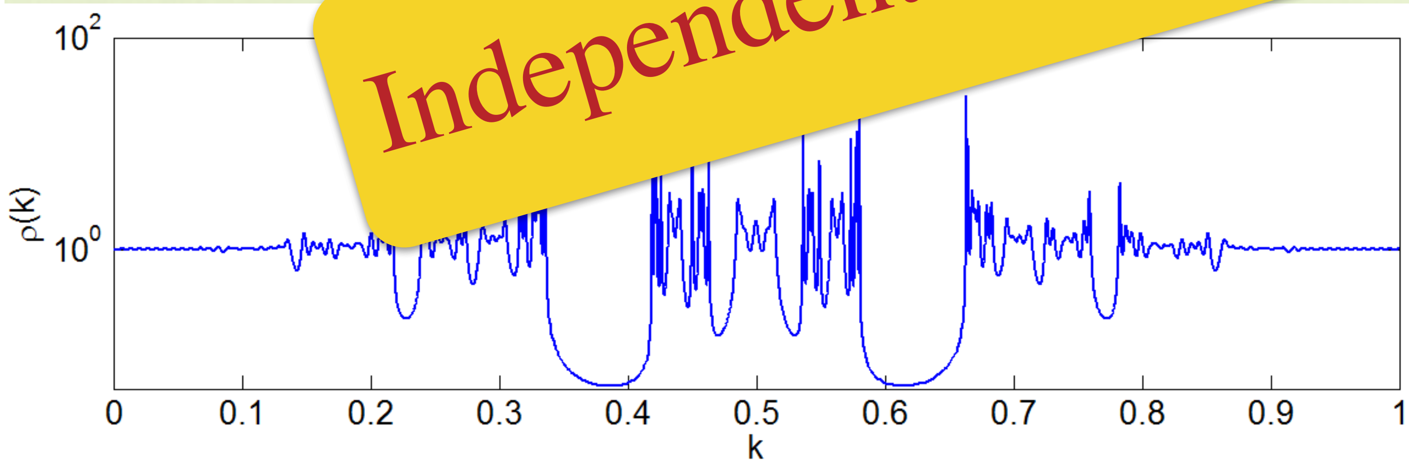
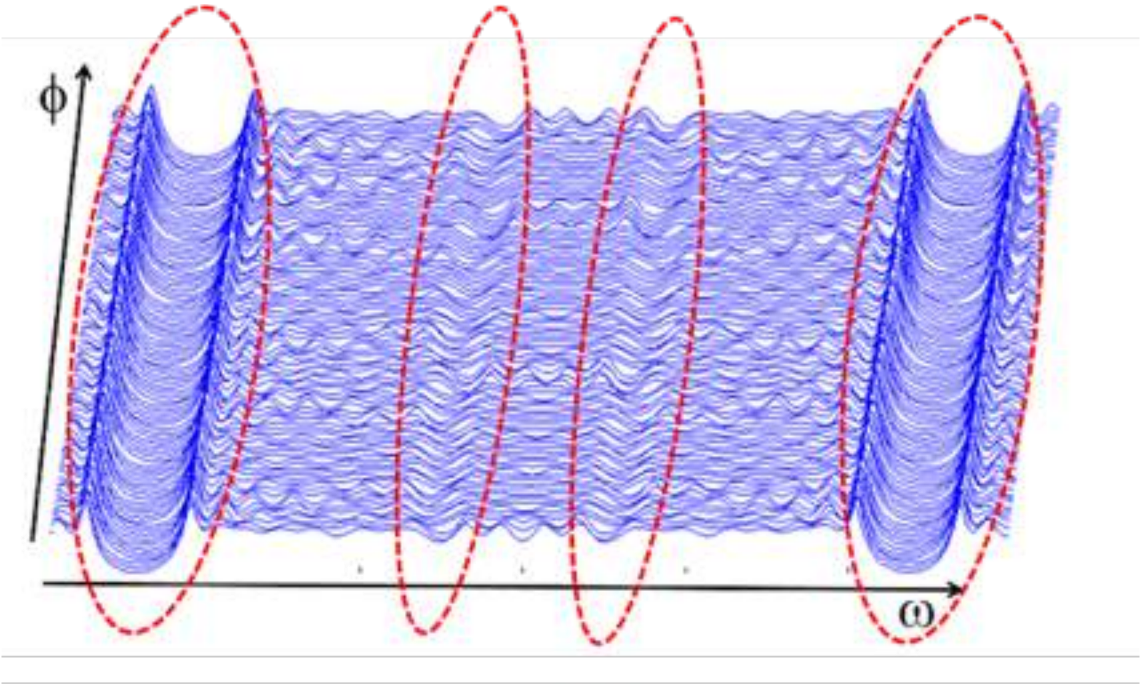
smission spectrum



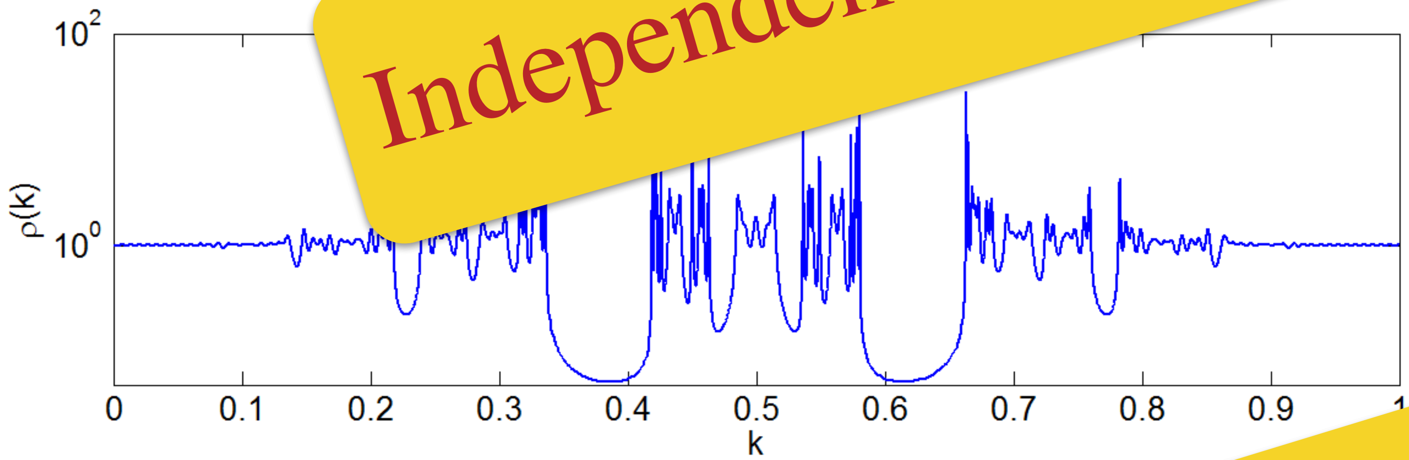
Density of modes



Spectral characteristics are independent of the phase $\Delta\phi$

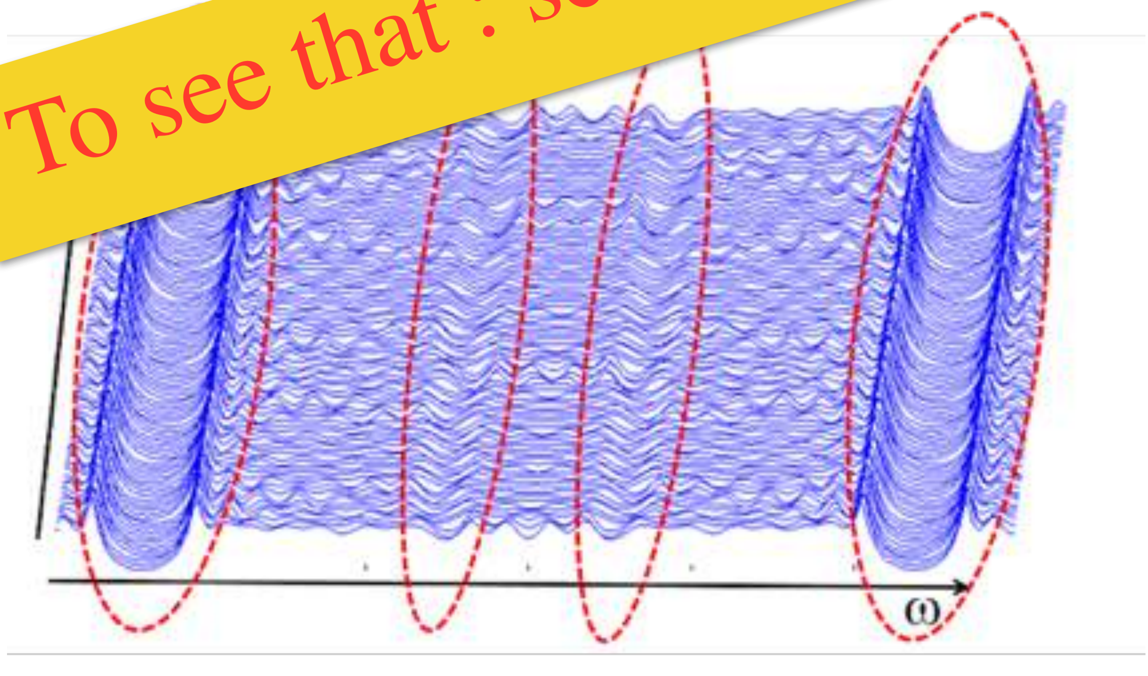
Independent of the phase $\Delta\phi$

smission spectrum



Density of modes

To see that : scattering formalism



Scattering formalism

offers a general and elegant framework to study spectral properties and transport (Landauer approach).

A (quantum,wave) system with a potential (defined w.r.t. a free part) is enclosed in a "black box". We probe it using scattering waves.

With obvious notations, the (unitary) scattering matrix is :

$$\begin{pmatrix} o_L \\ o_R \end{pmatrix} = \begin{pmatrix} r & t \\ t & r' \end{pmatrix} \begin{pmatrix} i_L \\ i_R \end{pmatrix} \equiv S \begin{pmatrix} i_L \\ i_R \end{pmatrix}$$

It can be diagonalised as $\begin{pmatrix} e^{i\phi_1} & 0 \\ 0 & e^{i\phi_2} \end{pmatrix}$

Defining the total phase shift : $\delta(k) \equiv (\phi_1(k) + \phi_2(k))/2$

There is a relation between the total phase shift and the change of density of states :

$$\rho(k) - \rho_0(k) = \frac{1}{\pi}\frac{d\delta(k)}{dk}$$
$$= \frac{1}{2\pi}\,\mathrm{Im}\,\frac{\partial}{\partial k}\ln\det S(k)$$

since $\det S(k) = e^{2i\delta(k)} = -\frac{t}{t^*}$

G. Dunne, E. Levy, E.A.,"Optics of Aperiodic Structures: Fundamentals and Device Applications", L. dal Negro ed., Pan Stanford Publishing, (2013)

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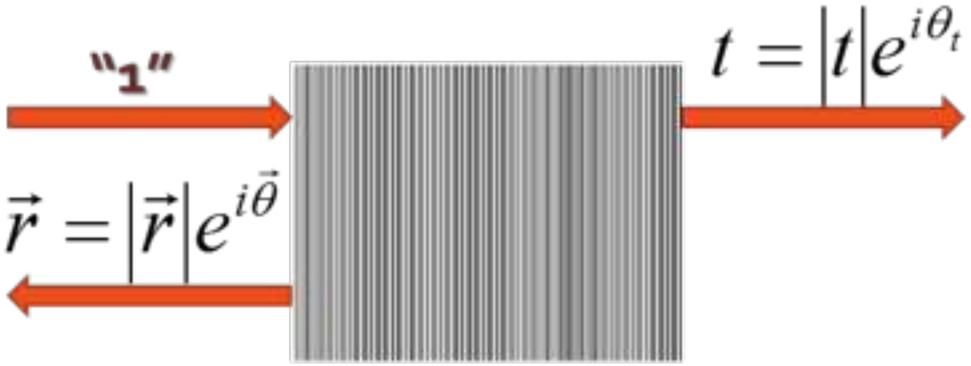
Fibonacci chain embedded between free spaces:



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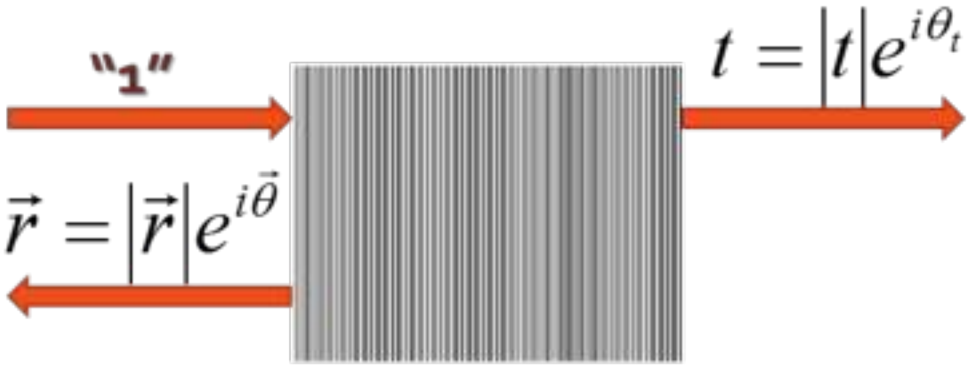

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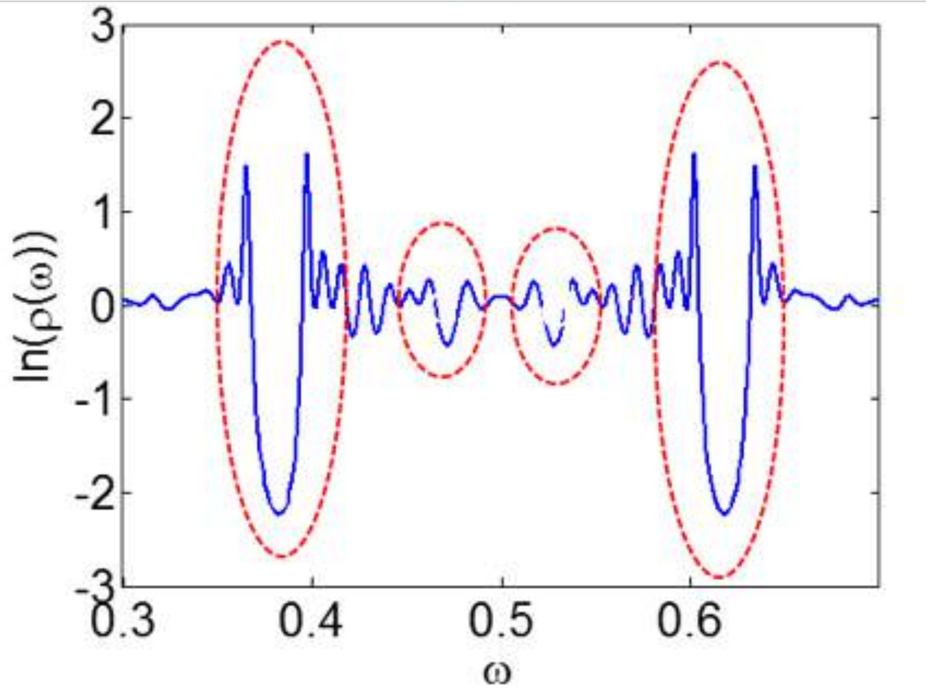
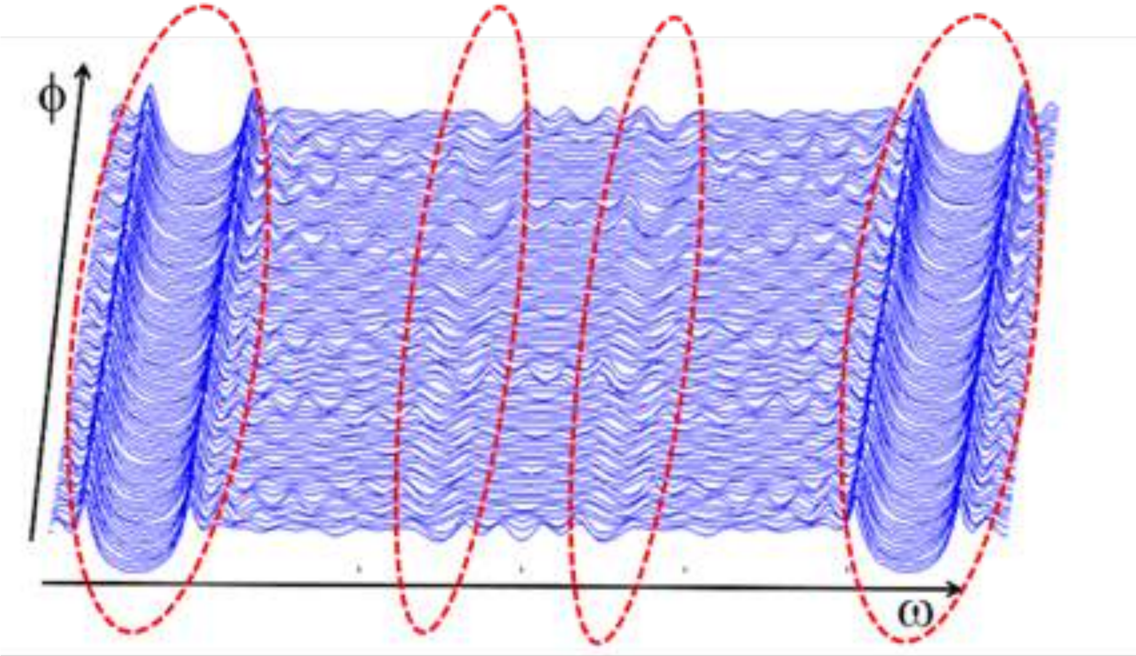




$\delta(k)=\theta_t(k)+\dfrac{\pi}{2}$ is independent of the modulation phase $\Delta\phi$

How to observe a $\Delta\phi$ dependence ?

How to observe a $\Delta\phi$ dependence ?

On edge states since scattering states are independent of the structural phase $\Delta\phi$

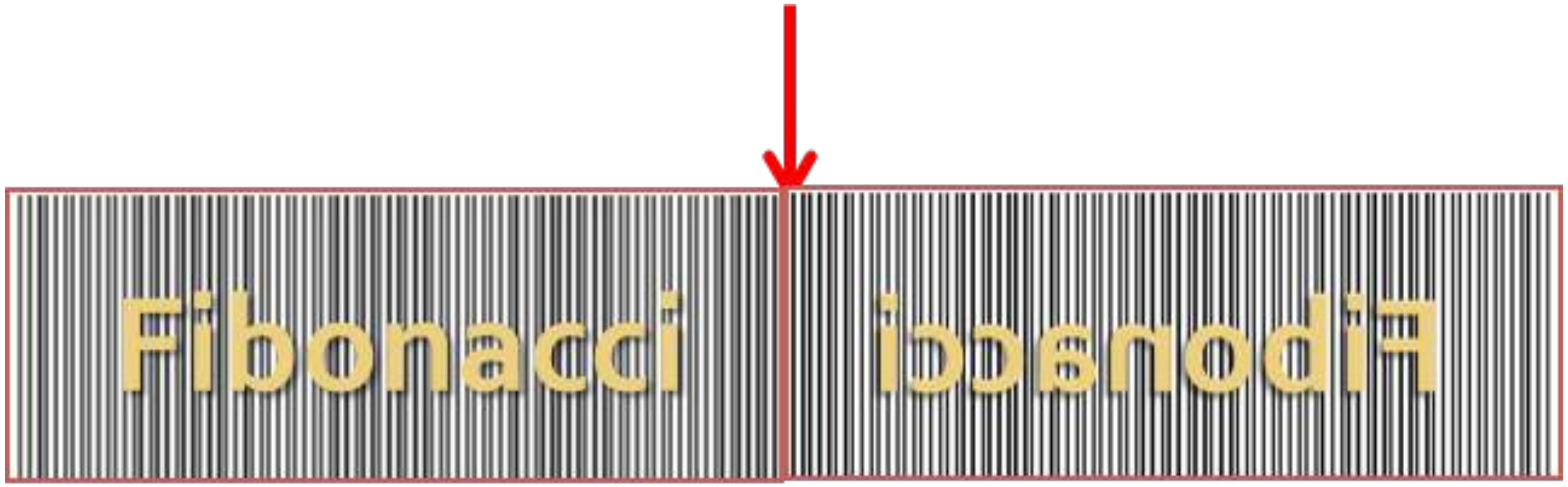
How to create edge states and relate them to the scattering formalism ?

Impose a closed boundary

How to create edge states and relate them to the scattering formalism ?

Impose a closed boundary

Equivalent to

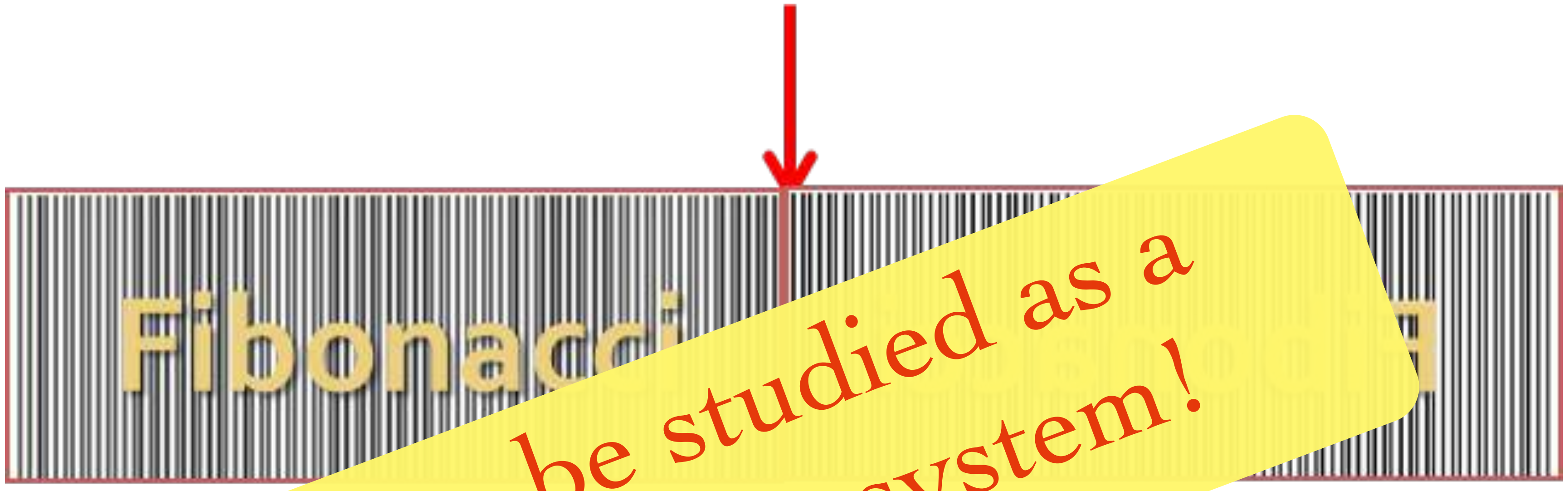


$$\overrightarrow{F}_N \overleftarrow{F}_N$$

How to create edge states and relate them to the scattering formalism ?

Impose a closed boundary

Equivalent to

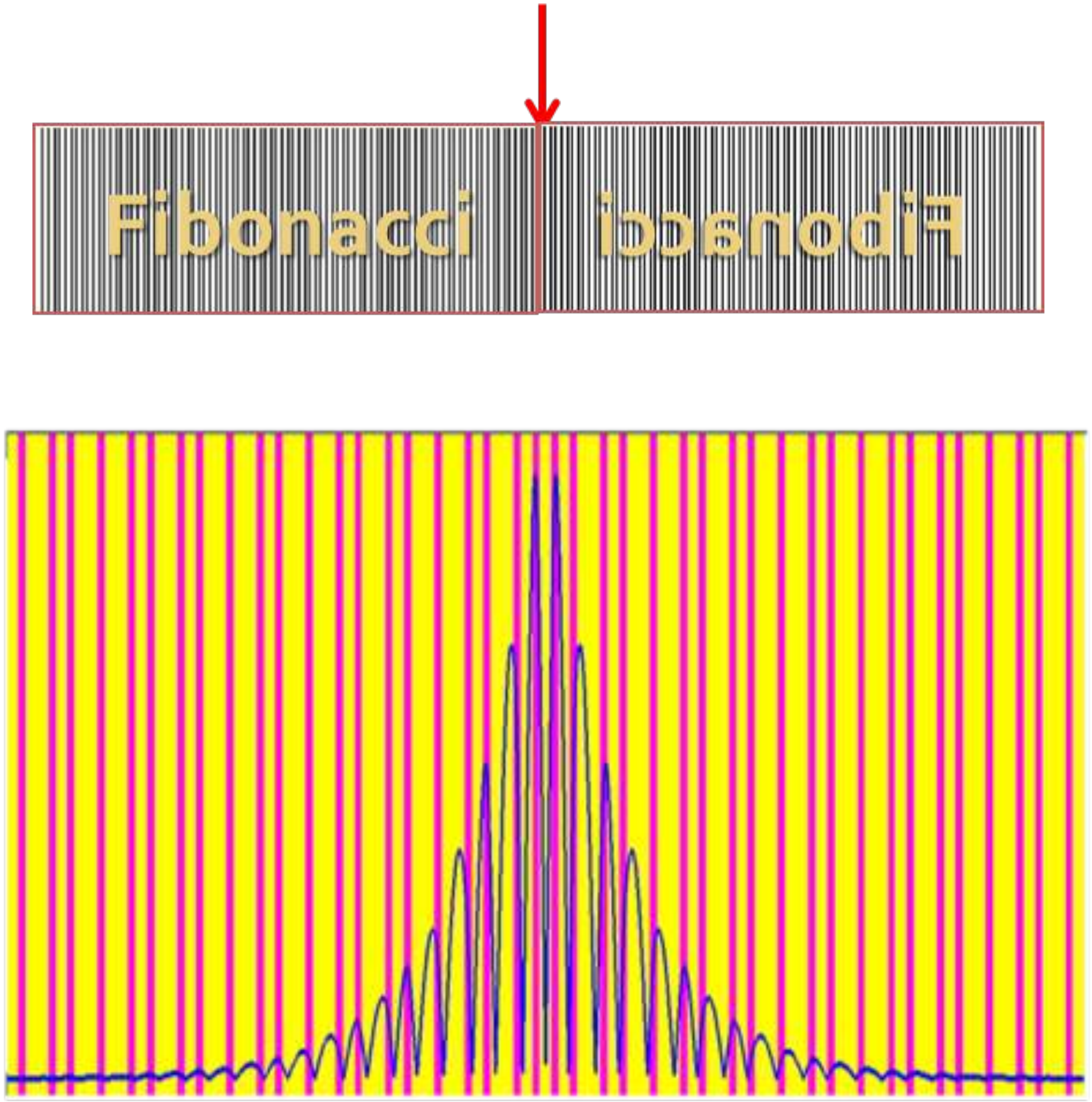


can be studied as a scattering system!

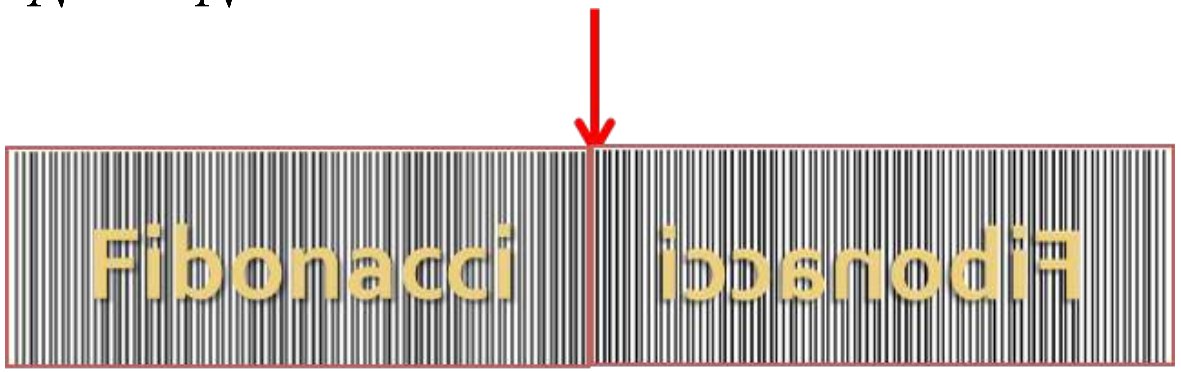
$$\vec{F}_N \overline{F}_N$$

Edge states of $\overrightarrow{F}_N \overleftarrow{F}_N$

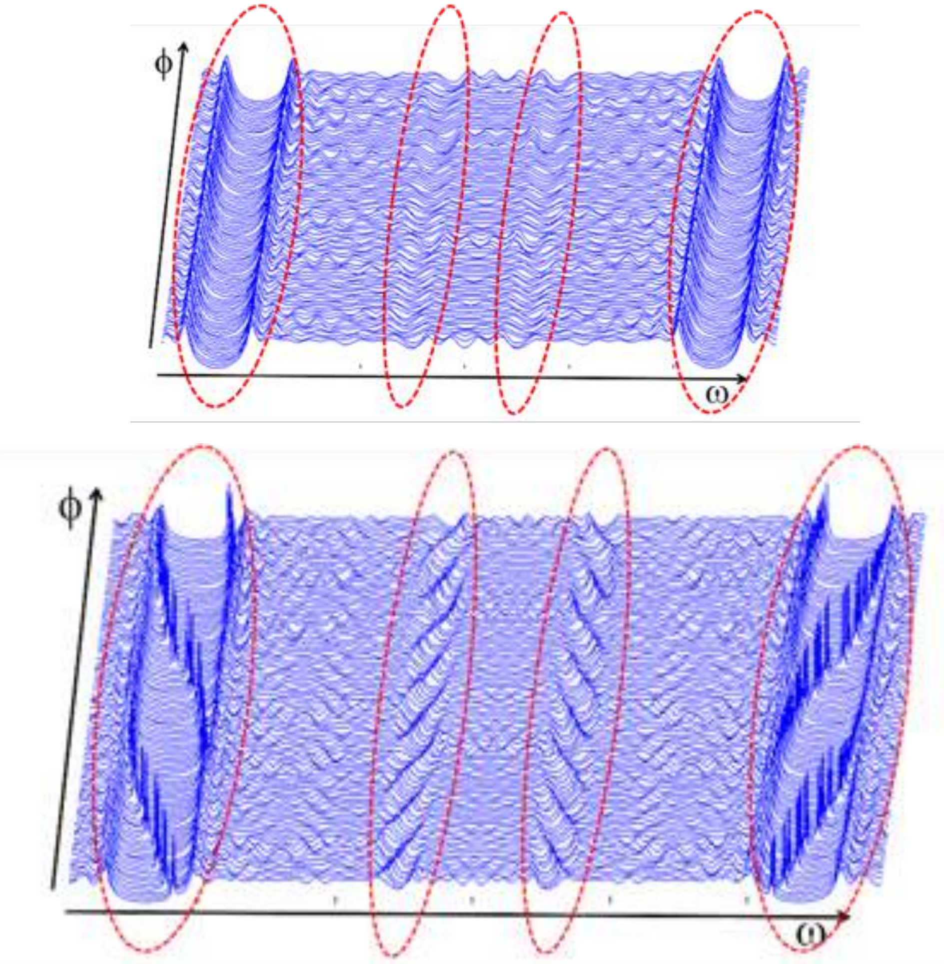
- This structure isn’t Fibonacci : it displays additional modes in the gaps



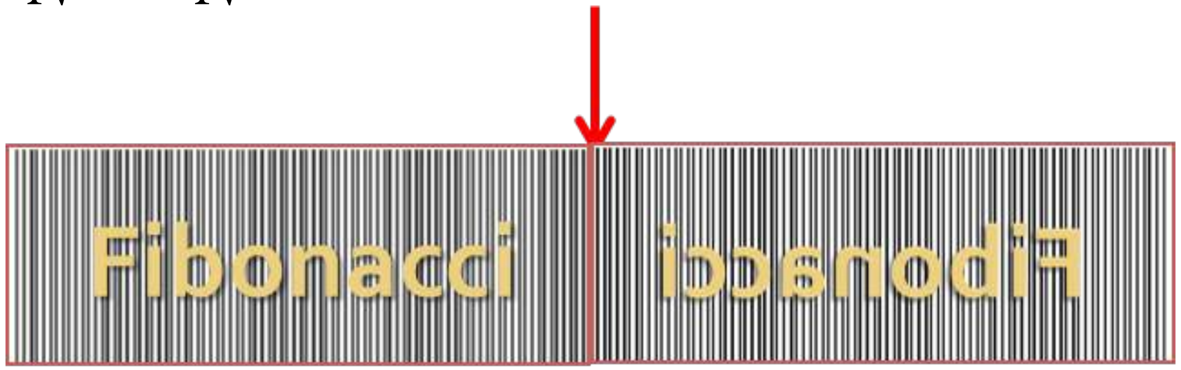
Edge states of $\overrightarrow{F}_N \overleftarrow{F}_N$



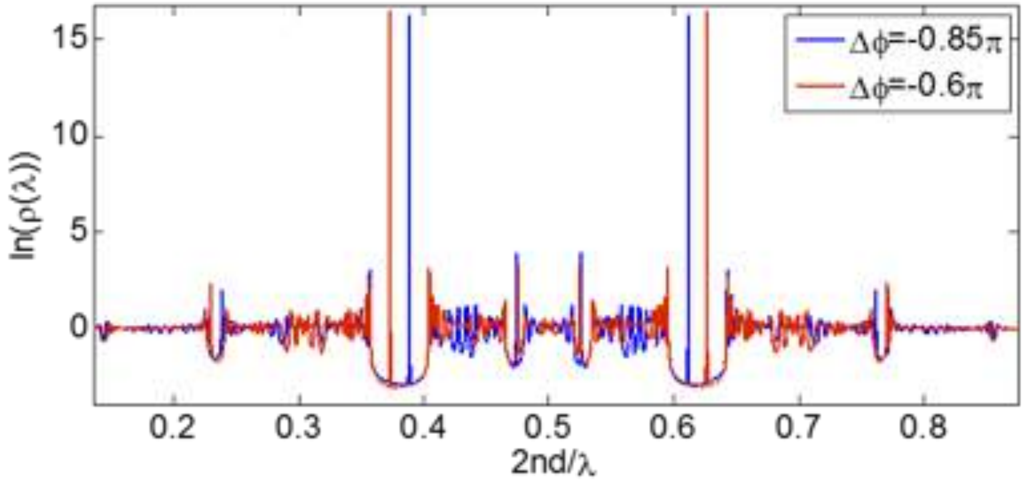
- This structure isn't Fibonacci : it displays additional modes in the gaps
- Gap locations remain unchanged w.r.t. the original structure

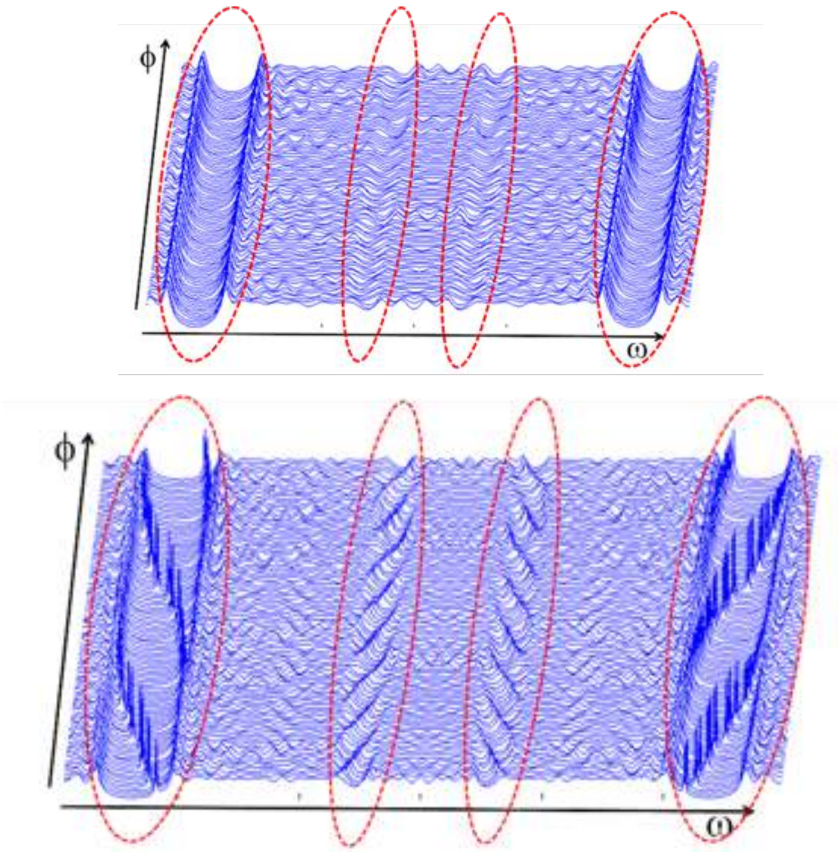


Edge states of $\overrightarrow{F}_N \overleftarrow{F}_N$

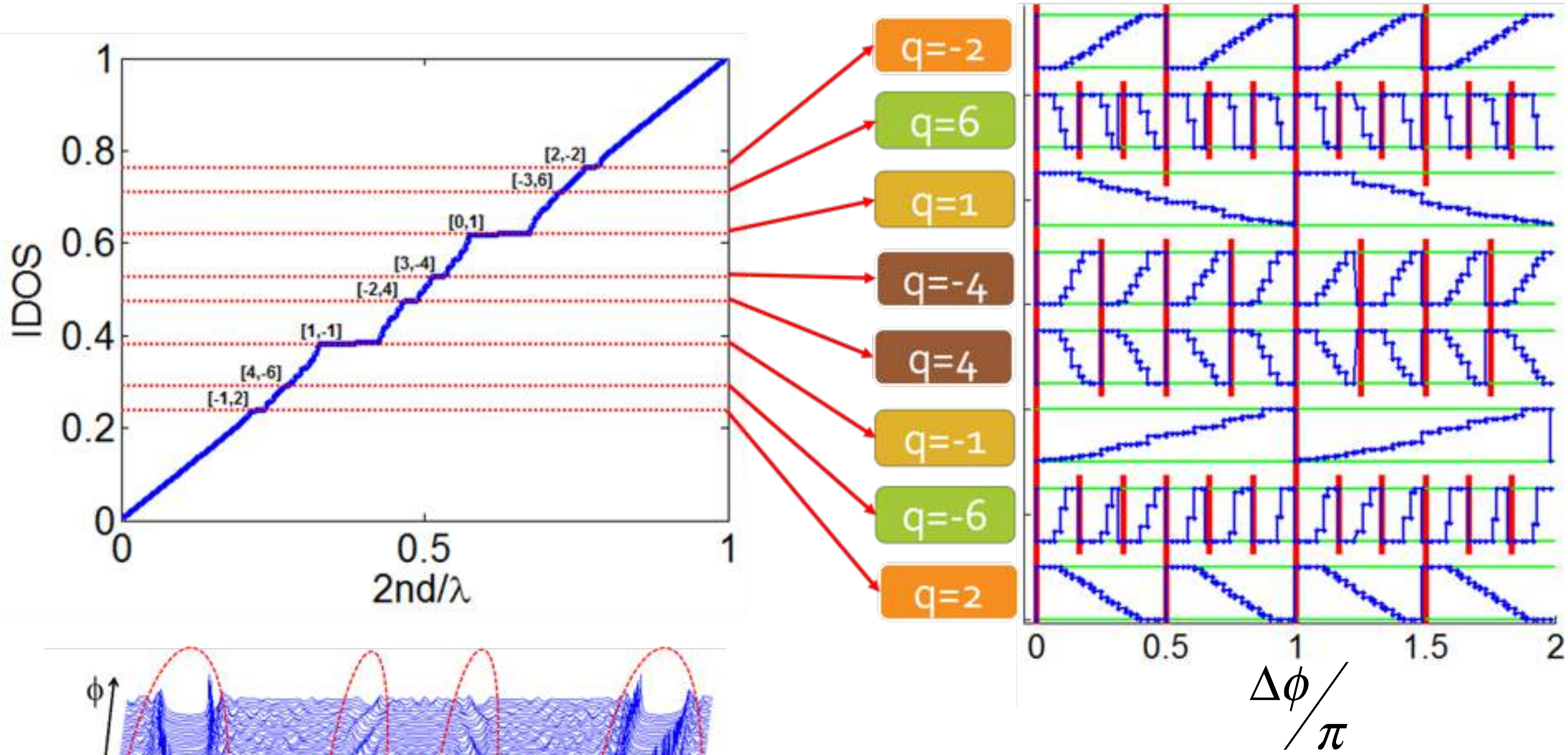


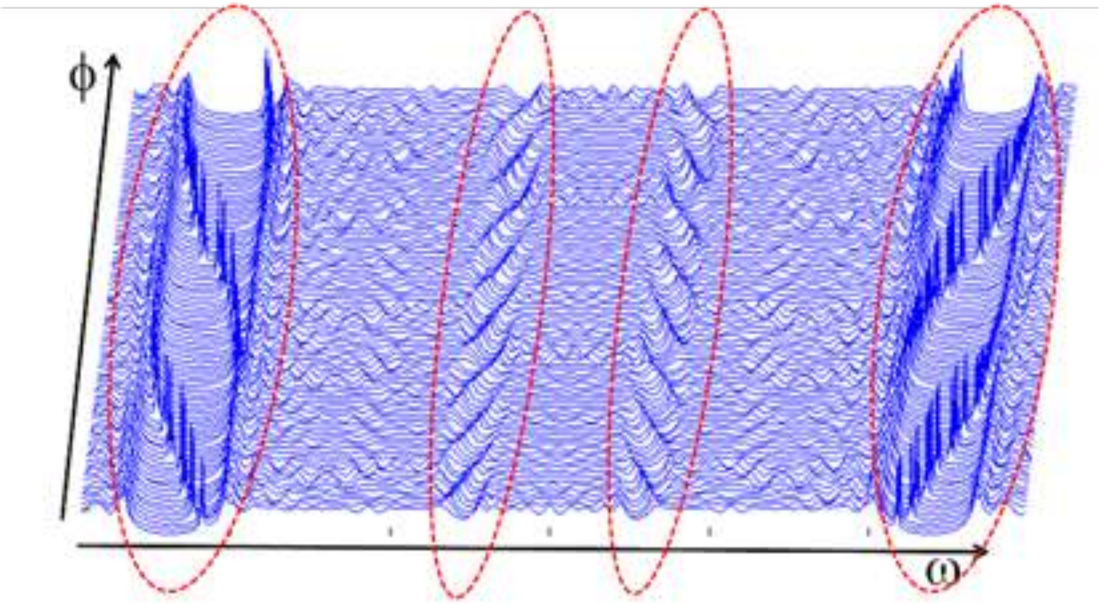
- This structure isn't Fibonacci : it displays additional modes in the gaps
- Gap locations remain unchanged w.r.t. the original structure
- Frequencies of the gap modes depend on the structural modulation phase $\Delta\phi$





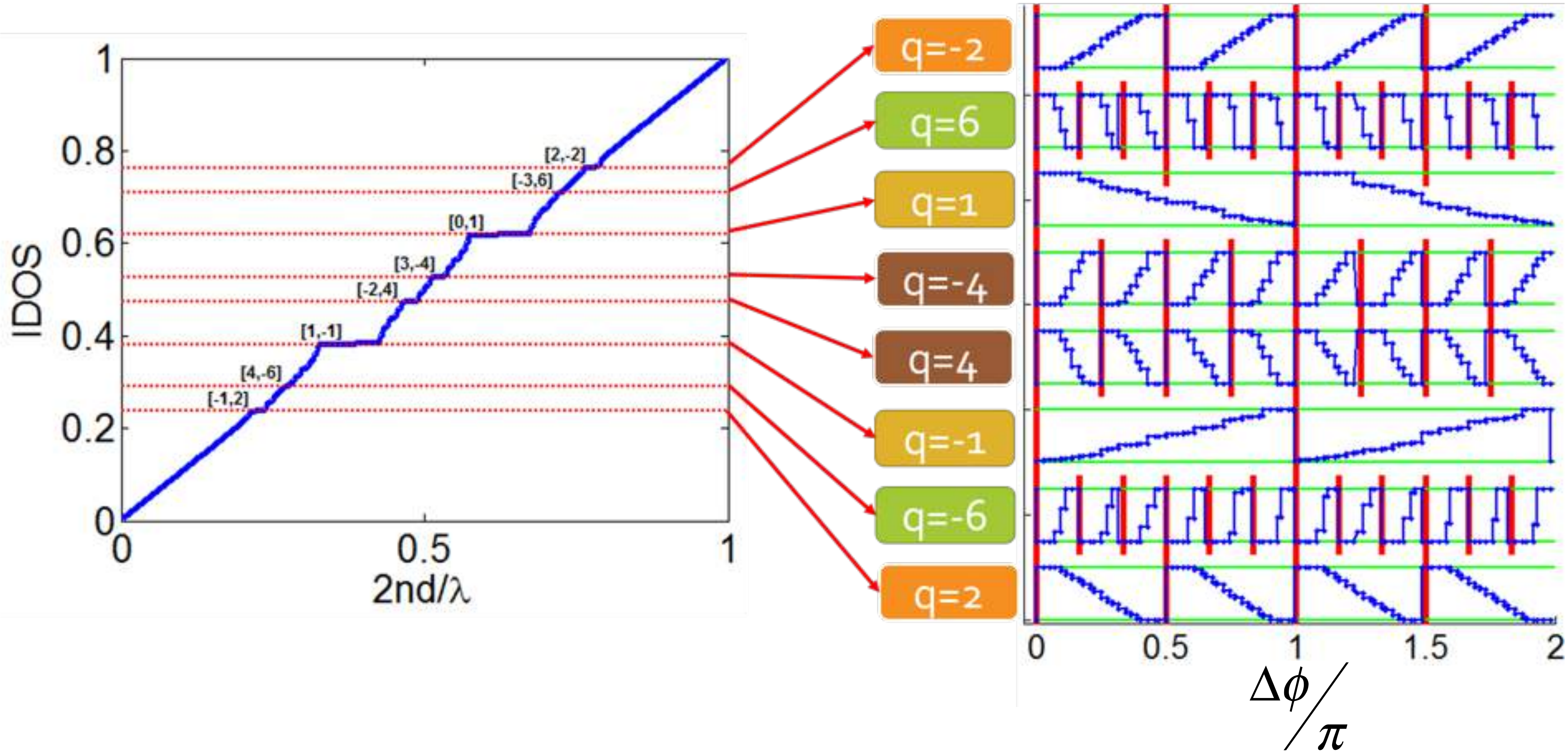
Relation to gap labelling and Chern numbers





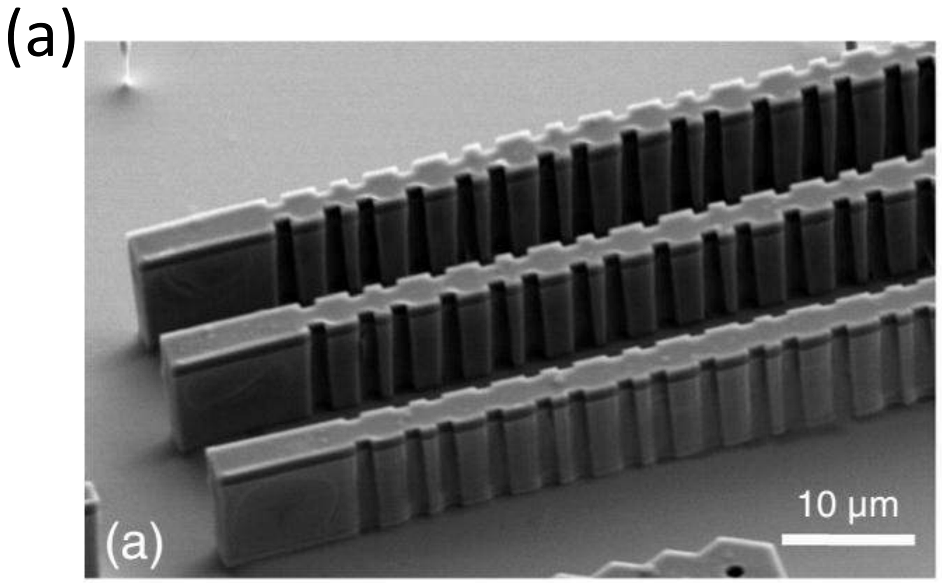
$$N\left(\omega_{(p,q)gap}\right)=p+q\,\tau^{-1}$$

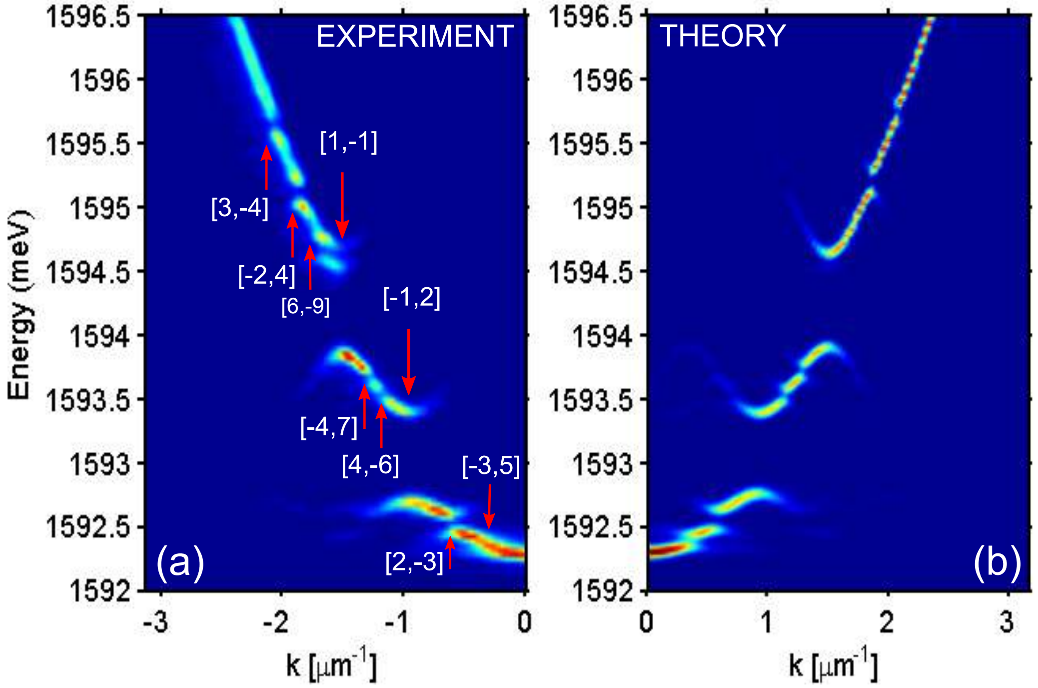
Relation to gap labelling and Chern numbers

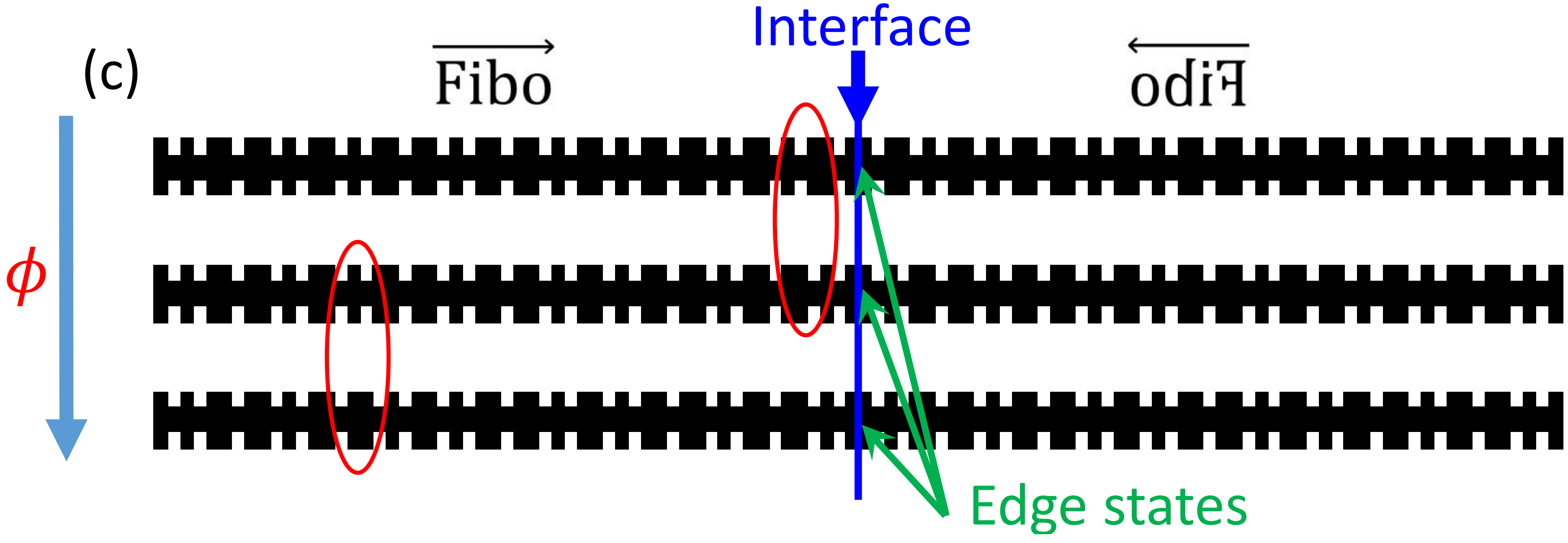


Chern numbers [p,q] describe the topological behaviour of edge states in the gaps when changing the structural phase angle $\Delta\phi$

Recently measured using cavity polaritons

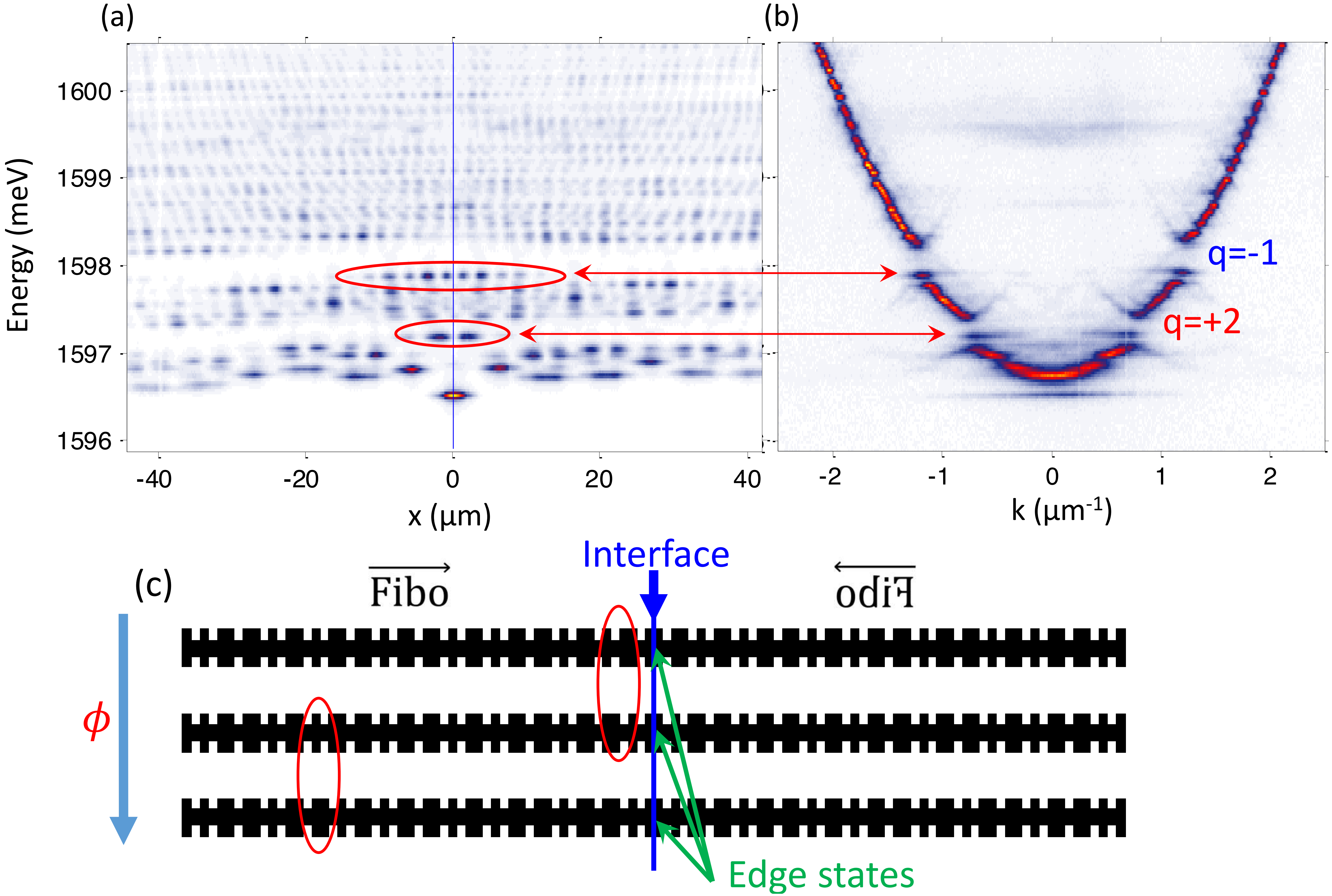




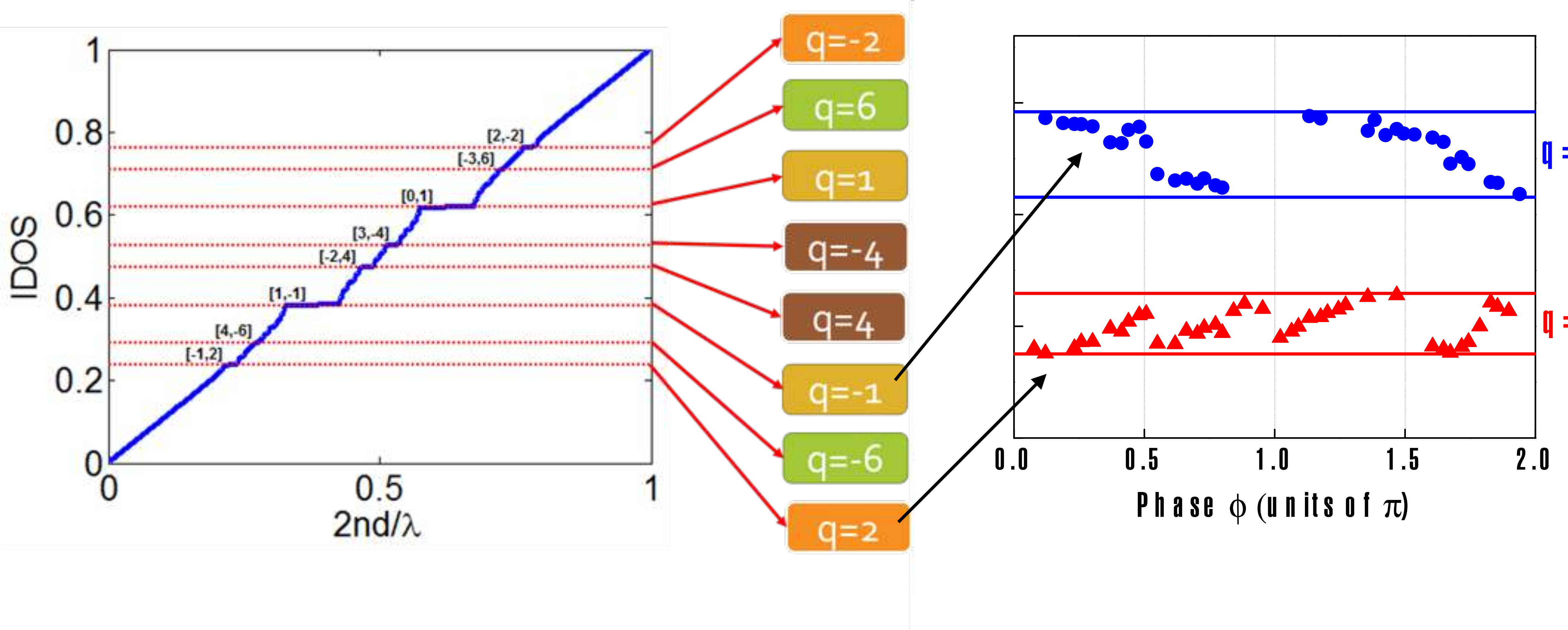


F. Baboux, E. Levy, J. Bloch, E.A, 2016

Recently measured using cavity polaritons



F. Baboux, E. Levy, J. Bloch, E.A, 2016



F. Baboux, E. Levy, J. Bloch, E.A, 2016

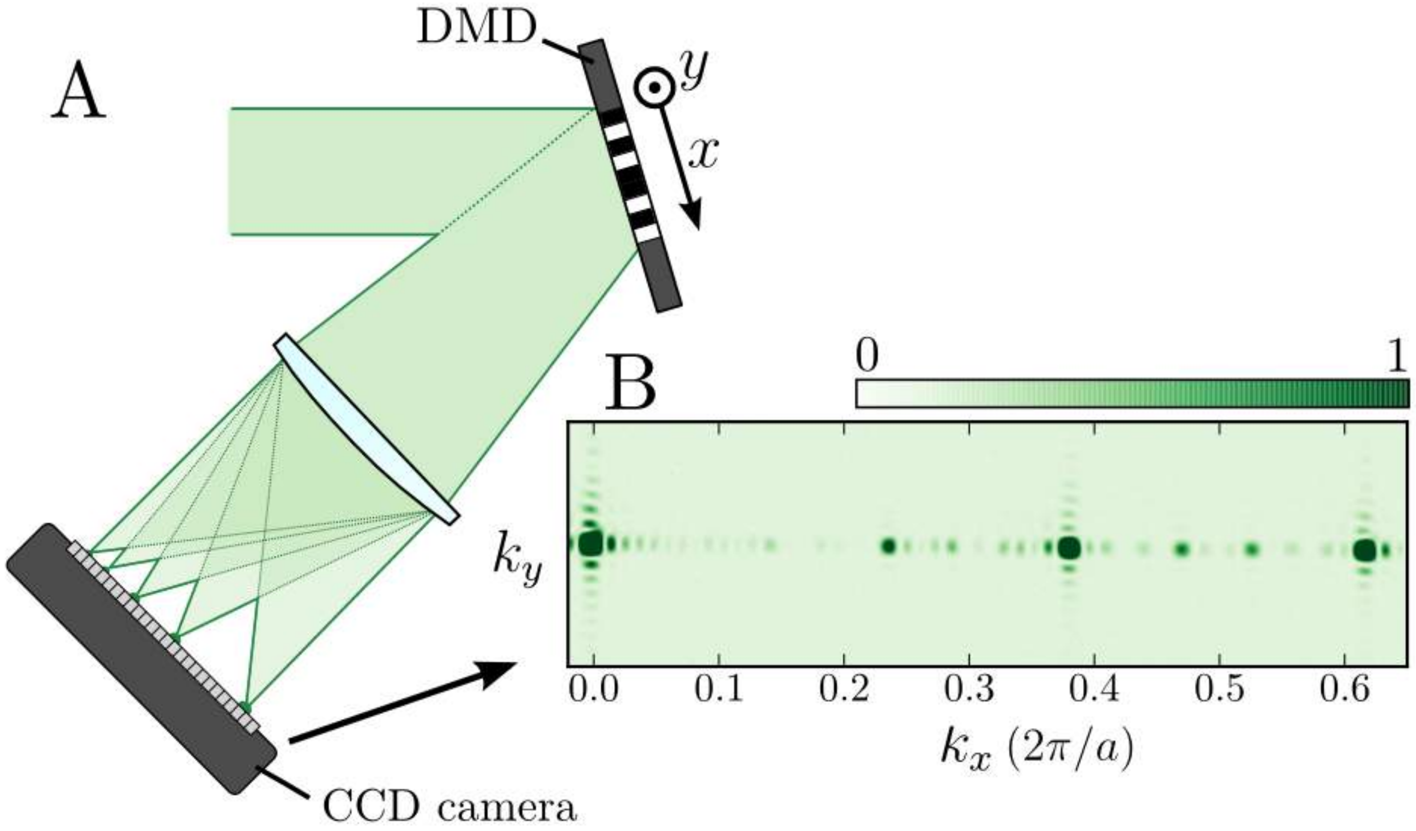
Topological Chern numbers in quasicrystals are of spectral origin !

Topological Chern numbers in quasicrystals are of spectral origin !

Not really

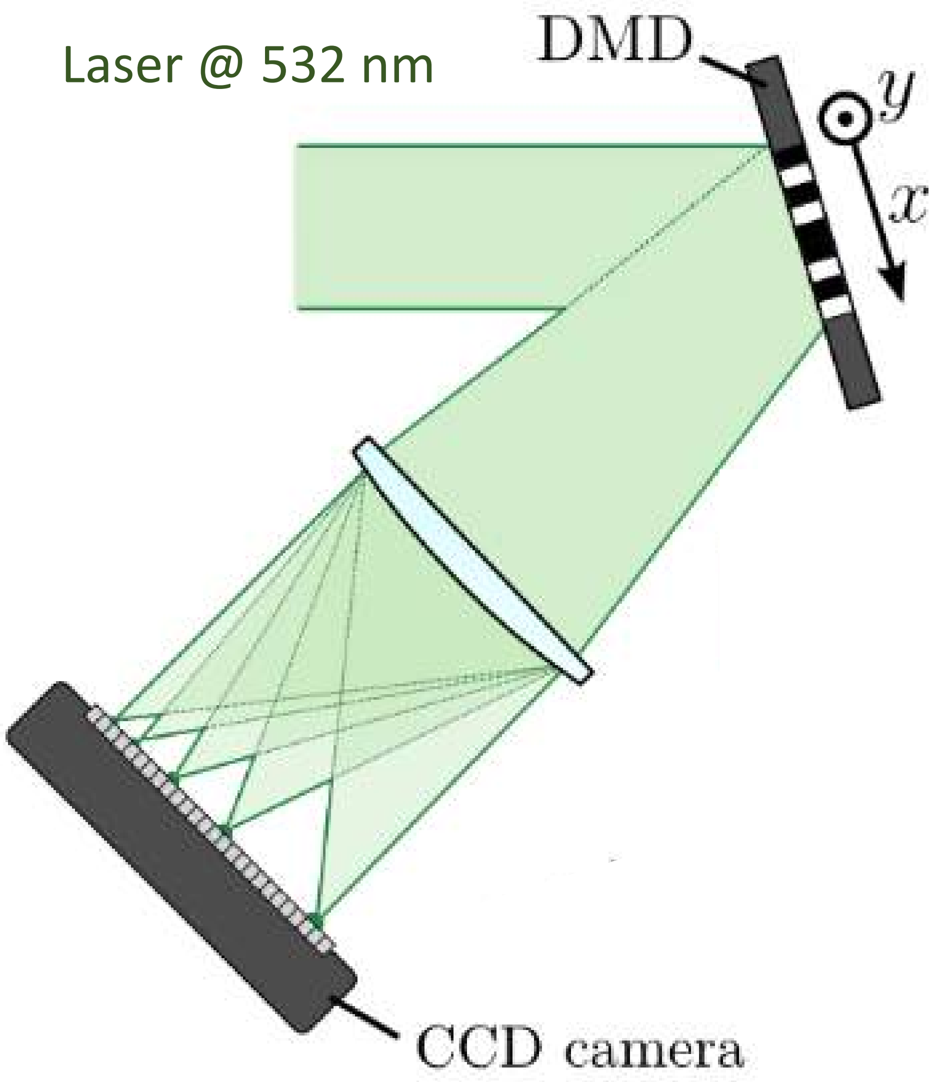
A diffraction measurement of Chern numbers

Fibonacci finite string



A. Dareau, E. Levy, E.A, F. Gerbier and J. Beugnon, 2016

Optical setup

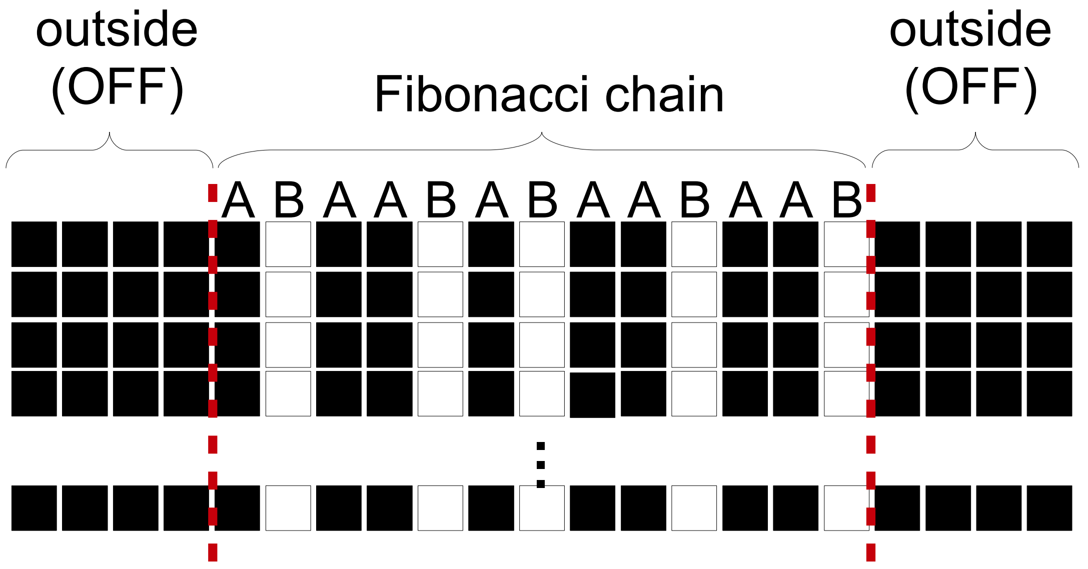


Digital Micromirror Device (DMD)

- – mirror (“pixel”) size a ~ 14 µm
- – 1024 × 768 pixels

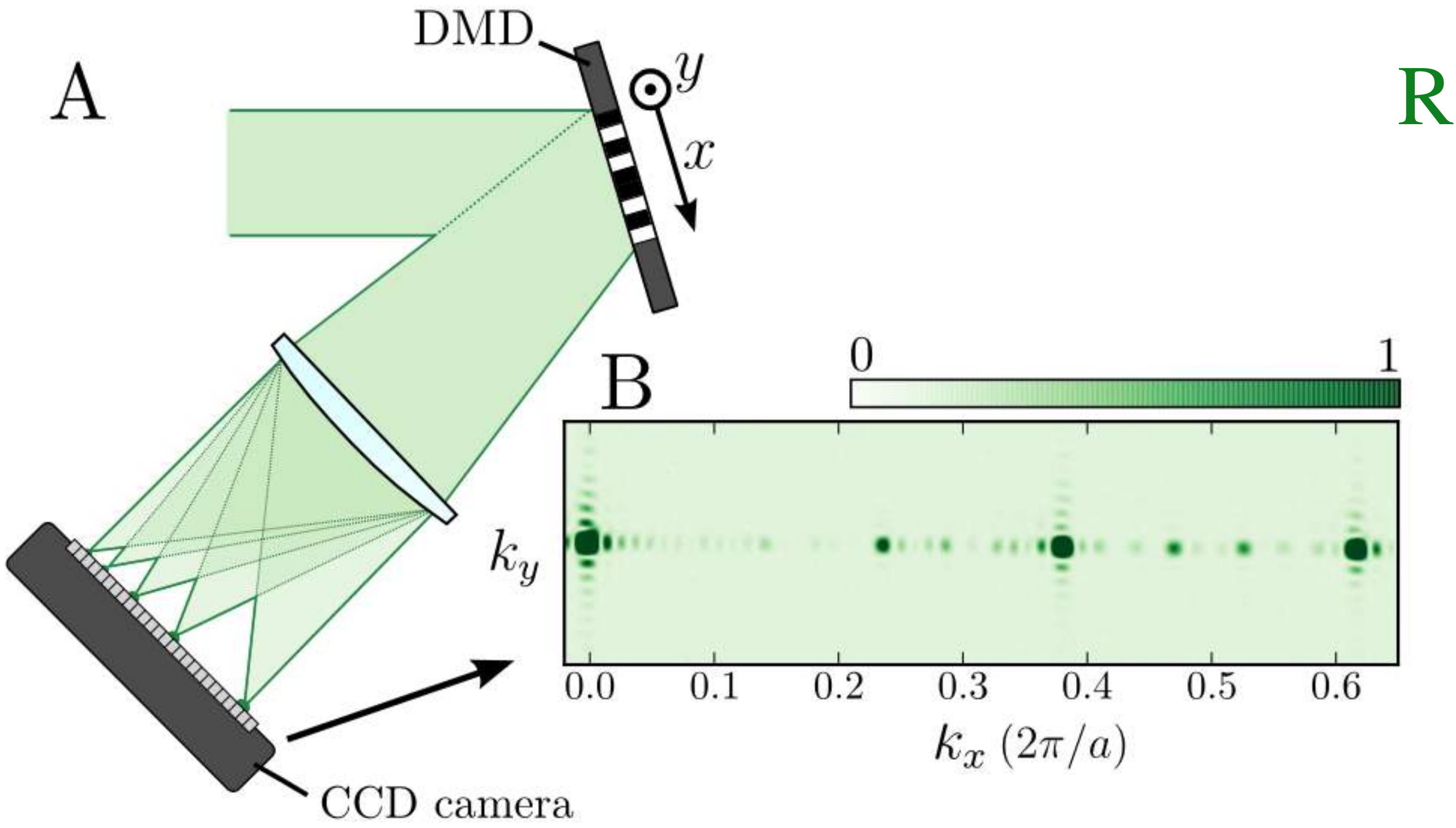
Fibonacci encoding :
A ■
B □

DMD front view

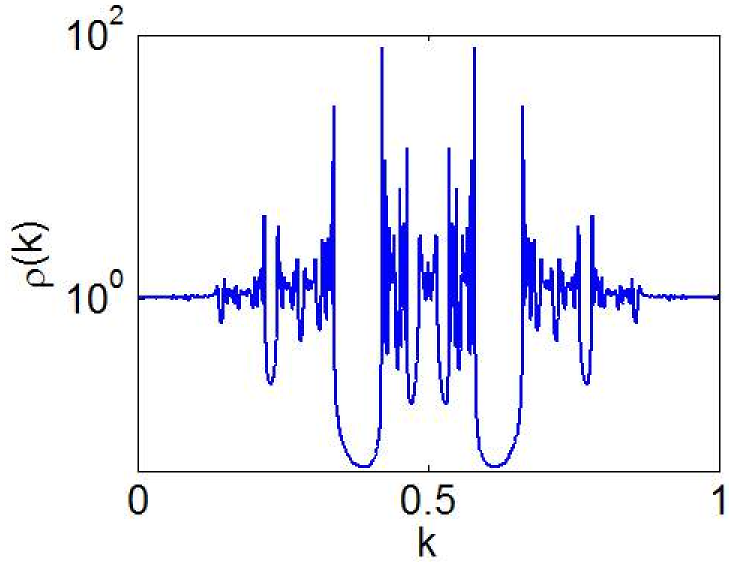


A diffraction measurement of Chern numbers

Fibonacci finite string



Related to the transmission spectrum ?



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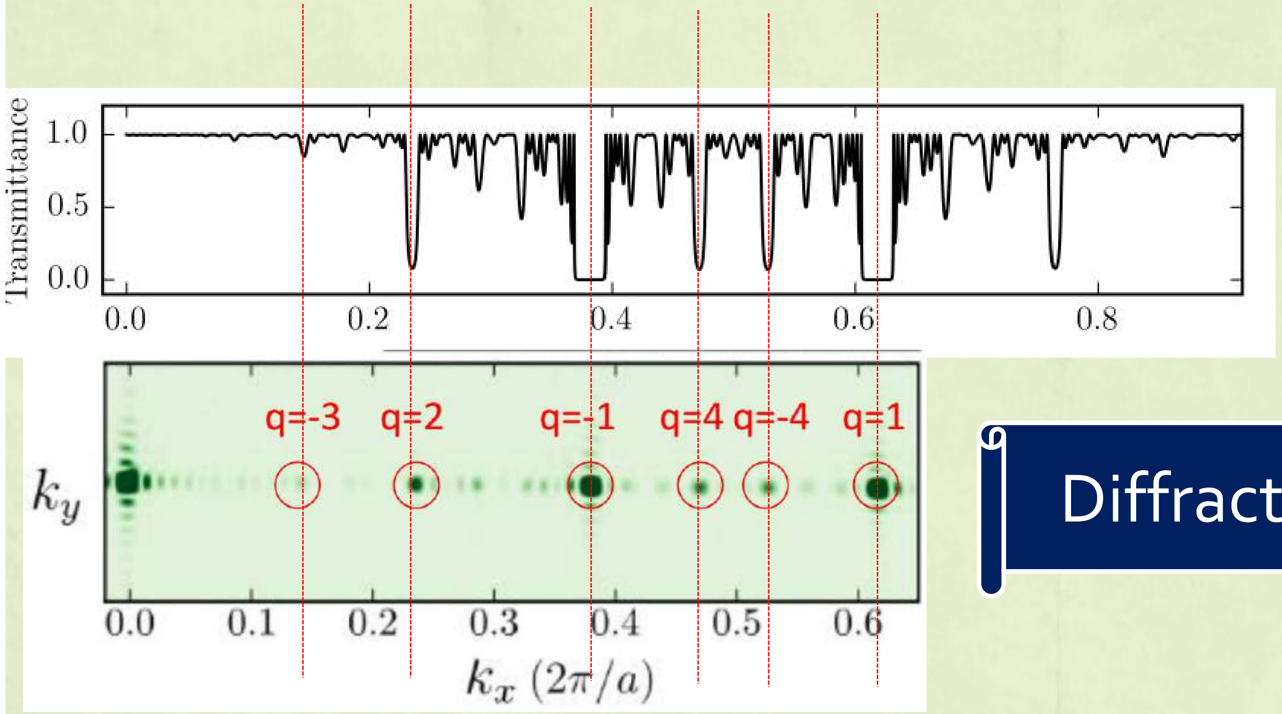
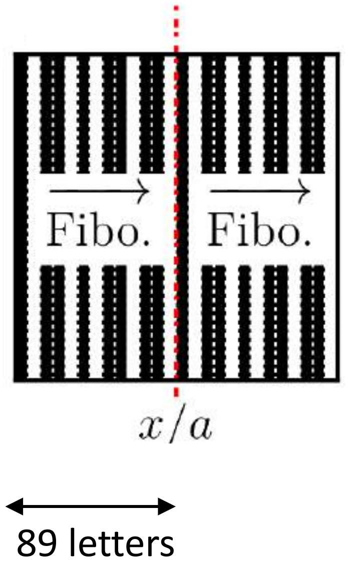
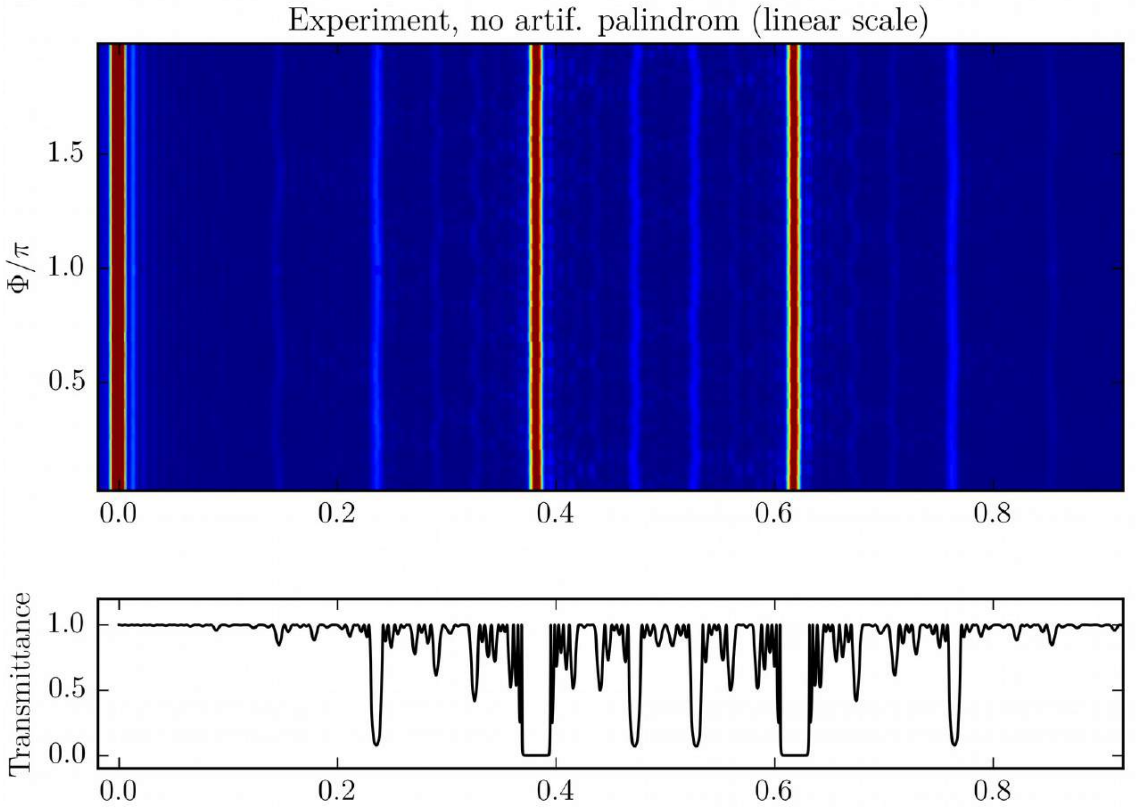


Transmission spectrum

Diffraction pattern

DMD Pattern

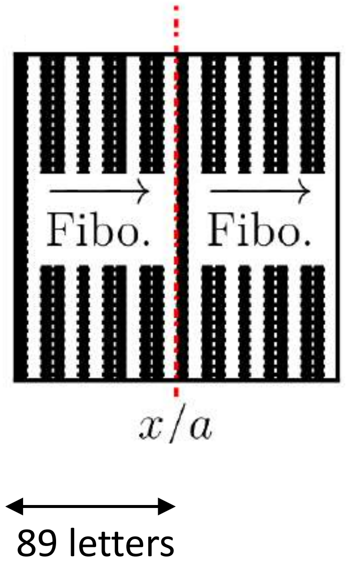


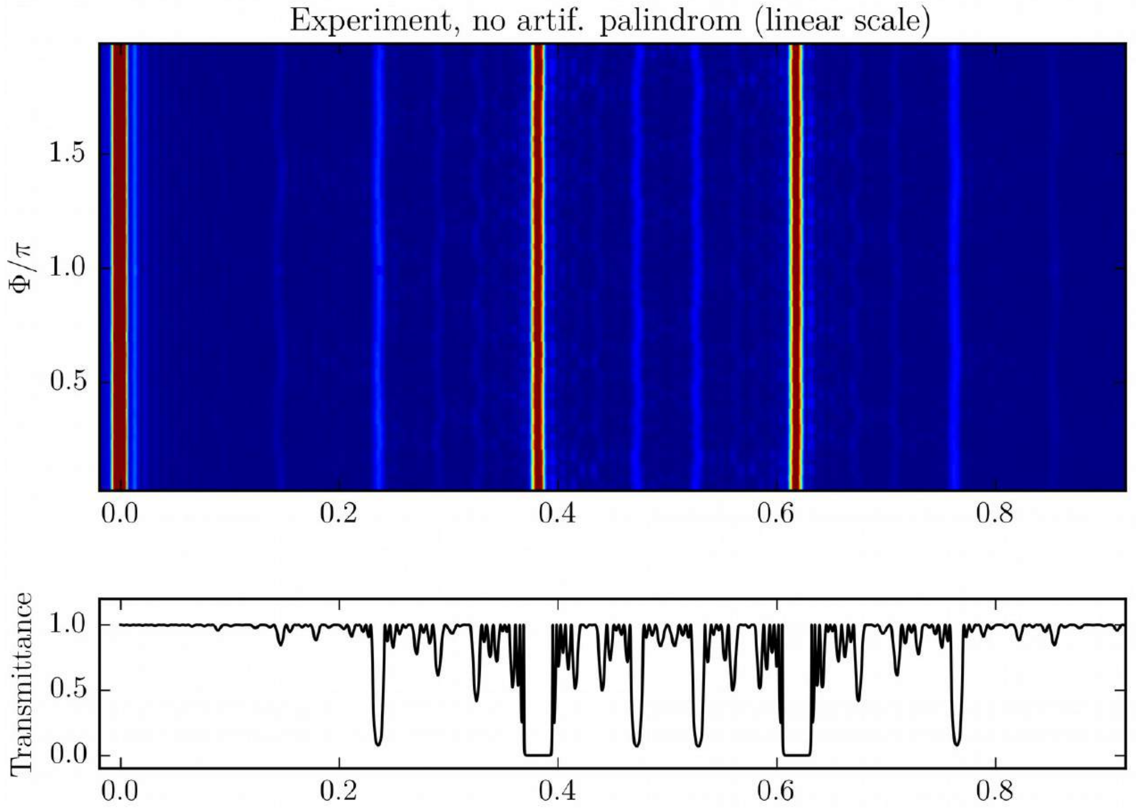


No effect of ϕ

Creating edge states

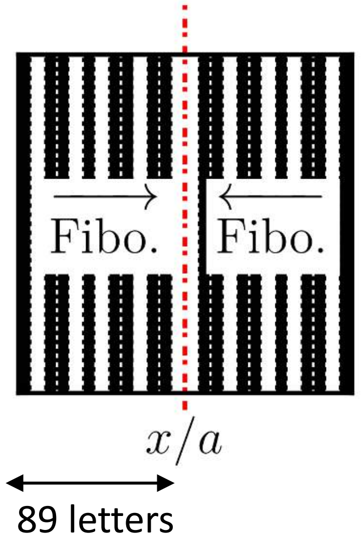
DMD Pattern

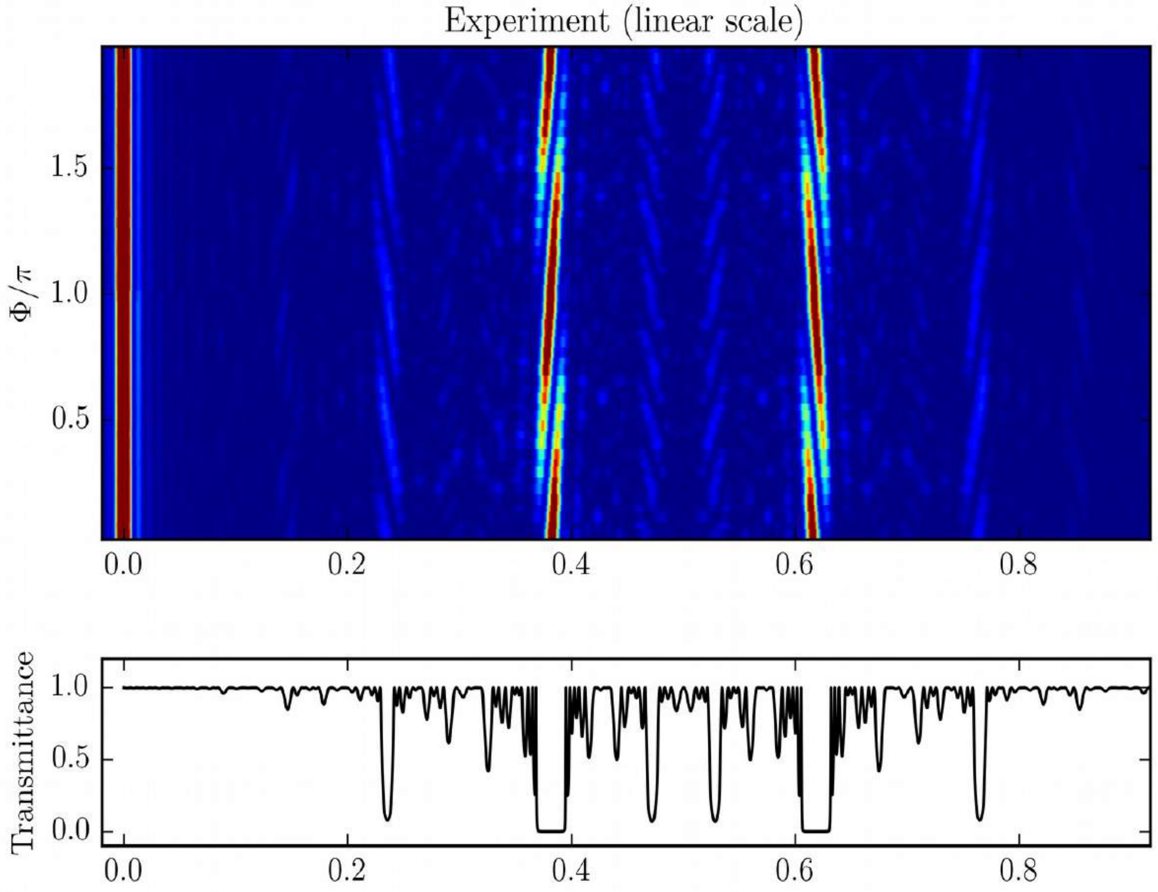




No effect of ϕ

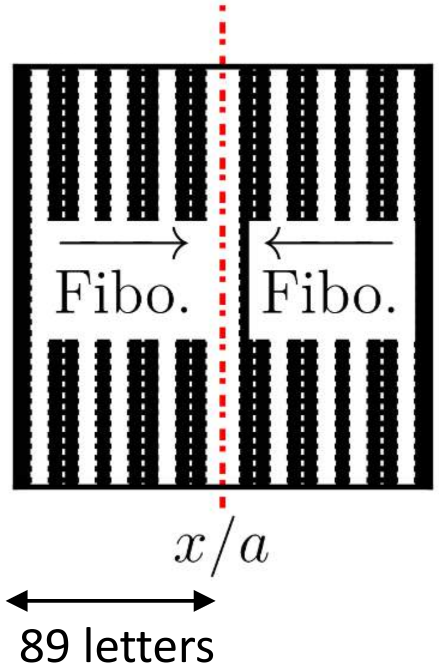
DMD Pattern



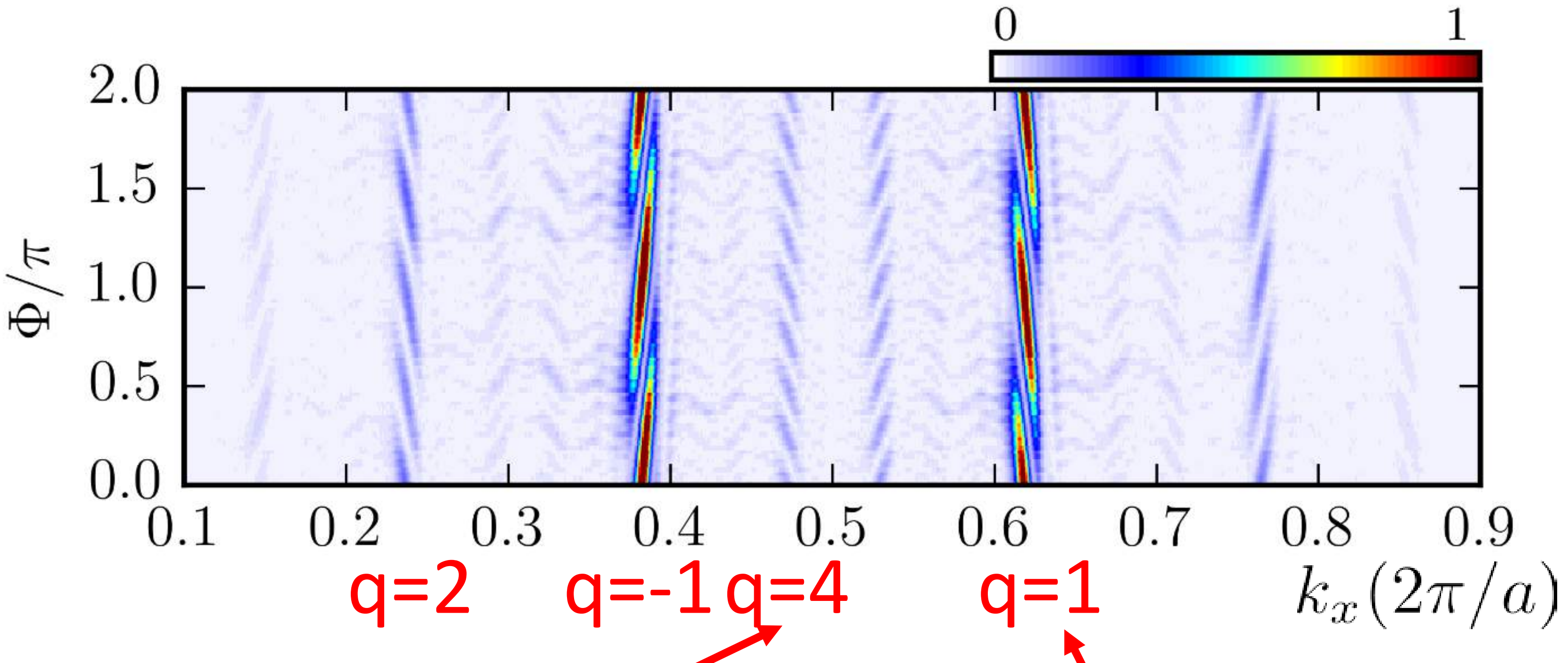


There is an effect of the phason ϕ

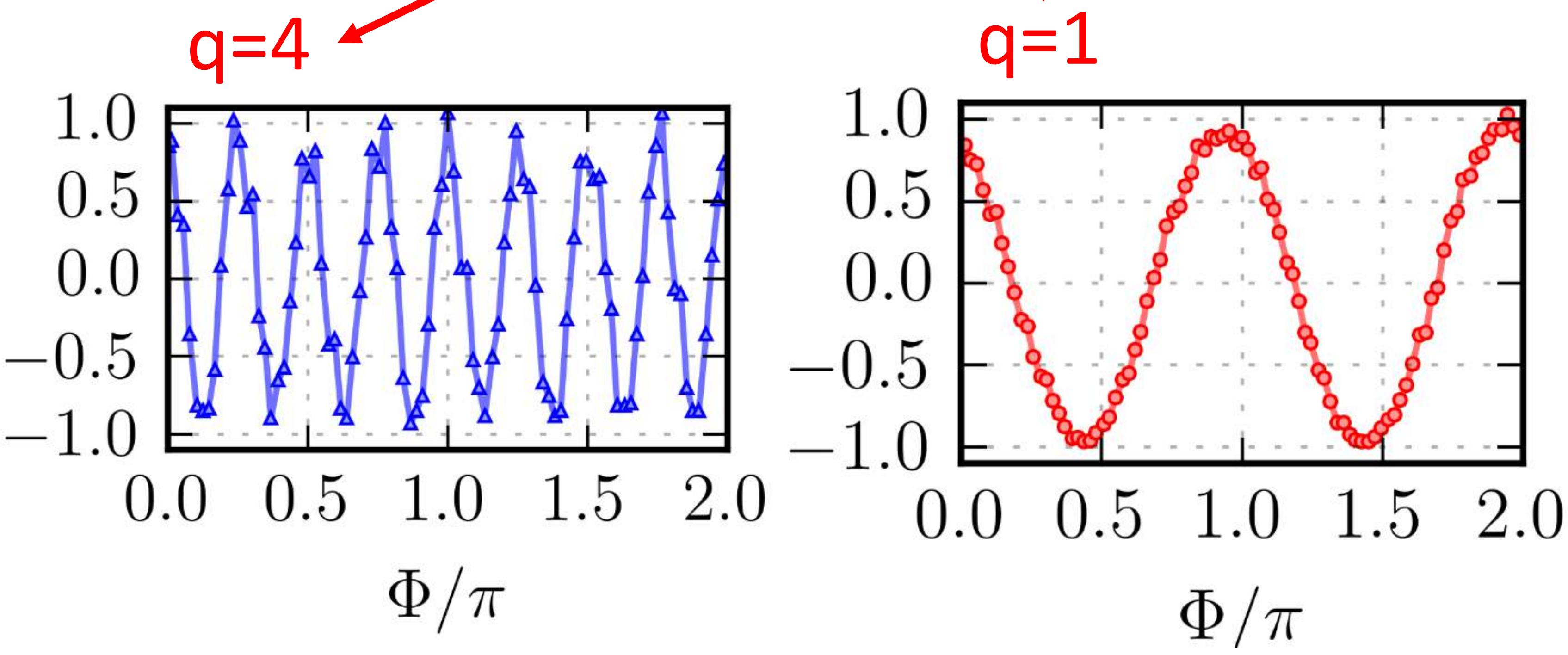
DMD Pattern



Diffraction pattern



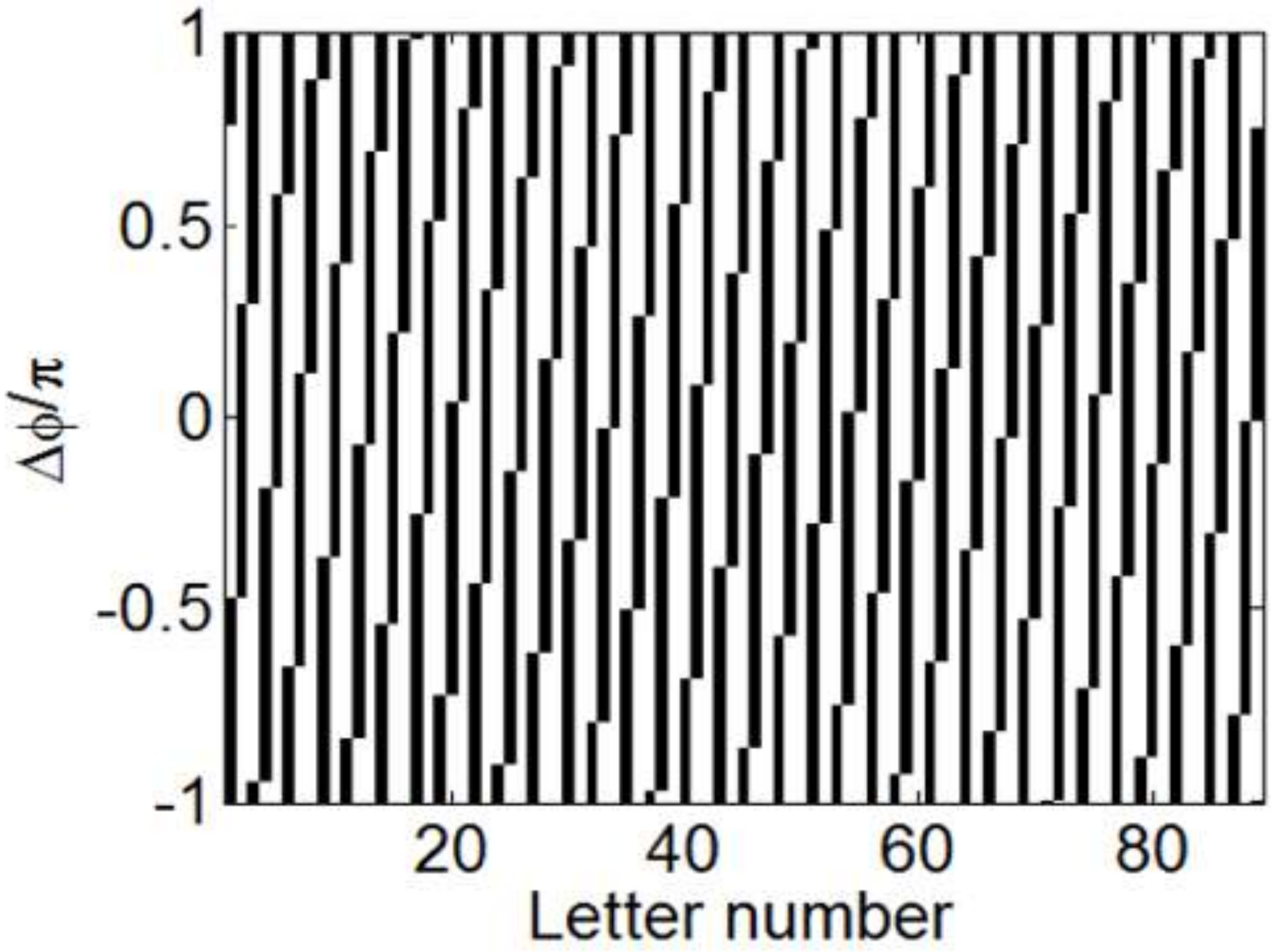
Cuts at a given k_x



2D diffraction experiment

Instead of

consider all realisations



2D diffraction experiment

Instead of

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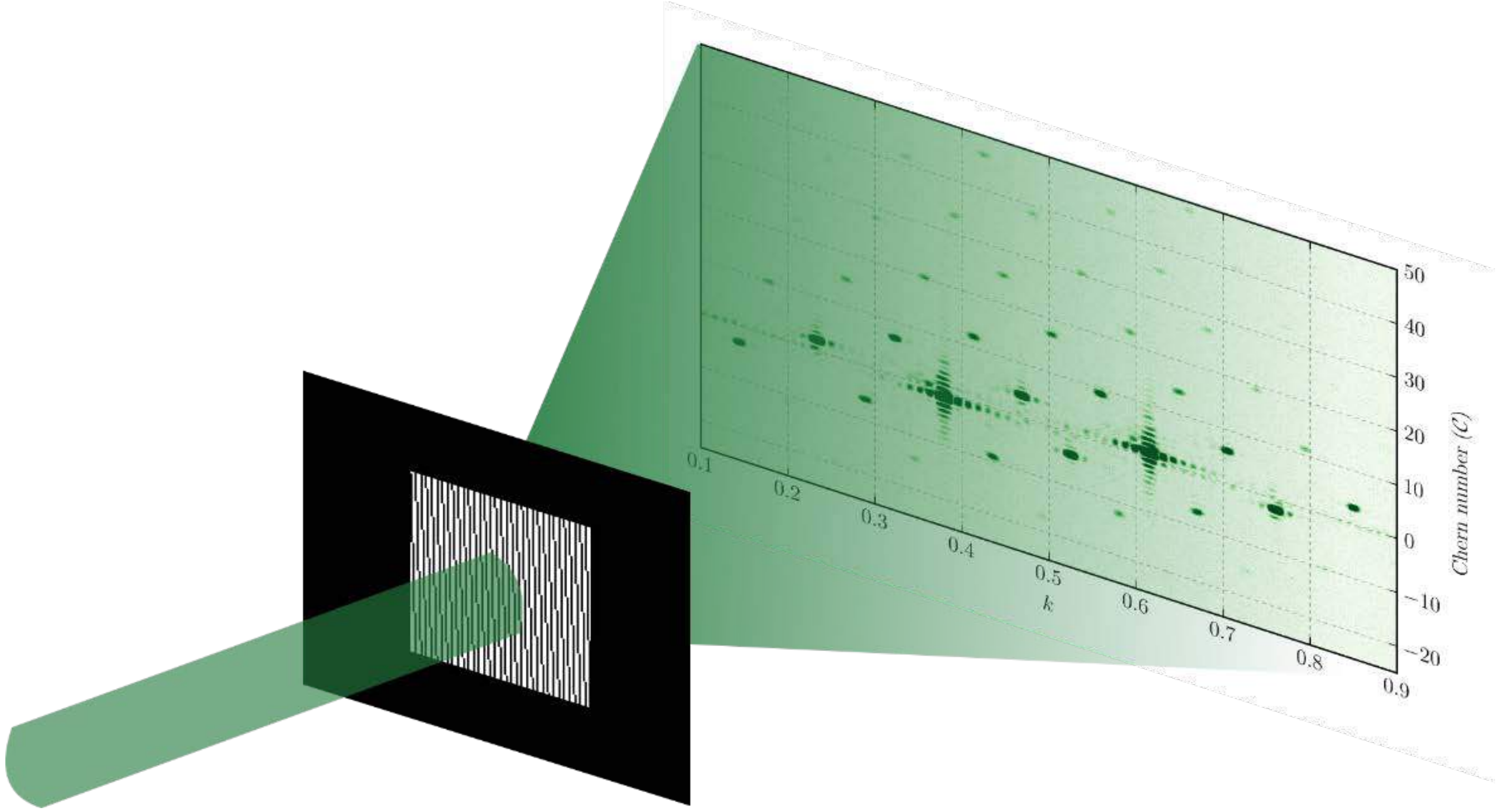
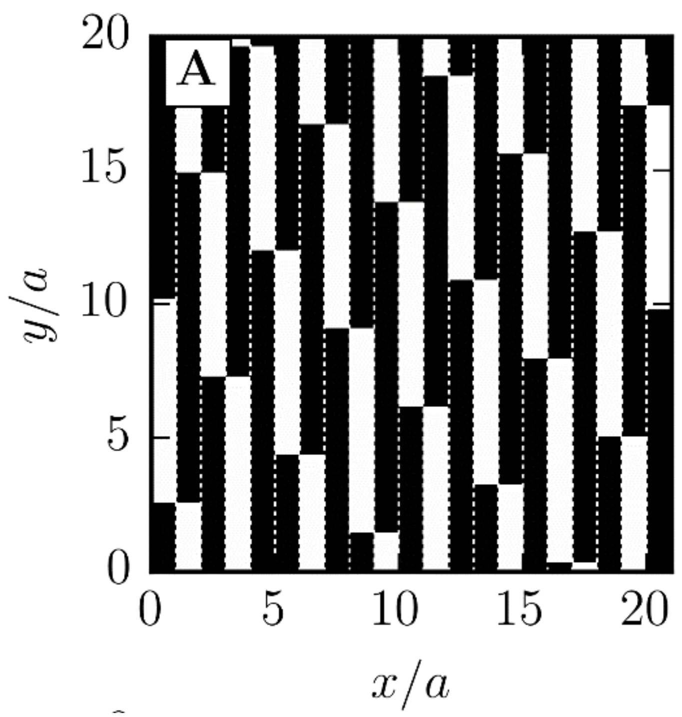
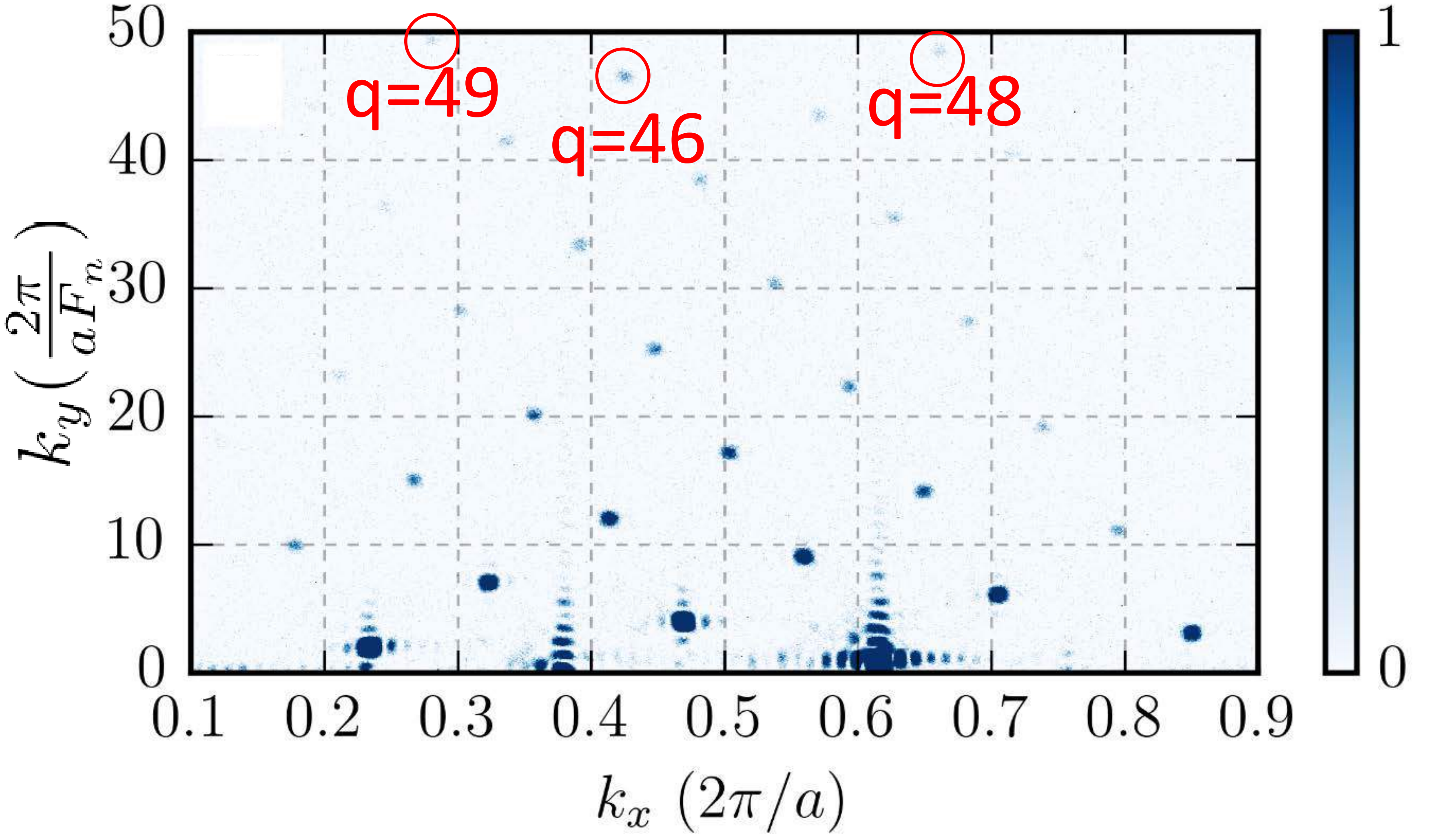


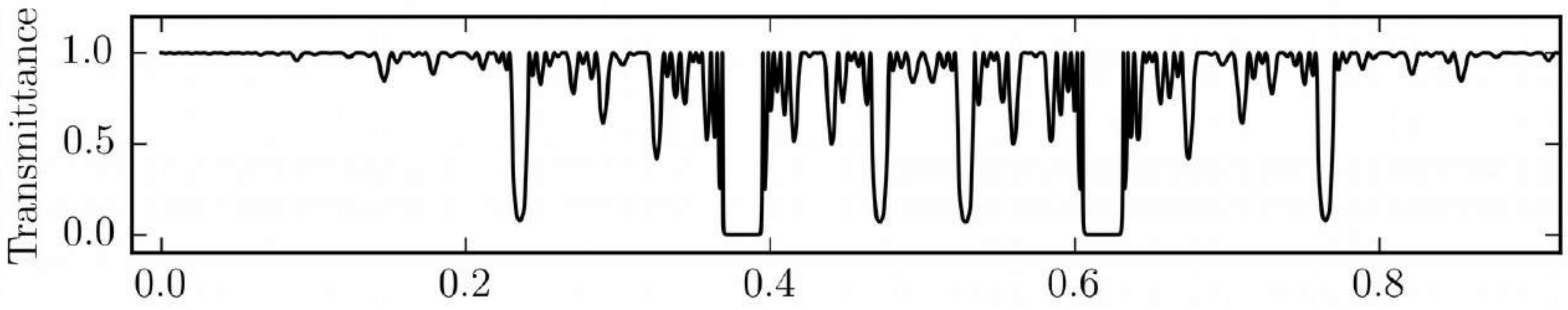
DMD Pattern



y axis is associated with Φ

Diffraction pattern





Summary-Further directions

- Demonstrate the fractal structure of the energy spectrum using polaritons in a Fibonacci cavity.

- Gap Chern numbers are the winding numbers of the chiral reflection phase.
- Gap traversing of edge states is completely determined by corresponding gap Chern numbers.
- Scattering theory gives a simple way to calculate/measure Chern numbers.
- Topological Chern numbers are also contained in structural data of the quasi-crystals. They can be retrieved from a Young-slit diffraction experiment. This approach allows for a measurement of high numbers even for short chains.

An algebraic description :

Consider the infinite chain W_∞ and a finite segment $F_N(0) \equiv W_N^0$

$$W_N^0 = \ell_1 \ell_2 \dots \ell_N .$$

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Shift operator (translation by one unit) :

$$\mathcal{T}\left(\ell_1 \ell_2 \ldots \ell_n \ldots\right) = \ell_2 \ell_3 \ldots \ell_{n+1} \ldots$$

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$$\boxed{W_N^i = \mathcal{T}^i\left(W_N^0\right)}$$

Each word W_N^i is a valid segment of the Fibonacci chain corresponding to a specific value of ϕ

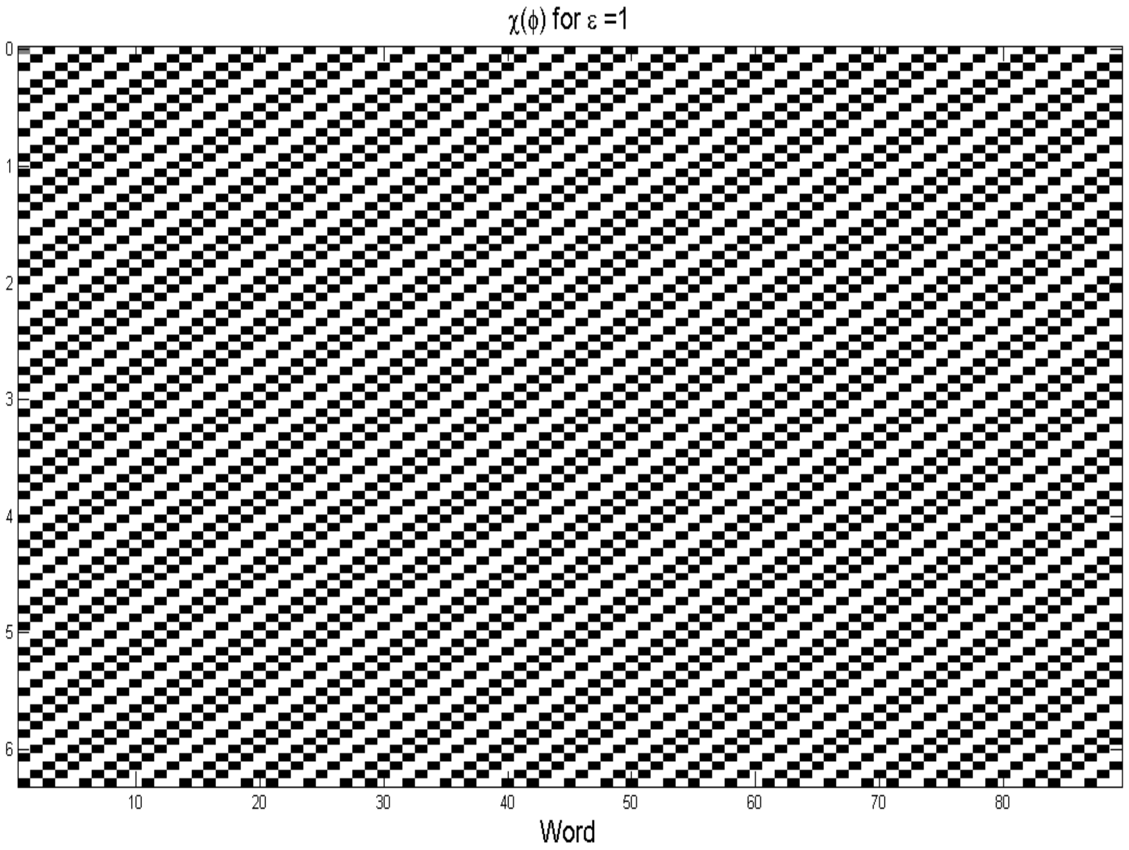
$$\mathcal{T}^N\left(W_N^0\right) = W_N^0$$

(torus)

Define the symmetric matrix

$$\Sigma_0^N = \begin{pmatrix} W_N^0 \\ W_N^1 \\ \vdots \\ W_N^{N-1} \end{pmatrix}.$$

Define the symmetric matrix



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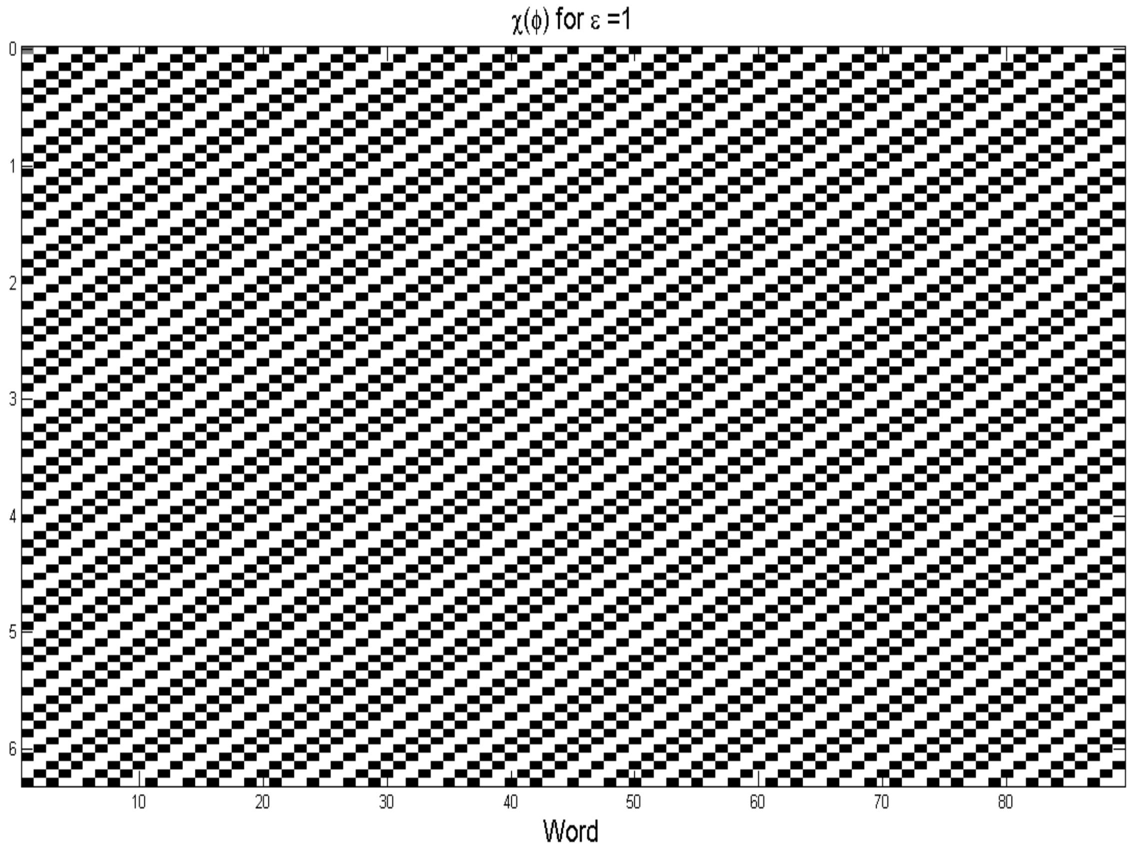


$$\Sigma_0^N = \begin{pmatrix} W_N^0 \\ W_N^1 \\ \vdots \\ W_N^{N-1} \end{pmatrix}.$$

Define the unitary transformations :

$$U_q \Sigma_0 \equiv \Sigma_q$$

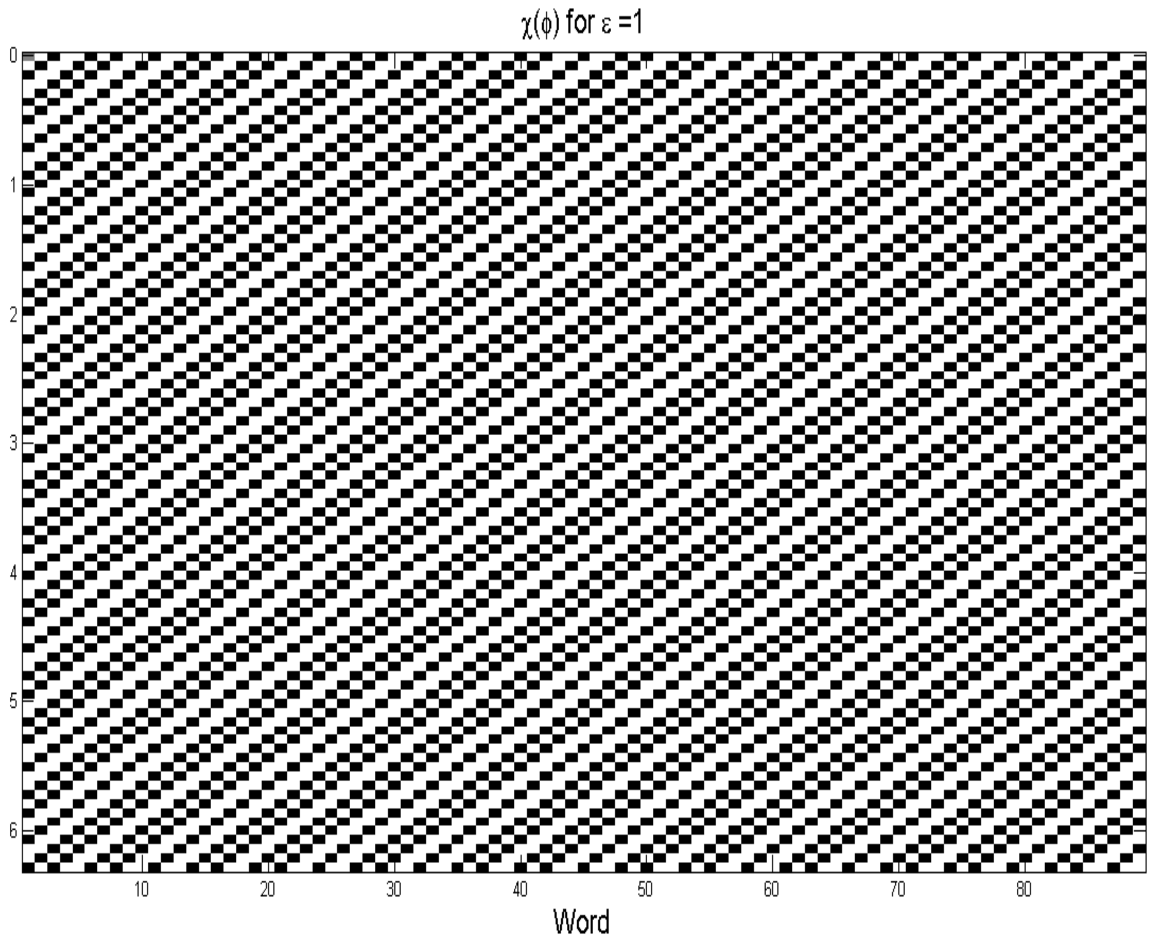
with
$$U_q\,(m', m) = \begin{cases} 1 & m' = \frac{m\,q\,F_{n-2}}{\gcd(F_{n-2}, F_n)} \pmod{F_n} \\ 0 & \text{otherwise} \end{cases}$$

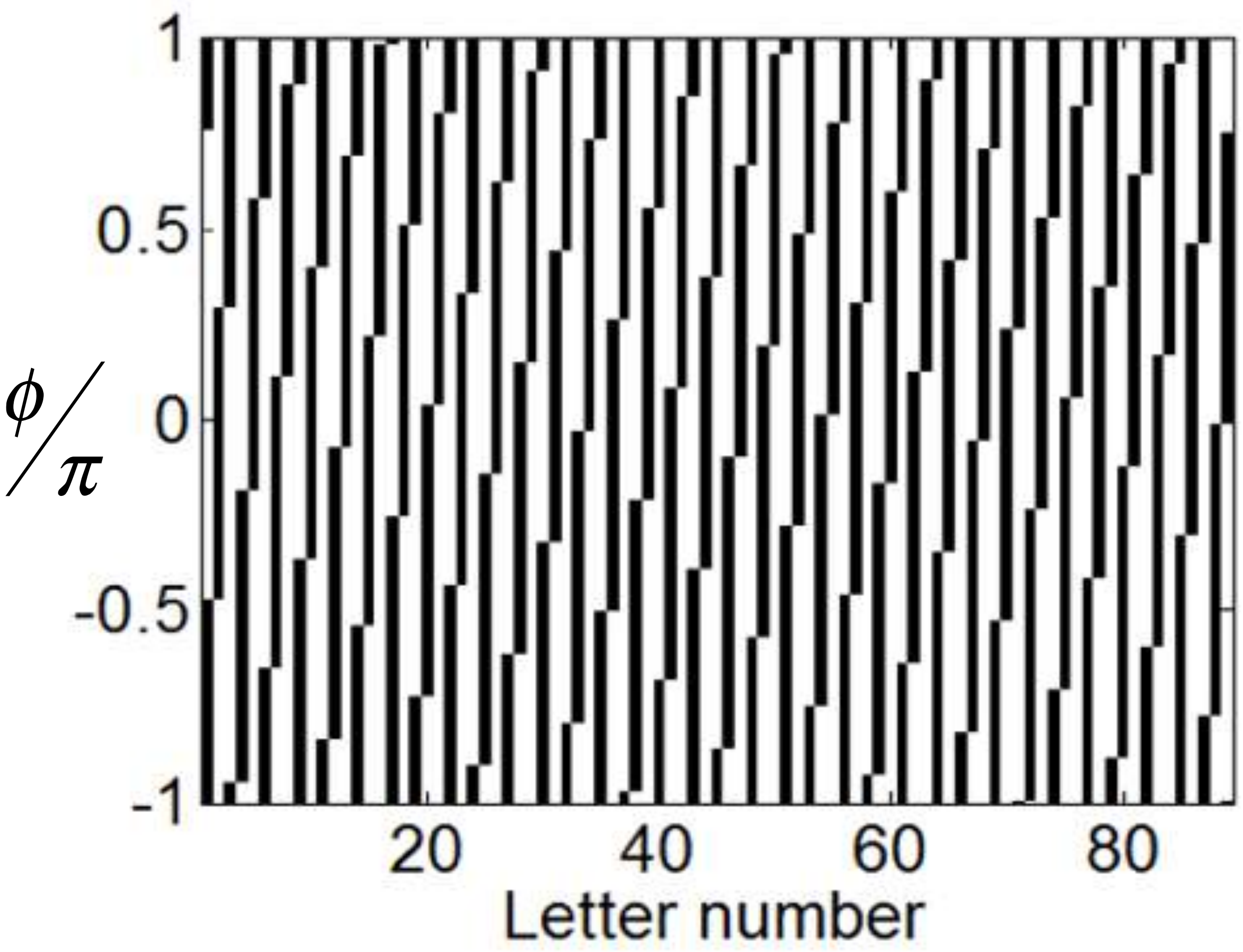
$$q \in 0 \ldots F_n - 1$$

Σ_0

$$U_q \Sigma_0 \equiv \Sigma_q$$

Σ_q





The unitary matrices U_q form an abelian group $\mathcal{U} = \{U_q\}_{q=0}^{F_n - 1}$

$$U_q U_r = U_{L(q,r)}$$

$$L\,(q,r) = q\,r\,F_{n-2} \pmod{F_n}$$

F_n are Fibonacci numbers

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F_n are Fibonacci numbers

The Fourier transform matrices $\tilde{U}_q$ form also an abelian group

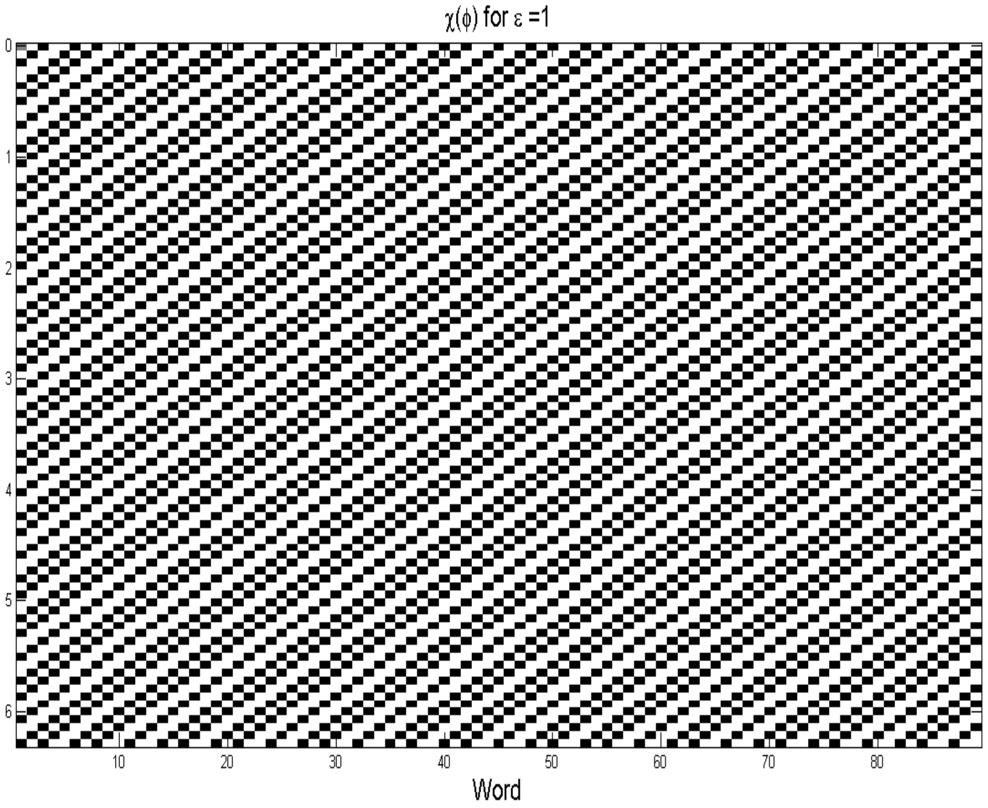
$$\tilde{U}_q \tilde{U}_s = \tilde{U}_s \tilde{U}_q = \tilde{U}_{F_N - (qsF_{N-2})} \mod (F_N)$$

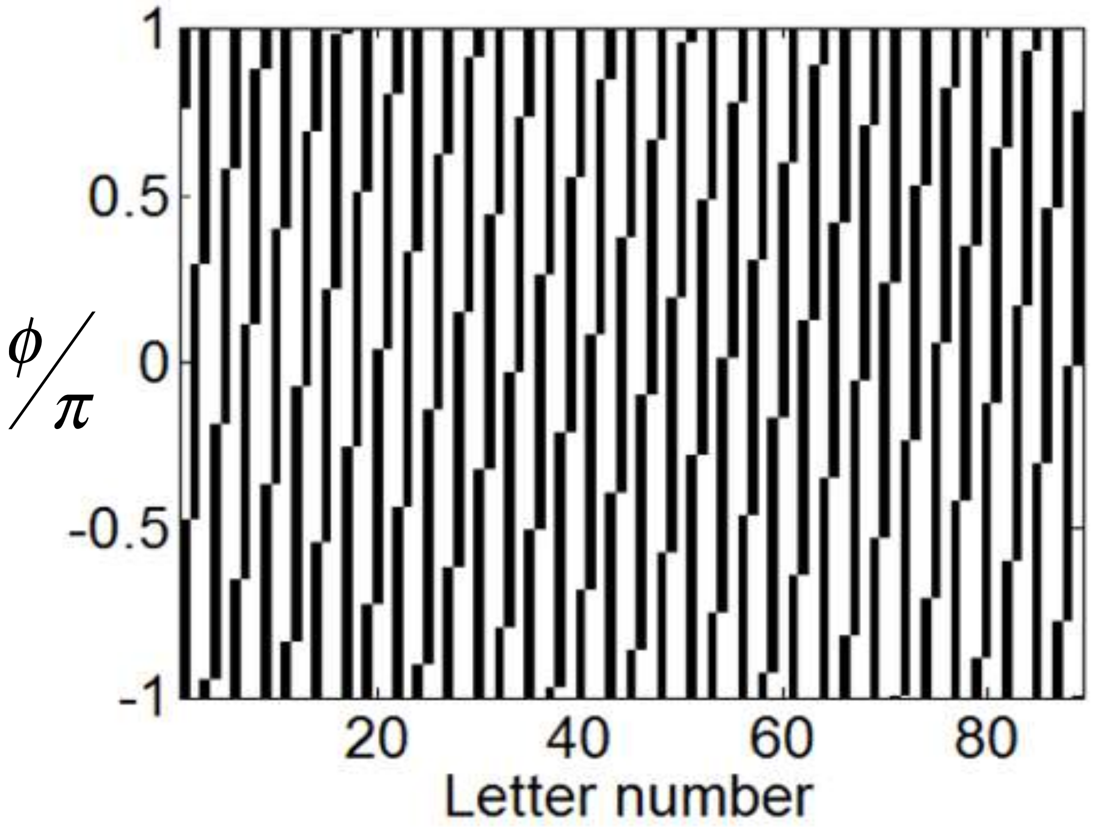
$$\Lambda_{qs} \equiv F_N - \left[qs\,F_{N-2} \mod\left(F_N\right) \right]$$

Σ_0

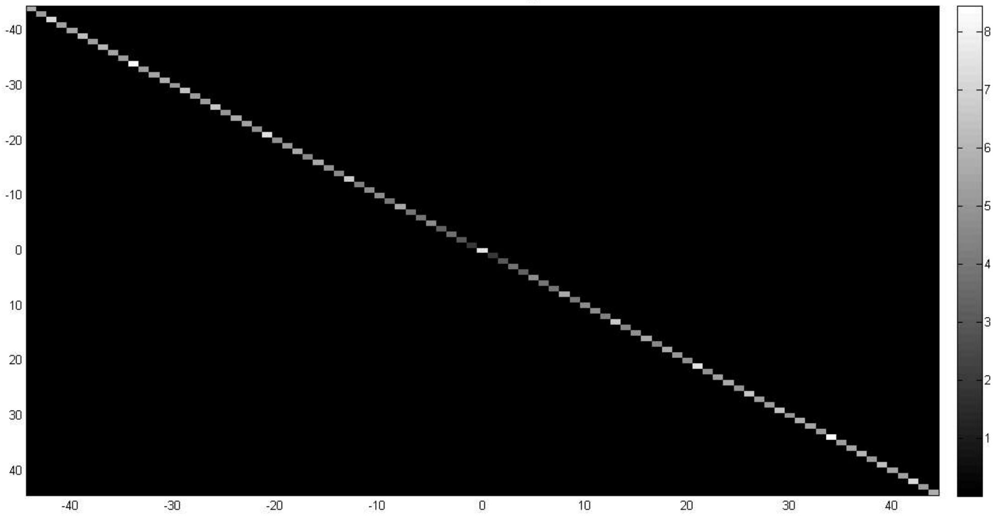
$$U_q \Sigma_0 \equiv \Sigma_q$$

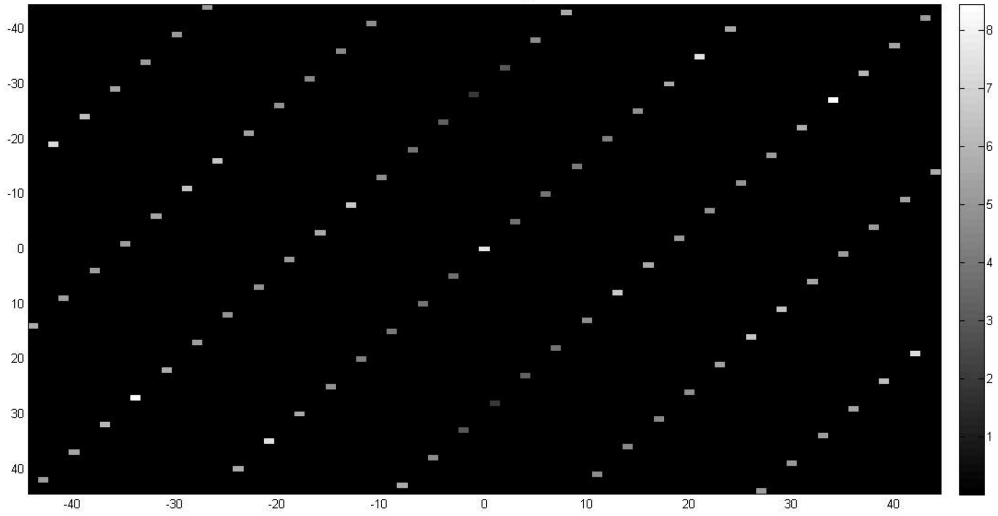
Σ_q





$\tilde{U}_q$





$$\boxed{\Lambda_{qs} \equiv F_N - \left[qs\, F_{N-2} \quad \mathrm{mod}\left(F_N\right)\right]}$$

The matrices Λ_{qs} defines an abelian group isomorphic to $\mathbb{Z}/F_N\mathbb{Z}$

An example for $F_5 = 5$:

$$\begin{array}{cccccc}
 & 2 & 4 & 1 & 3 & 0 \\
 & 4 & 3 & 2 & 1 & 0 \\
s & 1 & 2 & 3 & 4 & 0 \\
 & 3 & 1 & 4 & 2 & 0 \\
 & 0 & 0 & 0 & 0 & 0 \\
 & & & q & &
\end{array}$$

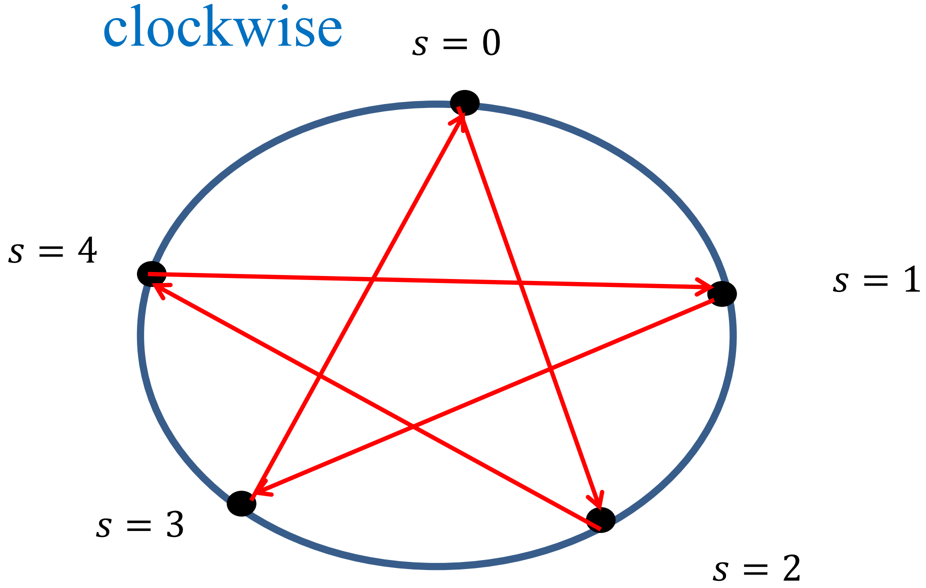
Winding numbers $\mathcal{W}_q$

$q = 1$

$$\begin{array}{cccccc}
 & 2 & 4 & 1 & 3 & 0 \\
 & 4 & 3 & 2 & 1 & 0 \\
s & 1 & 2 & 3 & 4 & 0 \\
 & 3 & 1 & 4 & 2 & 0 \\
 & 0 & 0 & 0 & 0 & 0 \\
 & & & q & &
\end{array}$$

$\mathcal{W}_q = 2$

clockwise



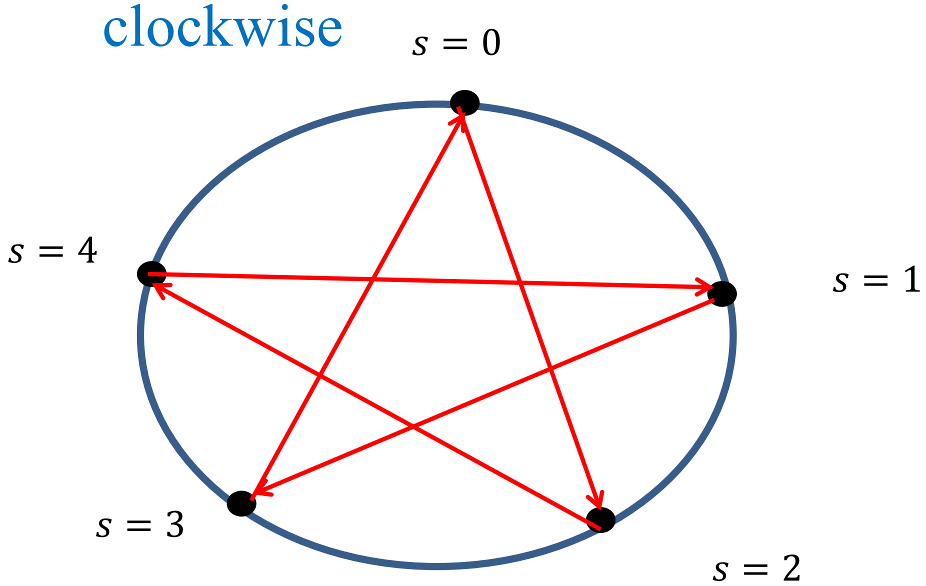
Winding numbers $\mathcal{W}_q$

$q = 1$

$$\begin{array}{cccccc} & 2 & 4 & 1 & 3 & 0 \\ & 4 & 3 & 2 & 1 & 0 \\ s & 1 & 2 & 3 & 4 & 0 \\ & 3 & 1 & 4 & 2 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & & q & & \end{array}$$

$\mathcal{W}_q = 2$

clockwise

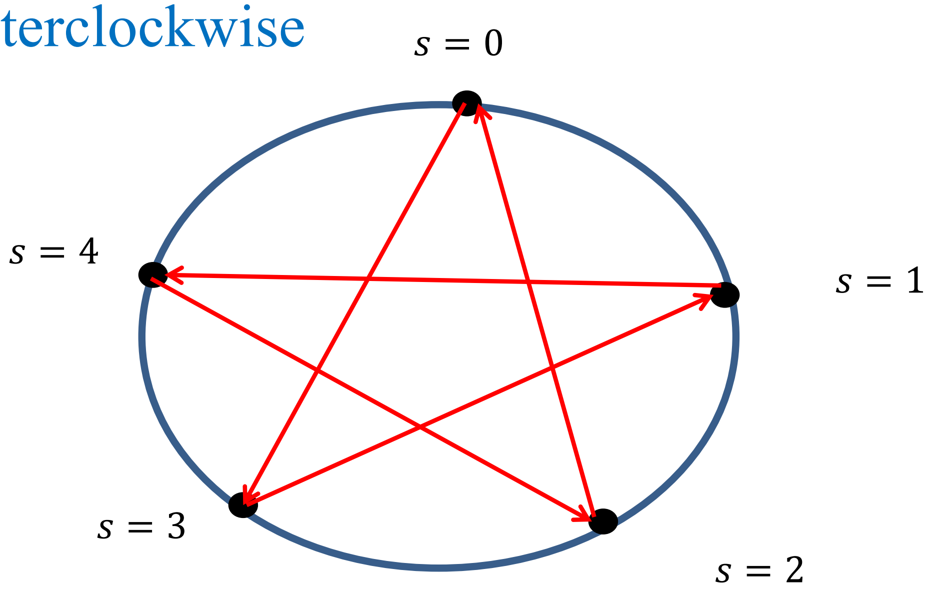


$q = 4$

$$\begin{array}{cccccc} & 2 & 4 & 1 & 3 & 0 \\ & 4 & 3 & 2 & 1 & 0 \\ s & 1 & 2 & 3 & 4 & 0 \\ & 3 & 1 & 4 & 2 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & & & q & \end{array}$$

$\mathcal{W}_q = -2$

counterclockwise



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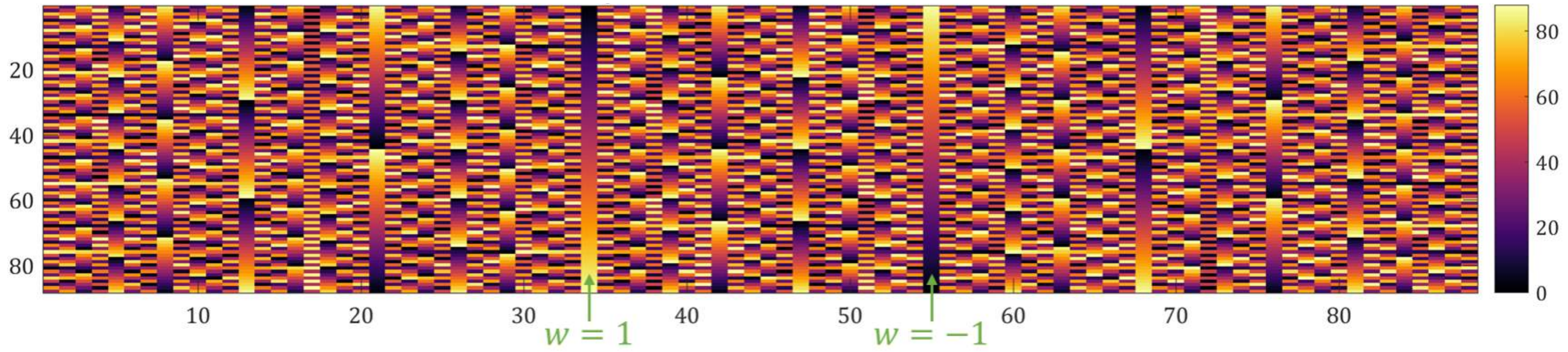
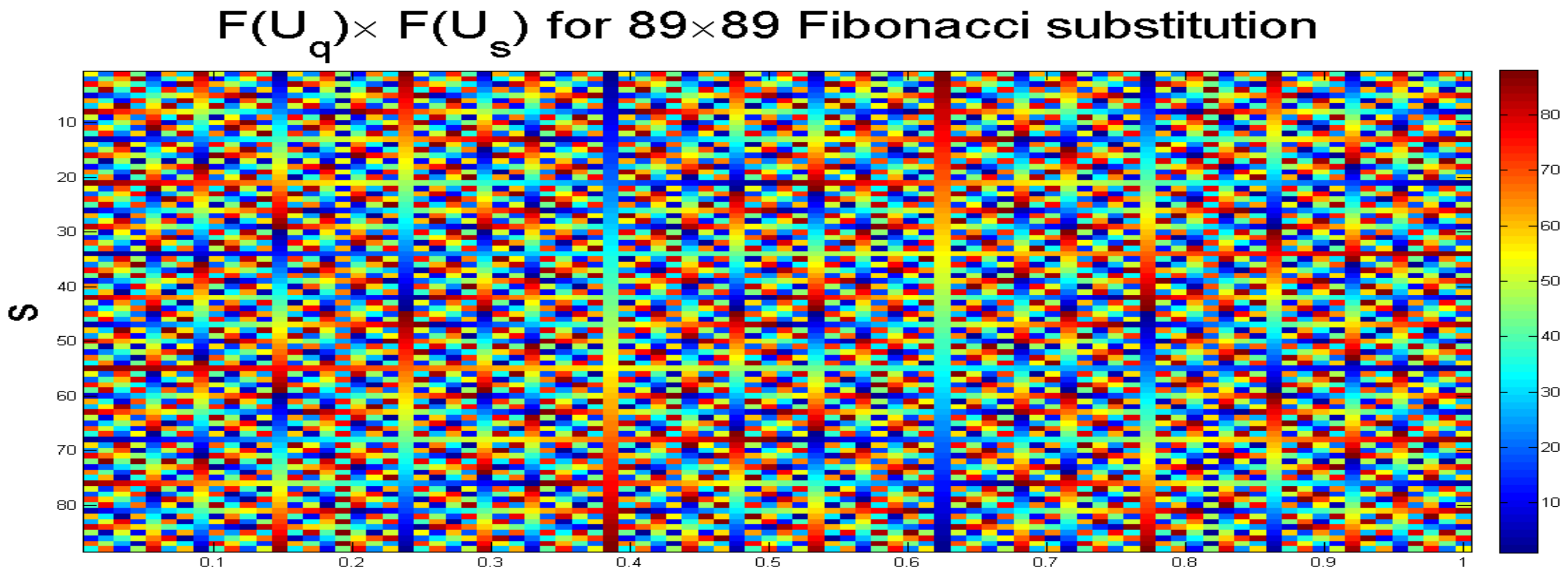


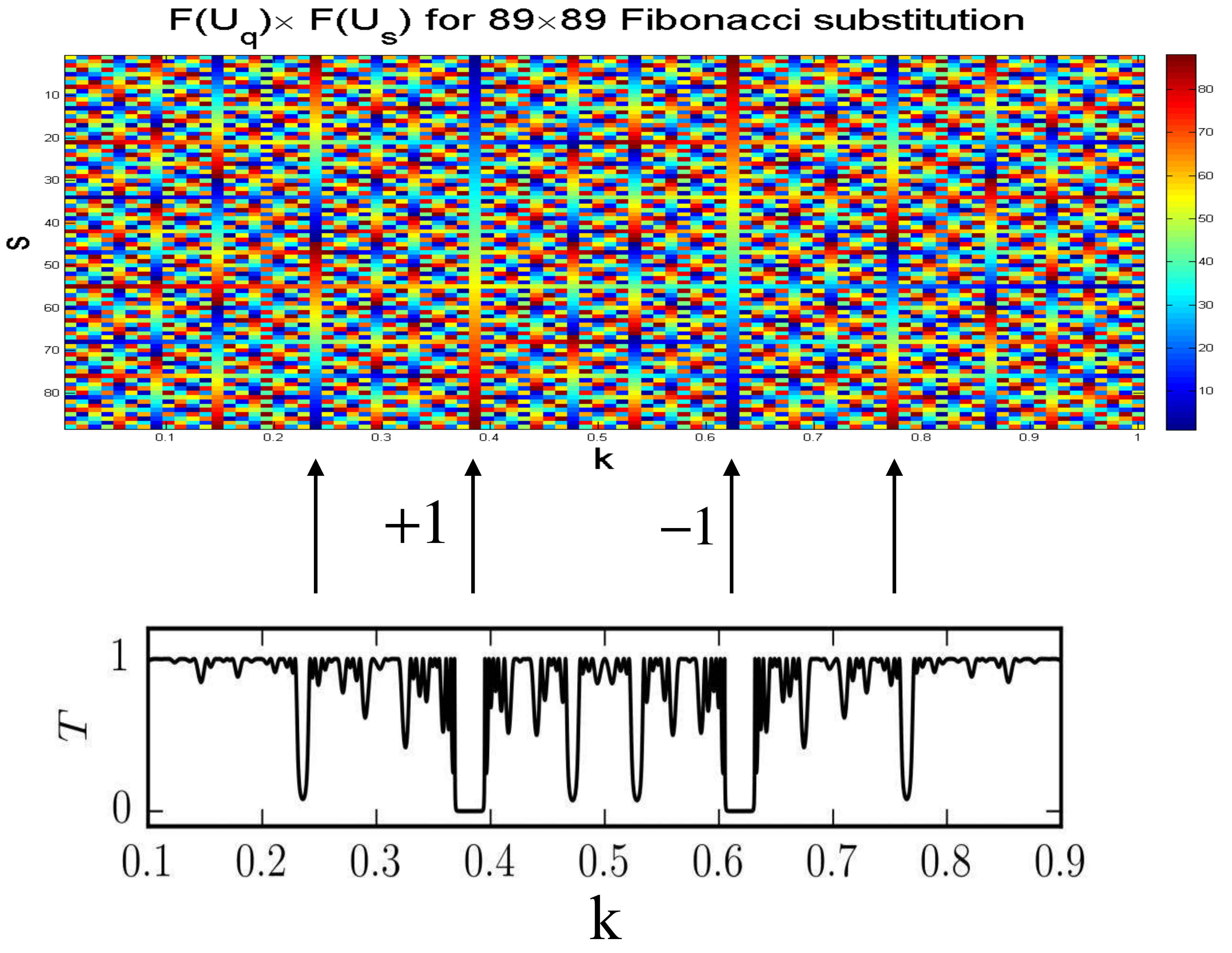
$$F_{10} = 89$$

Normalise $\Lambda_{qs} \equiv F_N - \left[qs\, F_{N-2} \mod\left(F_N\right)\right]$ between $[0,1]$



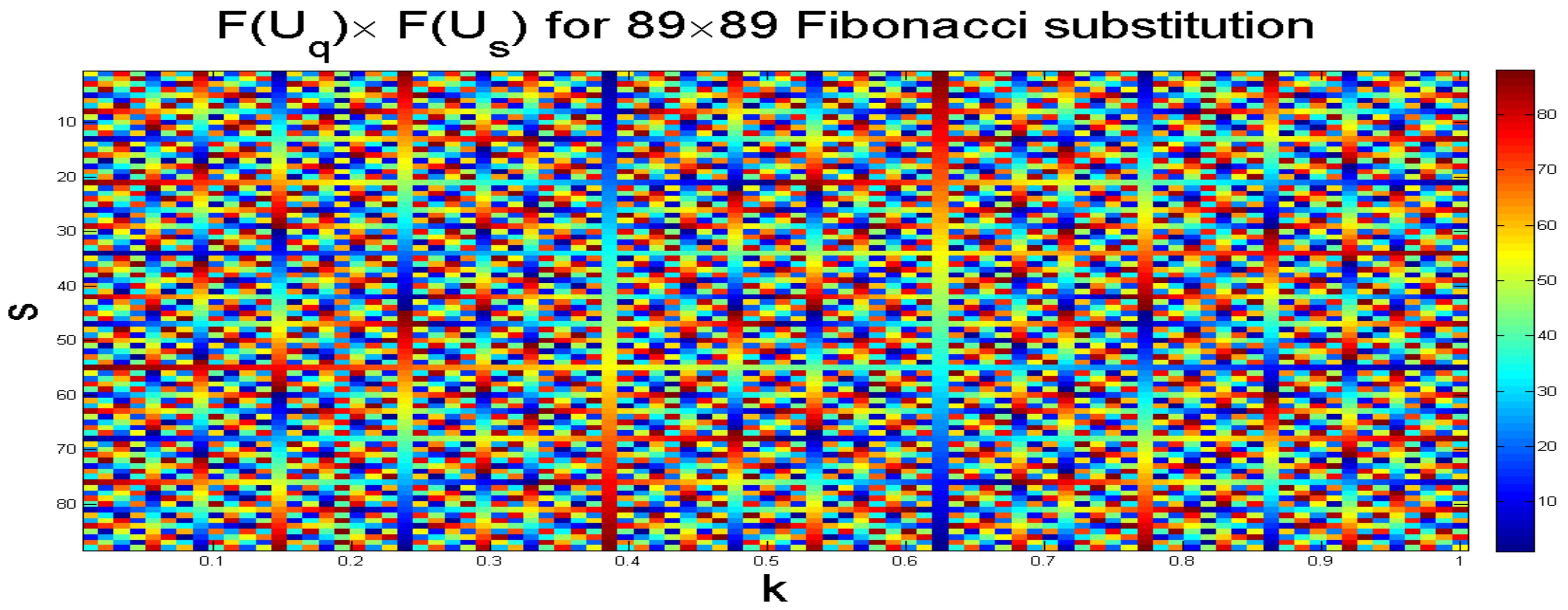
Normalise $\Lambda_{qs} \equiv F_N - \left[qs\,F_{N-2} \mod\left(F_N\right)\right]$ between $\left[0,1\right]$

$\mathcal{W}_q =$



Transmission spectrum

Normalise $\Lambda_{qs} \equiv F_N - \left[qs\, F_{N-2} \mod \left(F_N \right) \right]$ between $\left[0,1 \right]$



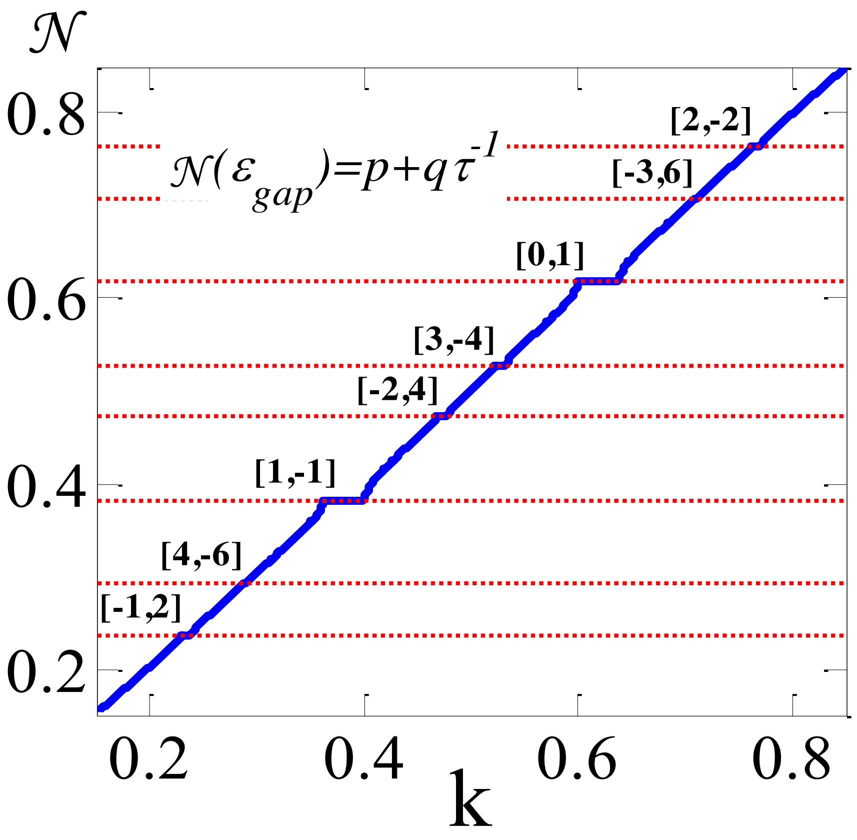
Locations of the gaps are given by

$$K\left(W_q, N\right) = \frac{F_{N-1} W_q}{F_N} \mod \left(F_N \right)$$

Gap Labeling Theorem (GLT)

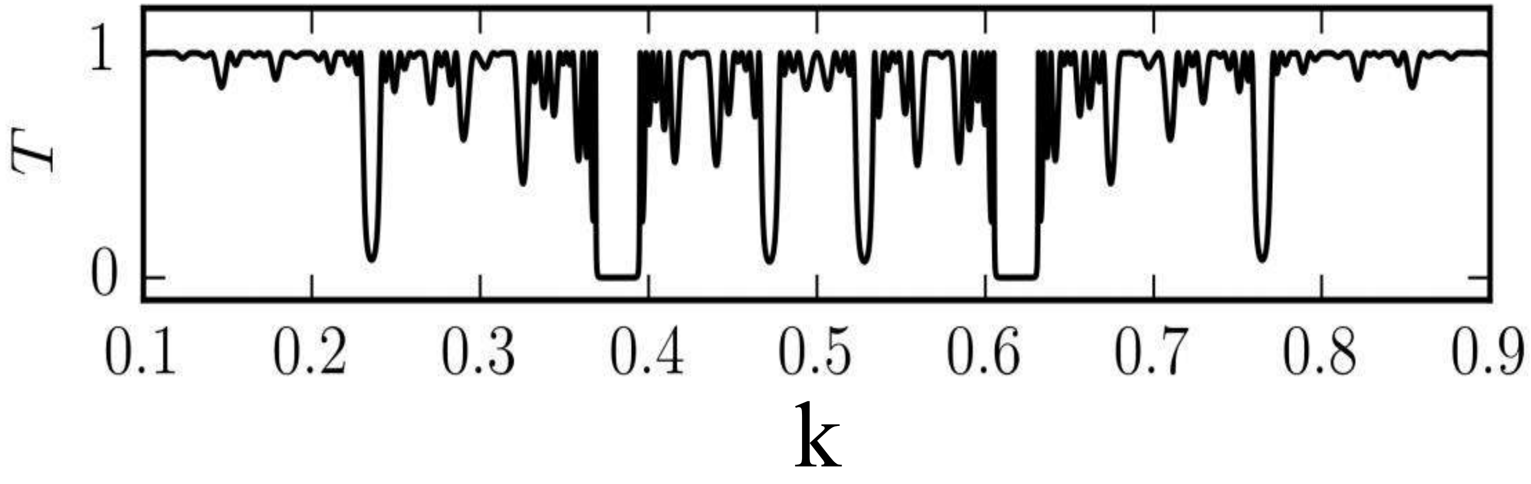
$$N\left(\varepsilon = E_{Q_{p,q}/2}\right) = p + q\,\tau^{-1}$$

$$K_0 = \{\mathrm{IDOS}(gap)\} = (\mathbb{Z} + \tau\mathbb{Z}) \cap [0,1] = \tau\mathbb{Z} \cap [0,1].$$



$$K\left(W_q, N\right) = \frac{F_{N-1}W_q}{F_N} \quad \mathrm{mod}\left(F_N\right)$$

$$\lim_{N\to\infty} K(W_q, N) \cap [0,1] = K_0$$



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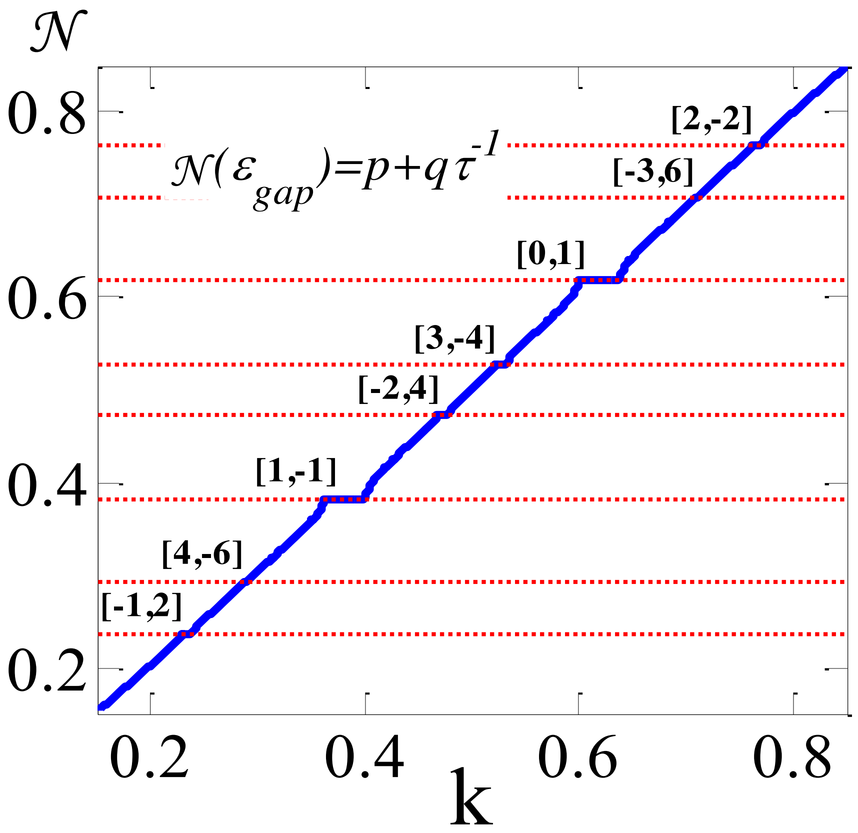


$$K_0 = \{\mathrm{IDOS}(gap)\} = (\mathbb{Z} + \tau\mathbb{Z}) \cap [0,1] = \tau\mathbb{Z} \cap [0,1].$$

$$K\left(W_q, N\right) = \frac{F_{N-1} W_q}{F_N} \quad \mathrm{mod}\left(F_N\right)$$

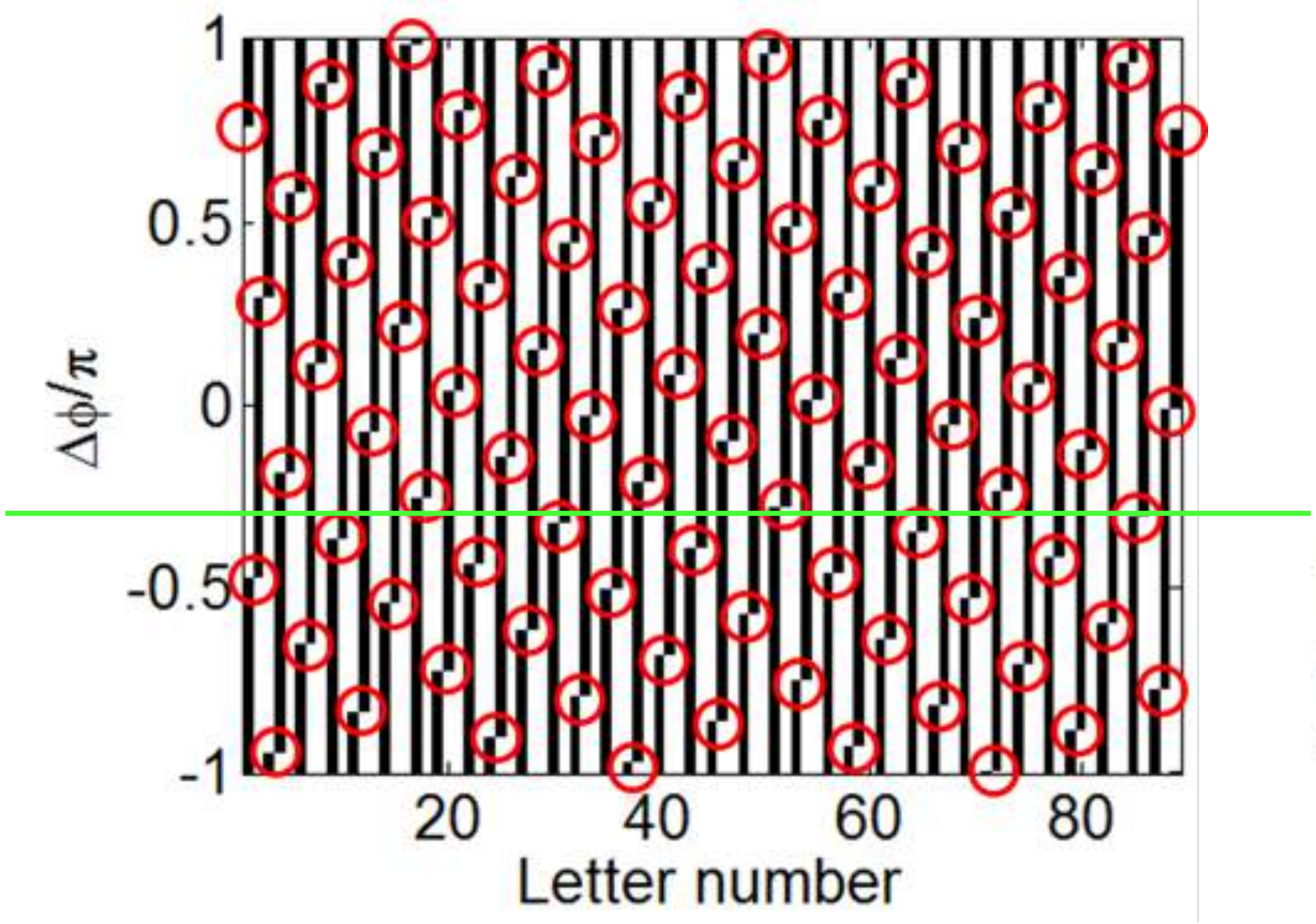
$$\lim_{N\to\infty} K(W_q, N) \cap [0,1] = K_0$$

Winding numbers

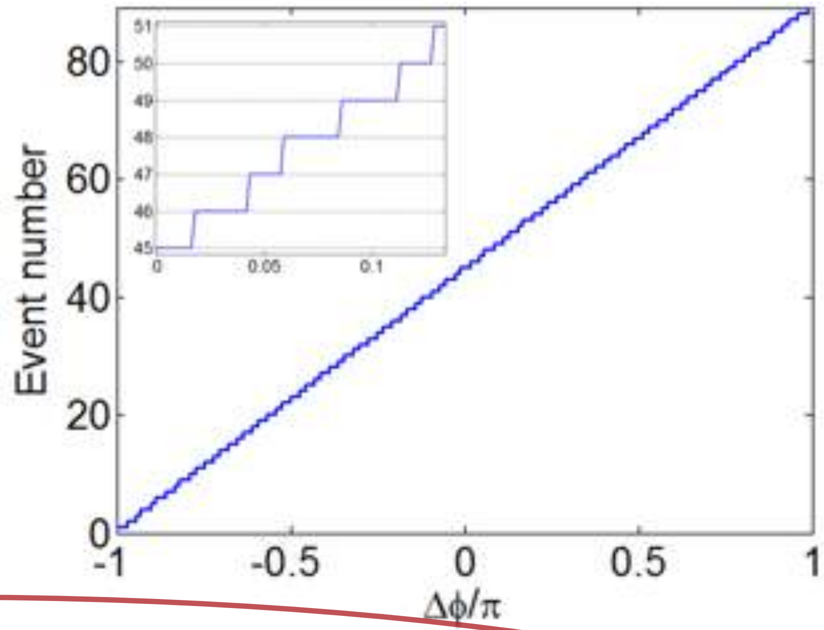
$$N\left(\varepsilon = E_{Q_{p,q}/2}\right) = p + q\tau^{-1}$$

How to observe these topological winding numbers ?

A structural degree of freedom



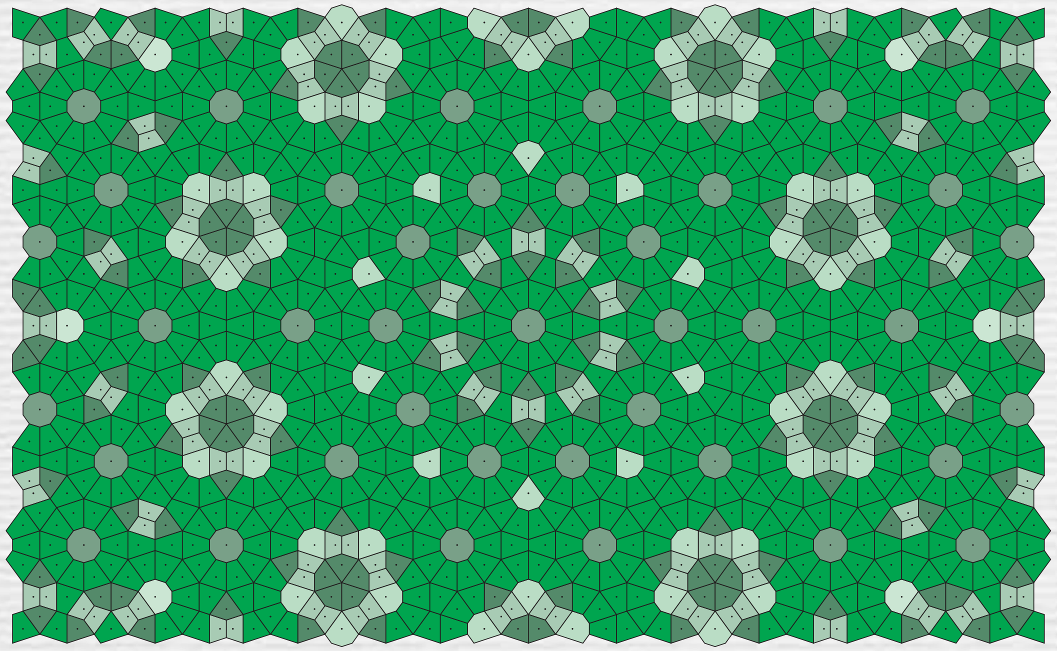
A single structural change for each value of $\Delta\phi$



Is there a symmetry of the Fibonacci chain probed when scanning the phase over a period ?

Cut & Project method

Very active branch in maths of tiling, dynamical systems

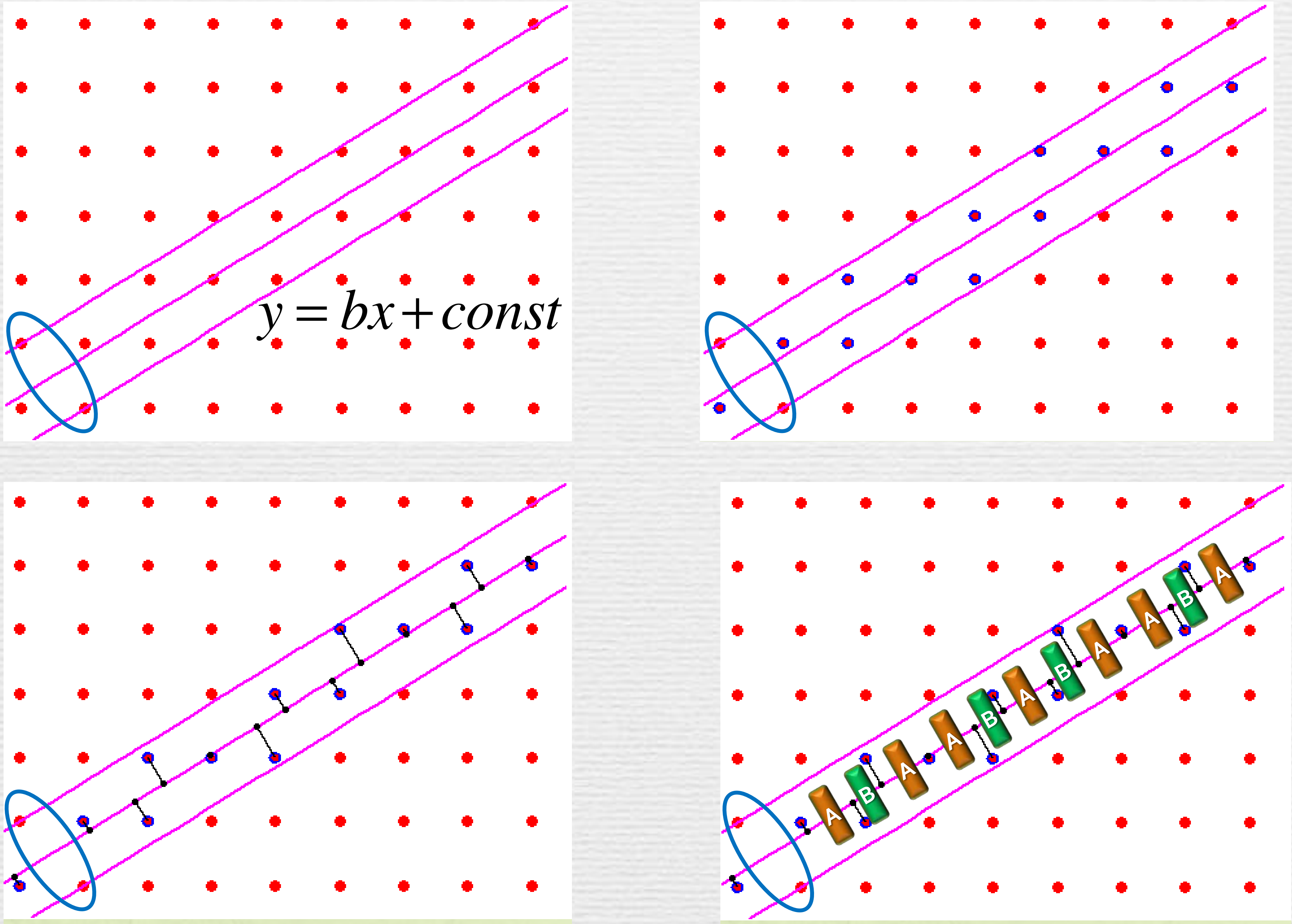


Duneau & Katz
Moody, Meyer
Pinsner, Voiculescu
Mendes-France, Allouche
Bombieri, Taylor,
Kellendonk, Grimm,
Queffelec, Bellissard, ……

Generate both periodic and quasi-periodic (quasicrystals) structures.

A brief tutorial for practical implementation.

Start from a 2D periodic lattice $L = \mathbb{Z}^2$



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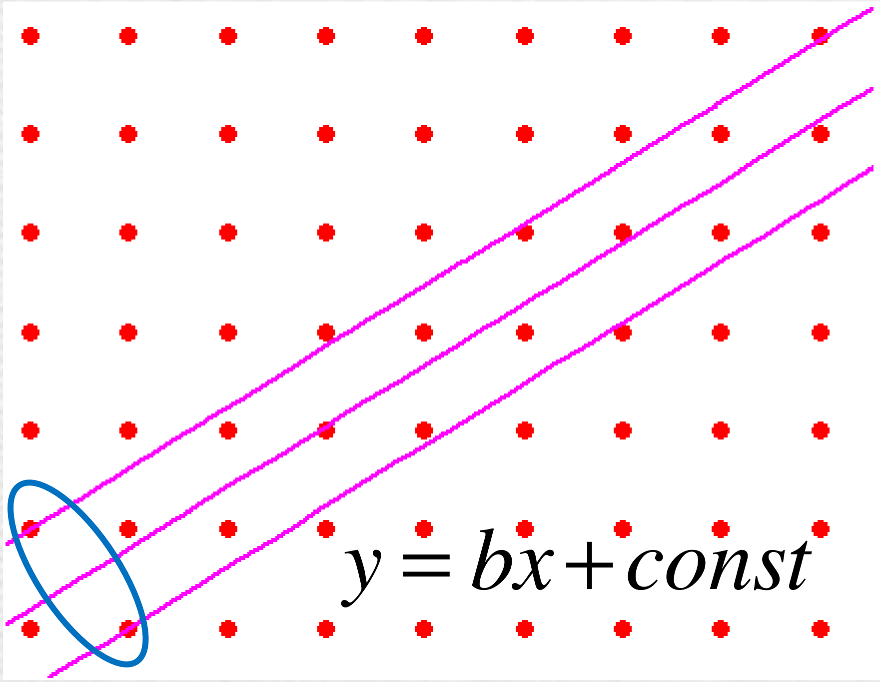

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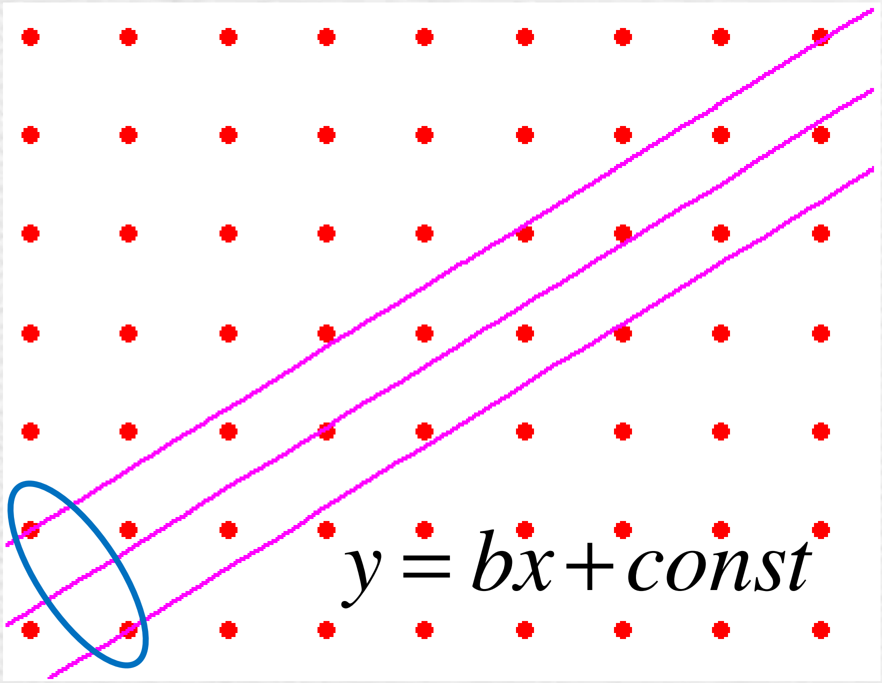

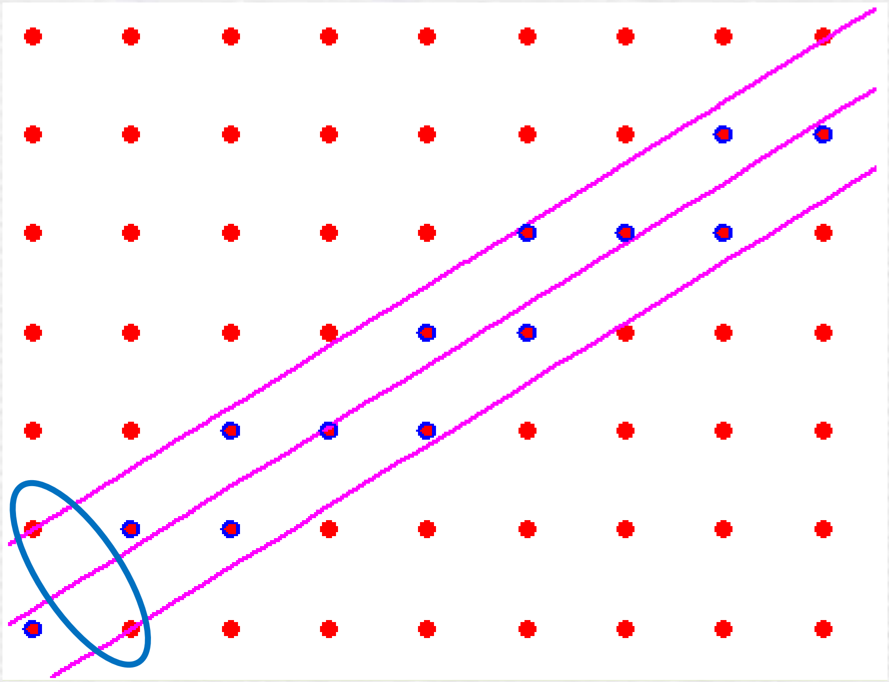

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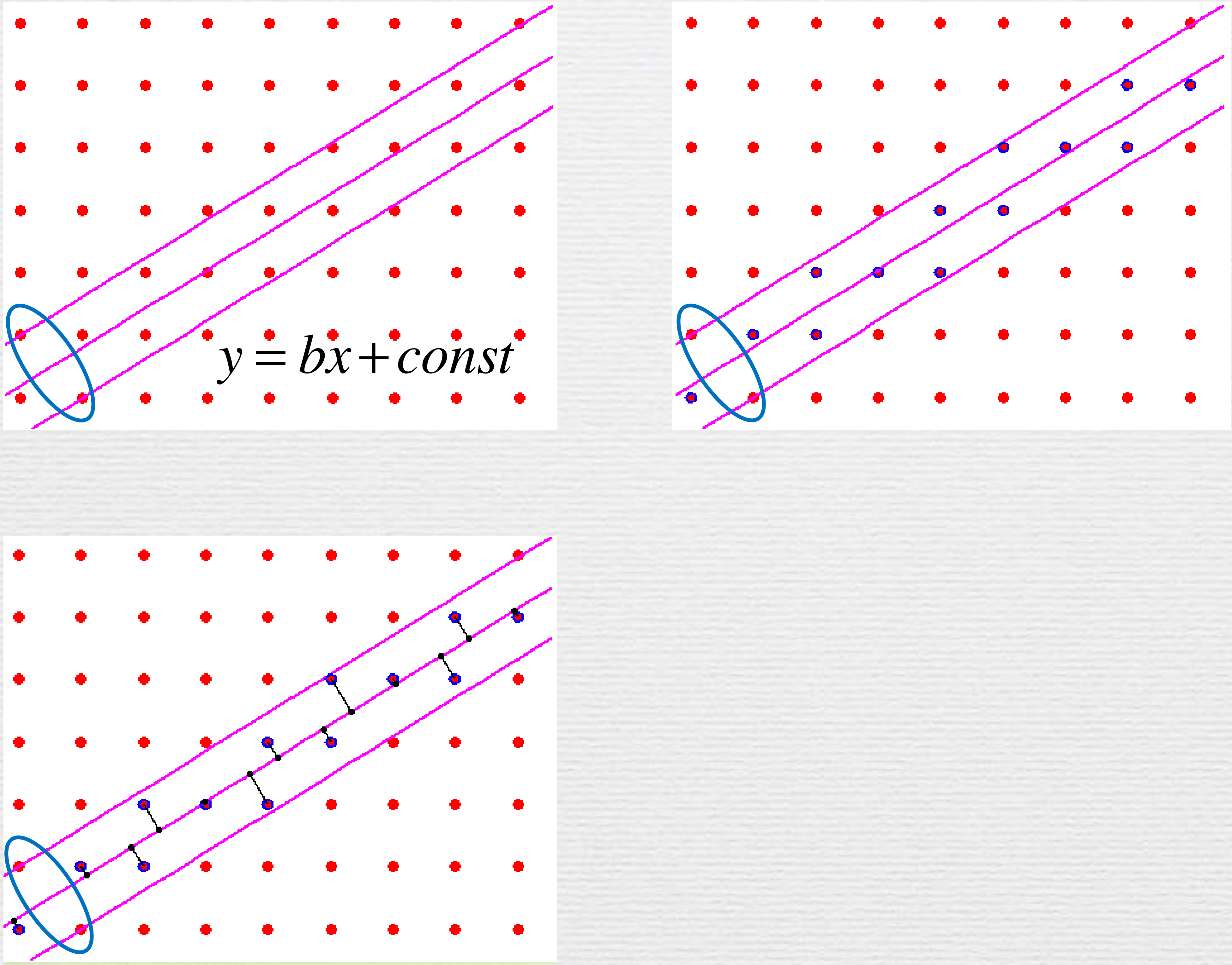

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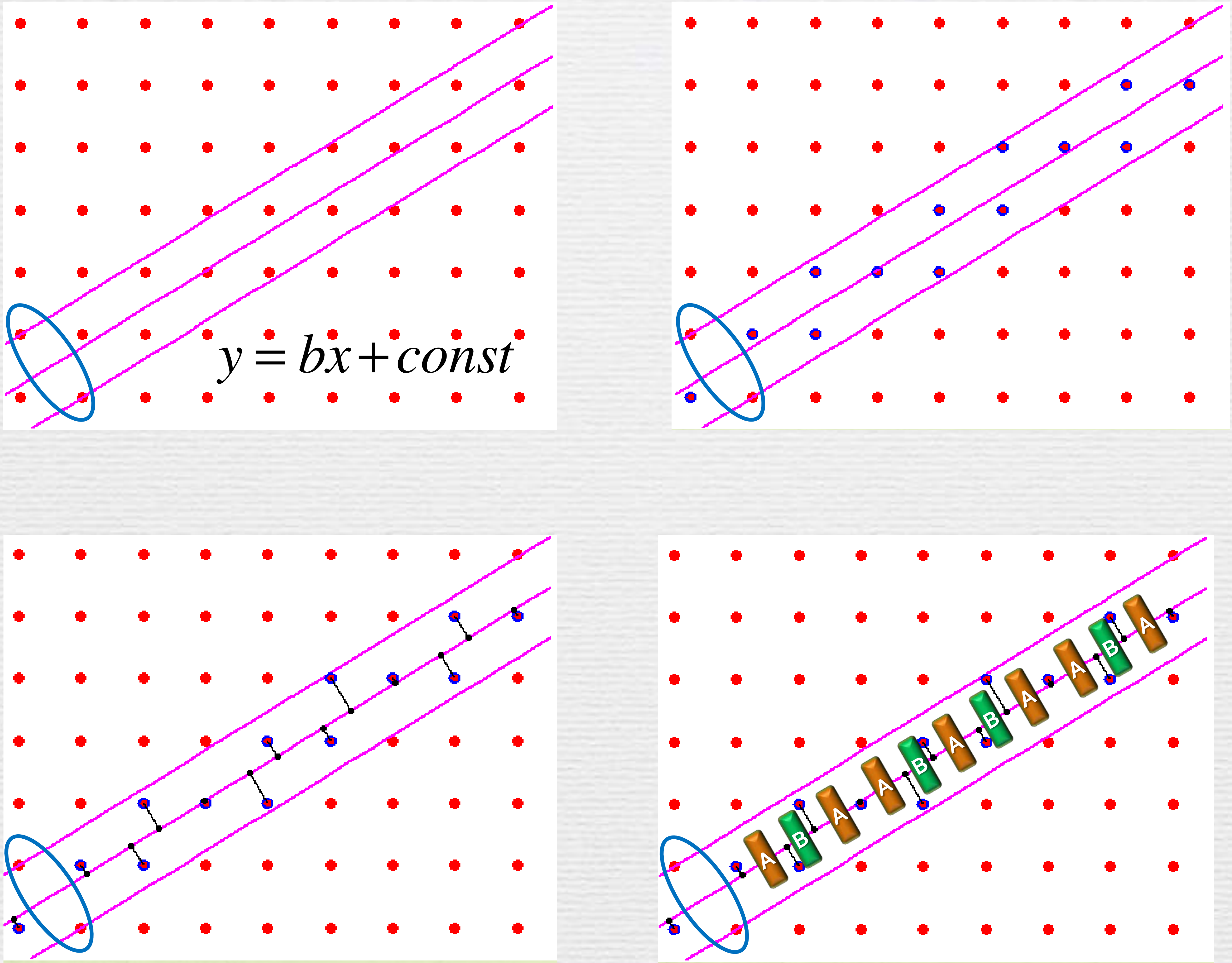

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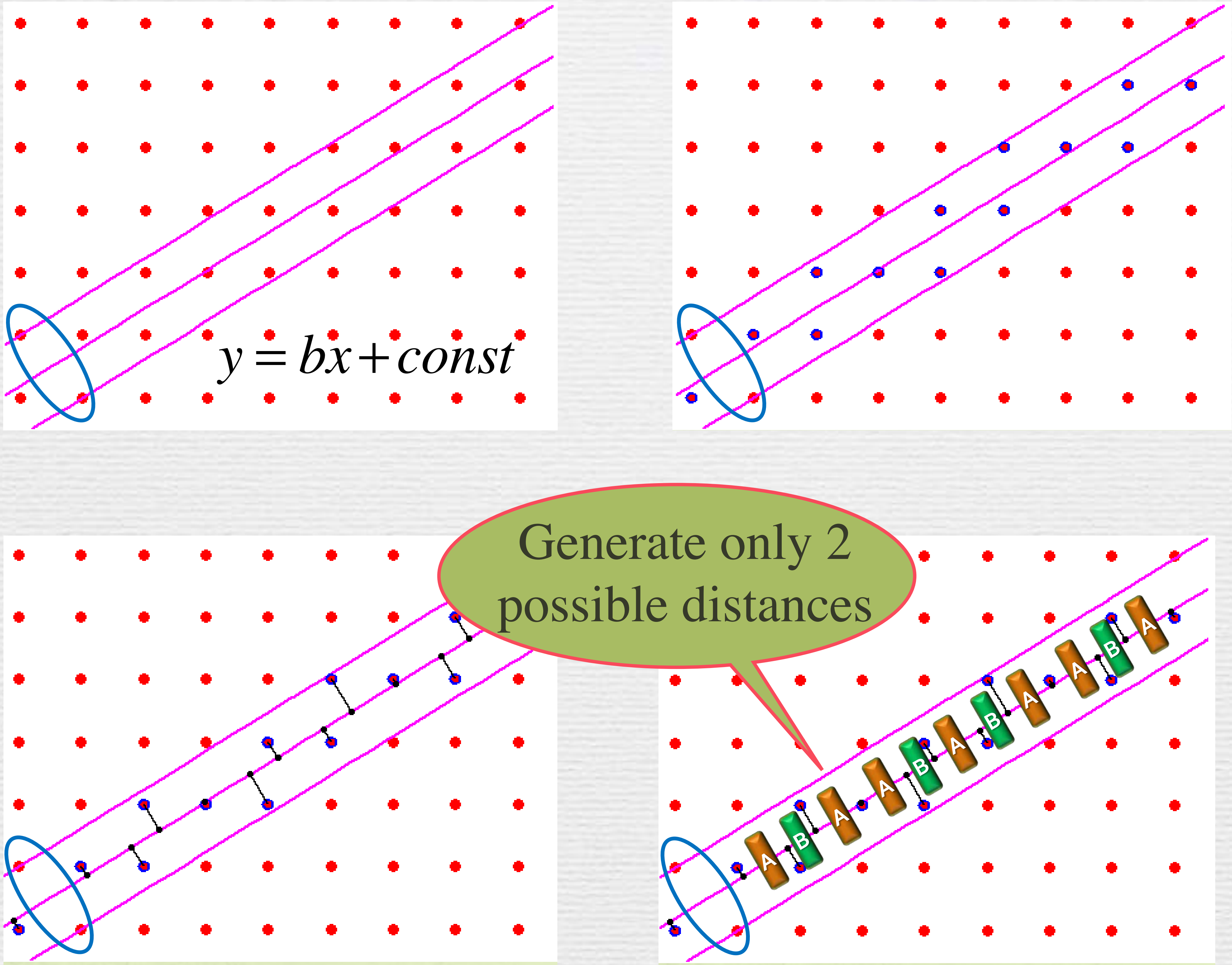
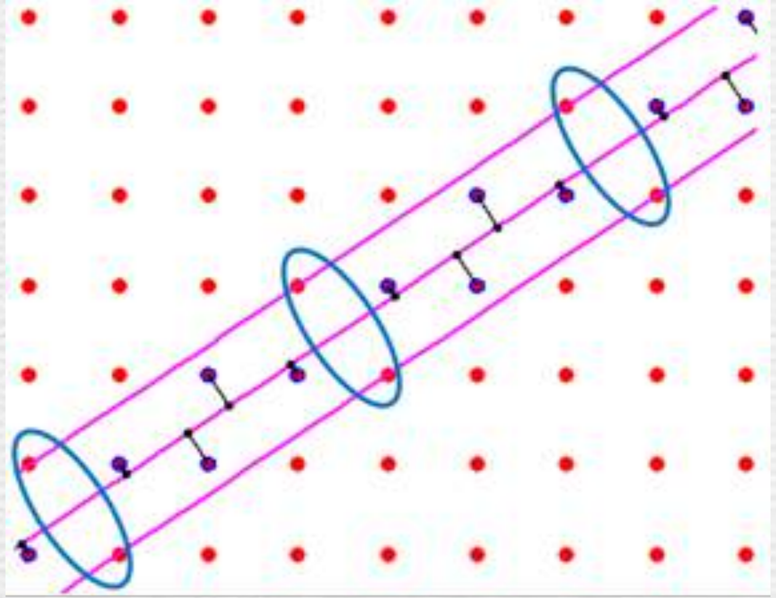


Generate only 2 possible distances

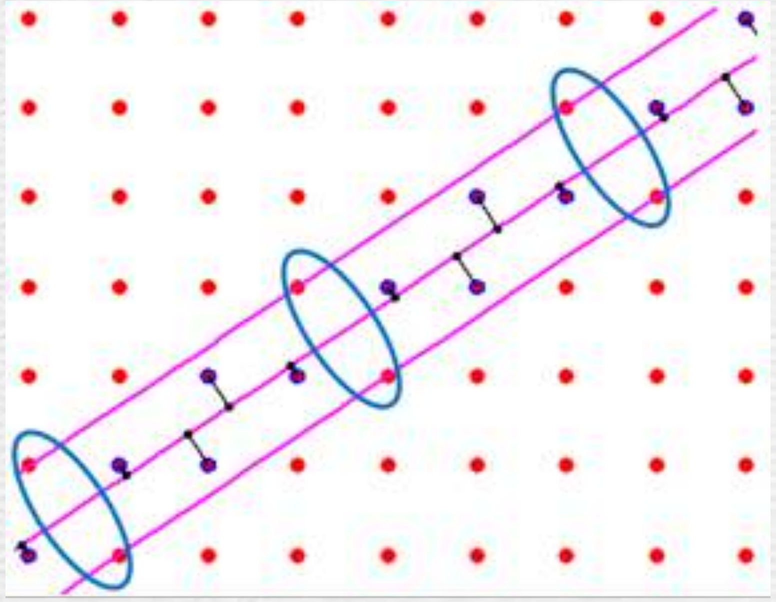
A B A A B A B A A B A ⋯

For a rational slope : periodic superlattice



$$y=\frac{2}{3}x+const$$

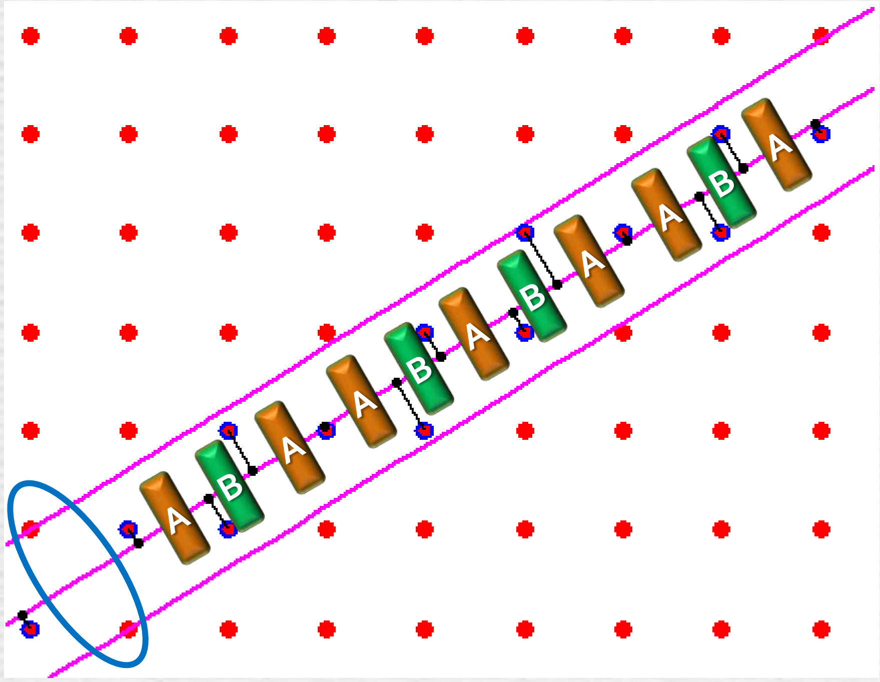
For a rational slope : periodic superlattice



$$y=\frac{2}{3}x+const$$

A B A A B A A B A ···

For an irrational slope : quasi-periodic structure



$$y=\tau^{-1}x+const$$

A B A A B A B A A B A A B ···

$$\tau = {\left(1+\sqrt{5}\right)}\Big/{2}$$

golden mean

Characteristic function

$$\chi_n = sign\left[\cos\left(2\pi n\,\tau^{-1} + \phi\right) - \cos\left(\pi\,\tau^{-1}\right)\right]$$

ϕ is an innocuous and thus ignored modulation phase.

For an infinite Fibonacci chain :

$$\phi_\infty = 3\pi\sigma = 3\pi\,\tau^{-1}$$

Define instead

$$\chi_n = sign\left[\cos\left(2\pi n\tau^{-1} + \phi_\infty + \Delta\phi\right) - \cos\left(\pi\tau^{-1}\right)\right]$$

$$\tau = {\left(1+\sqrt{5}\right)}\Big/{2}$$

golden mean

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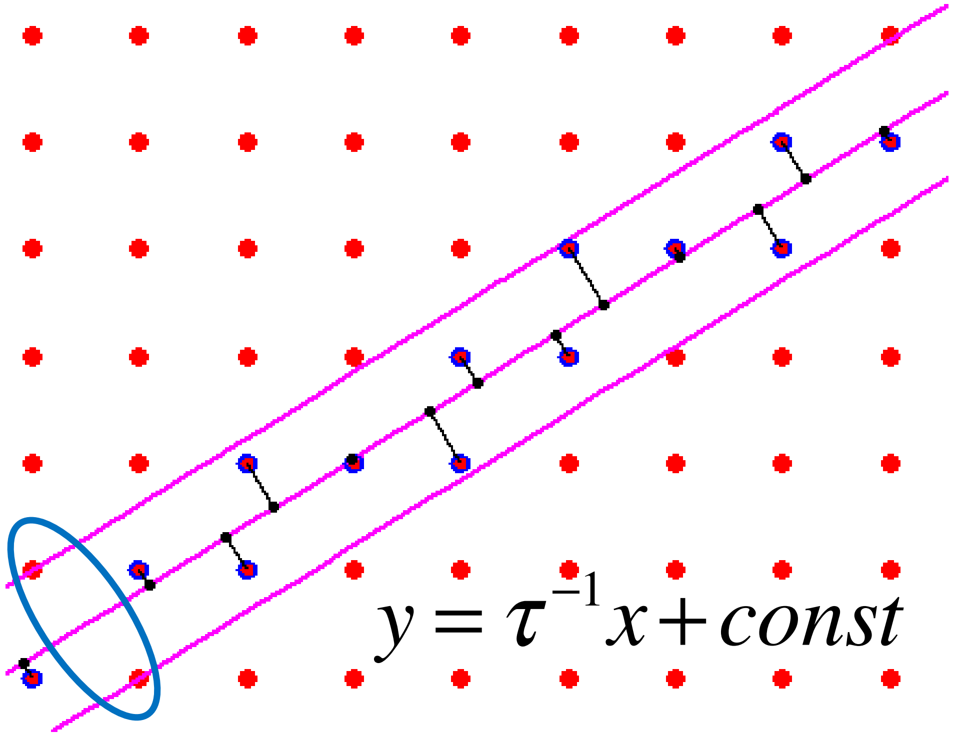
golden mean

C&P method

Is it possible to give a meaning to $\Delta\phi$ using the C&P method ?



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C&P method

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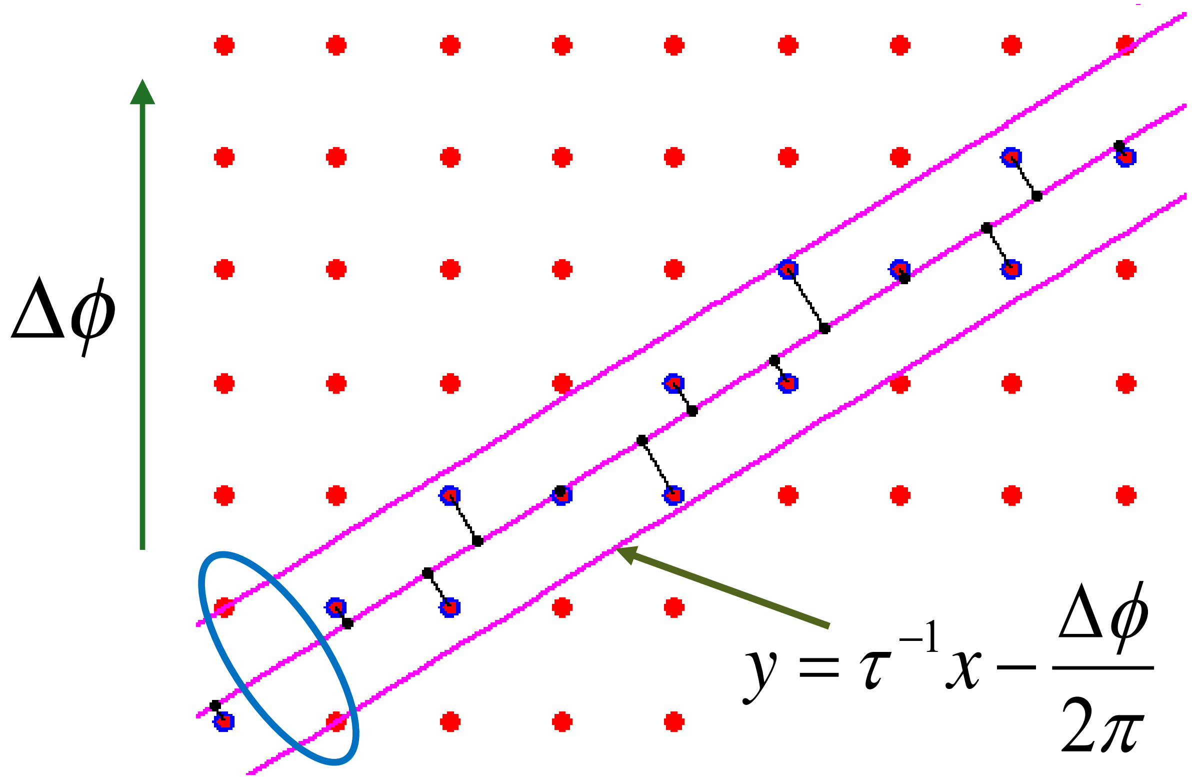
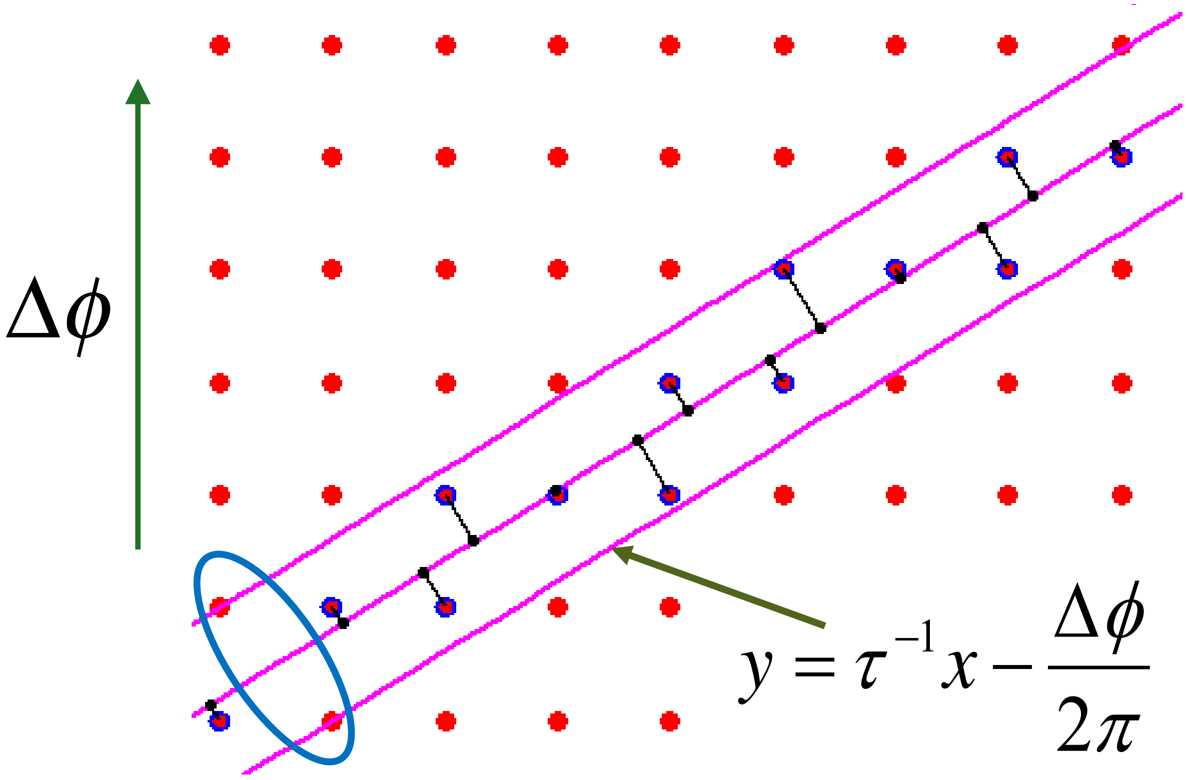


We understand the meaning of $\Delta\phi$

C&P method - Properties



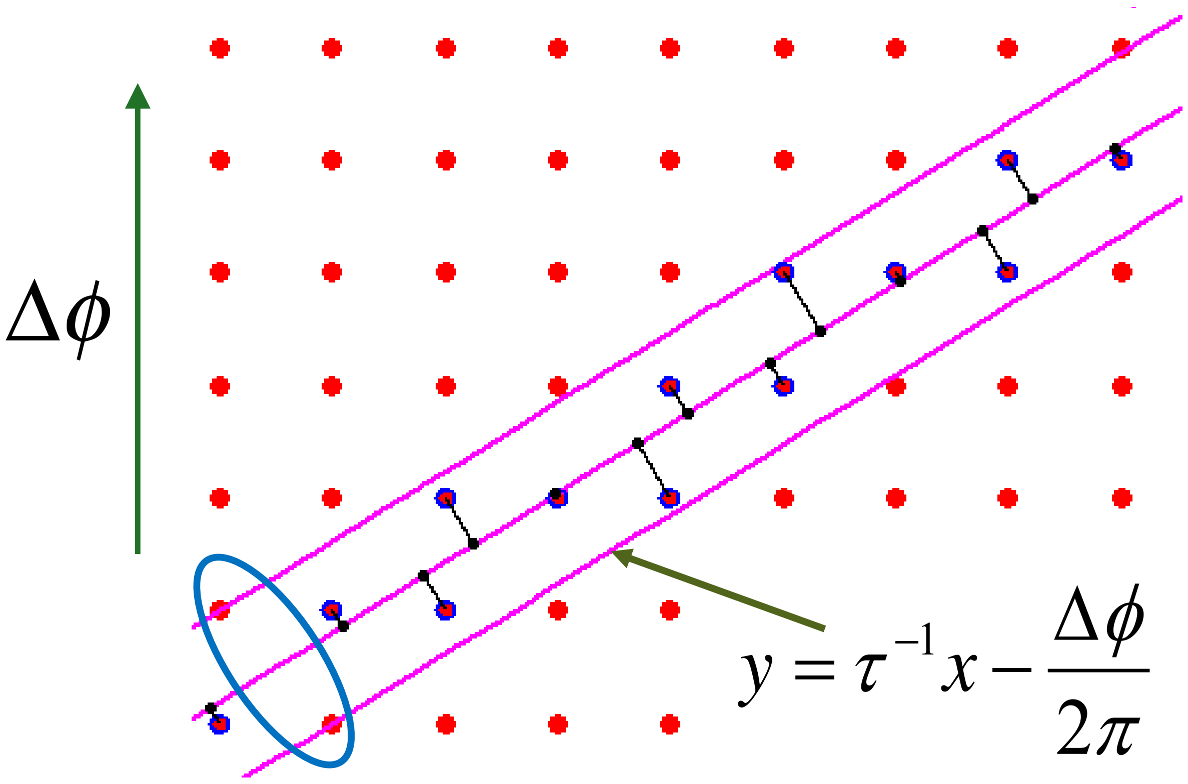
$$y = \tau^{-1}x - \frac{\Delta\phi}{2\pi}$$

- Each value of the phase $\Delta\phi$ accounts for an existing segment of the infinite Fibonacci chain.

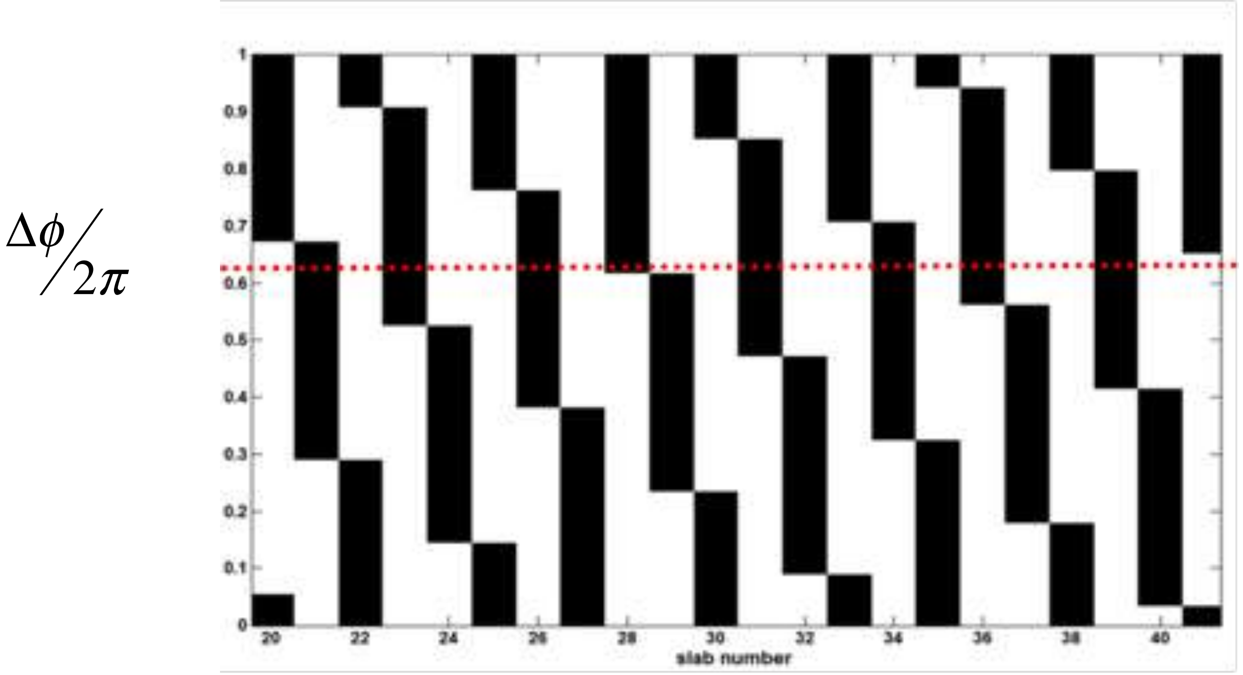
- $\Delta\phi$ is 2π-periodic.

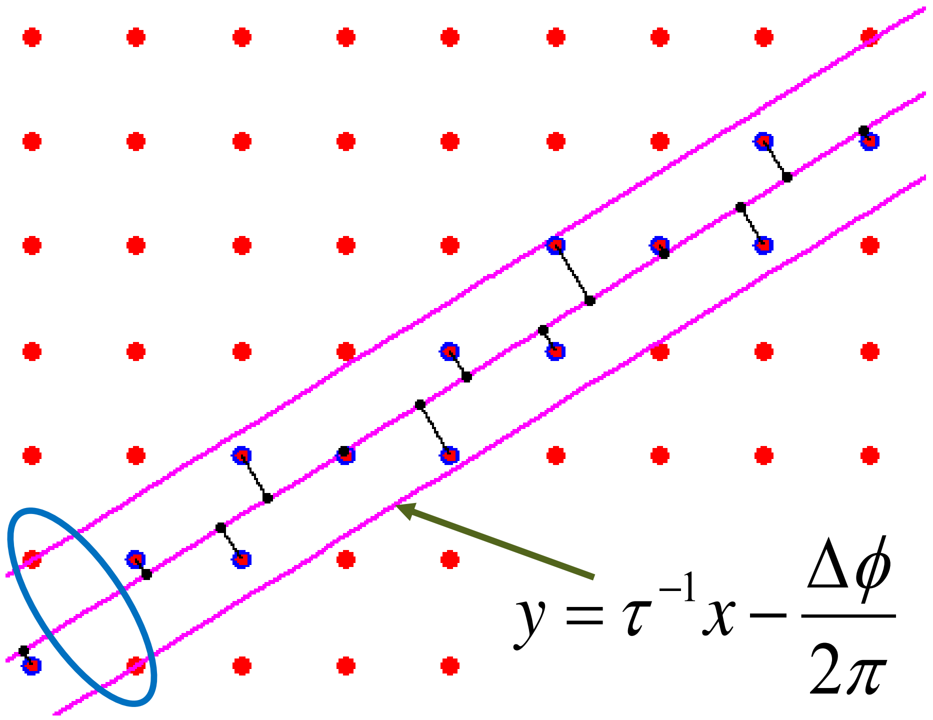
- $\Delta\phi$ corresponds to a translation (along the chain) cycle

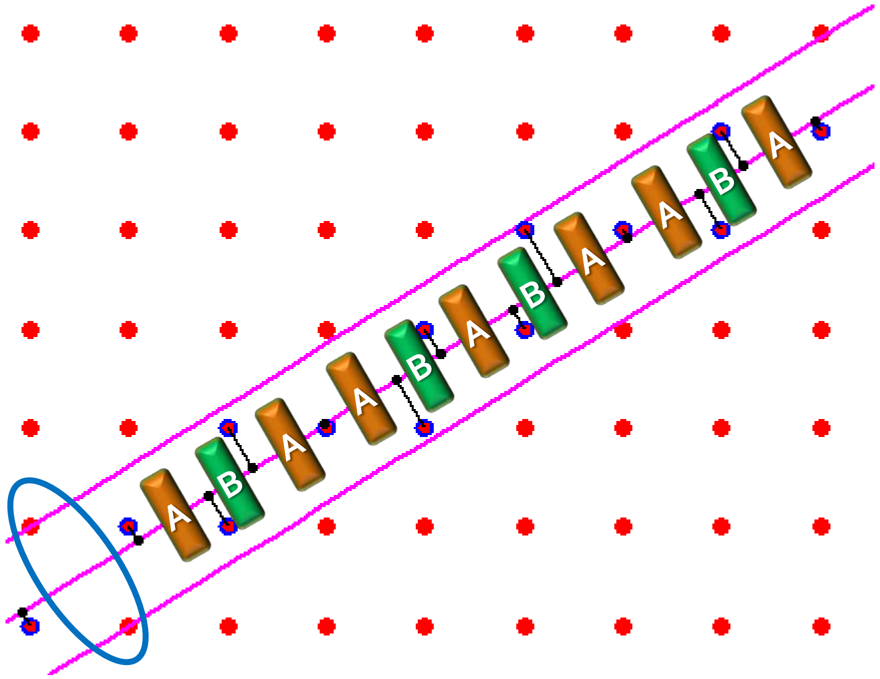
$$\Delta\phi = 2\pi\tau^{-1}\,\Delta n$$



How a change of phase is implemented along the chain ?







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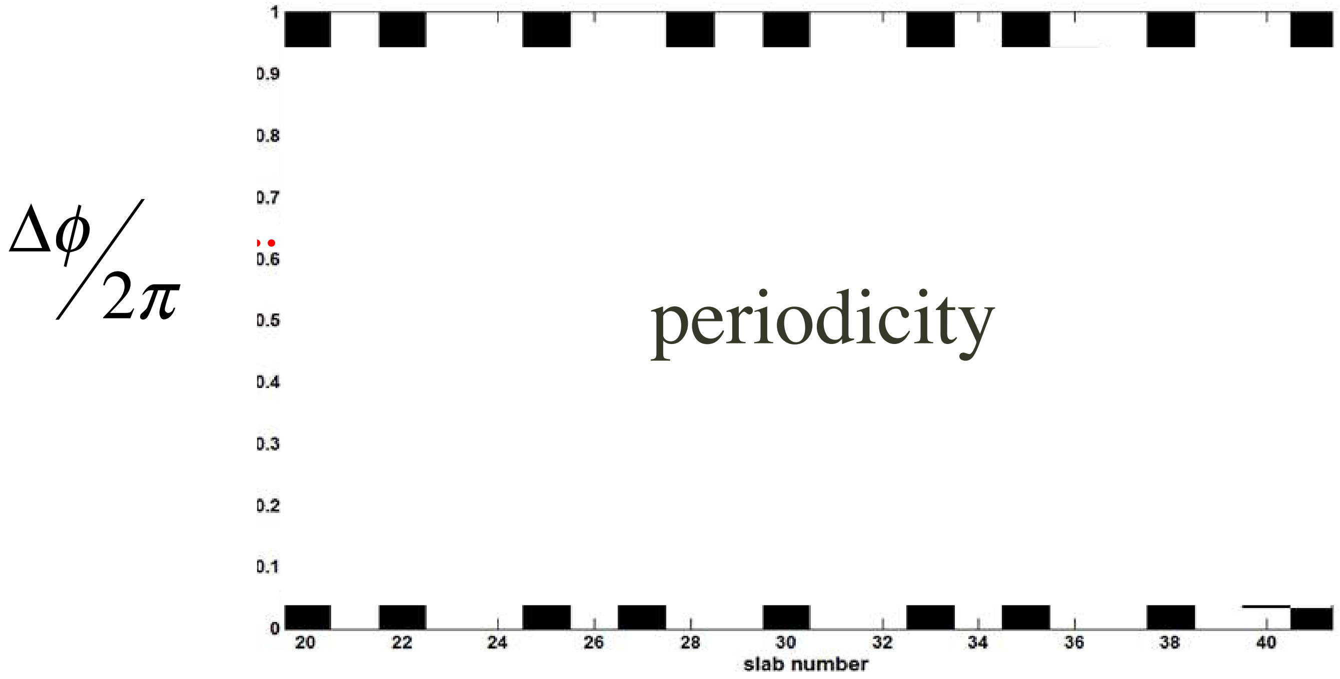
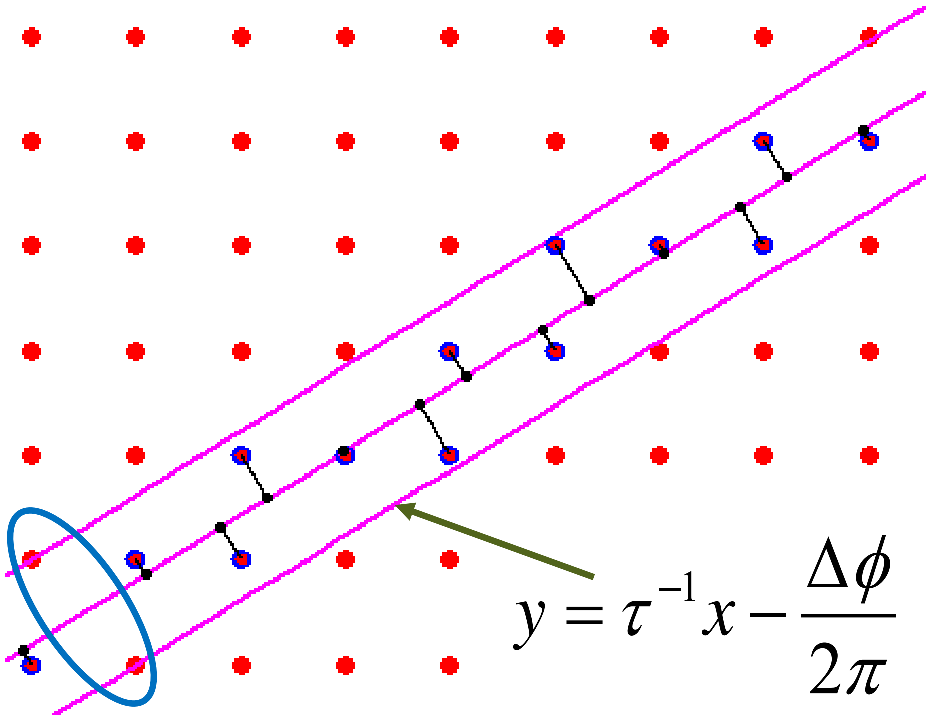
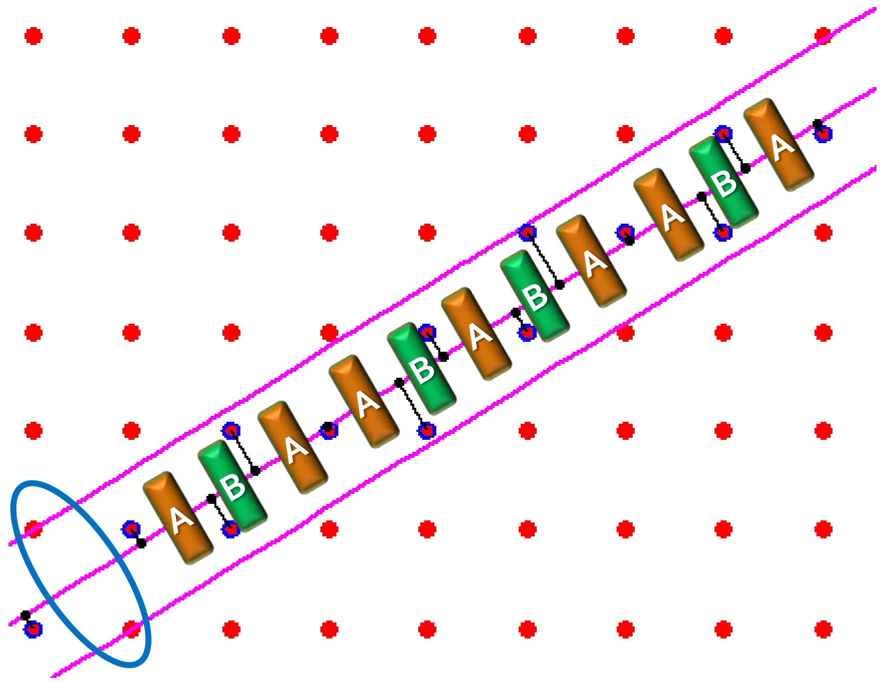


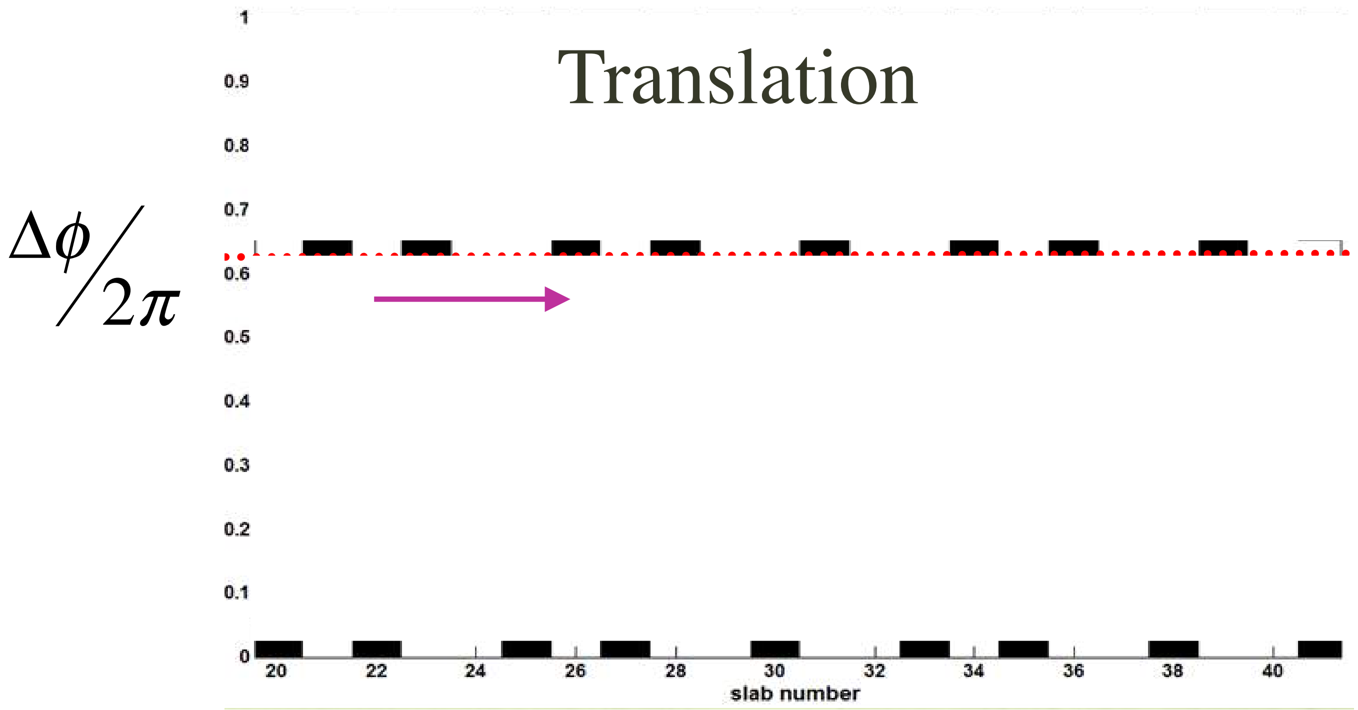
A B A A B A B A A B A A B ◆◆◆



$$y = \tau^{-1}x - \frac{\Delta\phi}{2\pi}$$



How a change of phase is implemented along the chain ?

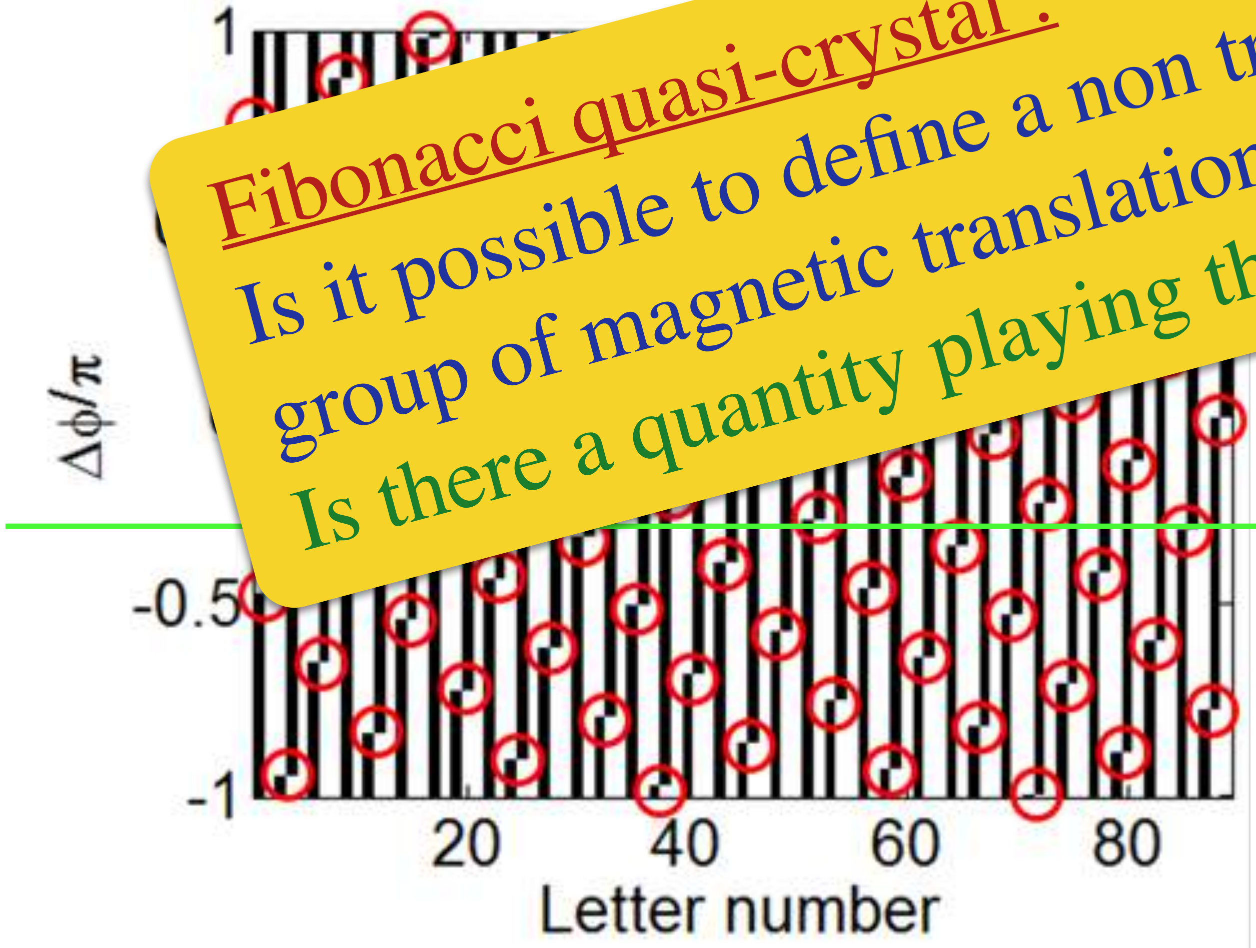


A structural degree of freedom

Fibonacci quasi-crystal :

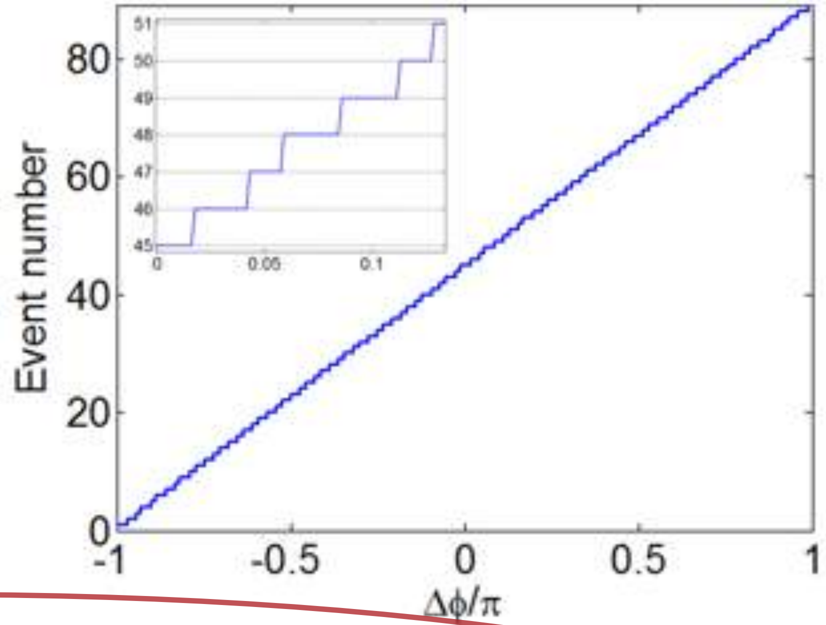
Is it possible to define a non trivial structure such as the group of magnetic translations in the Harper model ?

Is there a quantity playing the role of a magnetic field ?



ach

alue of $\Delta\phi$

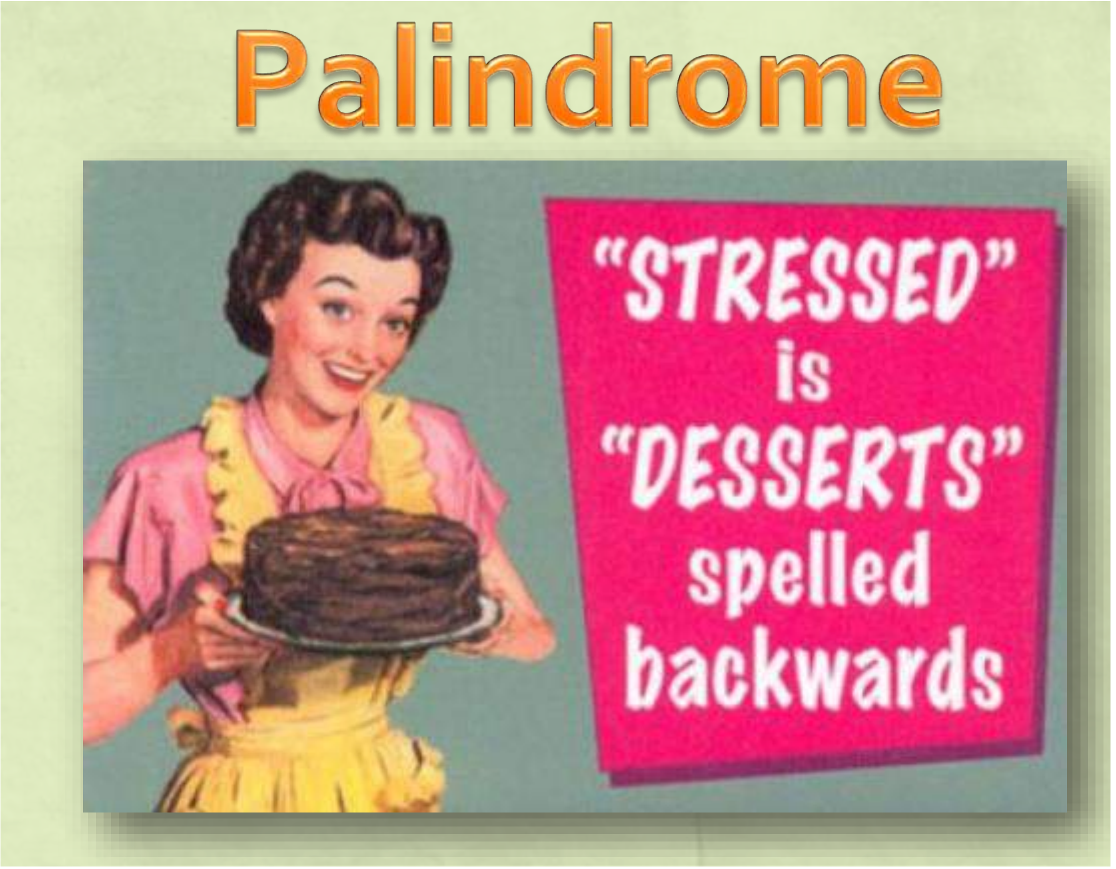


Is there a symmetry of the Fibonacci chain probed when scanning the phase over a period ?

Palindrome

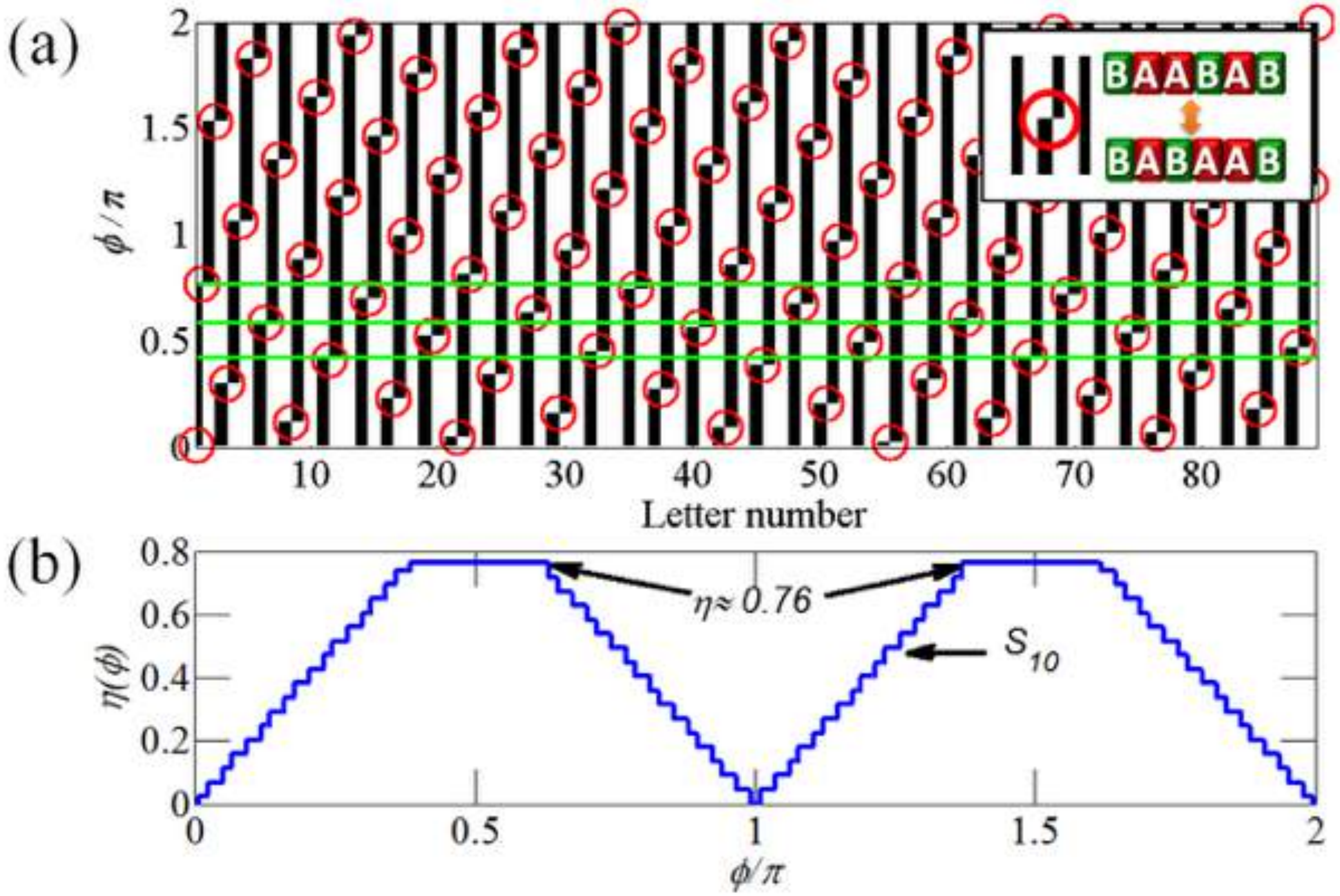
Scanning the phase $\Delta\phi$ drives the chain through a palindromic cycle.

Palindrome ?



Palindrome

Scanning the phase $\Delta\phi$ drives the chain through a palindromic cycle.



$$\eta(\Delta\phi) \equiv \frac{1}{N} \sum_{j=0}^{\left[\frac{N-1}{2}\right]} \left|\chi_j(\Delta\phi) - \chi_{N-j}(\Delta\phi)\right|$$

Remarks :

- Palindromic symmetry is (almost ?) ubiquitous in C&P quasicrystals.
- Counterexamples for non C&P cases : non Pisot substitutions are not palindromic.
- The phason ϕ cannot be defined for non C&P quasicrystals.
- But the Gap Labeling theorem and associated Chern numbers are well defined.

Palindromicity

Scanning the phase $\Delta\phi$ drives the chain through a palindromic cycle.

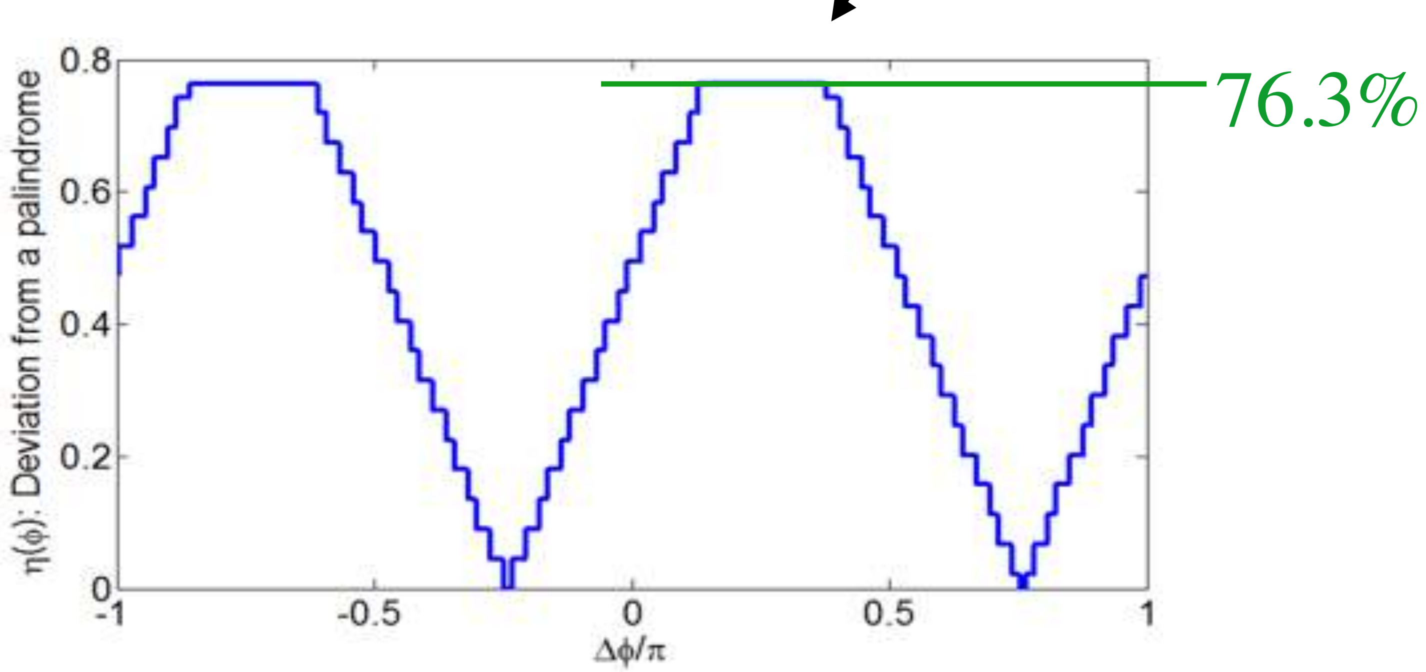
Fibonacci chains generated using substitution or concatenation are “almost” palindromic.

They correspond to $\Delta\phi = 0 \Leftrightarrow \phi = \phi_\infty = 3\pi\tau^{-1}$

and a length equal to a Fibonacci number N_F

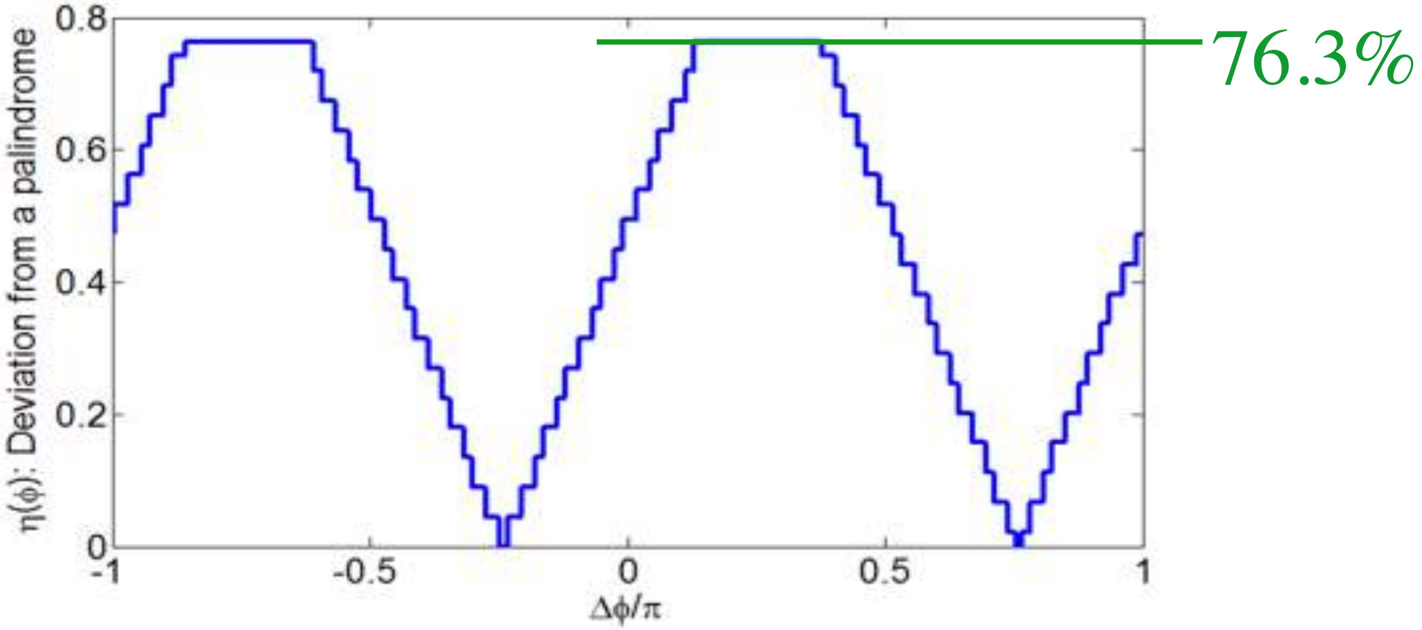
An infinite chain contains arbitrary long palindromic substructures

The deviation from palindromicity saturates.

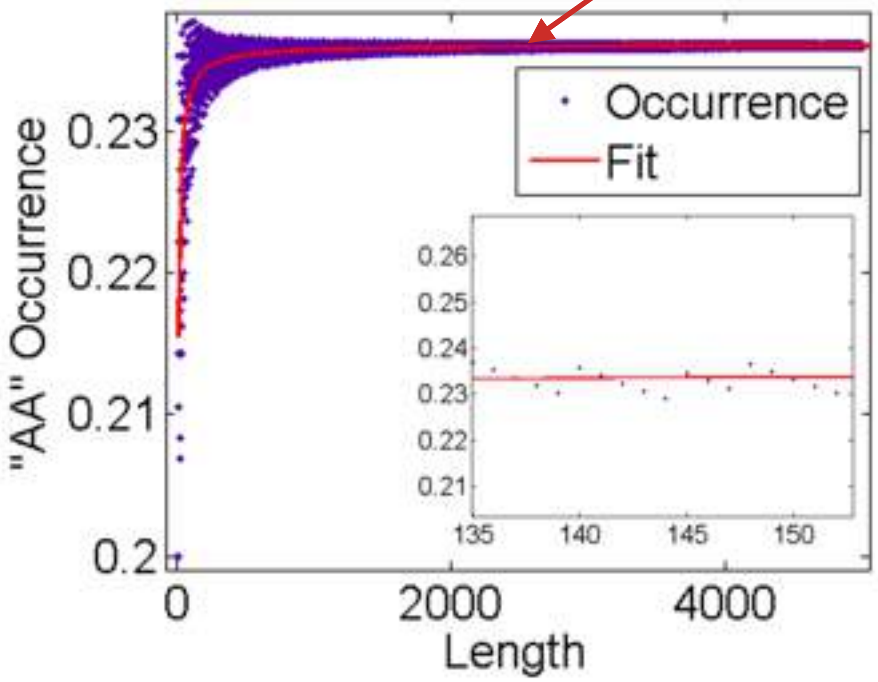


The saturation value depends on the C&P slope, i.e., on type of quasi-periodic potential.

The deviation from palindromicity saturates.



76.3%



23.7%

The saturation value depends on the C&P slope, i.e., on type of quasi-periodic potential.

For the Fibonacci chain, the saturation corresponds to the occurrence of [AA] doublets, knowing that [BB] doublets are forbidden.

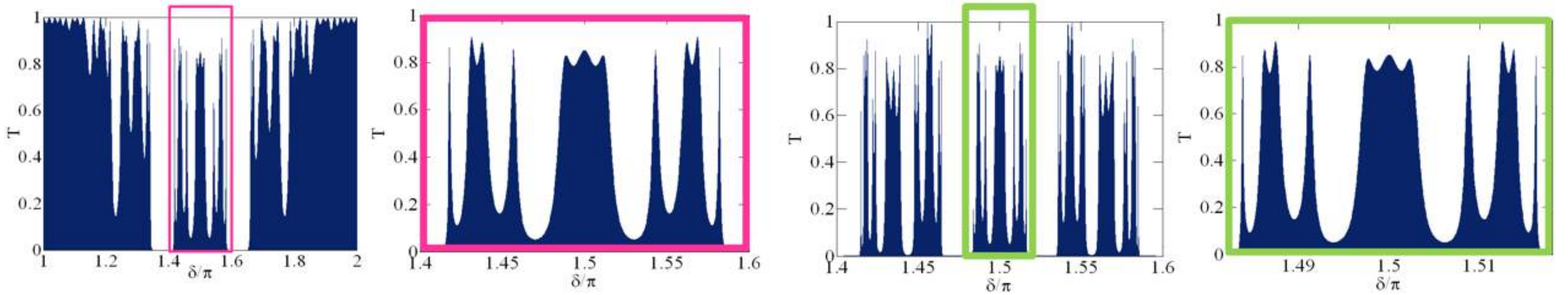
Are there spectral consequences
of these structural properties ?

Are there spectral consequences
of these structural properties ?

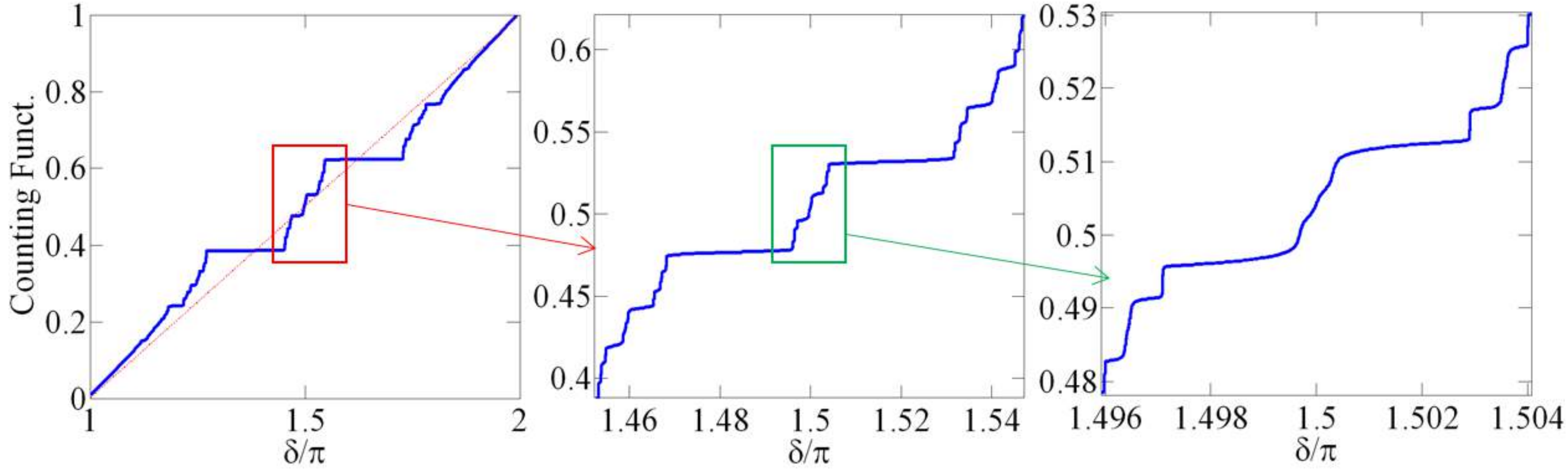
Almost No…

We have already calculated and measure the spectrum in details

Scaled finite size Fibonacci chains

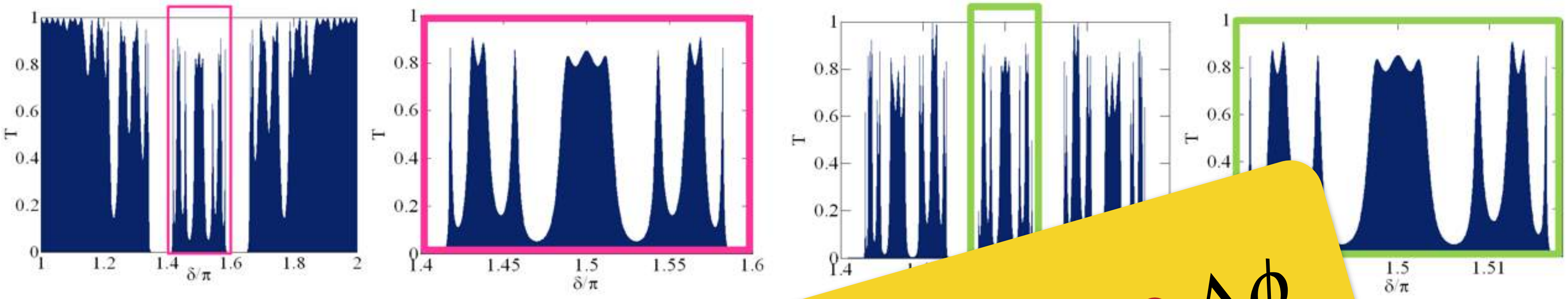


Density of modes



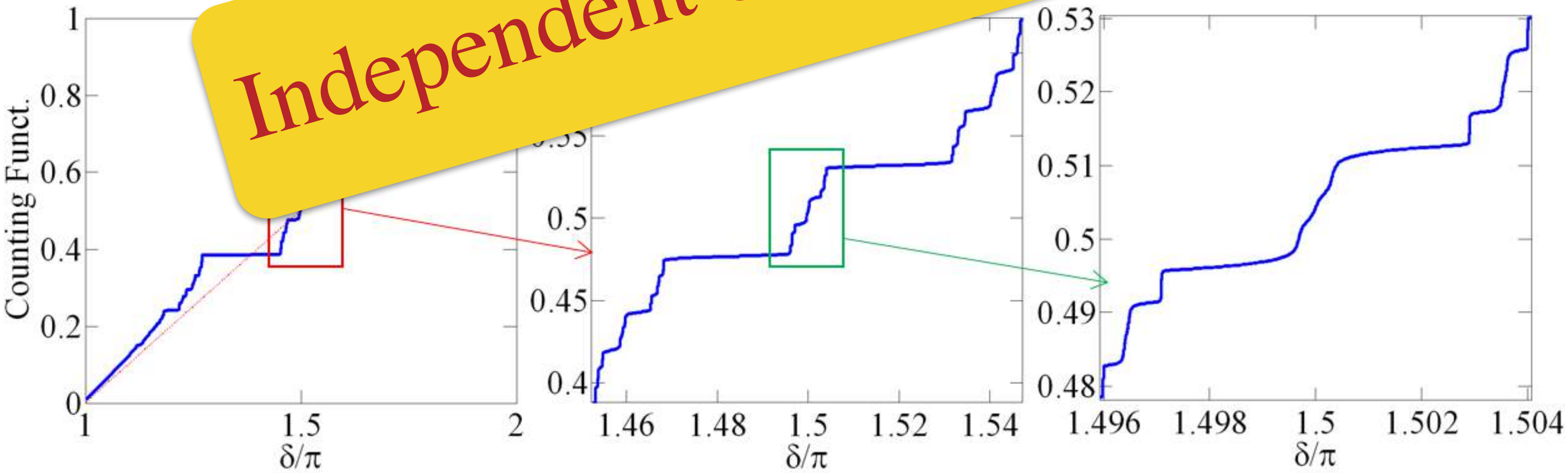
Integrated Density of

Scaled finite size Fibonacci chains



Independent of the phase $\Delta\phi$

D

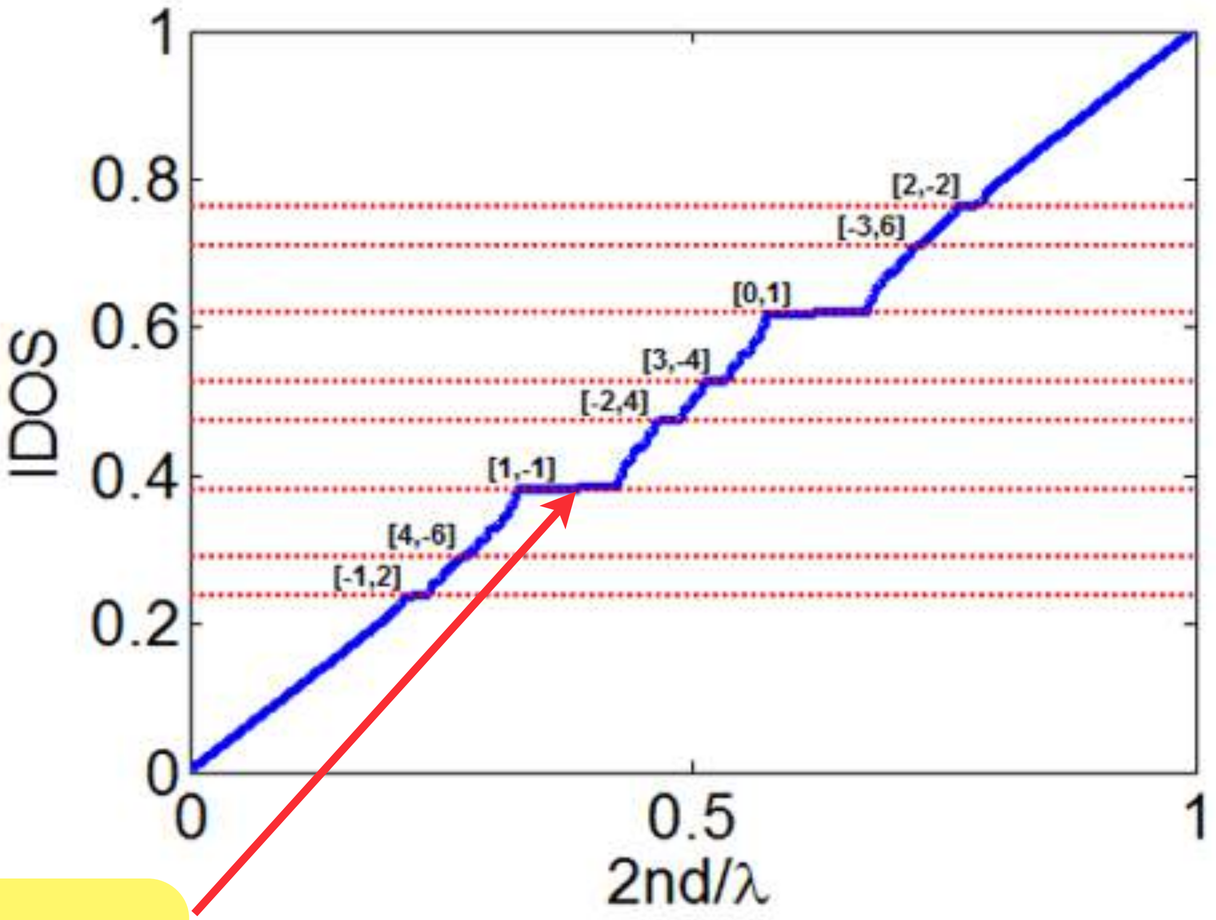


Integrated Density of

Integrated Density of States-Gap Labeling

$$\tau = {\left(1+\sqrt{5}\right)}\big/{2}$$

golden mean



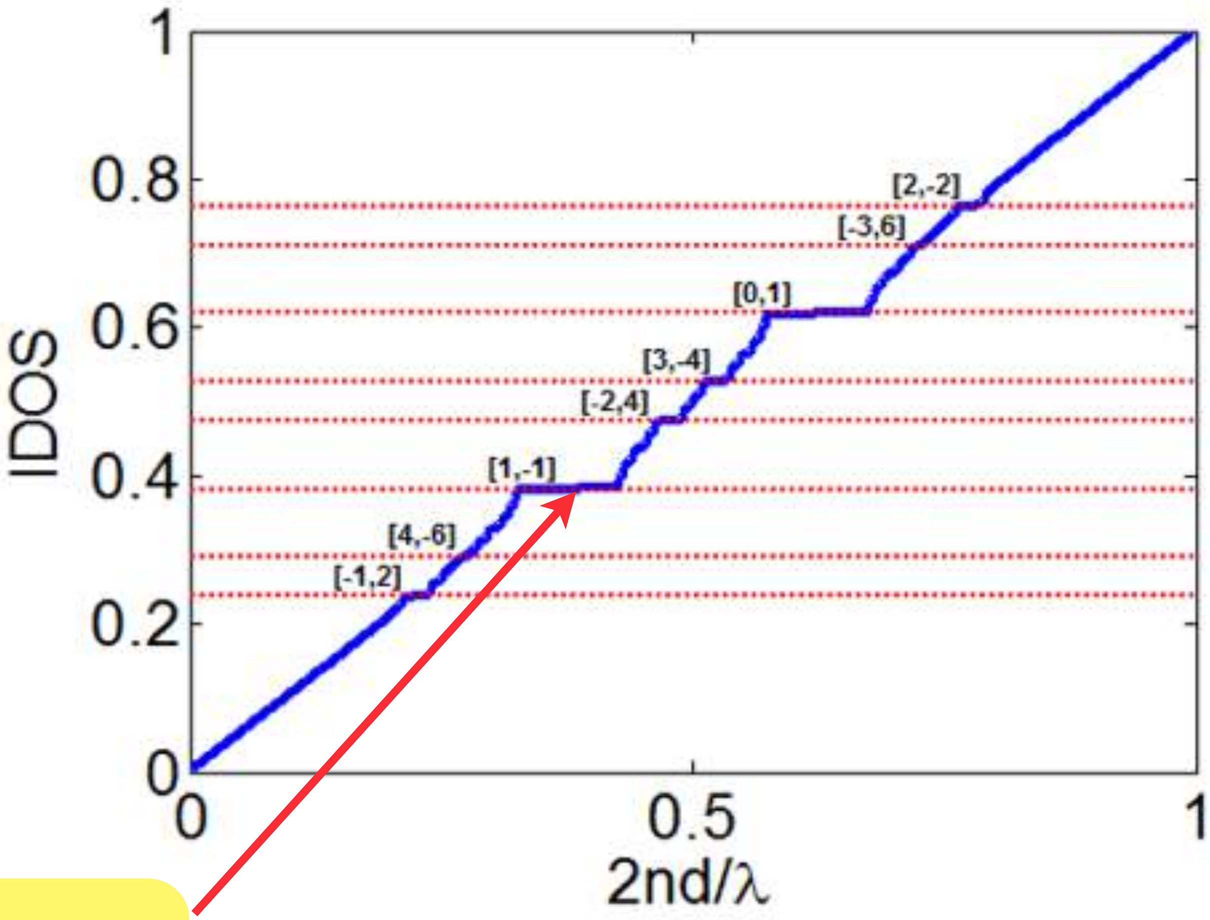
$$N\left(\omega_{gap}\right) = p + q\tau^{-1}$$

within a $\left[p,q\right]$ gap

$$\left(p,q\right) \in \mathbb{Z}$$

$$\tau = {\left(1+\sqrt{5}\right)}\Big/{2}$$

golden mean



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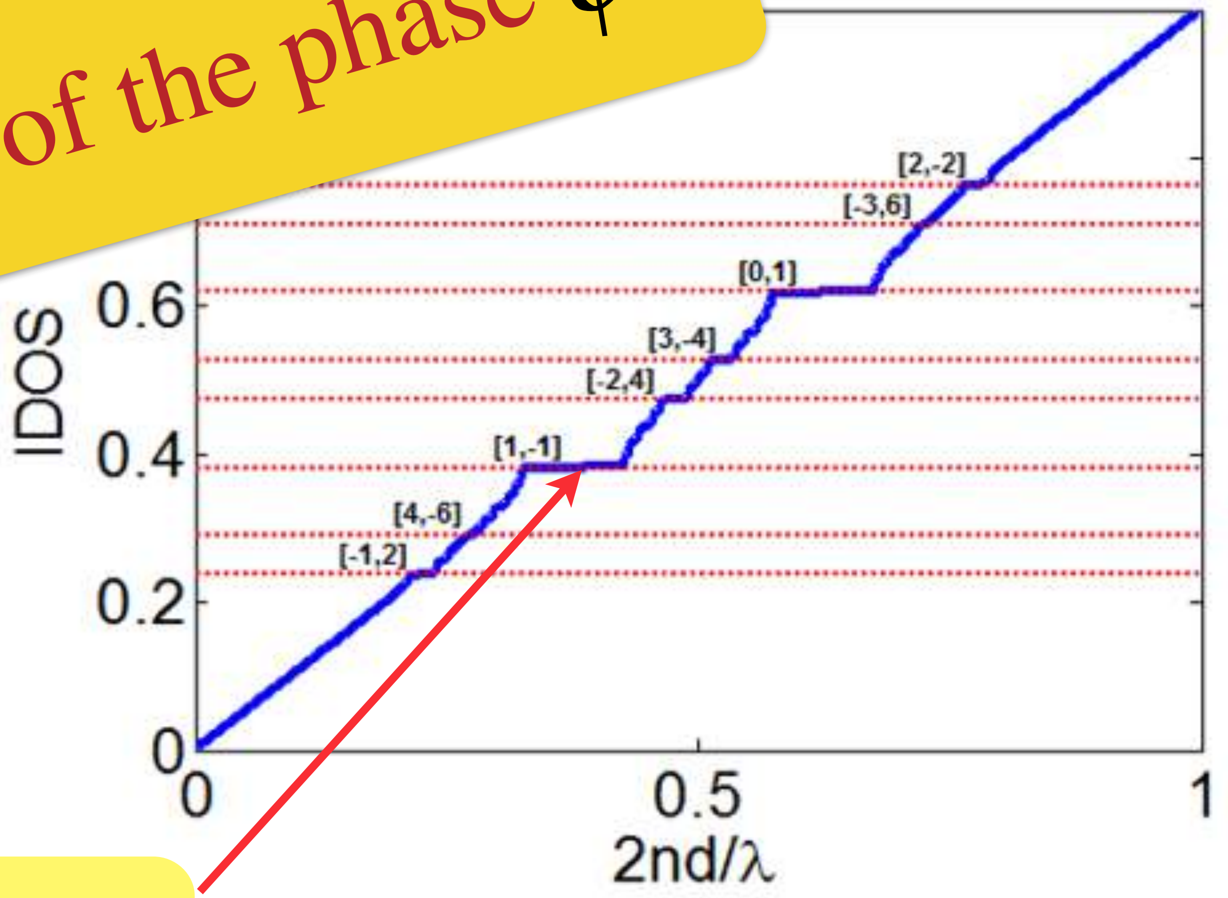
Integrated Density of States-Gap Labeling

Independent of the phase ϕ

golden mean



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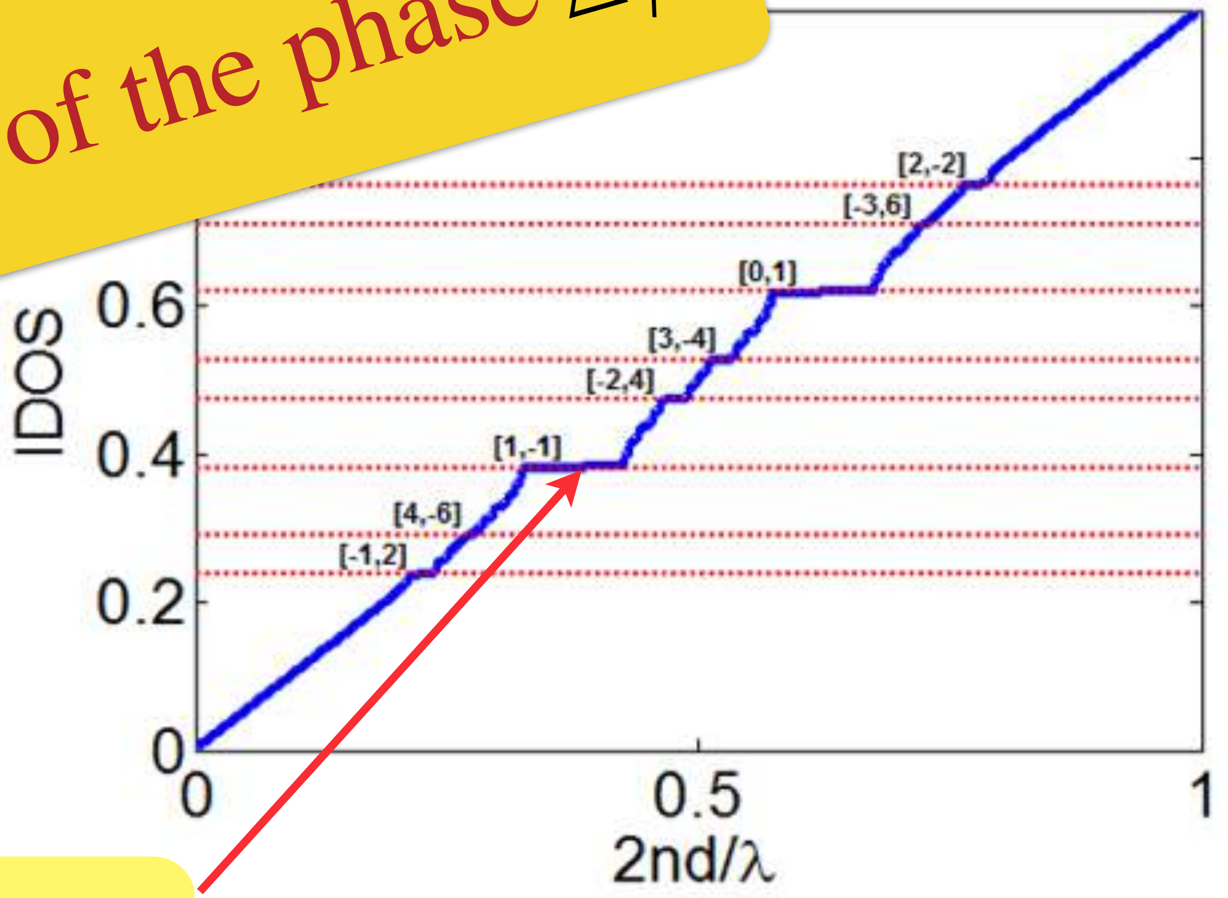
$$(p,q) \in \mathbb{Z}$$

Integrated Density of States-Gap Labeling

Independent of the phase $\Delta\phi$

golden mean



$$N\left(\omega_{gap}\right) = p + q\tau^{-1}$$

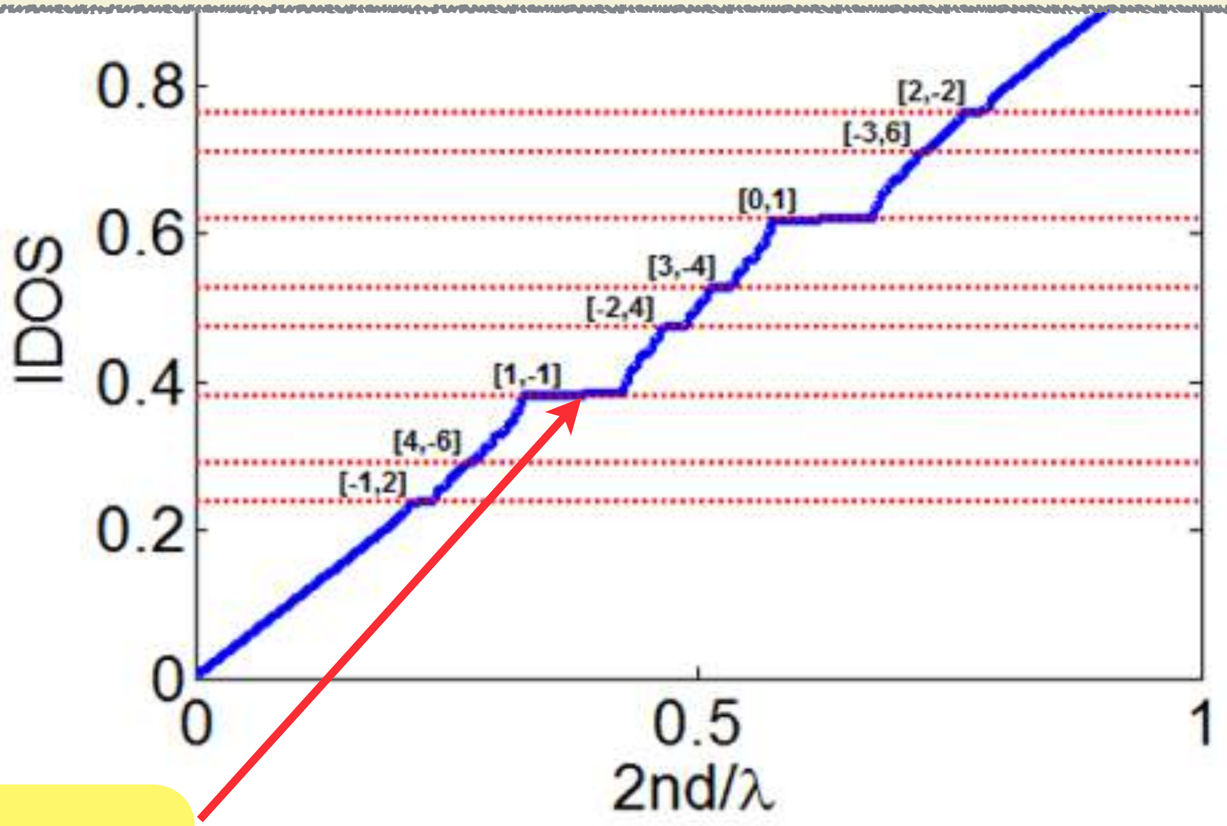
within a $\left[p,q\right]$ gap

$\left(p,q\right) \in \mathbb{Z}$ are topological invariants (Chern numbers).
Independent of the potential strength, inhomogeneity, ...

Existence of Chern numbers is independent of the palindromic symmetry and of the phason $\Delta\phi$

$$\tau = {\left(1+\sqrt{5}\right)}\big/{2}$$

golden mean



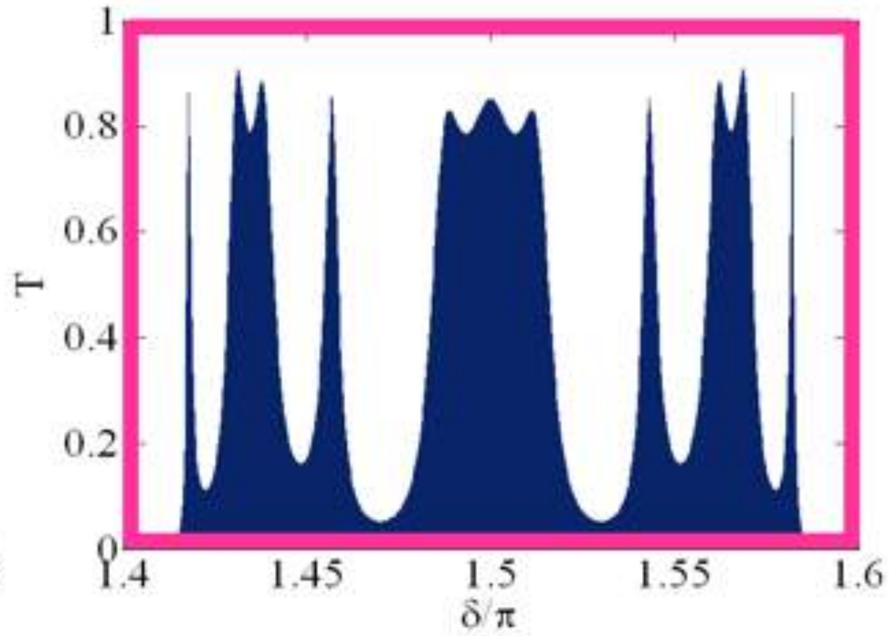
$$N\left(\omega_{gap}\right) = p + q\tau^{-1}$$

within a $\left[p,q\right]$ gap

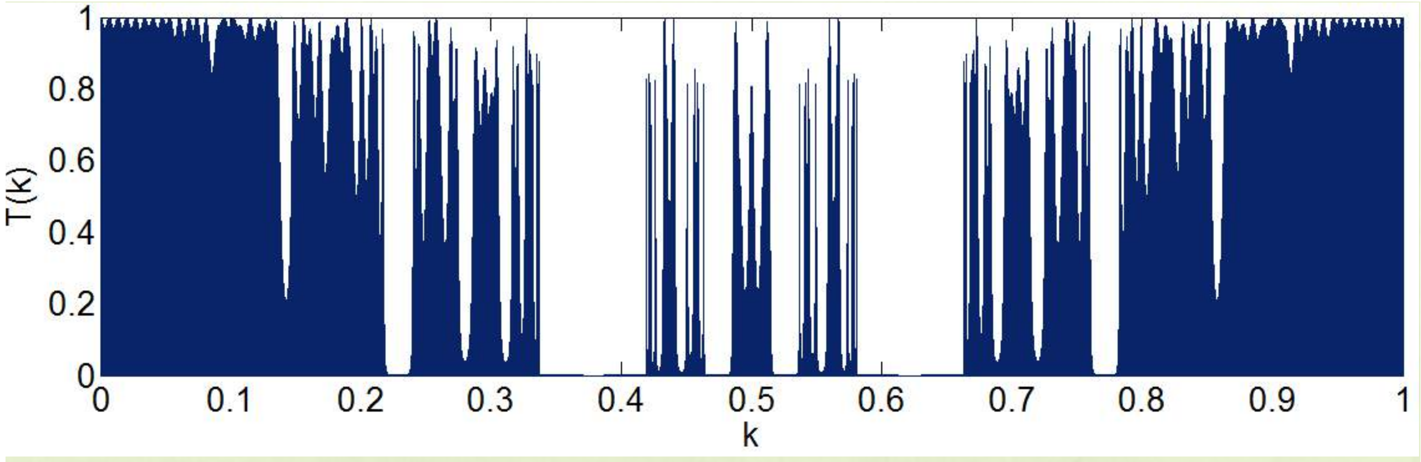
$\left(p,q\right) \in \mathbb{Z}$ are topological invariants (Chern numbers).

Independent of the potential strength, inhomogeneity, ...

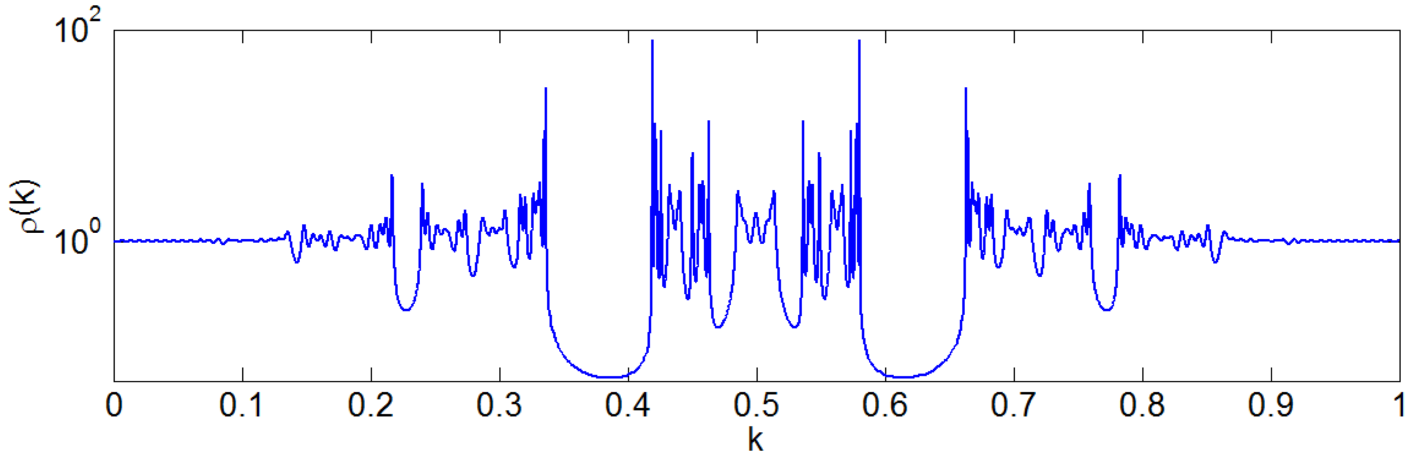
All these characteristics are independent of the phase $\Delta\phi$



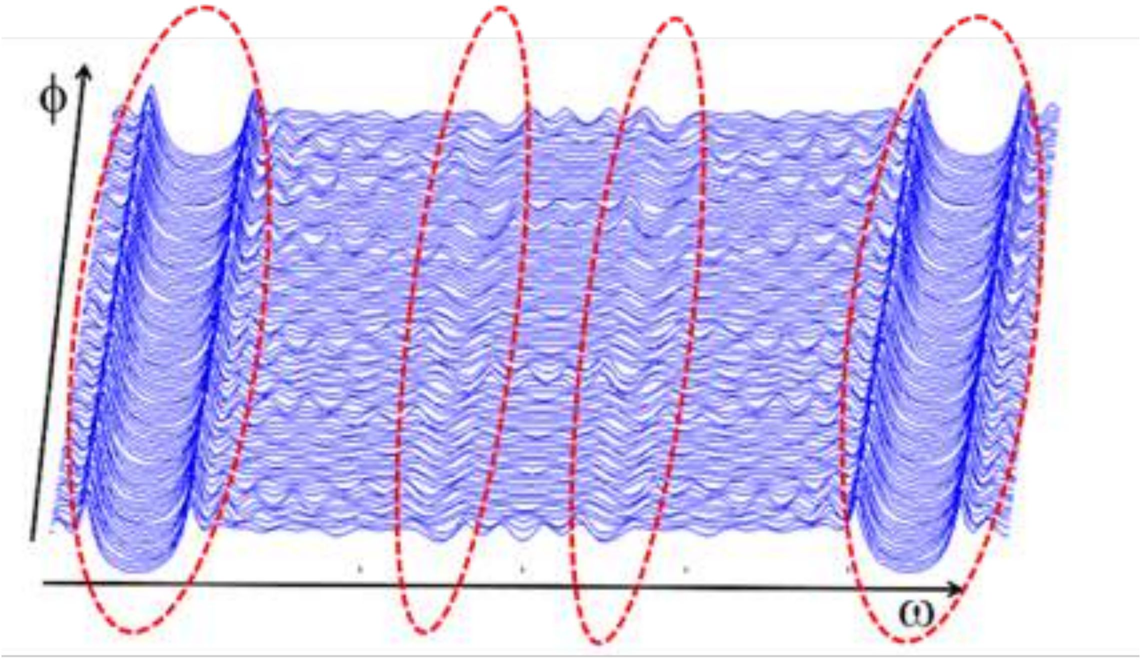
Spectral characteristics are independent of the phase ϕ



Transmission spectrum

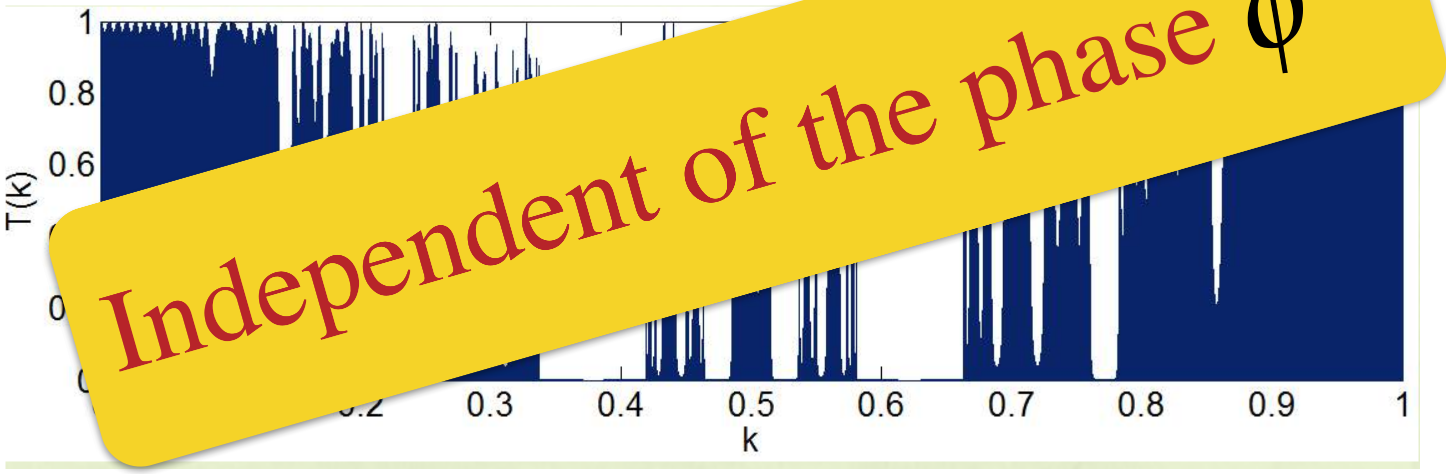


Density of modes

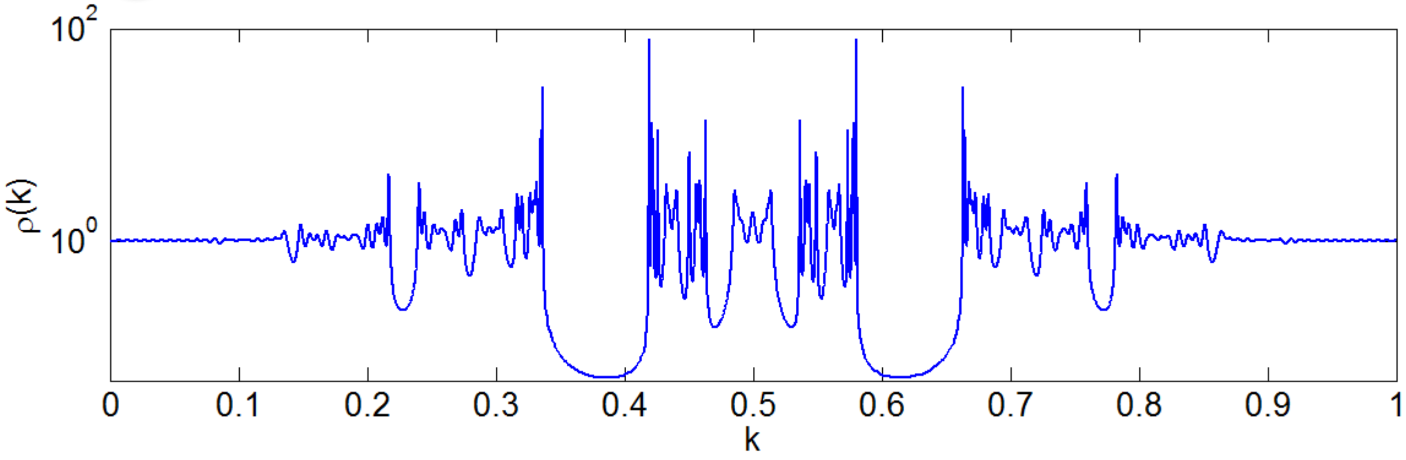


Spectral characteristics are independent of the phase $\Delta\phi$

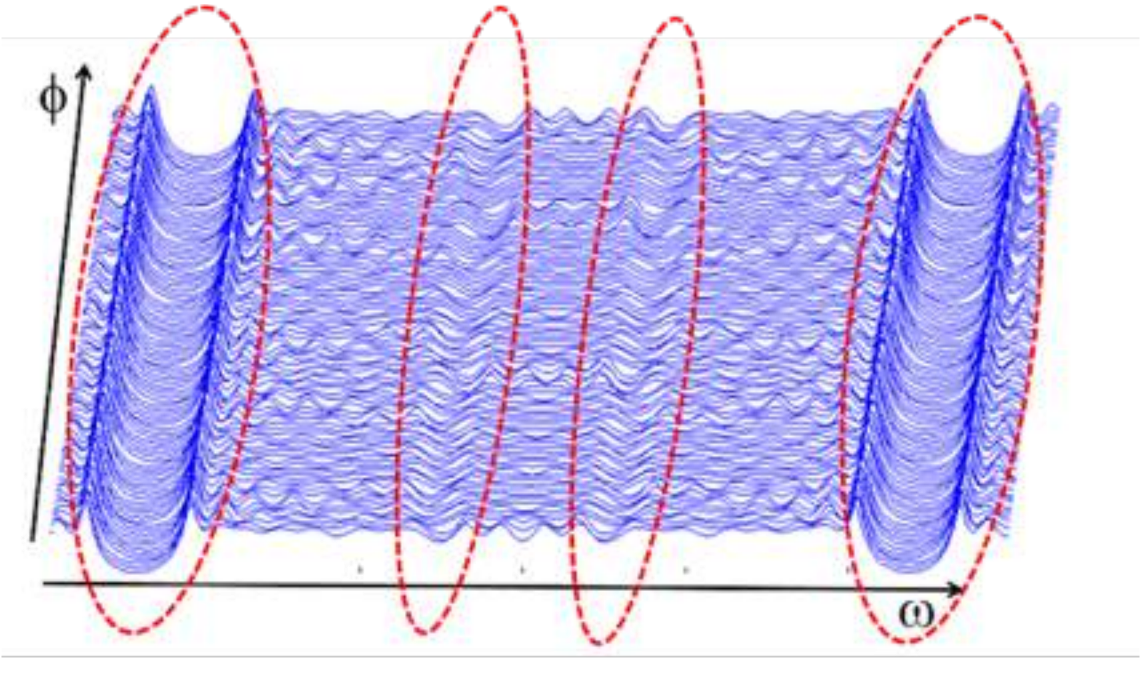
Independent of the phase ϕ

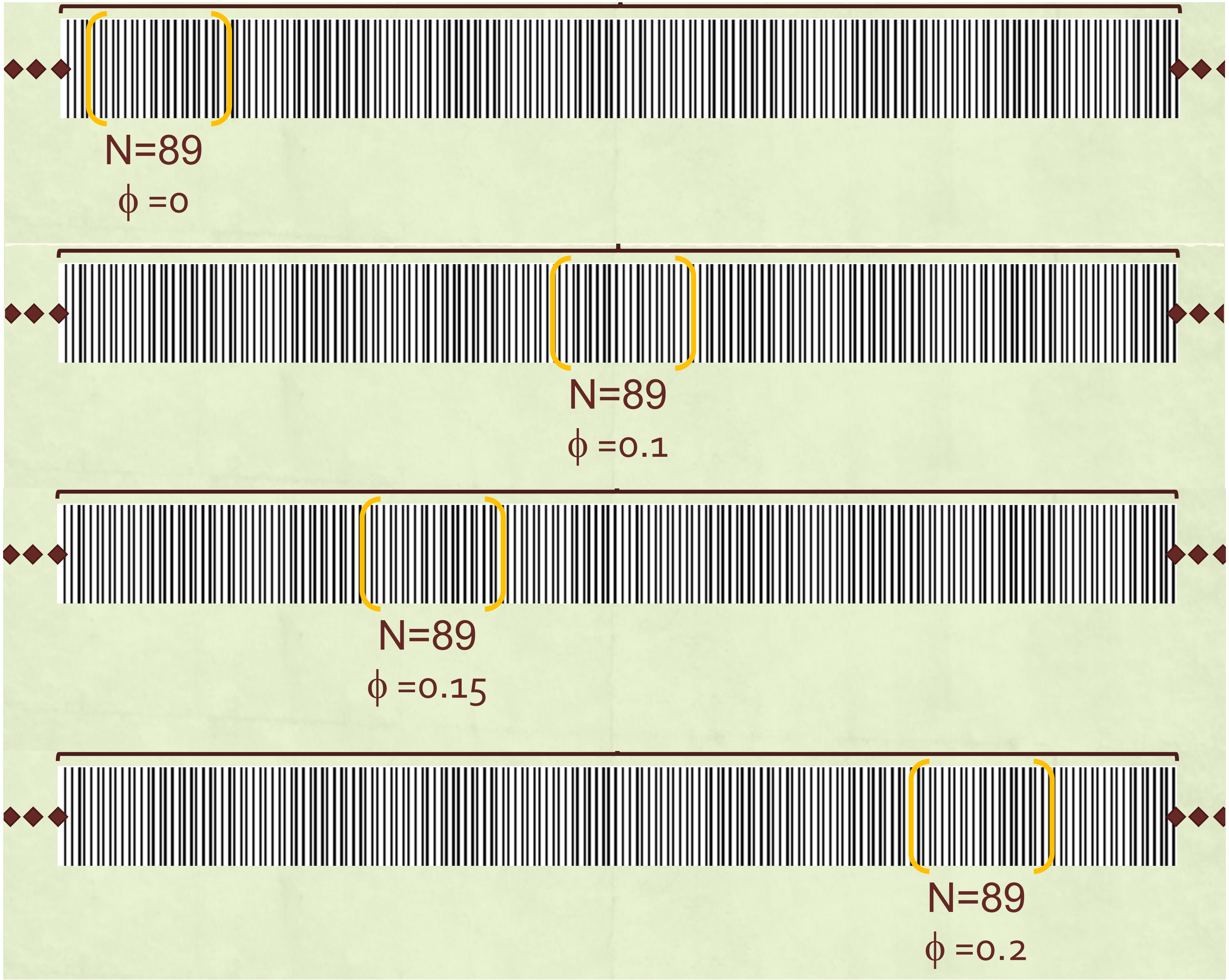


Transmission spectrum



Density of modes



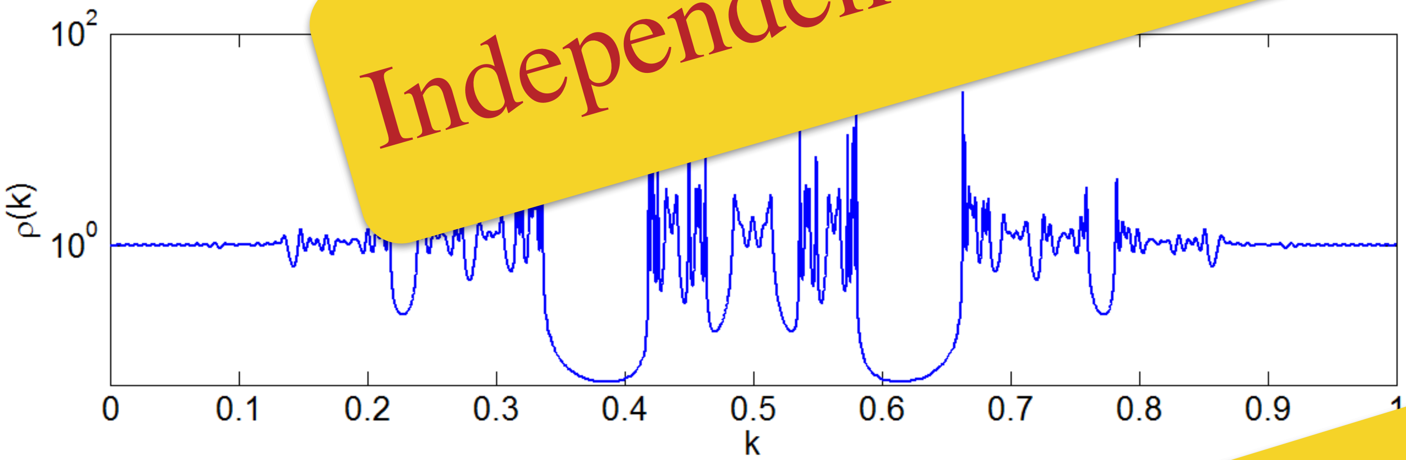


Each sequence has exactly the same spectrum.

Spectral characteristics are independent of the phase ϕ

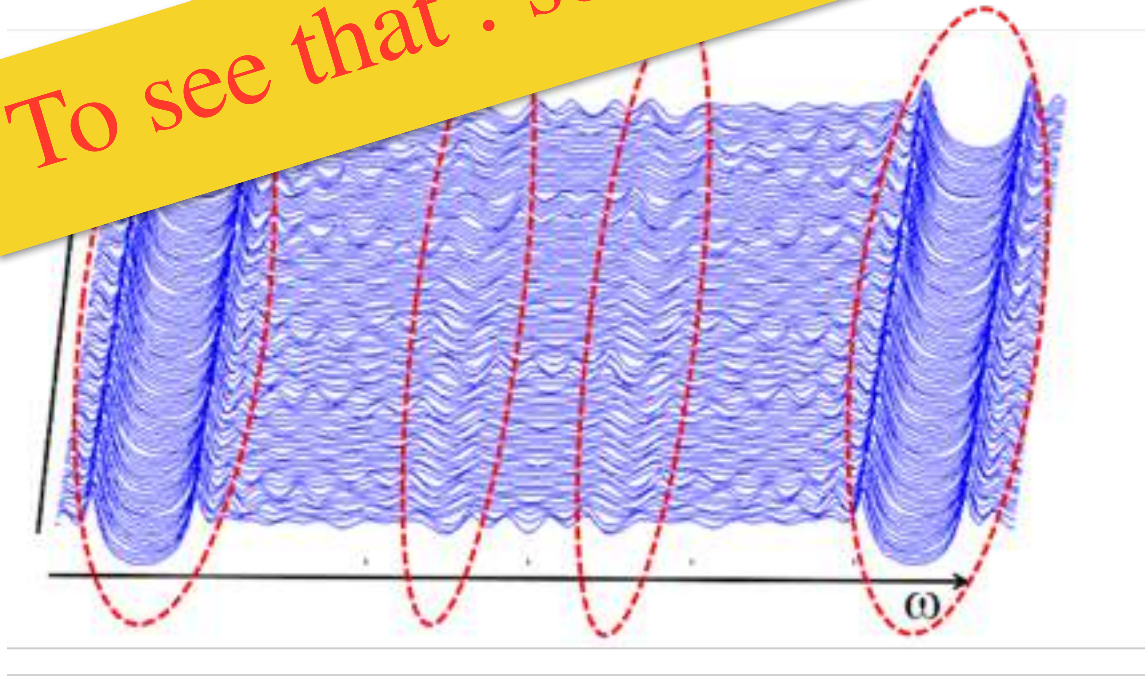
smission spectrum

Independent of the phase ϕ

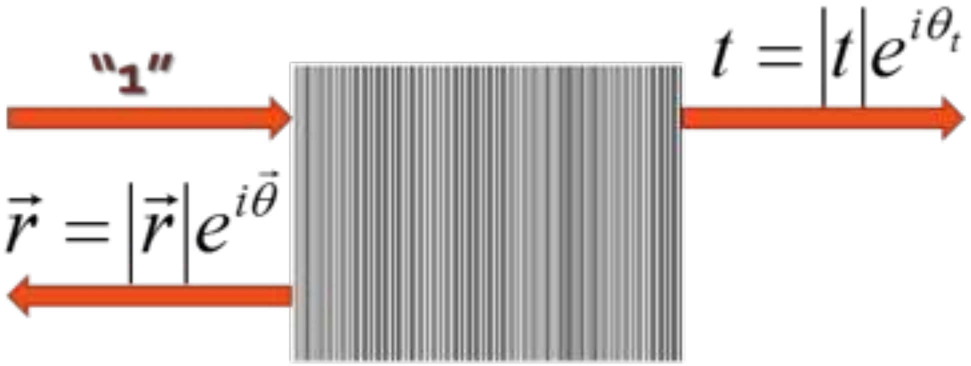


Density of modes

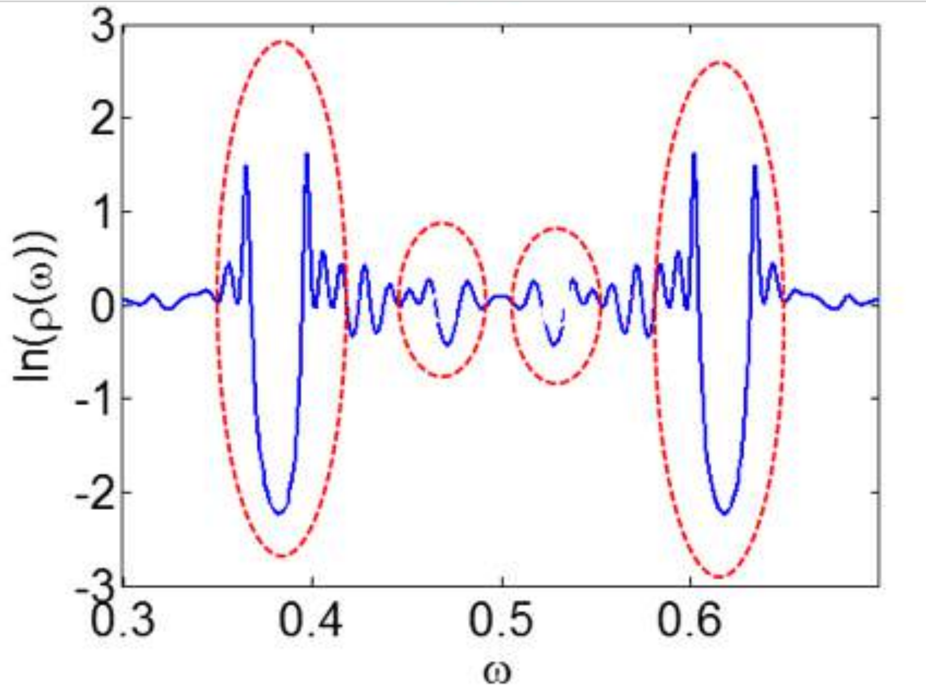
To see that : scattering formalism

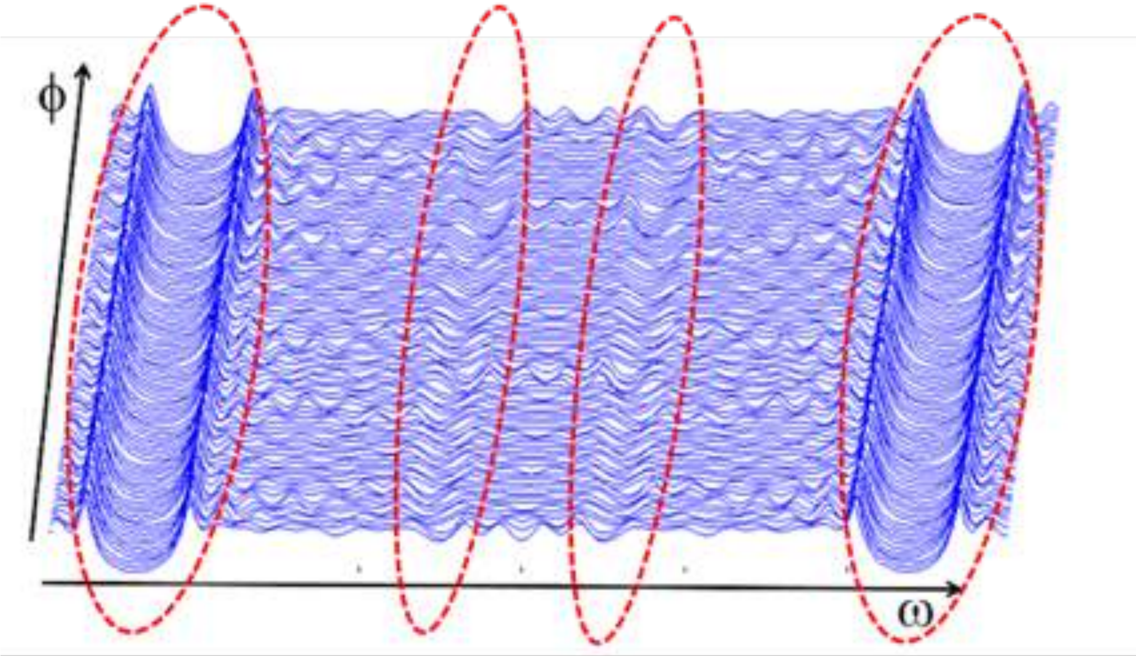


Fibonacci chain embedded between free spaces:



$$\rho(k)-\rho_0(k)=\frac{1}{\pi}\frac{d\delta(k)}{dk}$$
$$=\frac{1}{2\pi}\,\mathrm{Im}\,\frac{\partial}{\partial k}\ln\det S(k)$$



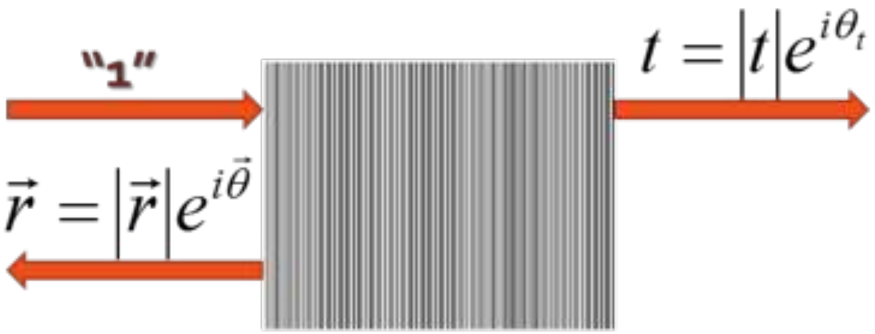


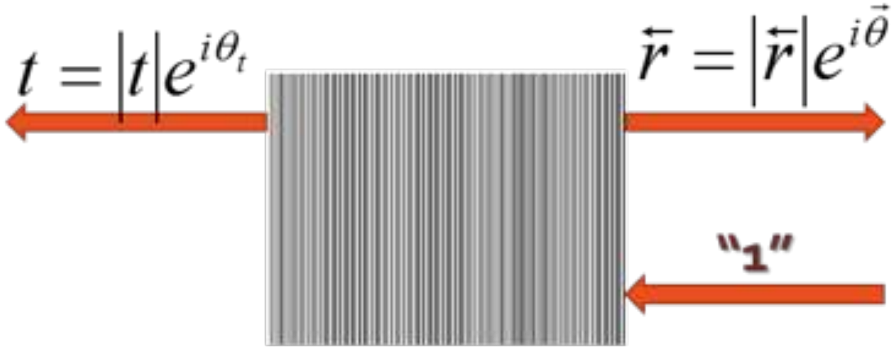
$\delta(k)=\theta_t(k)+\frac{\pi}{2}$ is independent of the modulation phase ϕ

But…since the phase $\Delta\phi$ measures the deviation from palindromicity, it should show up in the difference between the 2 scattering configurations,

$$\overleftarrow{r} = \overrightarrow{r}\, e^{i\alpha}$$

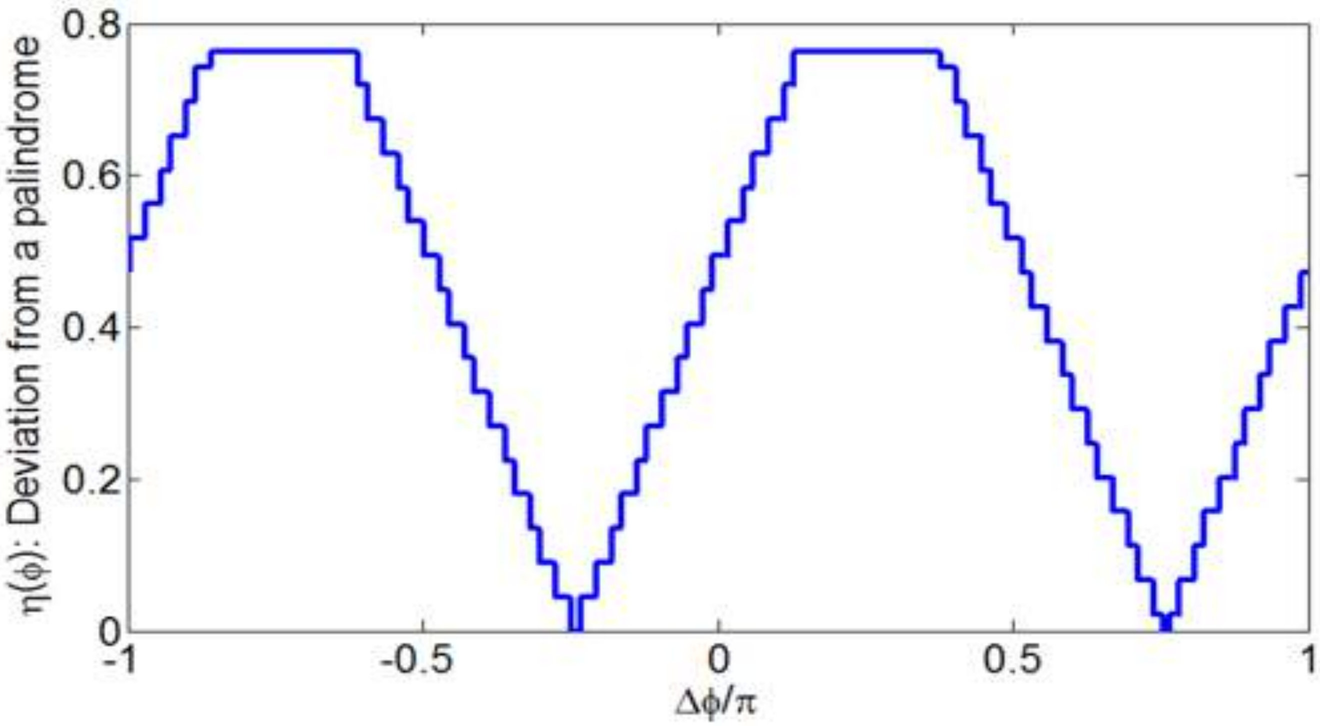
Since $\Delta\phi$ measures the deviation from a palindrome, it should show up in the difference between the 2 scattering configurations,





$$\overleftarrow{r} = \overrightarrow{r}\, e^{i\alpha}$$

The chiral angle α depends on the structural phase $\Delta\phi$



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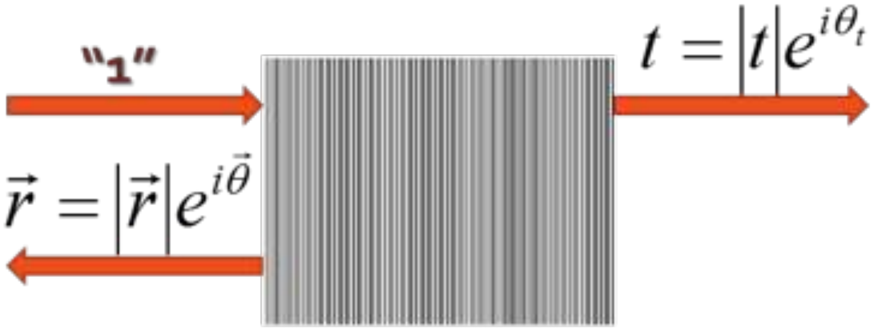

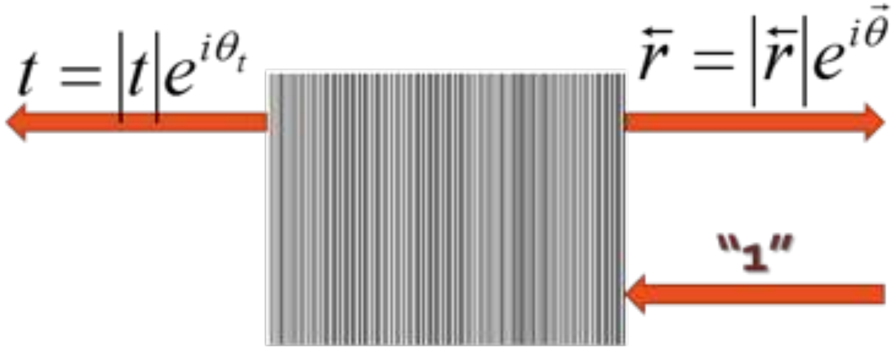

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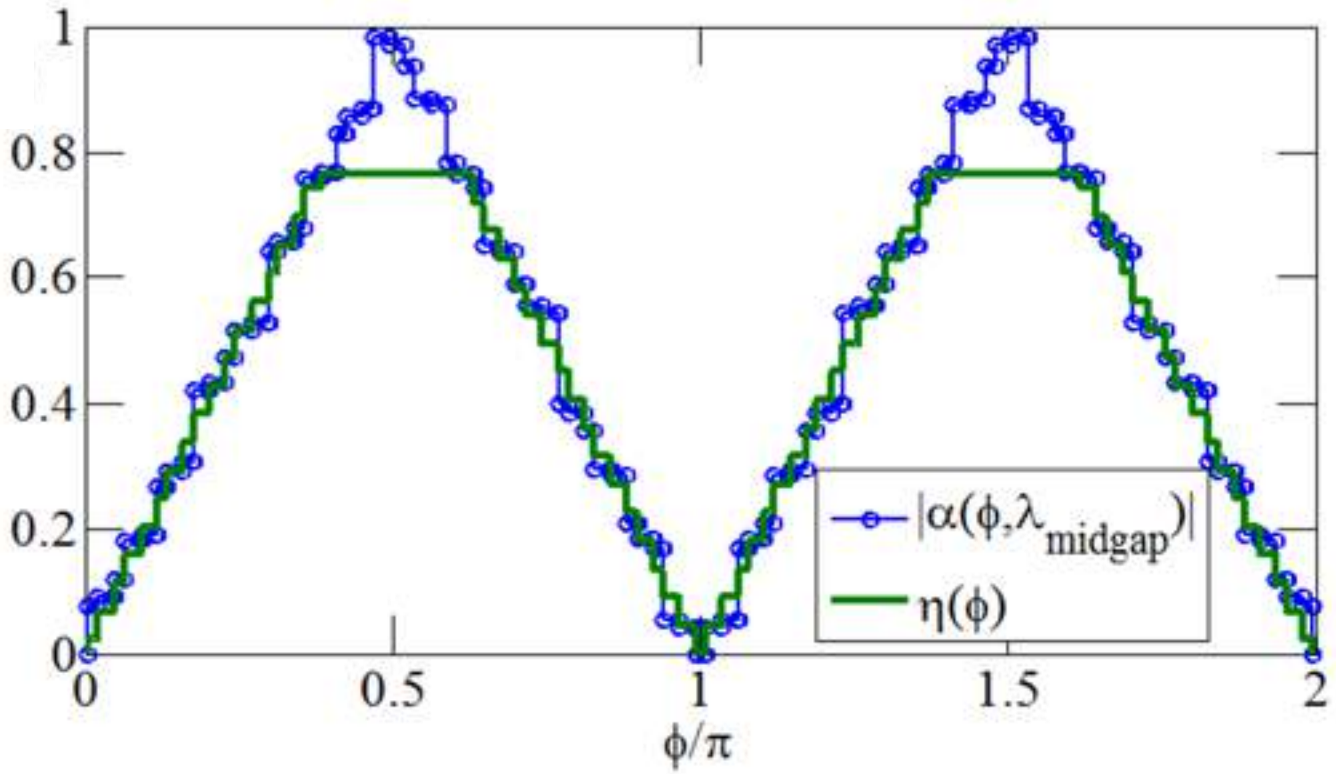


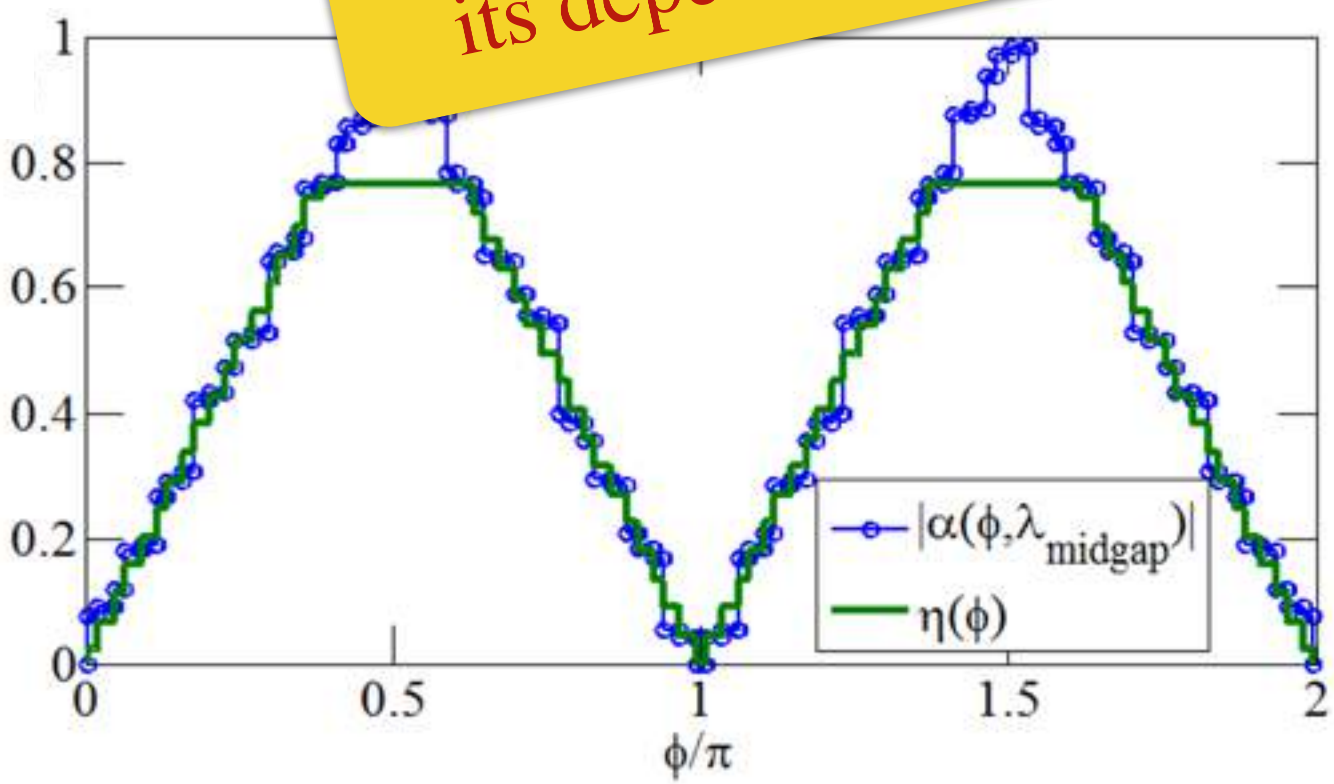
Since $\Delta\phi$ measures the deviation from a palindrome, it should show up in the difference between the 2 scattering configurations,

How to observe the chiral angle α,

its dependence on the structural phase $\Delta\phi$

The chi

al phase $\Delta\phi$



How to create edge states and relate them to the scattering formalism ?

Since $\Delta\phi$ measures the deviation from a palindrome, it should show up in the difference between the 2 scattering configurations,

The chi

phase $\Delta\phi$

How to observe the chiral angle α,

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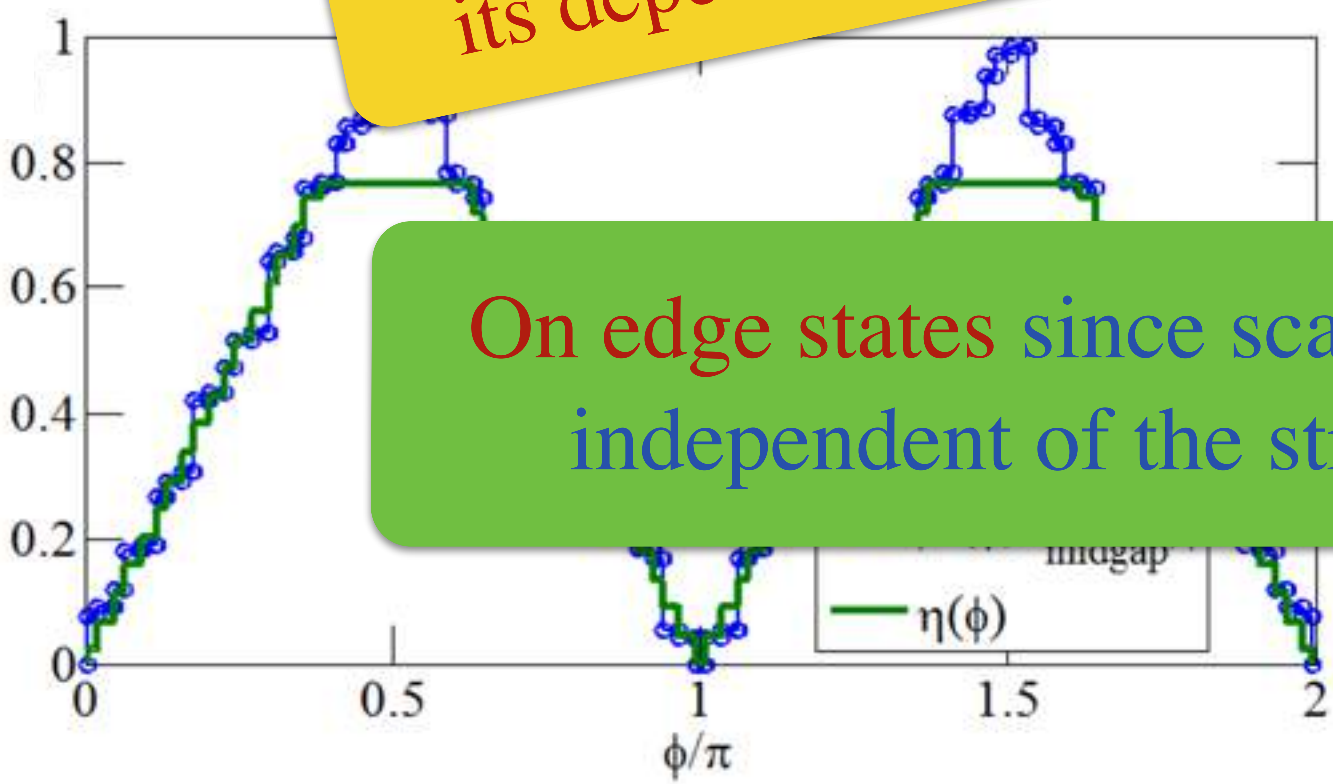
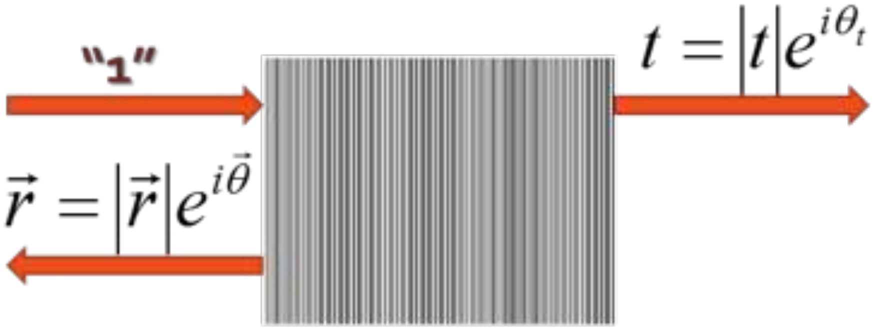


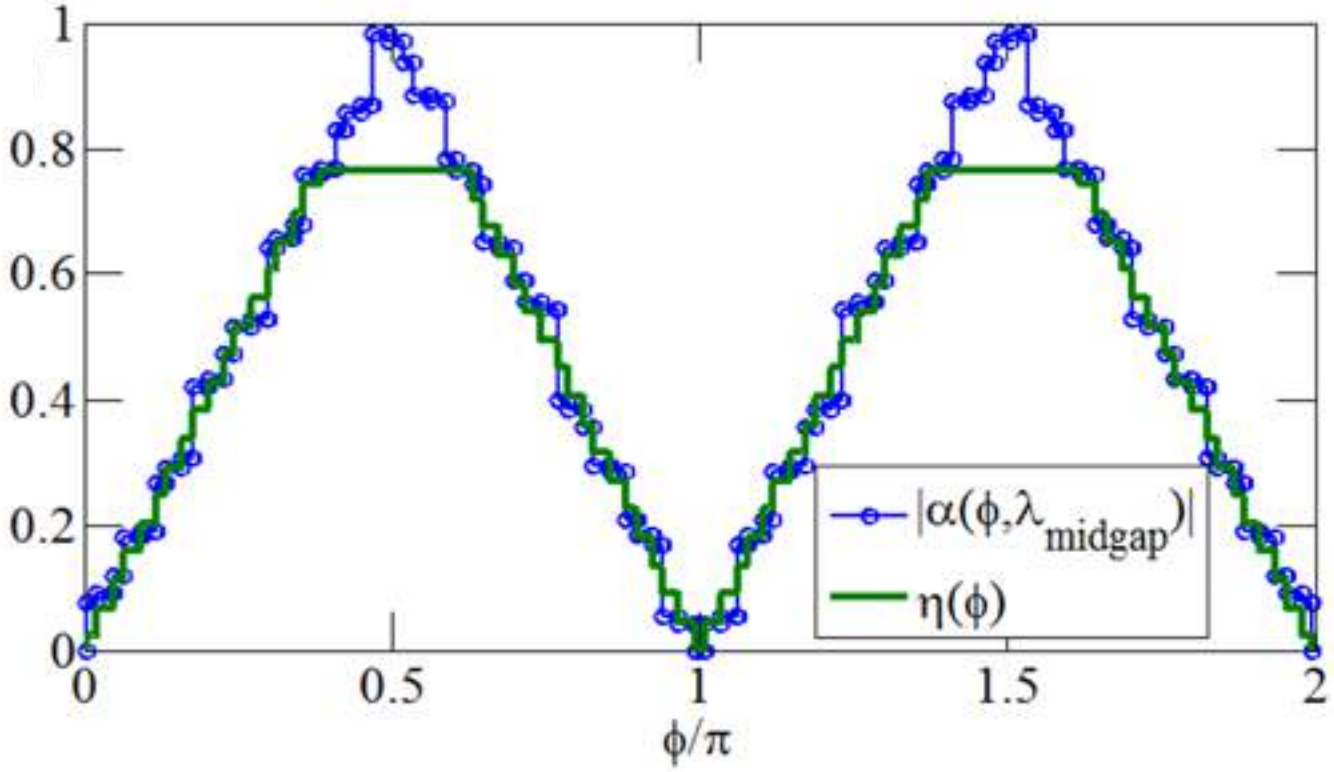
On edge states since scattering states are independent of the structural phase $\Delta\phi$

Relation between Chern numbers and the scattering matrix

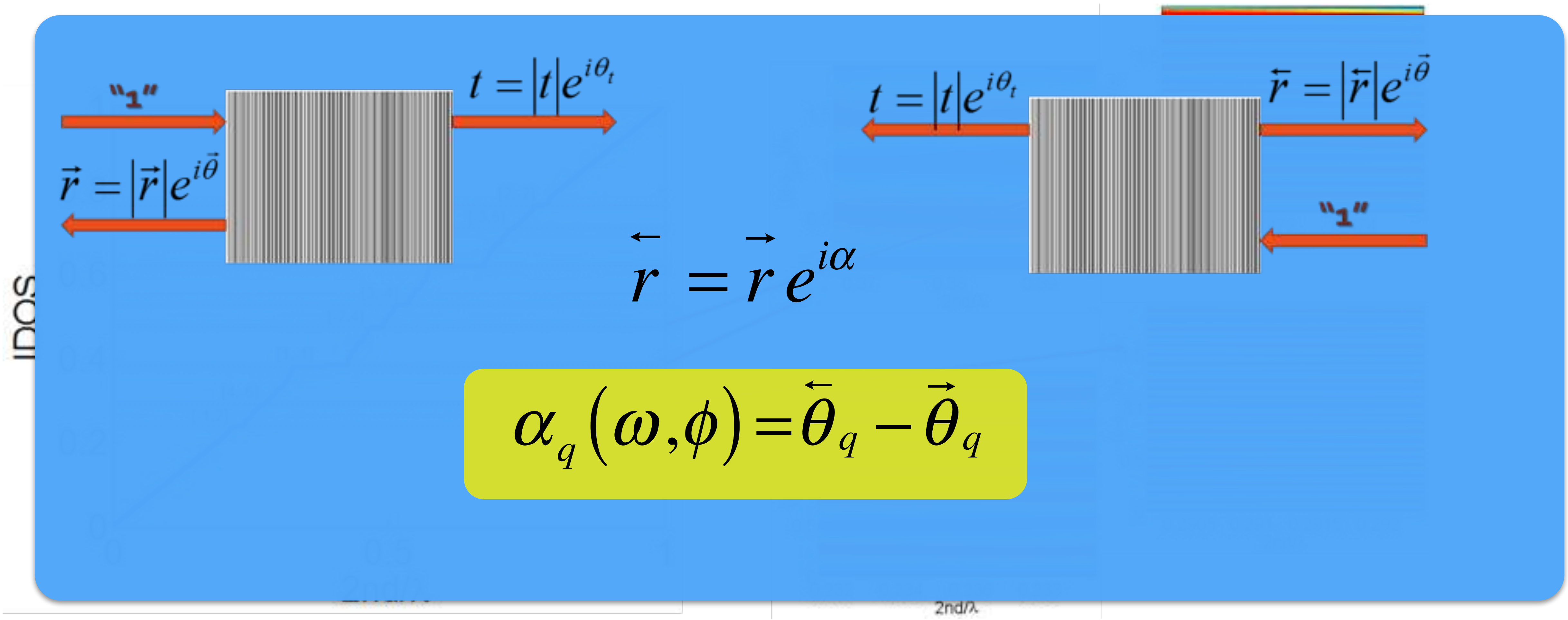


$$\overleftarrow{r} = \overrightarrow{r} e^{i\alpha}$$

$$\alpha_q(\omega,\phi) = \overleftarrow{\theta}_q - \overrightarrow{\theta}_q$$



Winding number of $\alpha_q(\omega,\phi)$



$$\overleftarrow{r}=\overrightarrow{r}\,e^{i\alpha}$$

$$\boxed{\alpha_q(\omega,\phi)=\overleftarrow{\theta}_q-\overrightarrow{\theta}_q}$$

The Chern number q is the winding number of the chiral phase $\alpha_q(\omega,\phi)$:

$$W(\alpha_q)\equiv\frac{1}{4\pi}\int_0^{2\pi}d\phi\frac{d\alpha_q(\omega,\phi)}{d\phi}=q$$

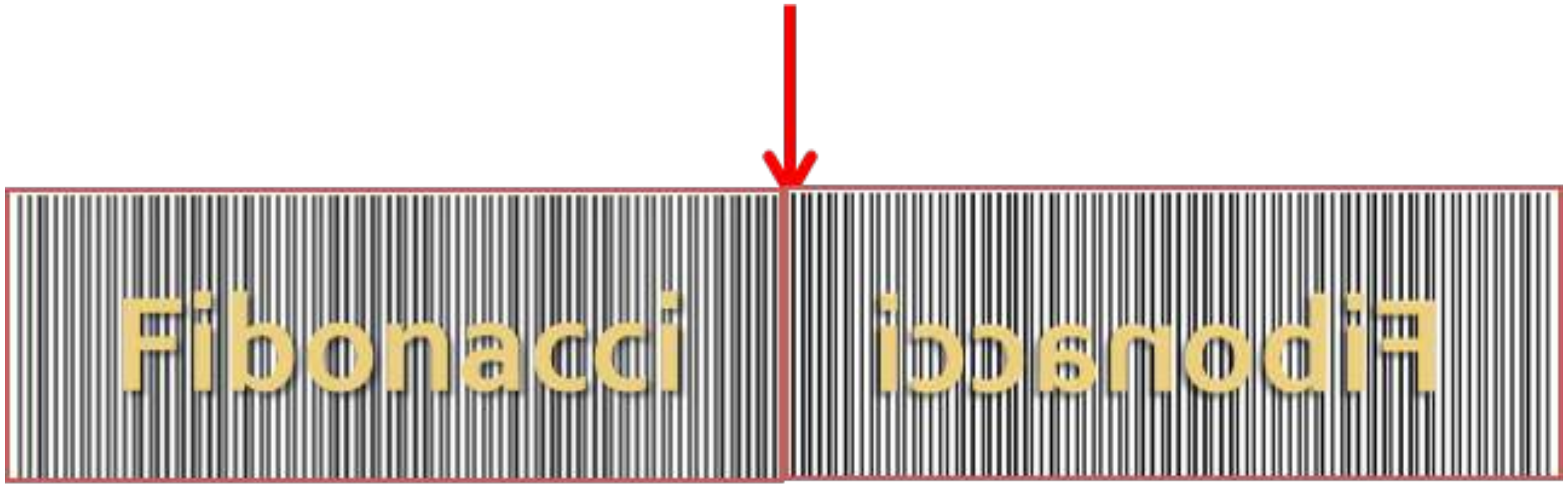
How to create edge states and relate them to the scattering formalism ?

Impose a closed boundary

How to create edge states and relate them to the scattering formalism ?

Impose a closed boundary

Equivalent to

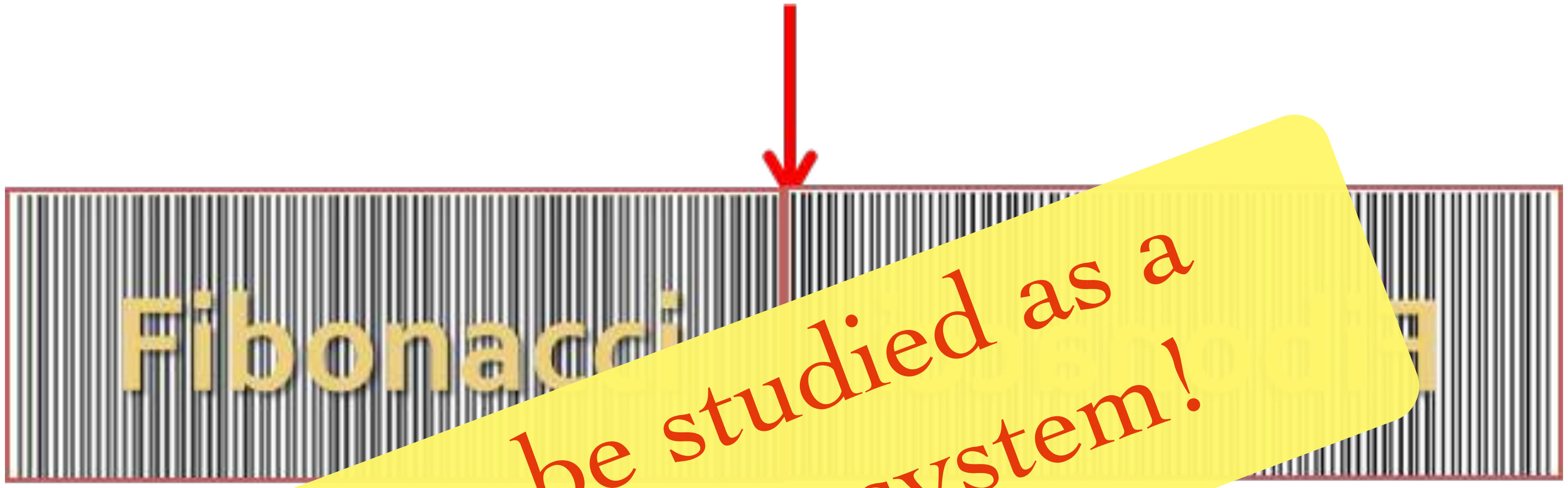


$$\overrightarrow{F}_N \overleftarrow{F}_N$$

How to create edge states and relate them to the scattering formalism ?

Impose a closed boundary

Equivalent to

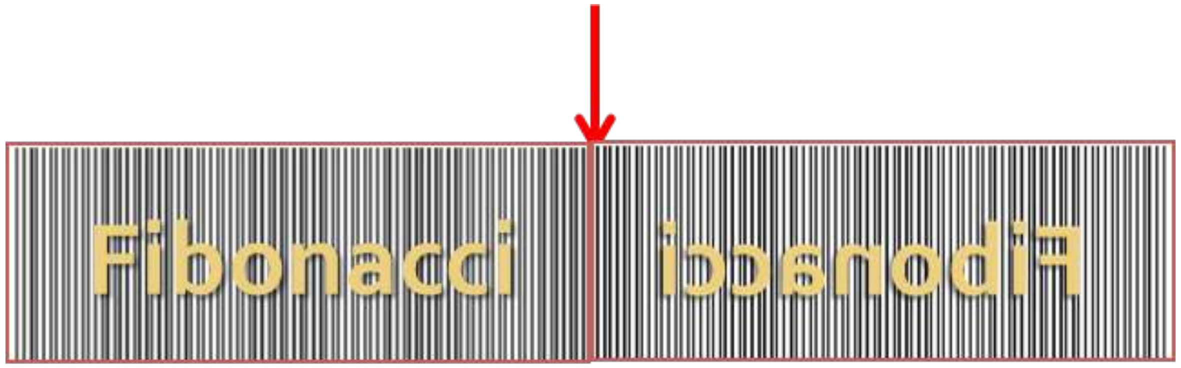


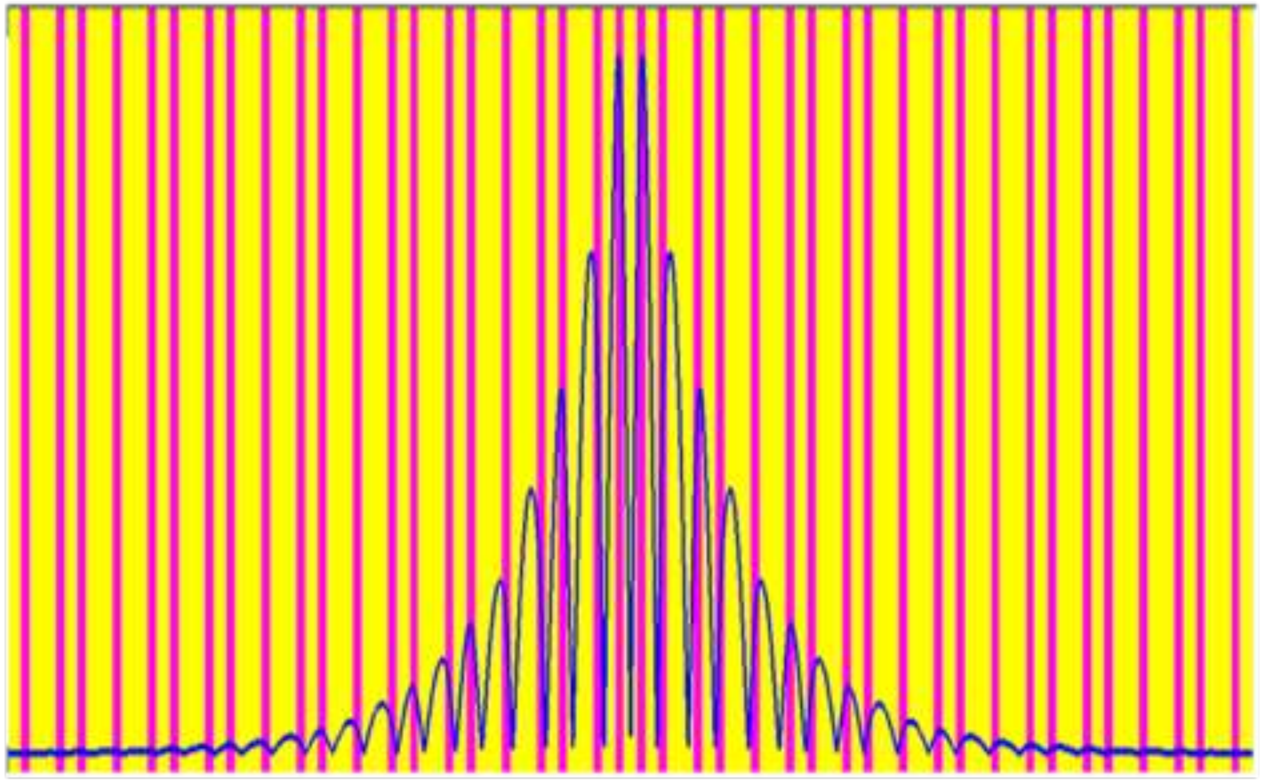
can be studied as a scattering system !

$\overrightarrow{F}_N \overline{F}_N$

Edge states of $\overrightarrow{F}_N \overleftarrow{F}_N$

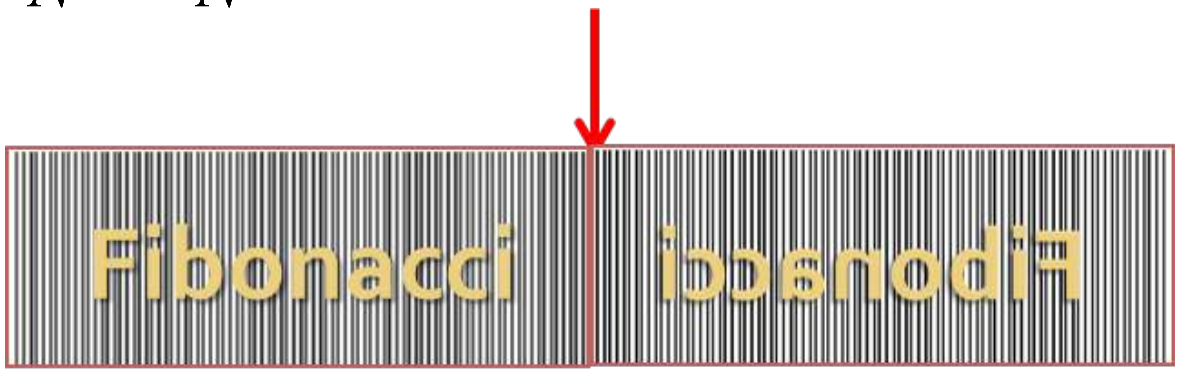
- This structure isn't Fibonacci : it displays additional modes in the gaps



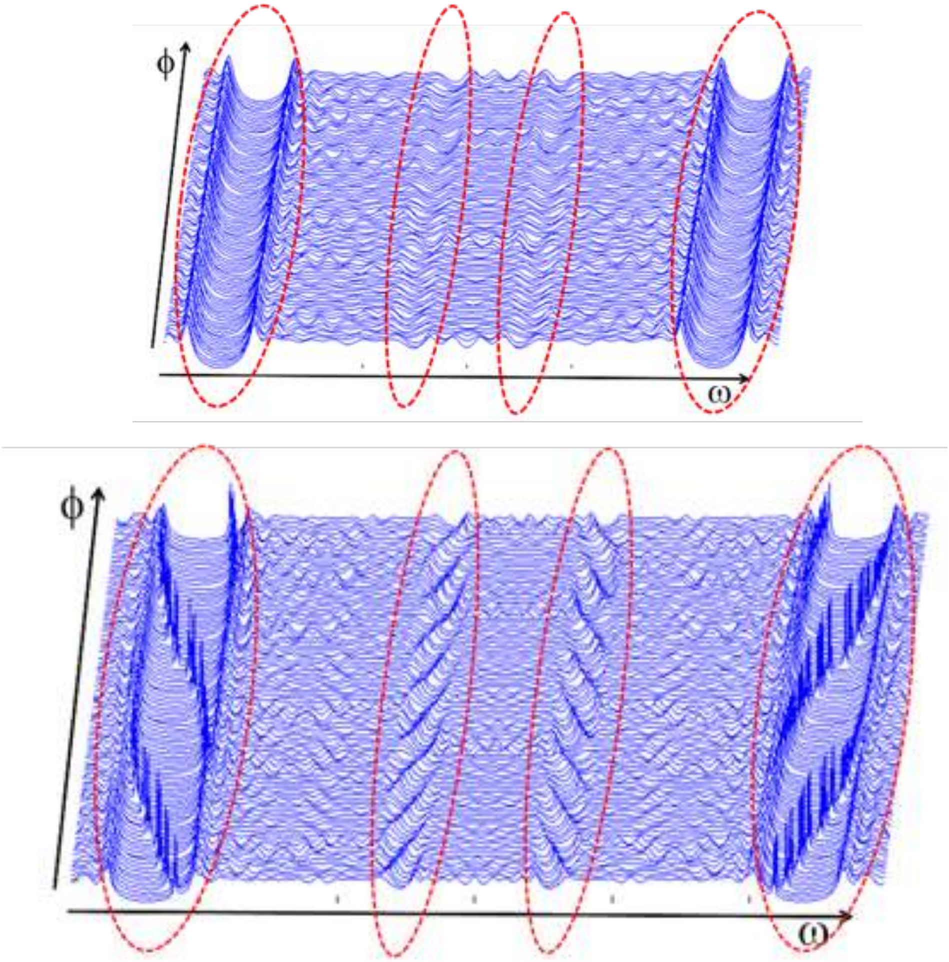


Spatial structure

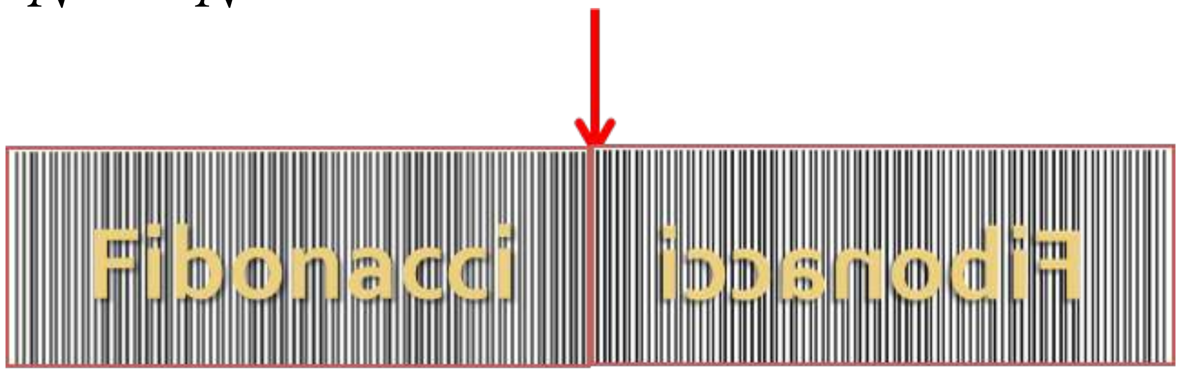
Edge states of $\overrightarrow{F}_N \overleftarrow{F}_N$



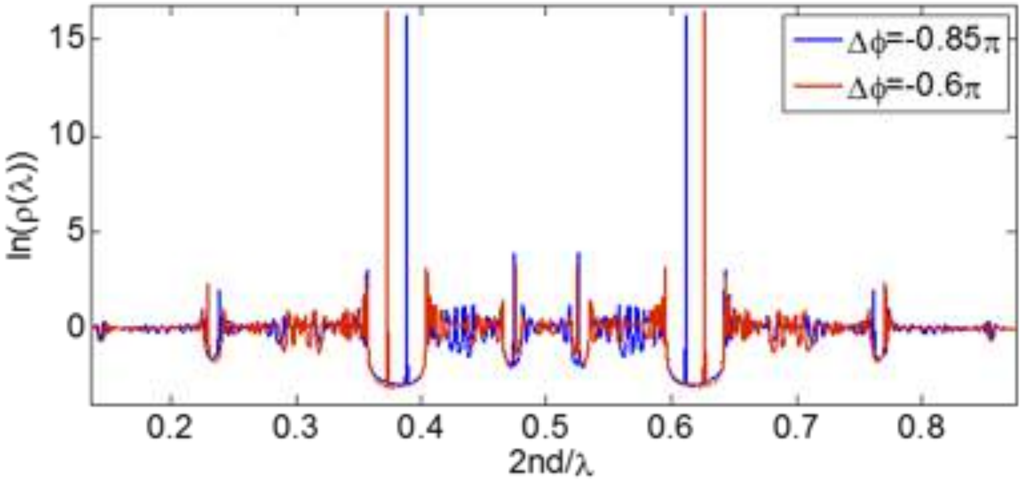
- This structure isn't Fibonacci : it displays additional modes in the gaps
- Gap locations remain unchanged w.r.t. the original structure

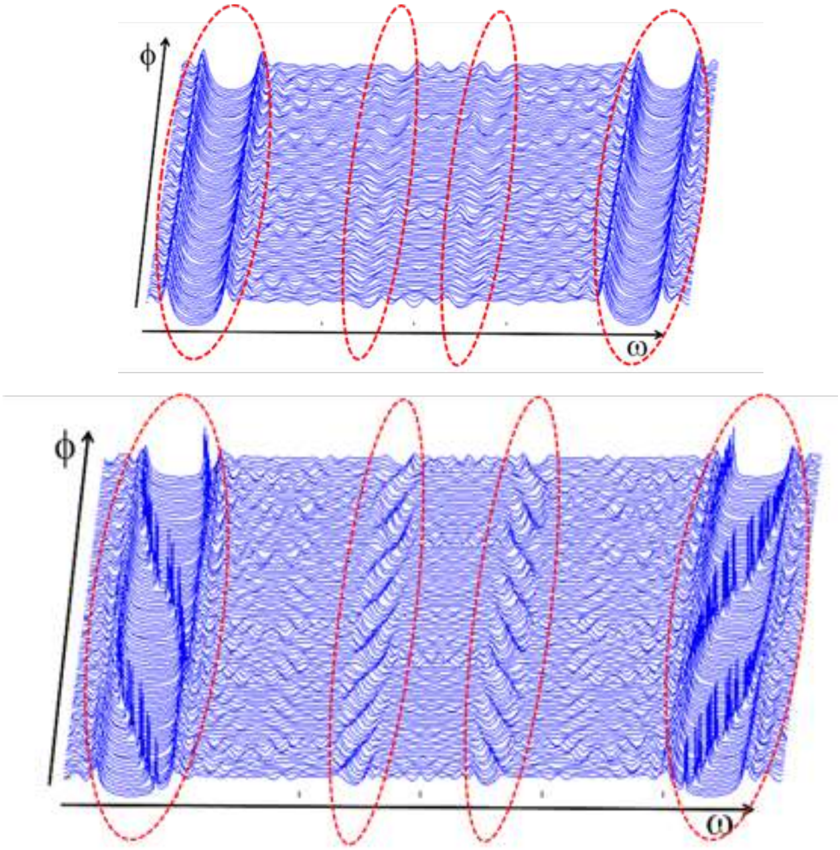


Edge states of $\overrightarrow{F}_N \overleftarrow{F}_N$



- This structure isn't Fibonacci : it displays additional modes in the gaps
- Gap locations remain unchanged w.r.t. the original structure
- Frequencies of the gap modes depend on the structural modulation phase $\Delta\phi$





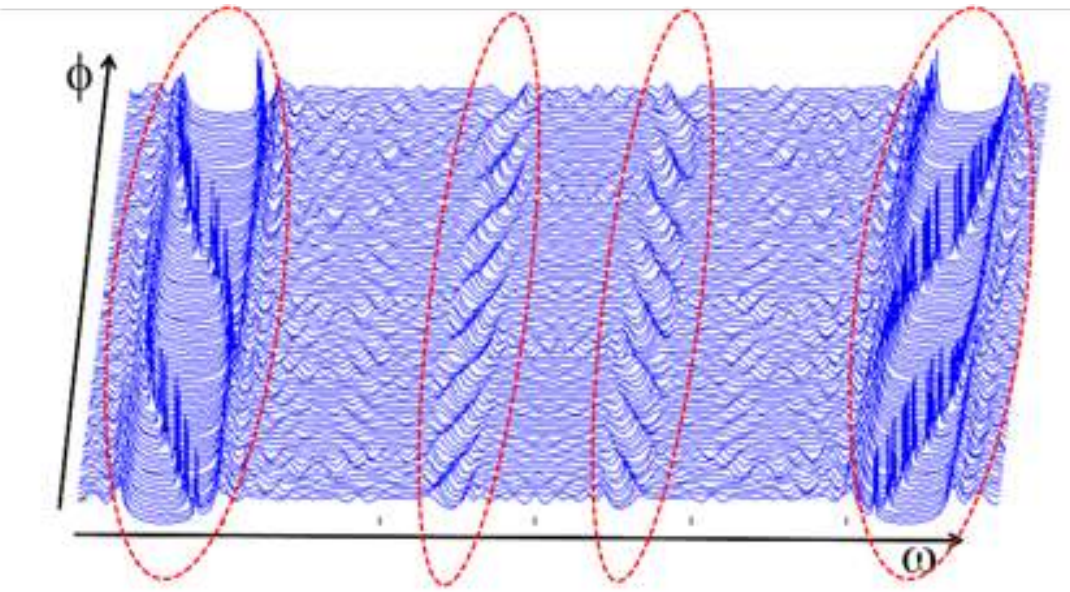
Creating edge modes

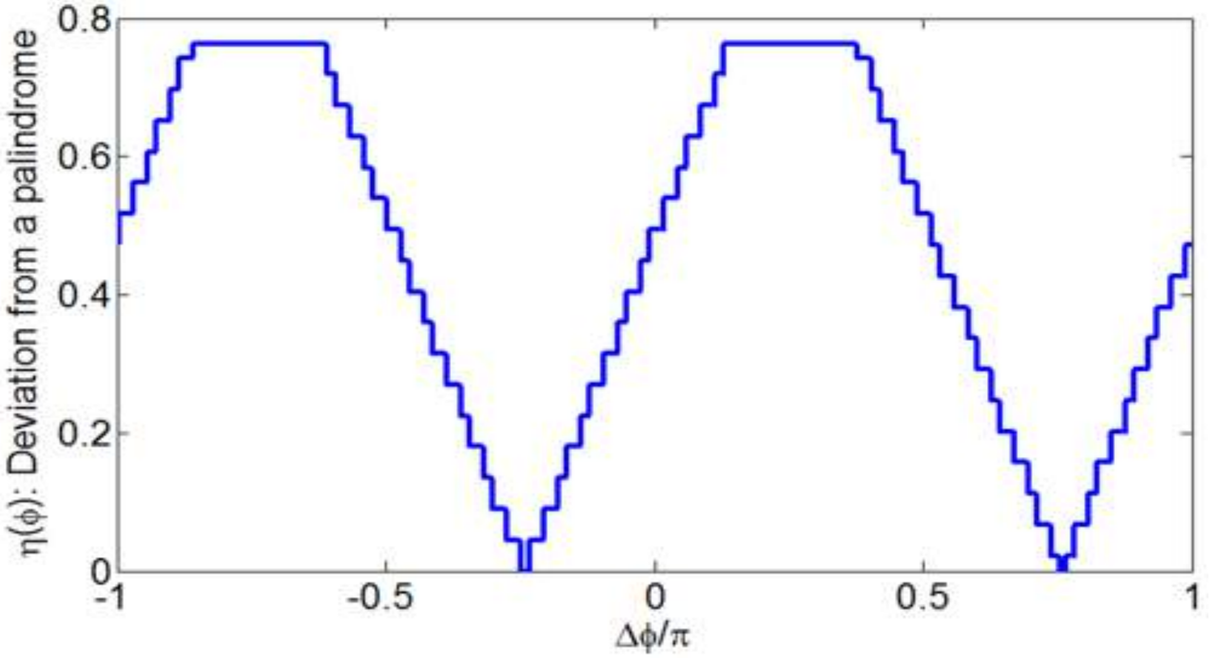
- Edge modes show up in most finite superlattices at gap frequencies which depend on boundary conditions:

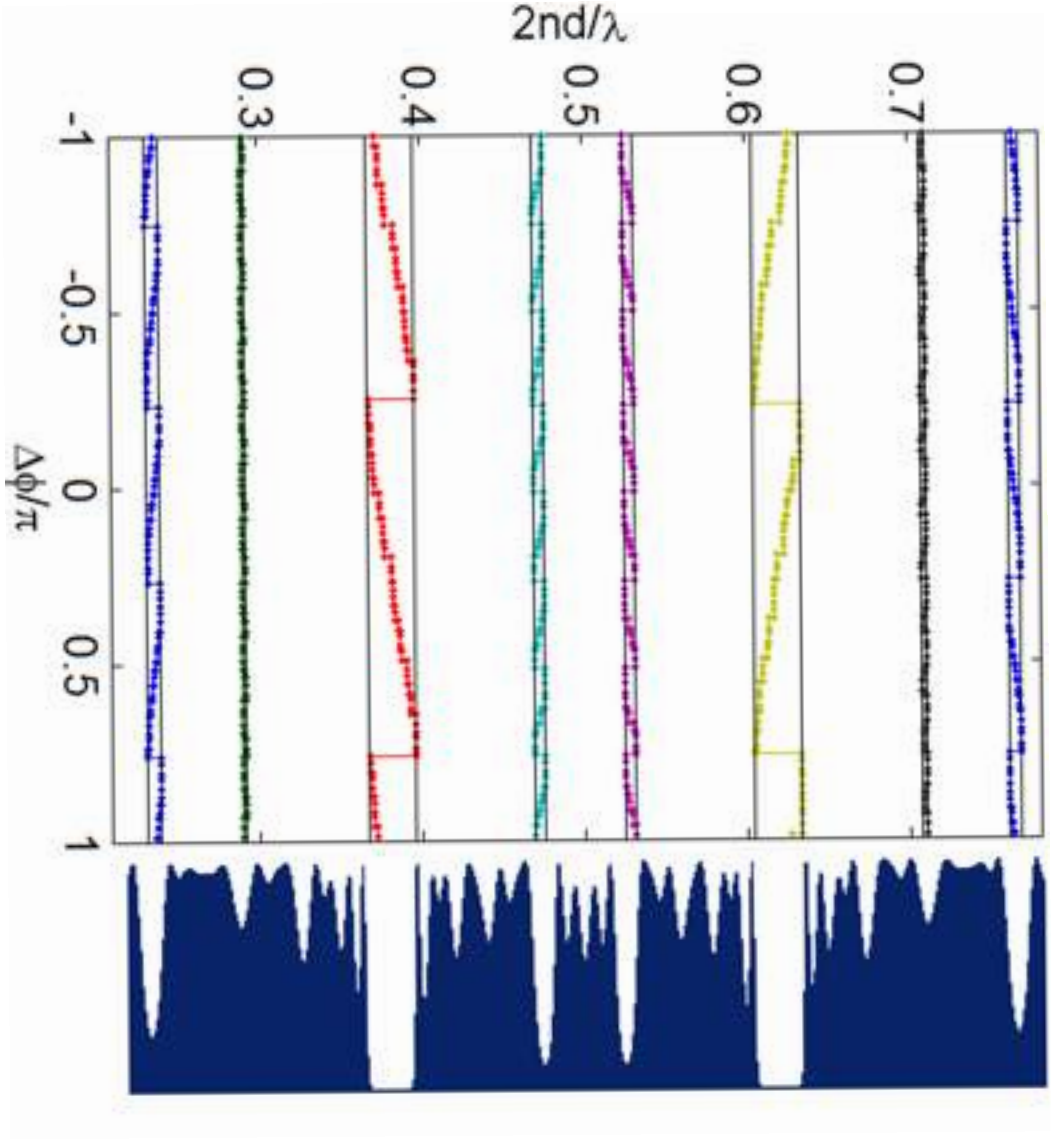
  A. Boundary states due to a "closed" boundary condition (*e.g.*, mirror)
  B. Interface modes
  C. Defect modes

- Edge modes have the same origin and are of topological nature.

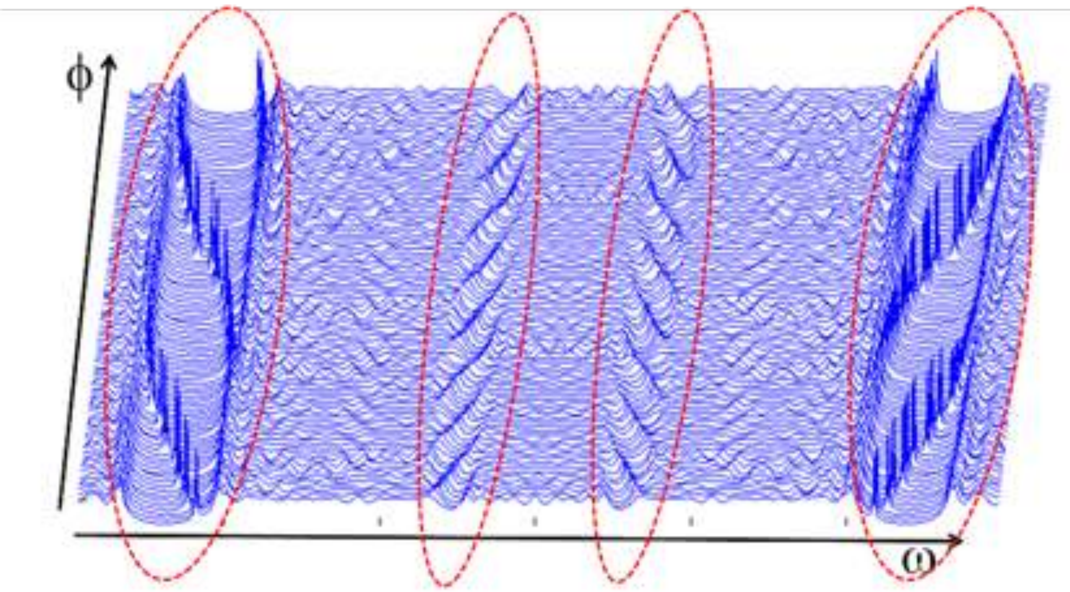
Relation to the palindromic cycle

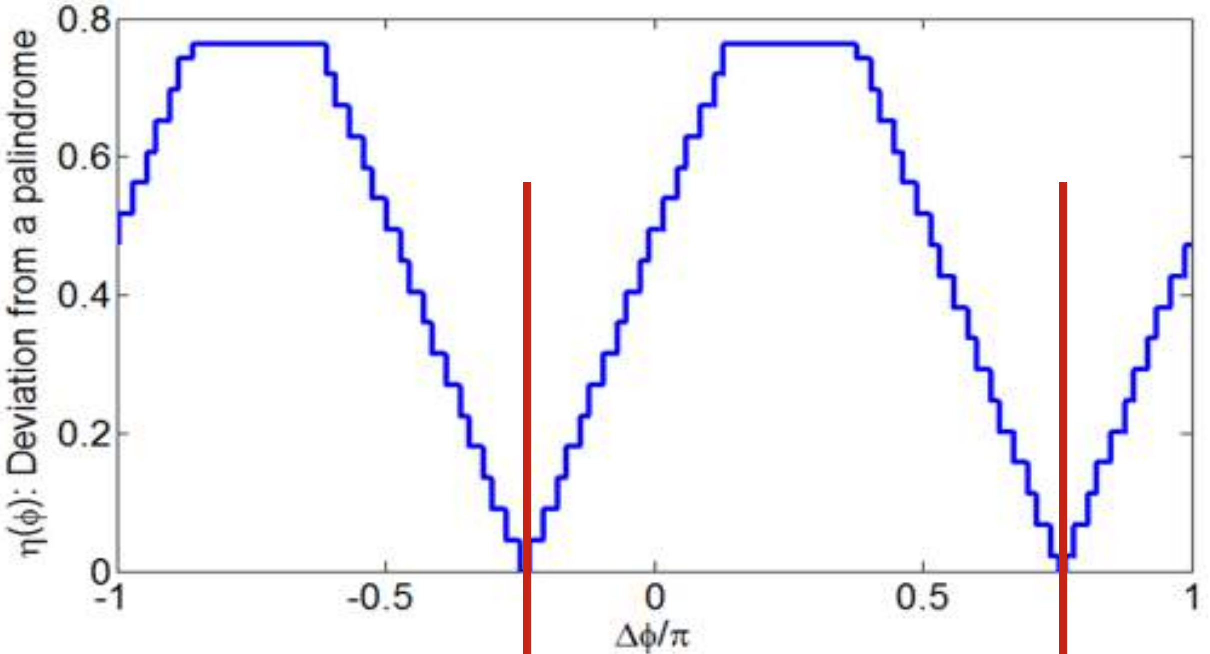


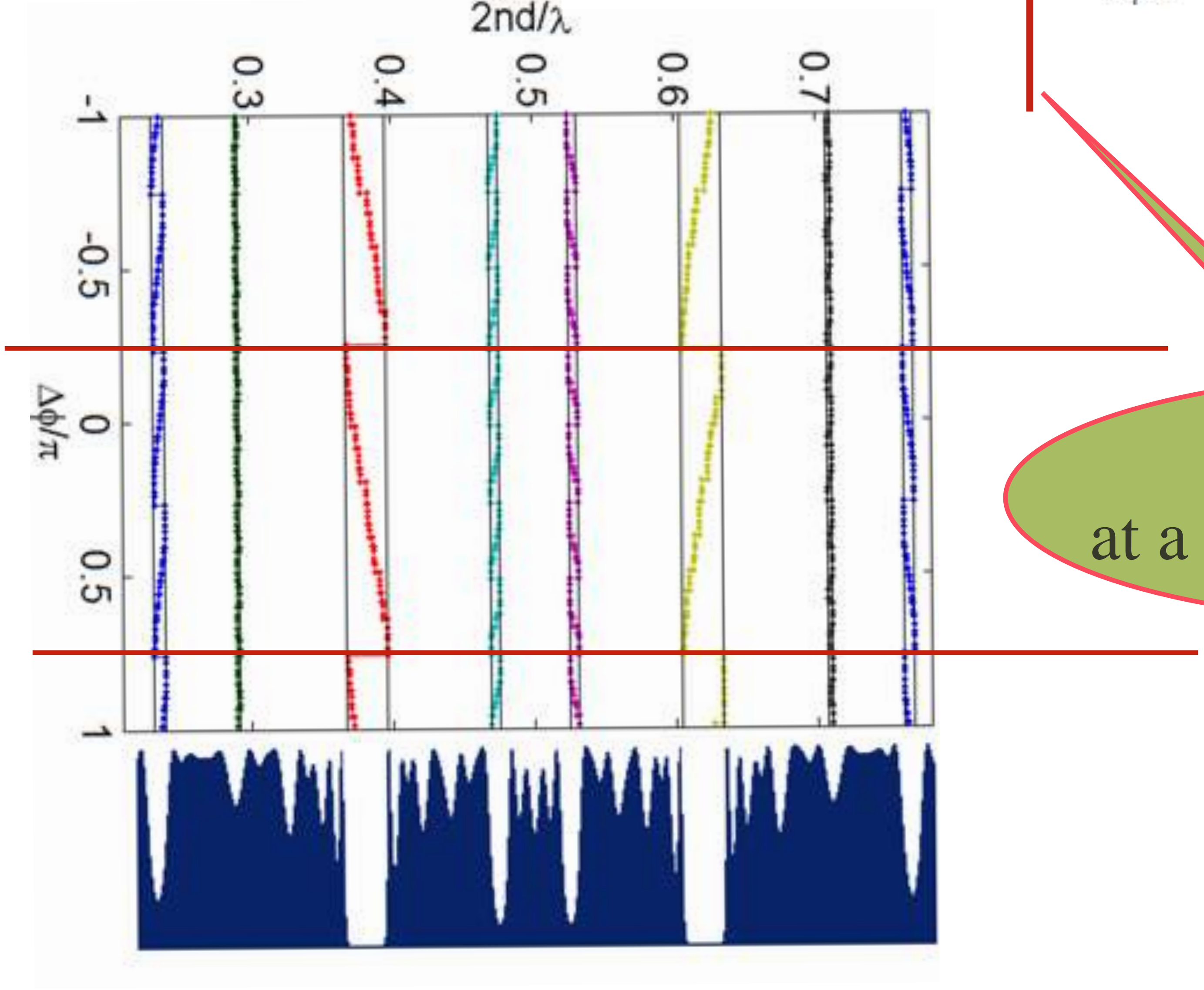




Relation to the palindromic cycle



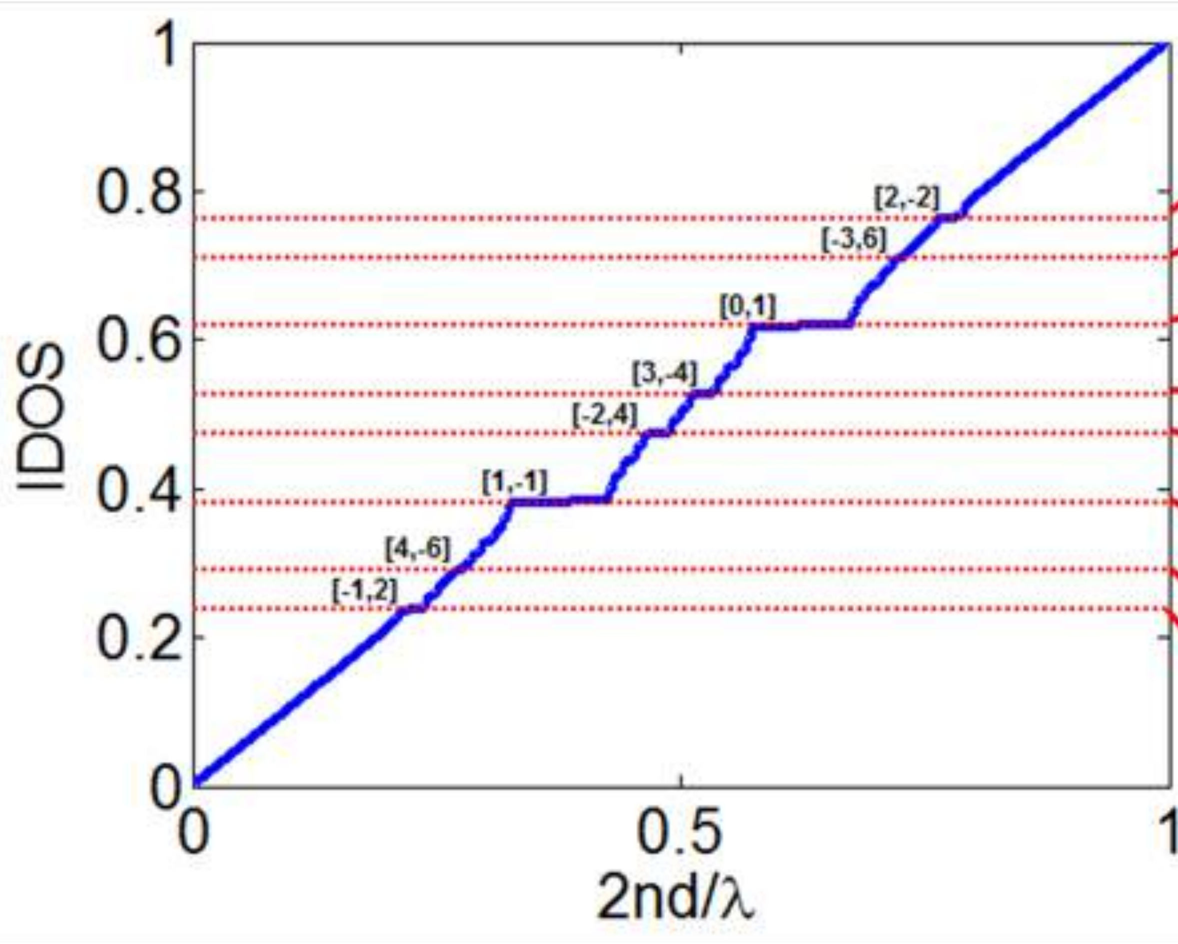


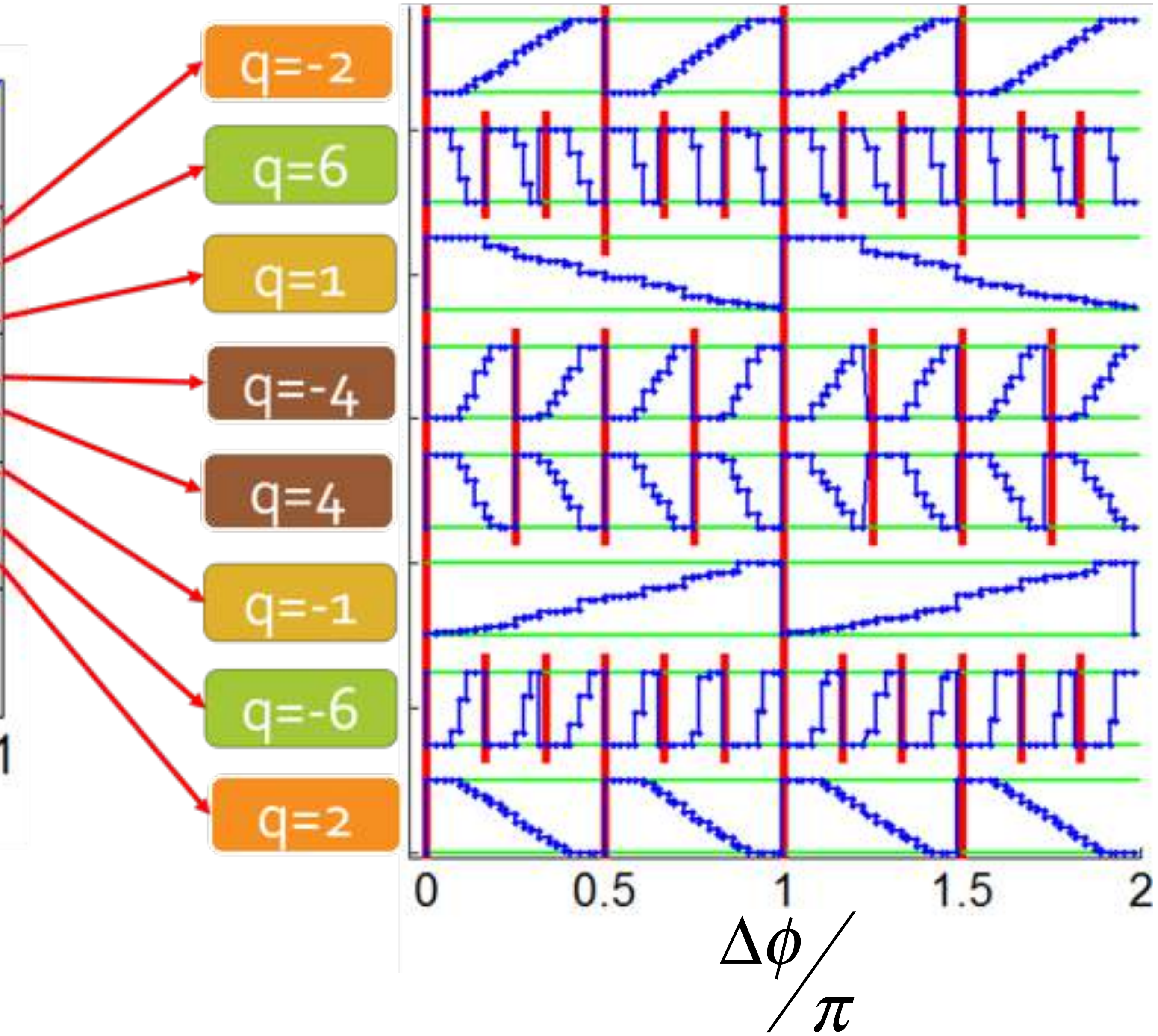


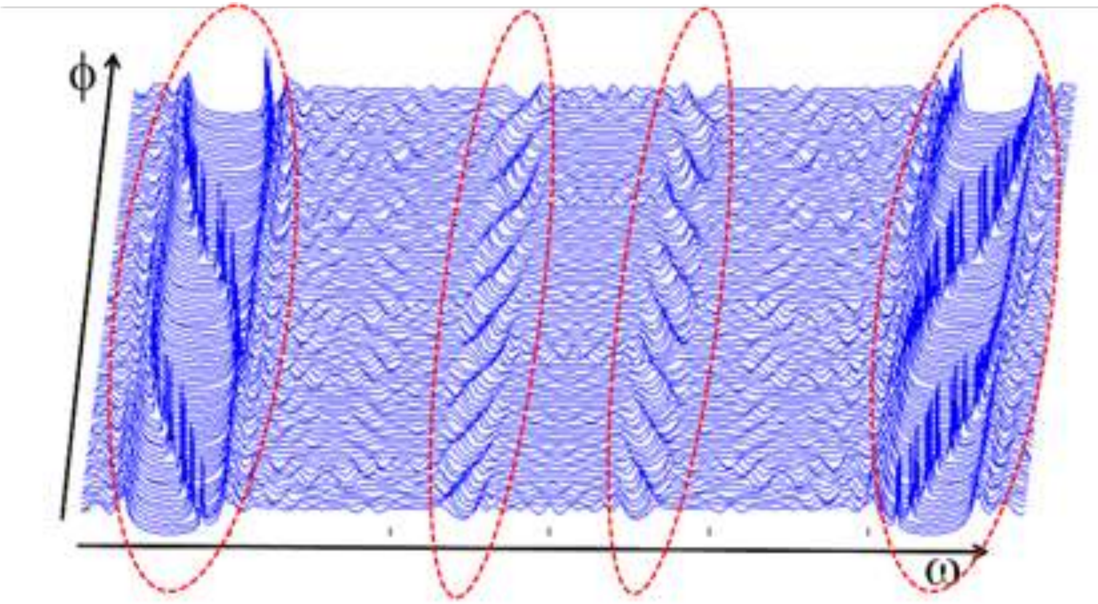
No edge states
at a structural palindrome

for all gaps

Relation to gap labelling and Chern numbers

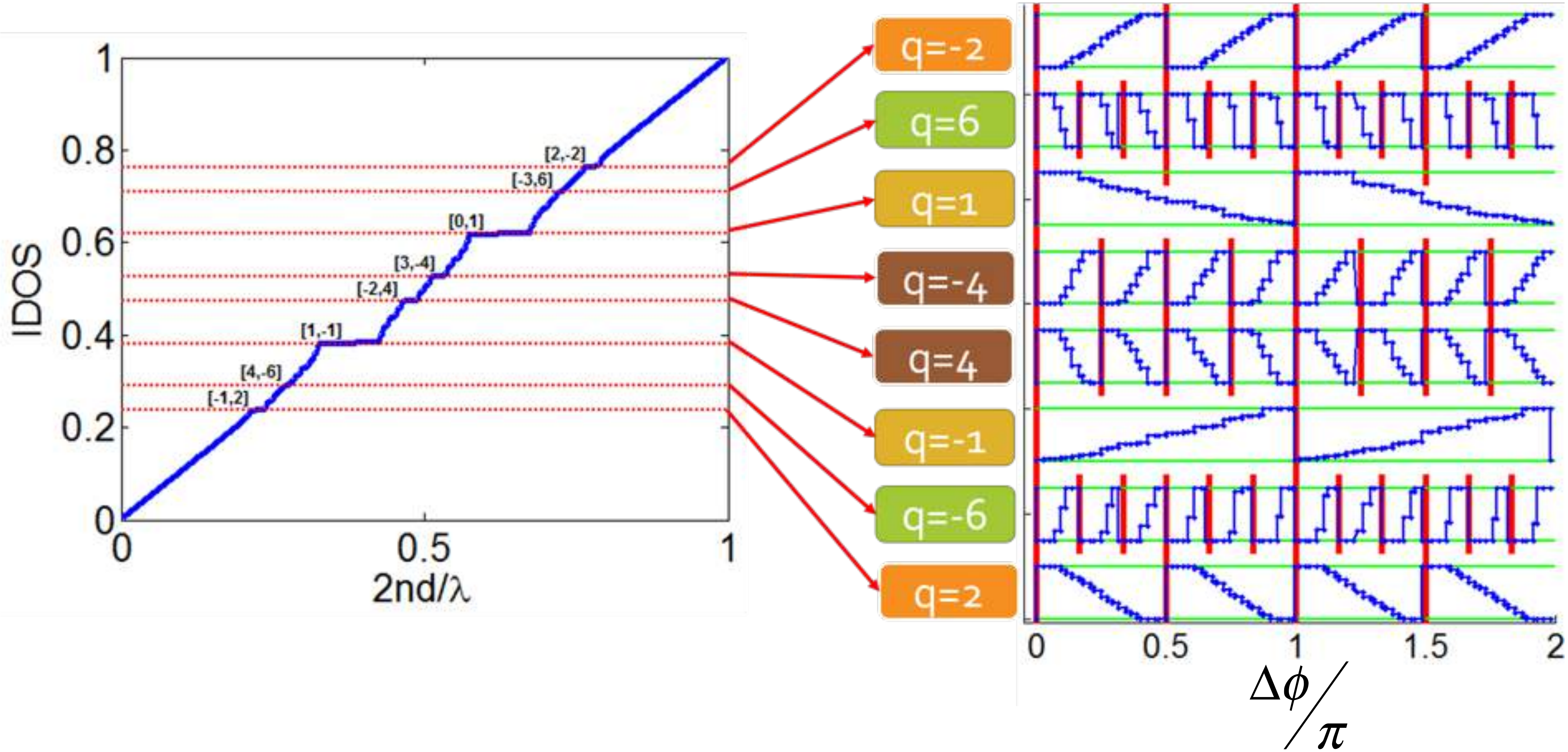






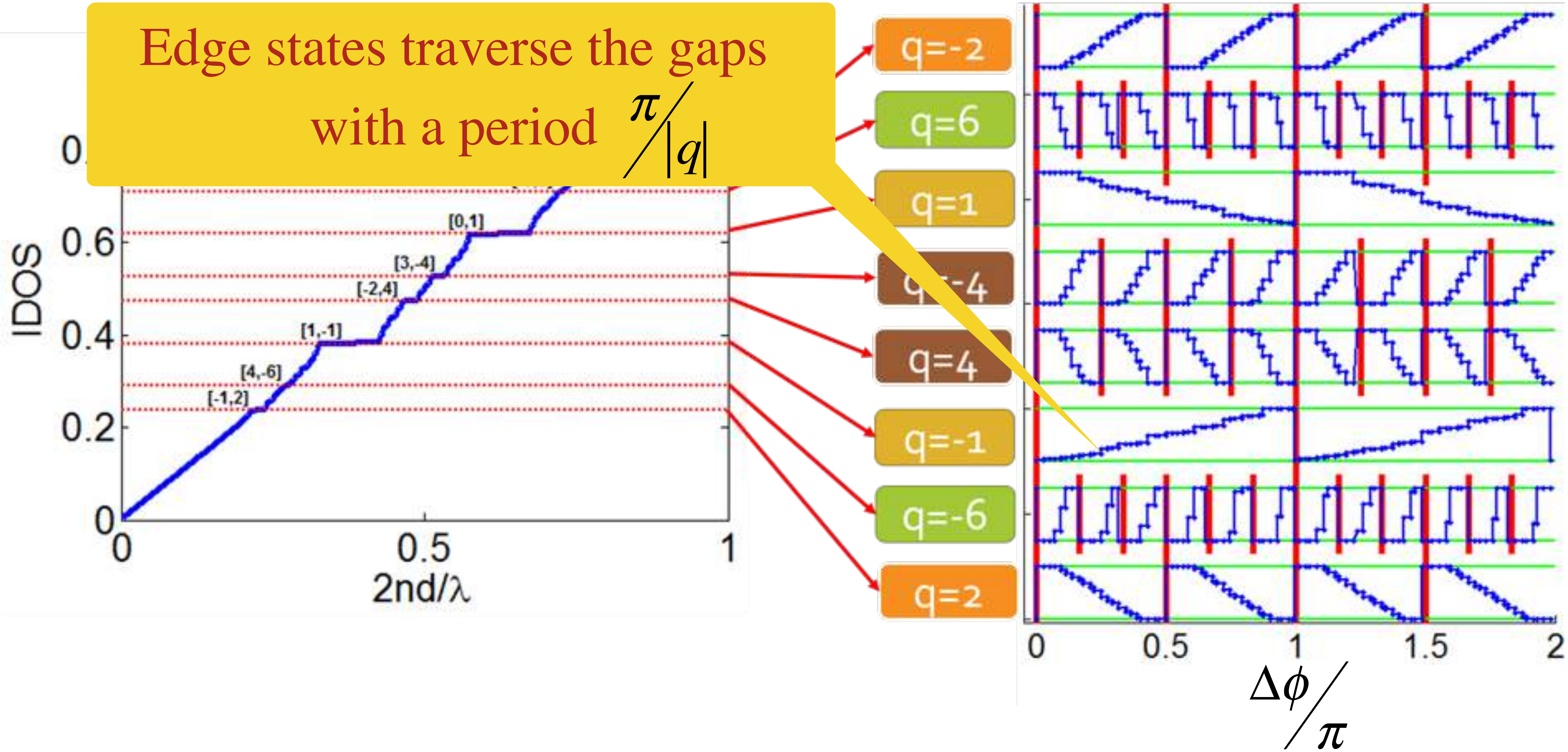
$$N\left(\omega_{(p,q)gap}\right) = p + q\,\tau^{-1}$$

Relation to gap labelling and Chern numbers



Chern numbers [p,q] describe the topological behaviour of edge states in the gaps when changing the structural phase angle $\Delta\phi$

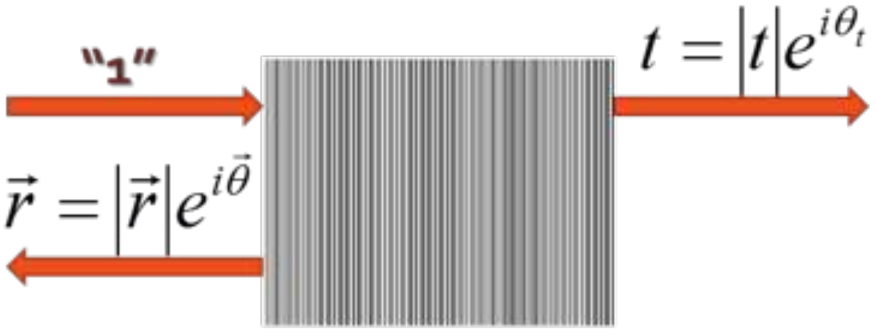
Relation to gap labelling and Chern numbers



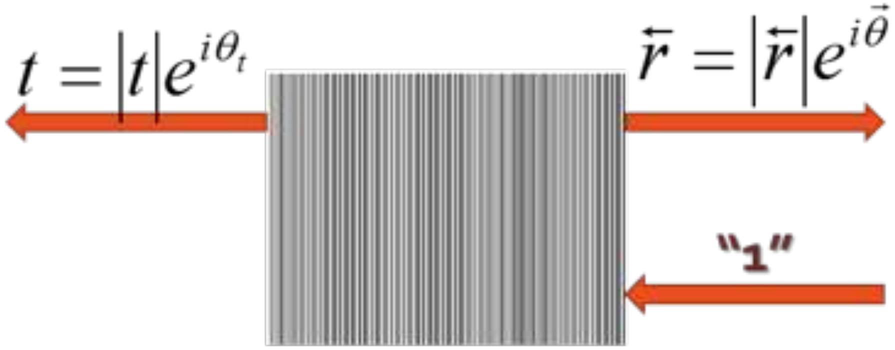
Edge states traverse the gaps with a period $\pi/|q|$

Chern numbers [p,q] describe the topological behaviour of edge states in the gaps when changing the structural phase angle $\Delta\phi$

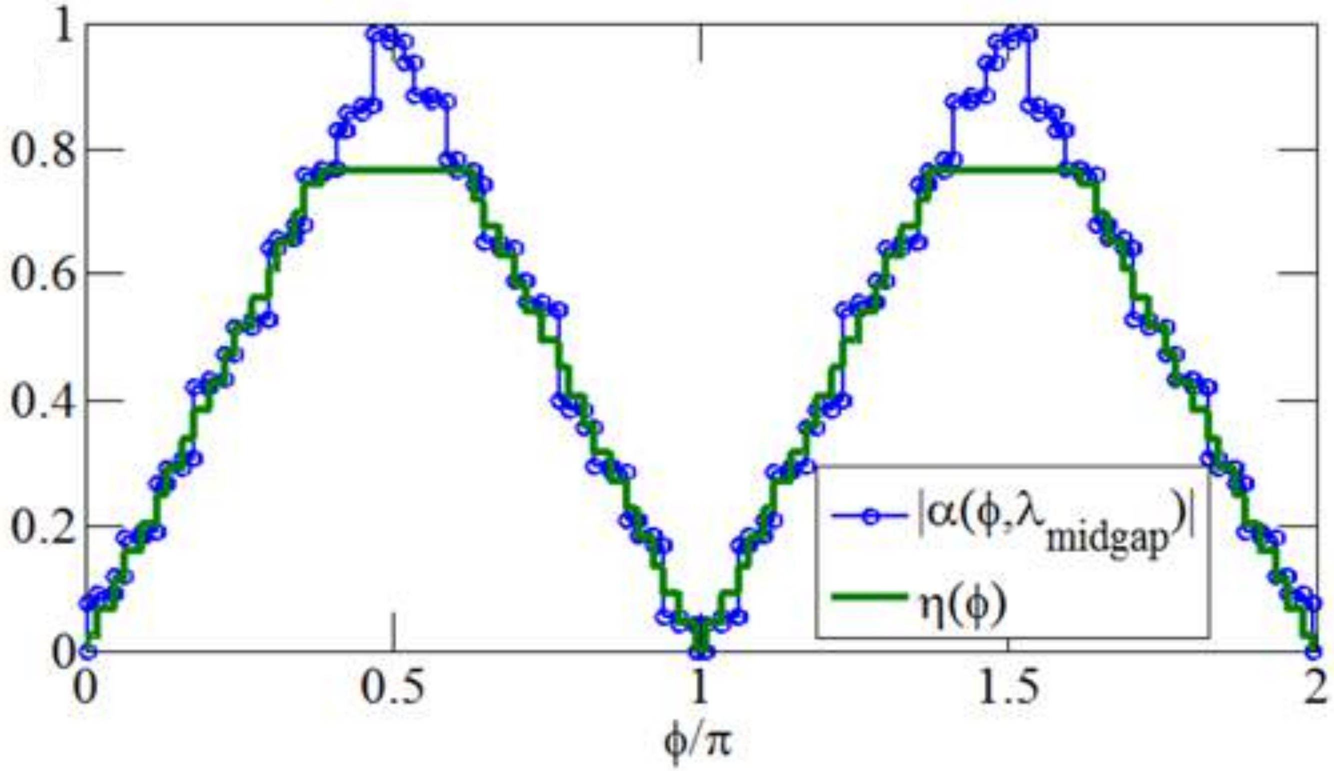
The chiral phase α depends on the structural phase $\Delta\phi$



$$\overleftarrow{r} = \overrightarrow{r} e^{i\alpha}$$



For edge state in a gap with Chern number q:

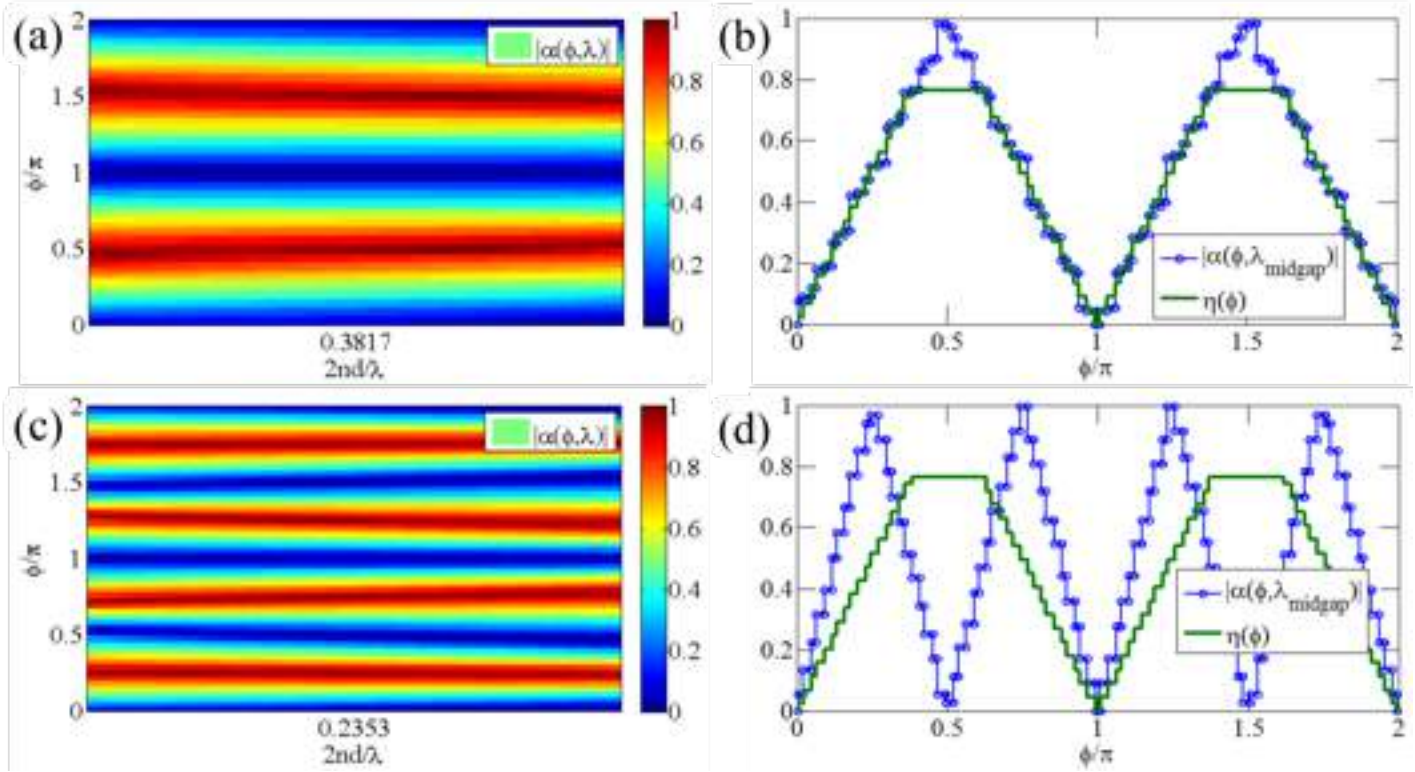


$$\alpha_q(\omega,\phi) = \overleftarrow{\theta}_q - \overrightarrow{\theta}_q$$

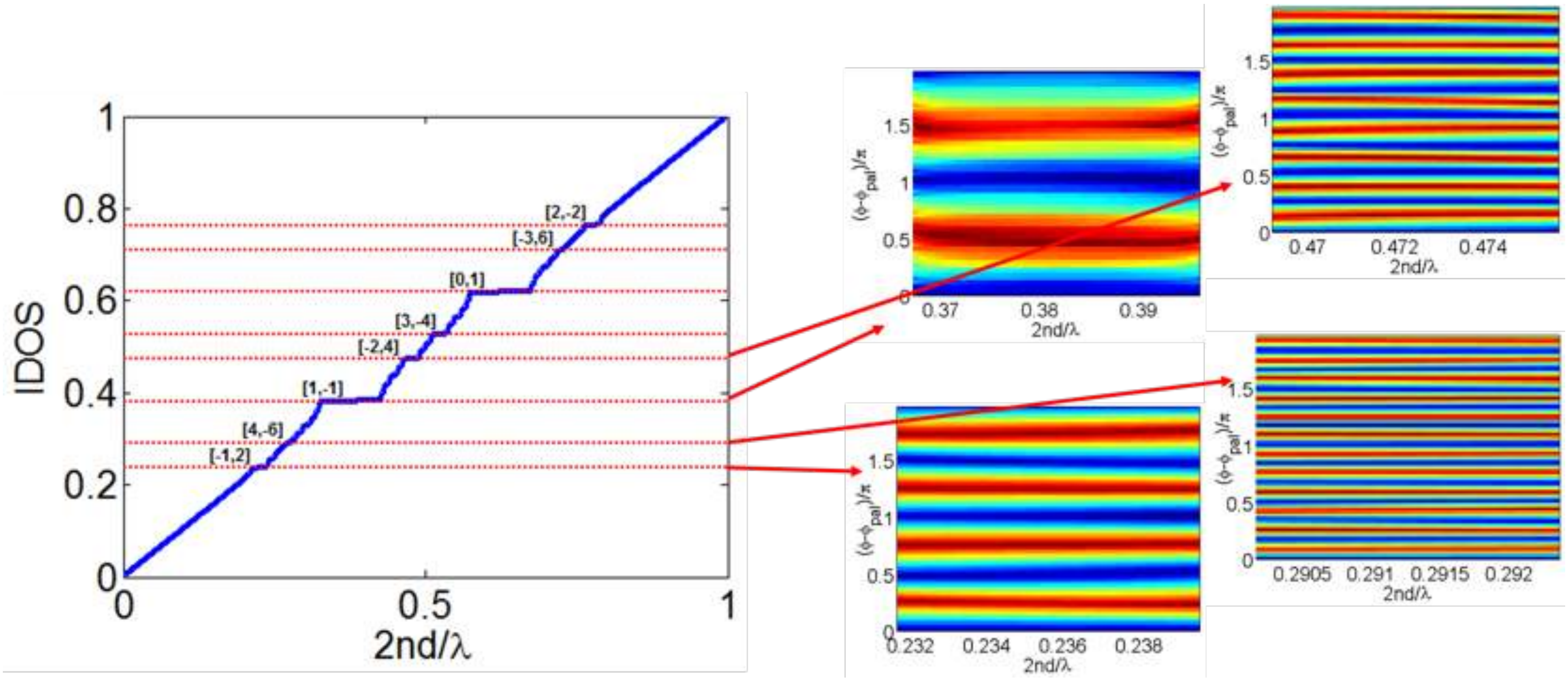
$\alpha_q(\omega,\phi)$ defines the spectral deviation from a palindrome

It depends on :

- the Chern number q in a gap
- the frequency ω in the gap
- the structural phase ϕ



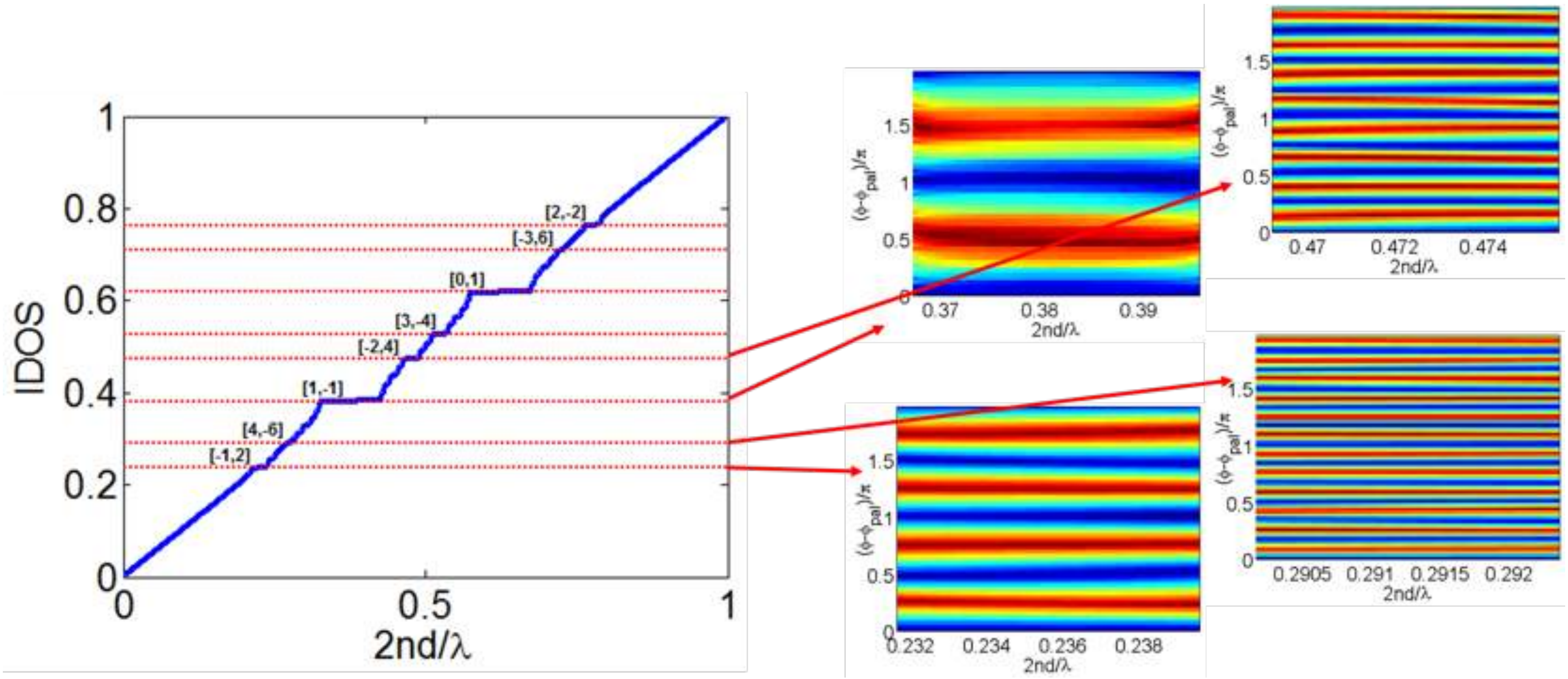
Winding number of $\alpha_q(\omega,\phi)$



The Chern number q is ***half*** the winding number of the chiral phase $\alpha_q(\omega,\phi)$:

$$W(\alpha_q) \equiv \frac{1}{2\pi}\int_0^{2\pi} d\phi \frac{d\alpha_q(\omega,\phi)}{d\phi} = 2q$$

Winding number of $\alpha_q(\omega,\phi)$



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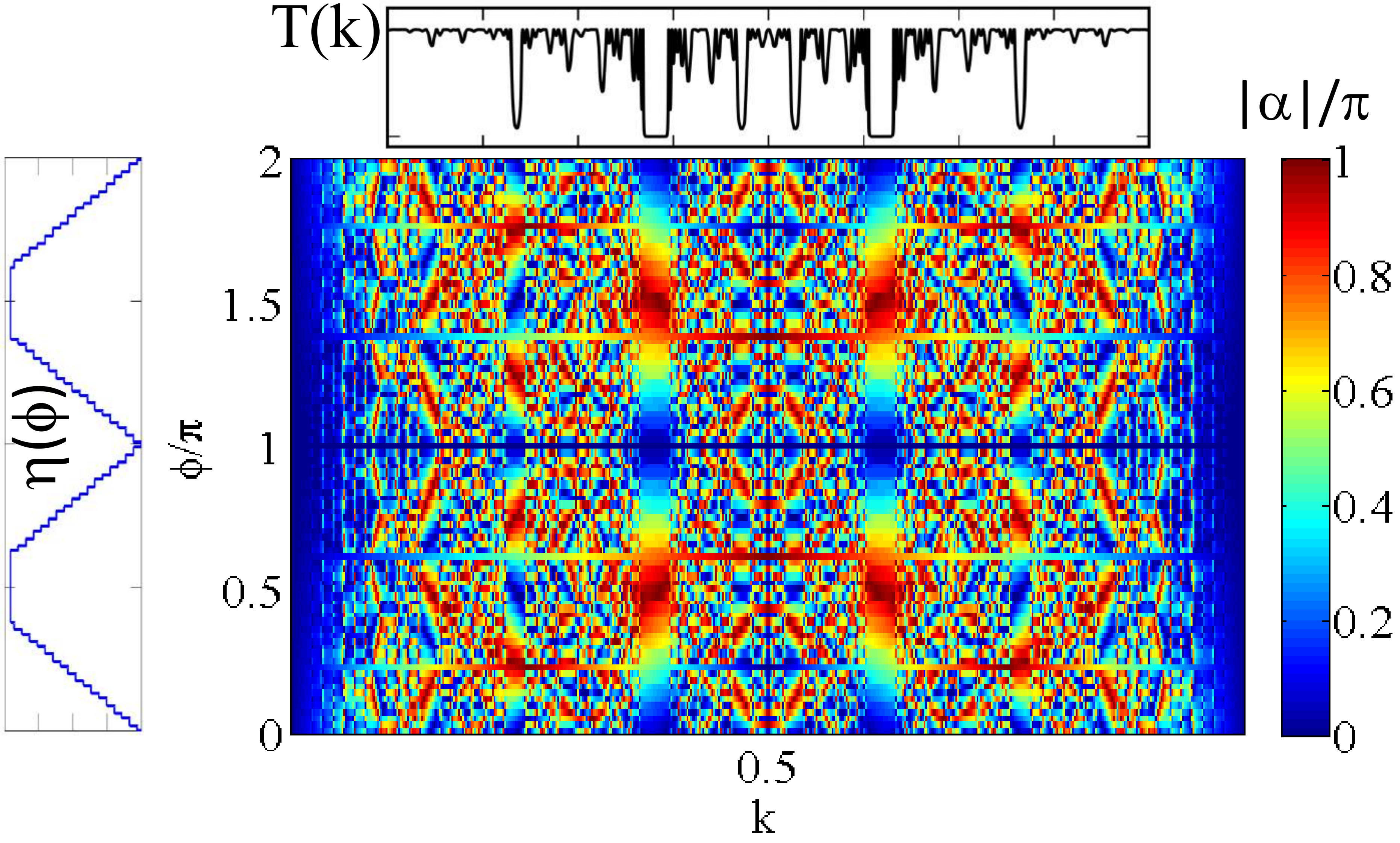
$$W(\alpha_q) \equiv \frac{1}{2\pi}\int_0^{2\pi} d\phi \frac{d\alpha_q(\omega,\phi)}{d\phi} = \boxed{2}q$$

This factor 2 is important : it accounts for the underlying palindromic symmetry.

$$W\left(\alpha_q\right) \equiv \frac{1}{2\pi}\int_0^{2\pi} d\phi \frac{d\alpha_q(\omega,\phi)}{d\phi} = 2q$$

C&P quasicrystal : phason degree of freedom + underlying palindromic symmetry - reflects in the additional factor 2 when expressing Chern numbers as the winding of a phase.

A Hofstadter butterfly for Fibonacci

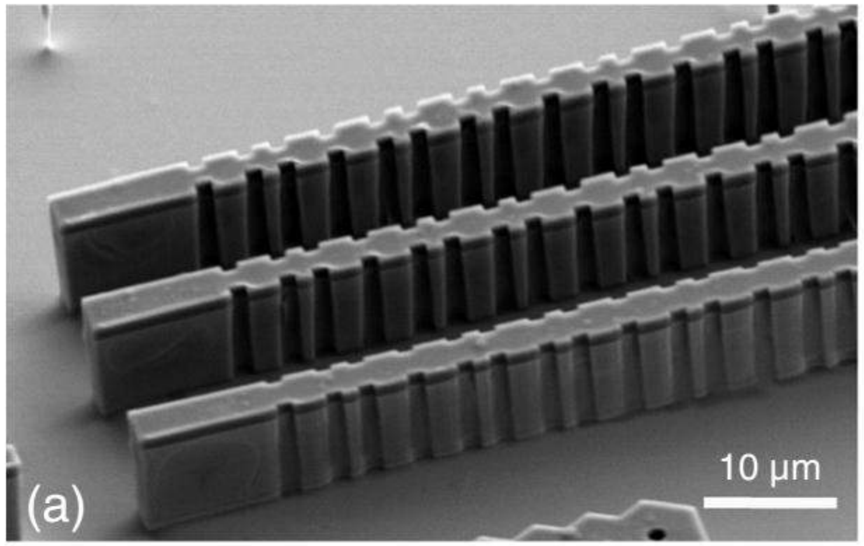


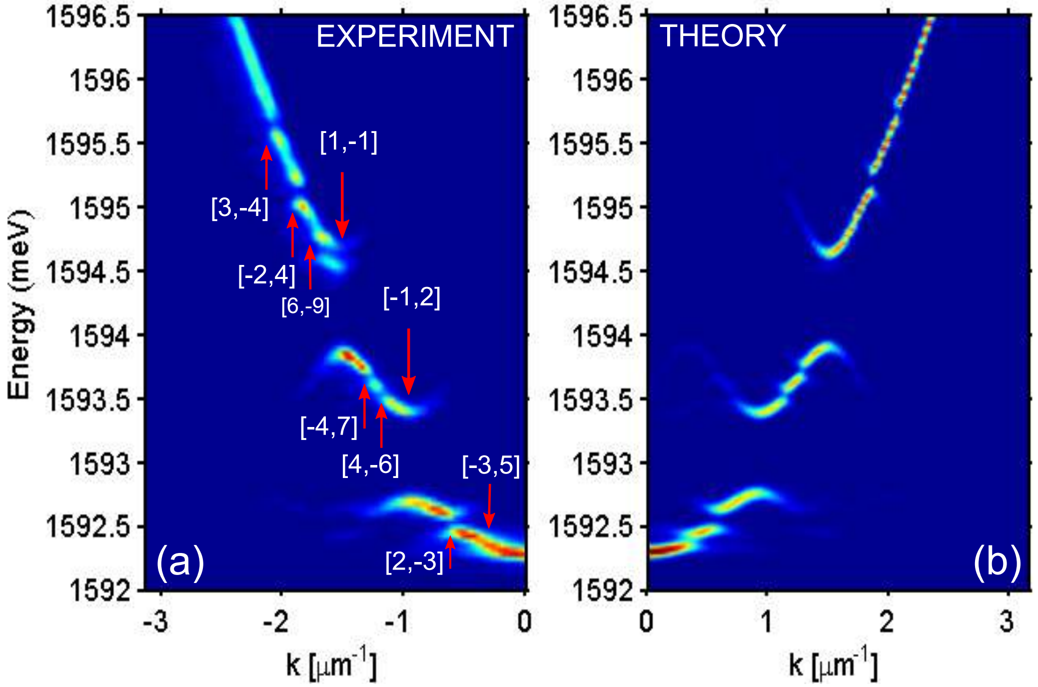
Experimental observation using cavity polaritons

F. Baboux, E. Levy, J. Bloch, E.A, 2016

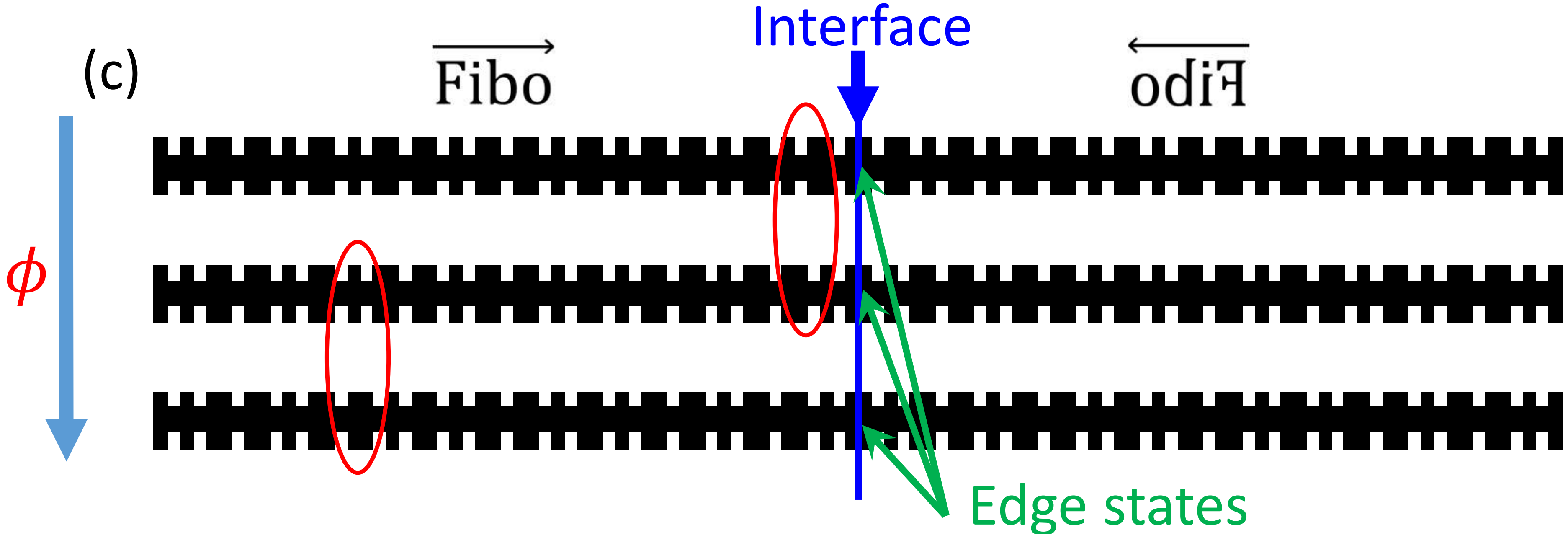
Recently measured using cavity polaritons

(a)



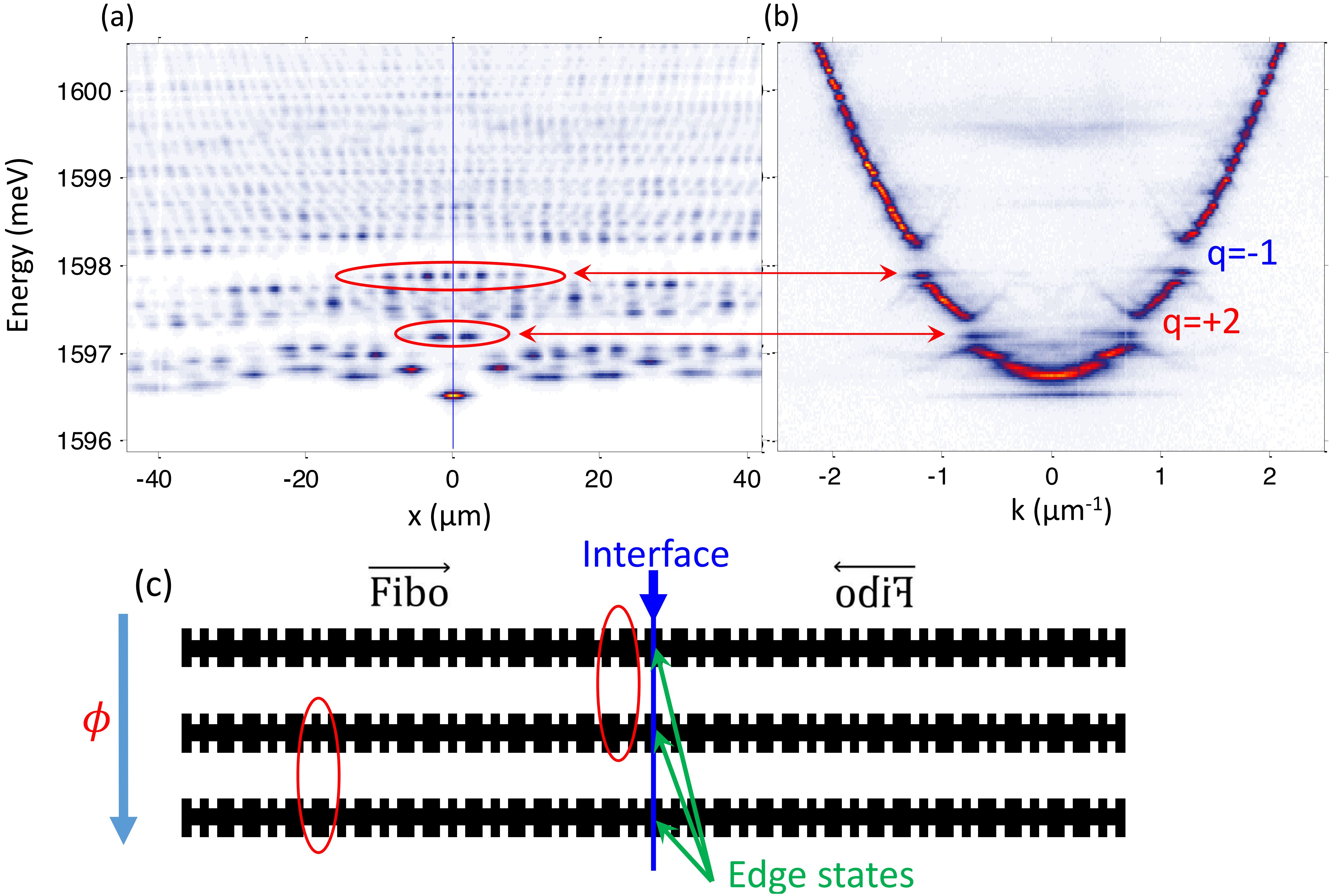


(c)



89 different samples i.e. 89 values of ϕ

Recently measured using cavity polaritons



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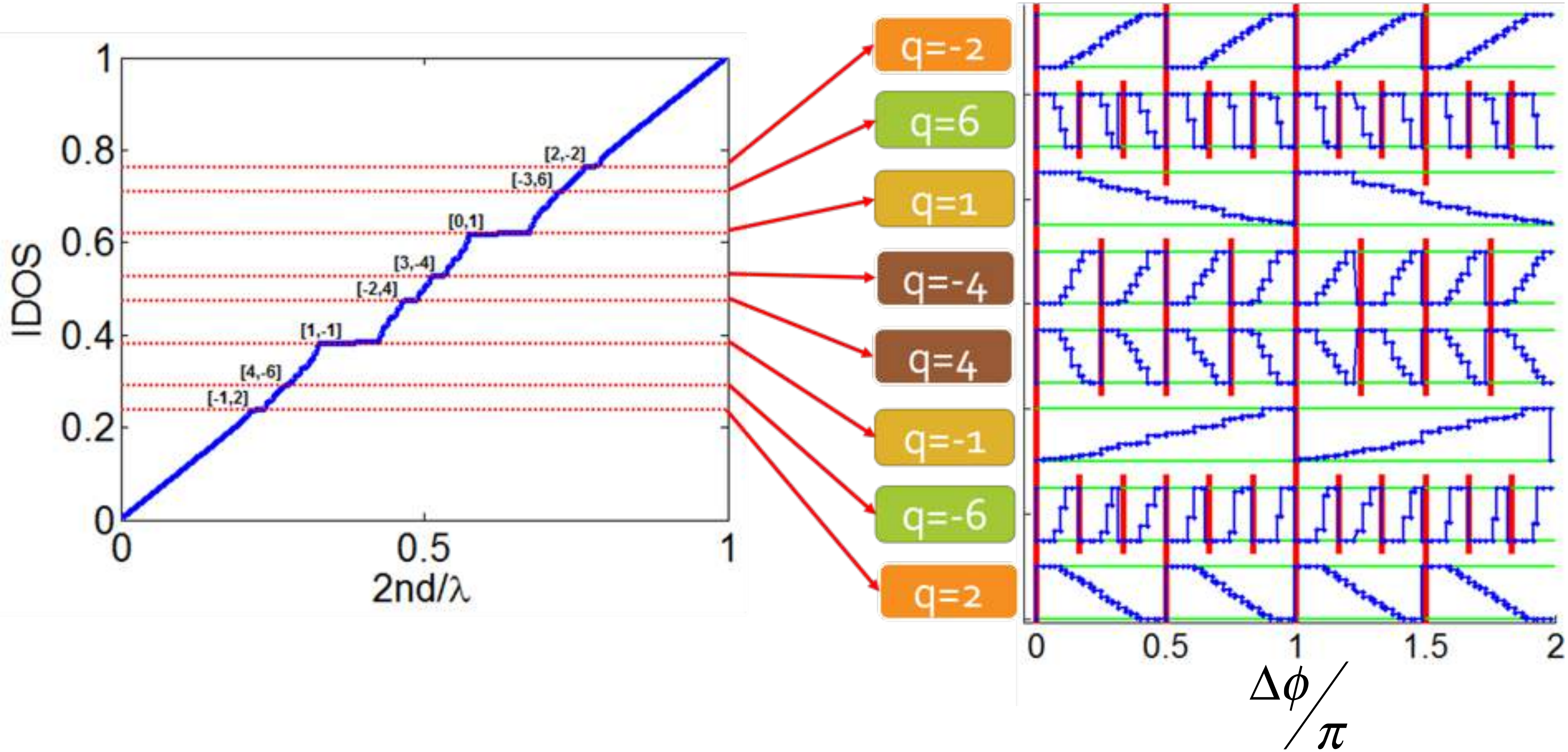
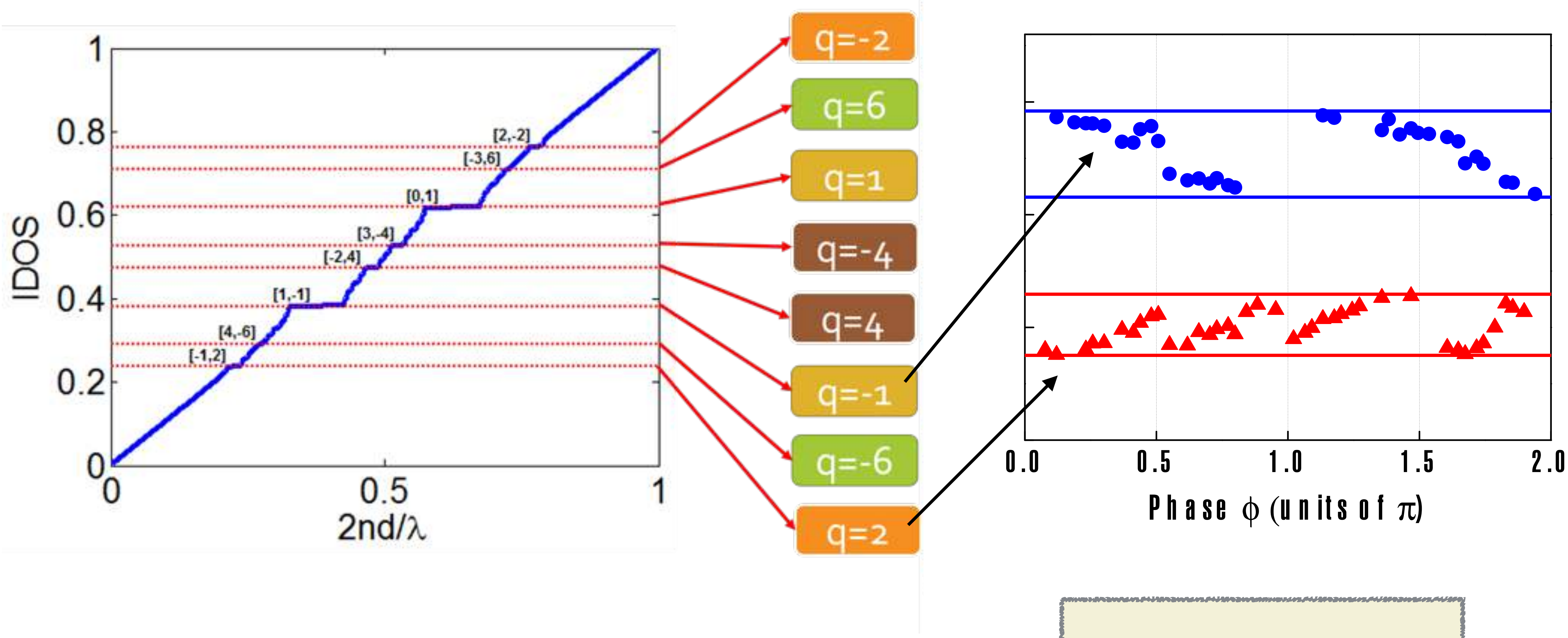


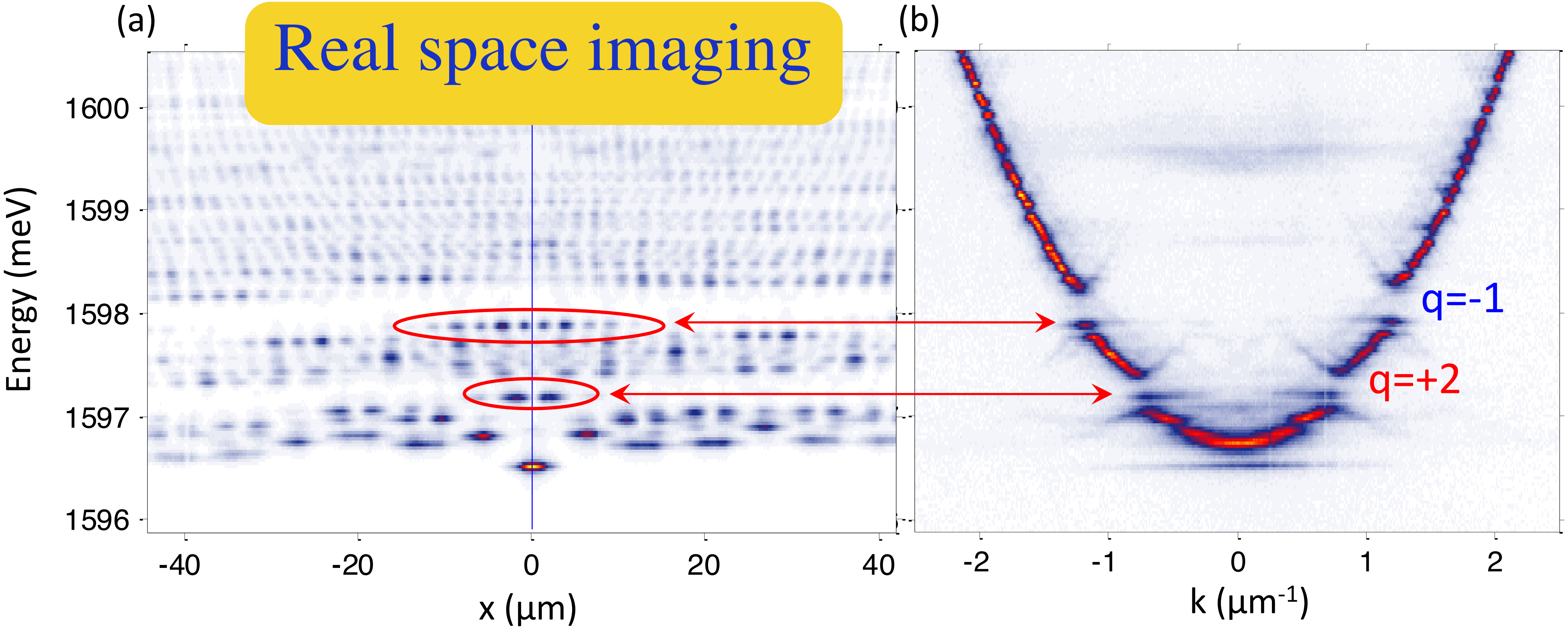
F. Baboux, E. Levy, J. Bloch, E.A, 2016



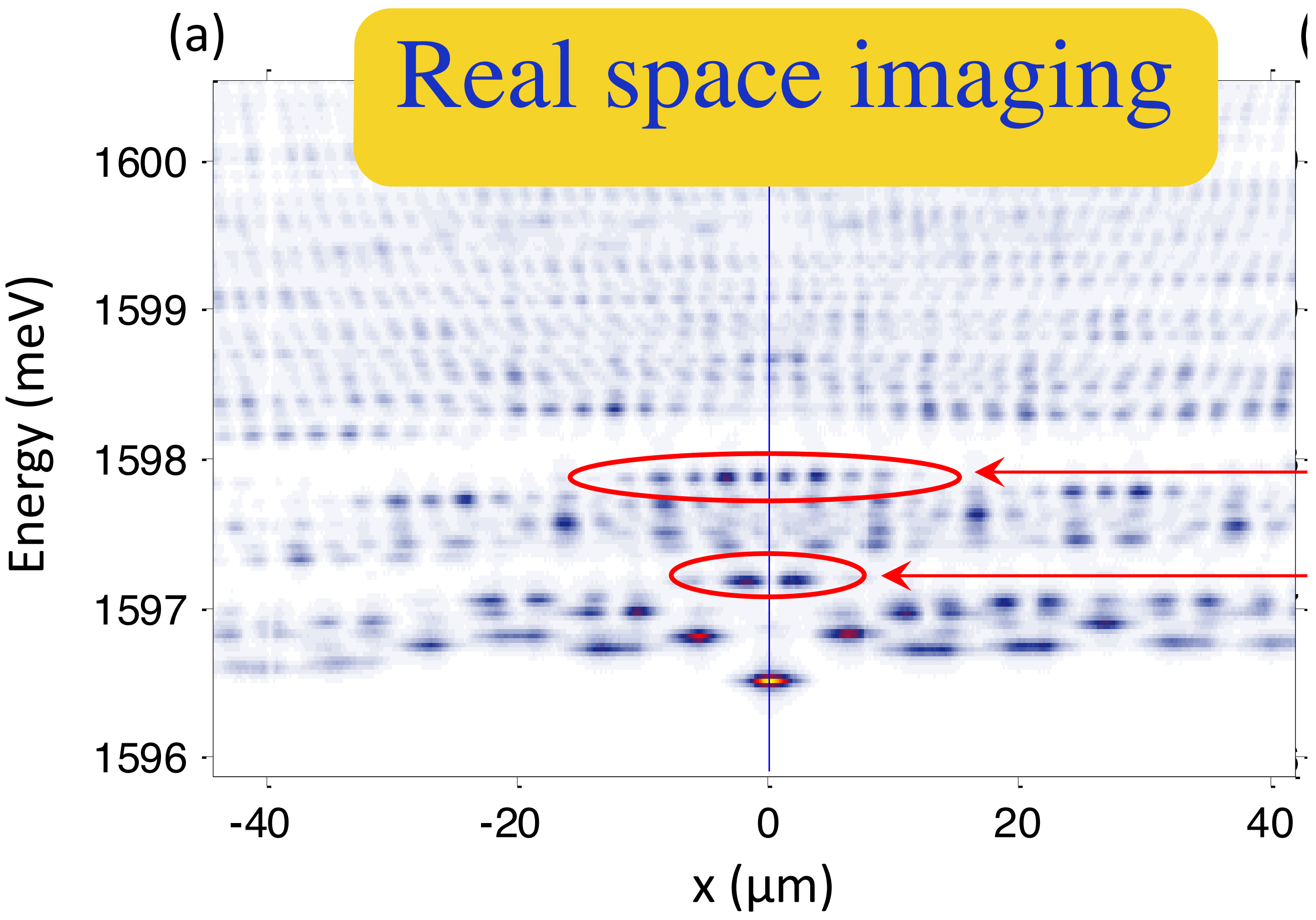
Note the factor 2 !

F. Baboux, E. Levy, J. Bloch, E.A, 2016

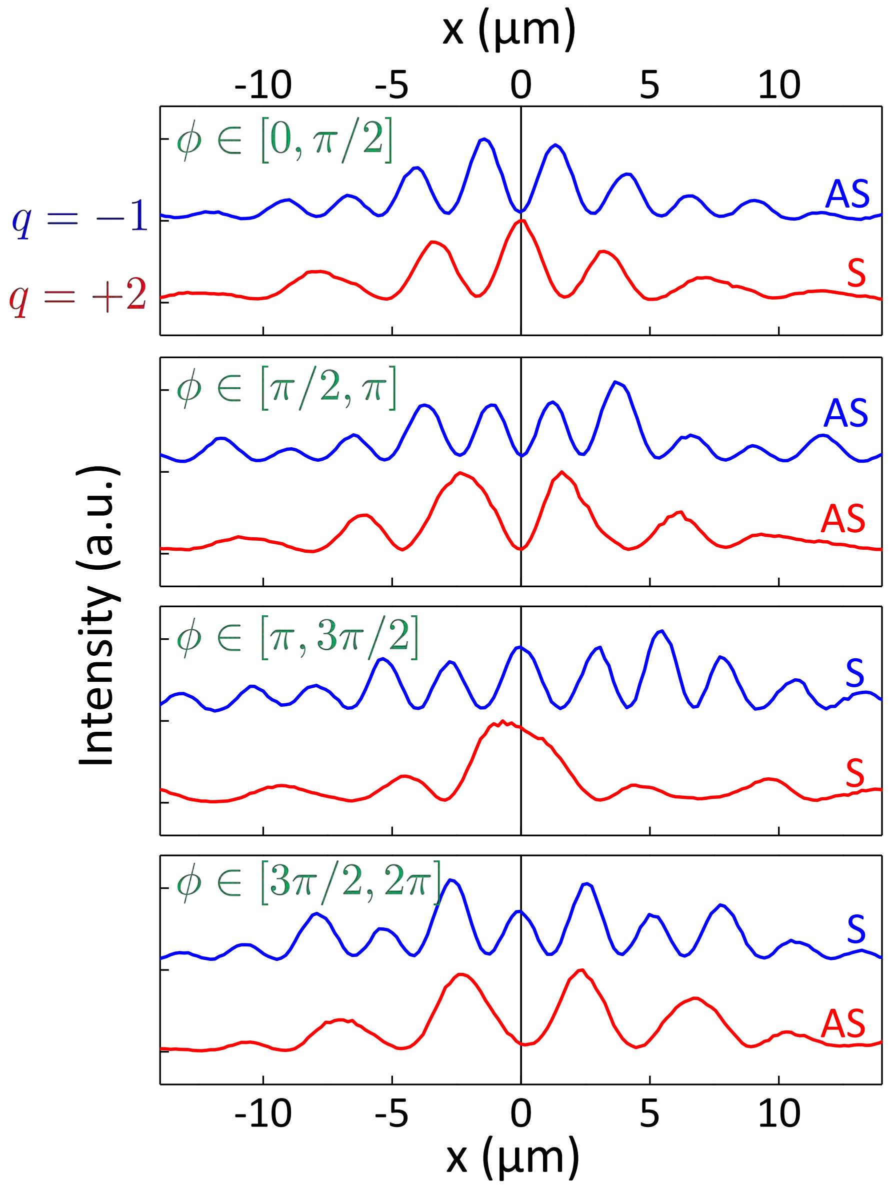
Measuring Chern numbers directly on the wave function



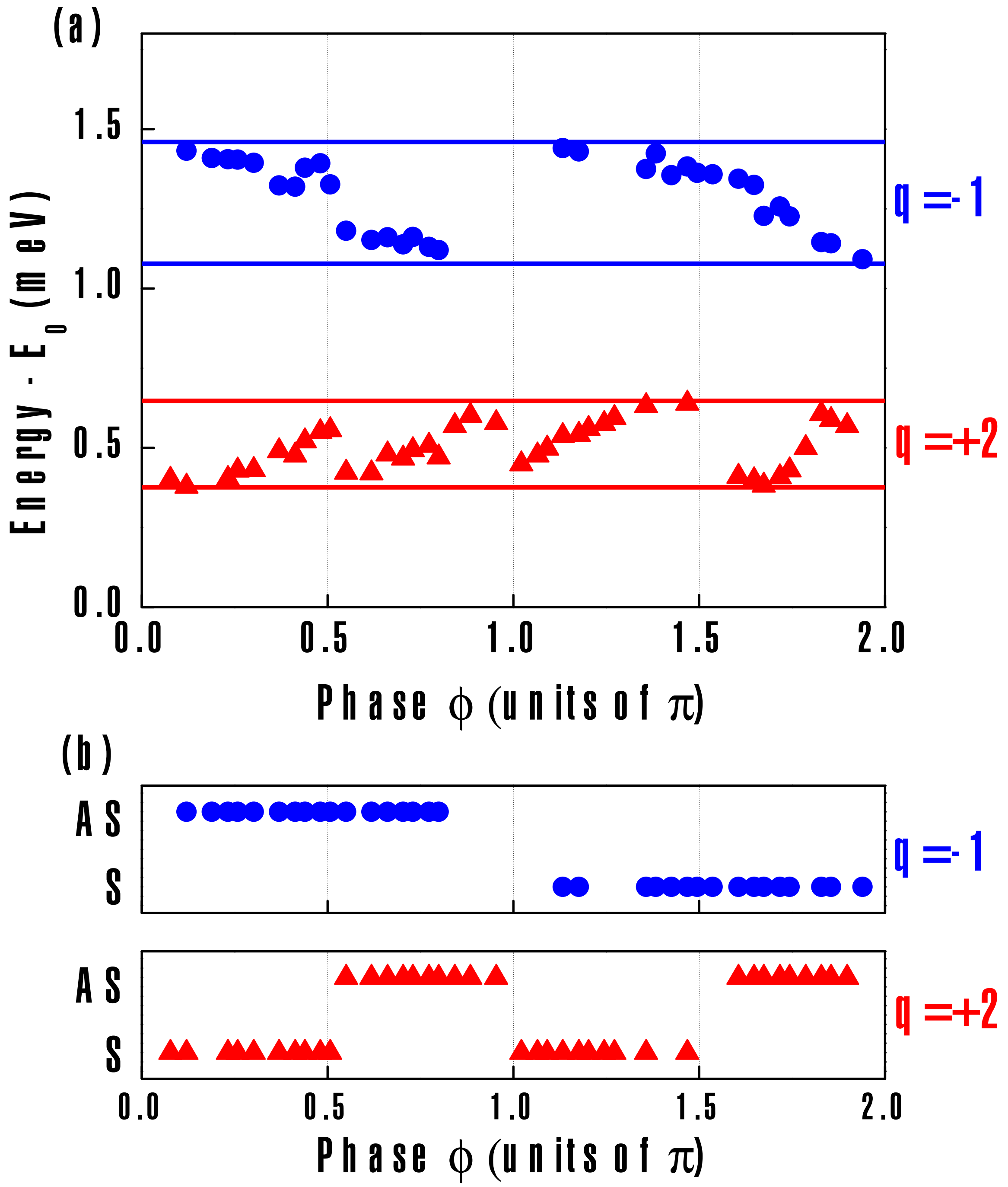
Measuring Chern numbers directly on the wave function

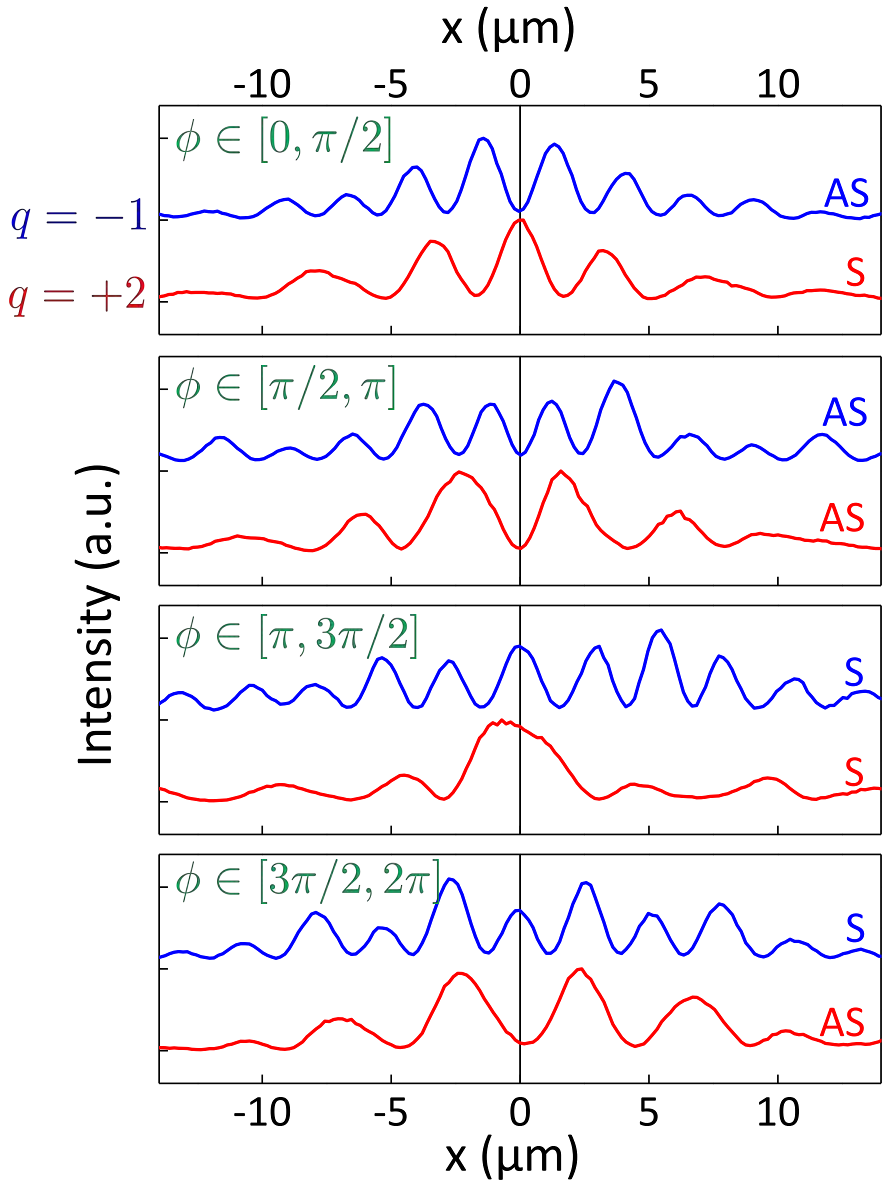


Real space imaging



Measuring Chern numbers directly on the wave function





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$\alpha_q(\omega, \Delta\phi)$ defines the spectral deviation from a palindrome

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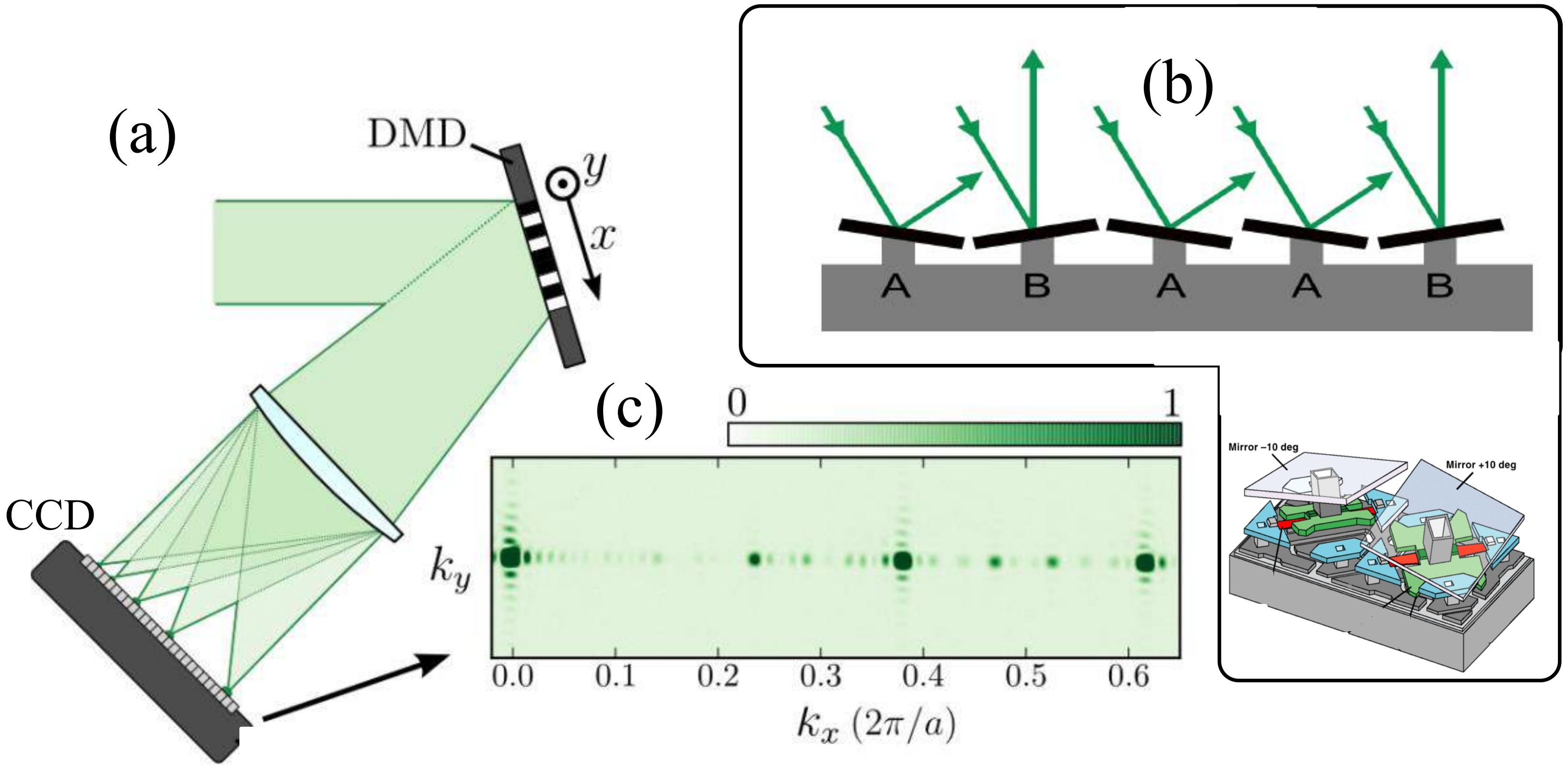
- the structural phase angle $\Delta\phi$

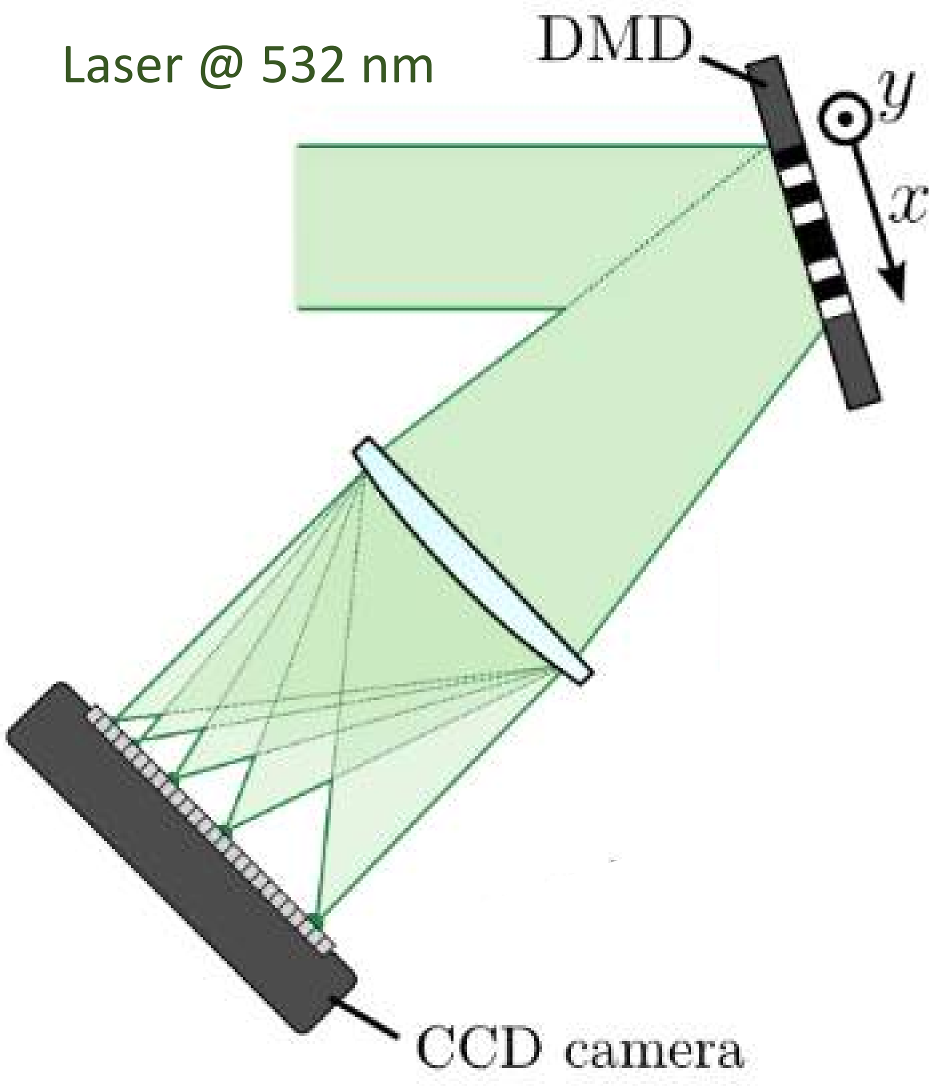
A diffraction measurement of winding numbers

Fibonacci finite string



A. Dareau, E. Levy, E.A, F. Gerbier and J. Beugnon, 2016

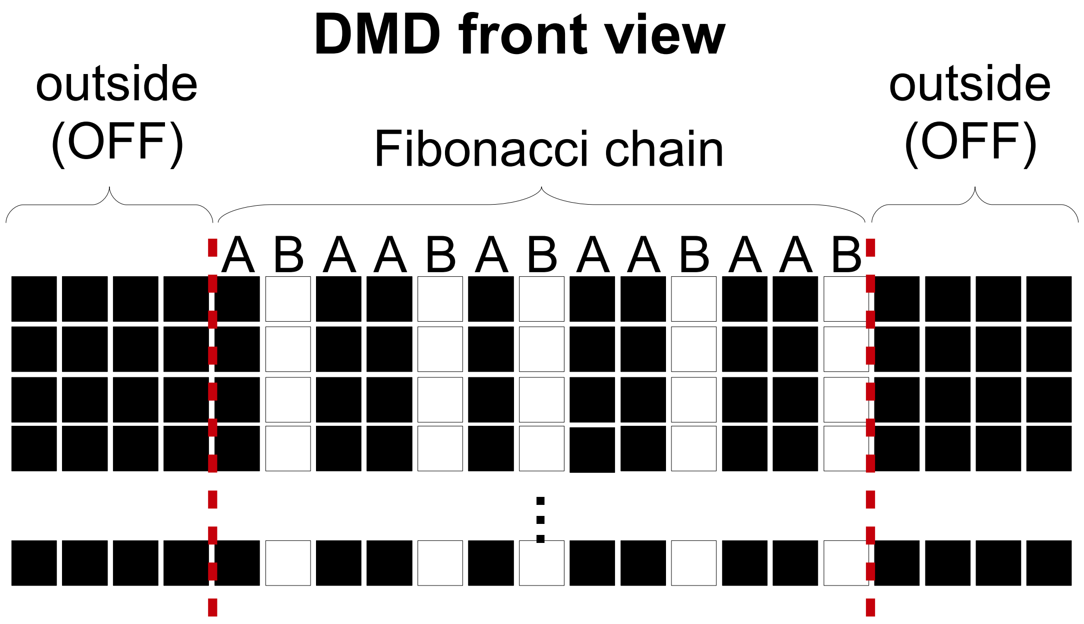
Optical setup



Digital Micromirror Device (DMD)

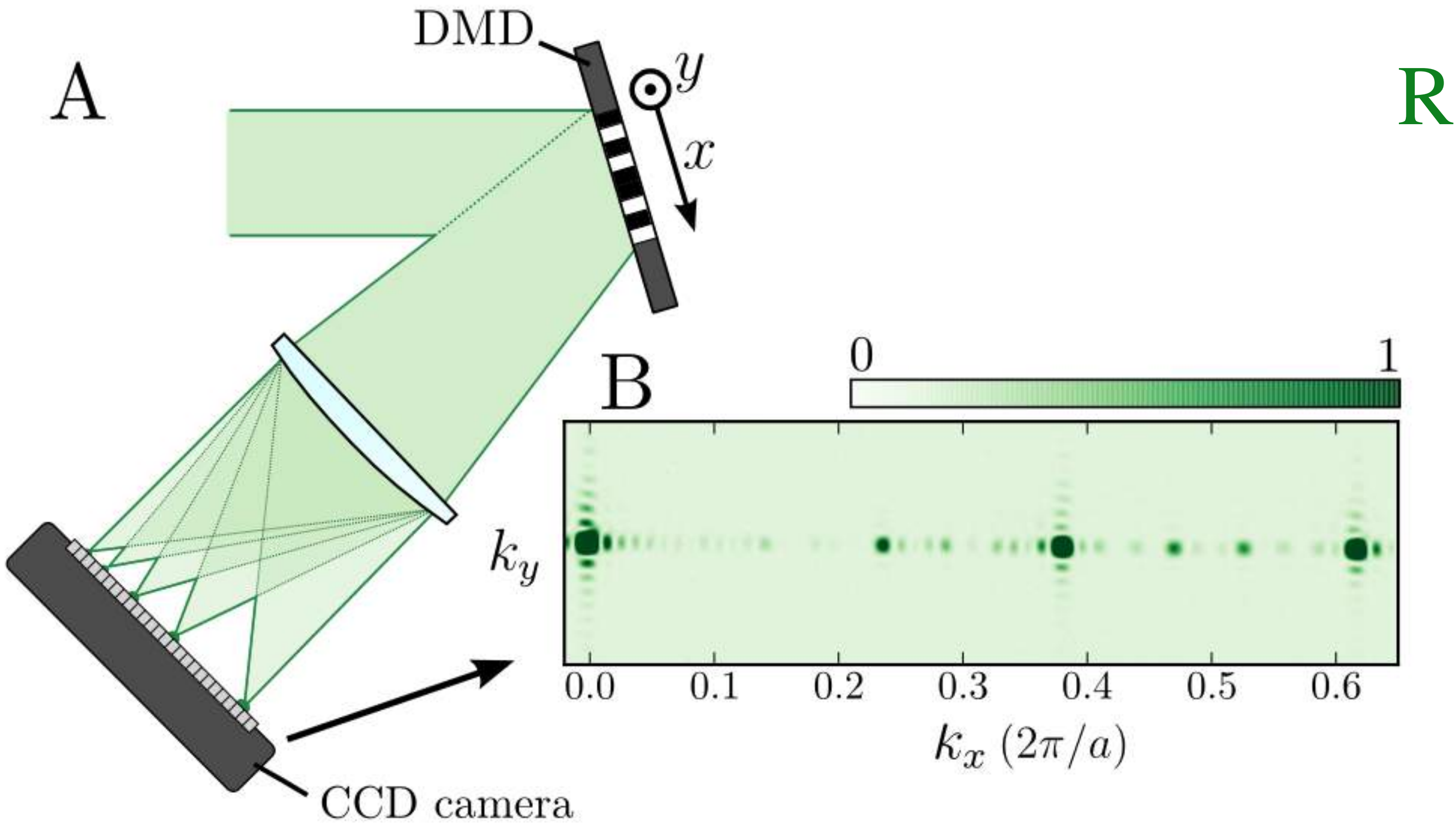
- mirror (“pixel”) size a ~ 14 µm
- 1024 × 768 pixels

Fibonacci encoding : A ■ B □

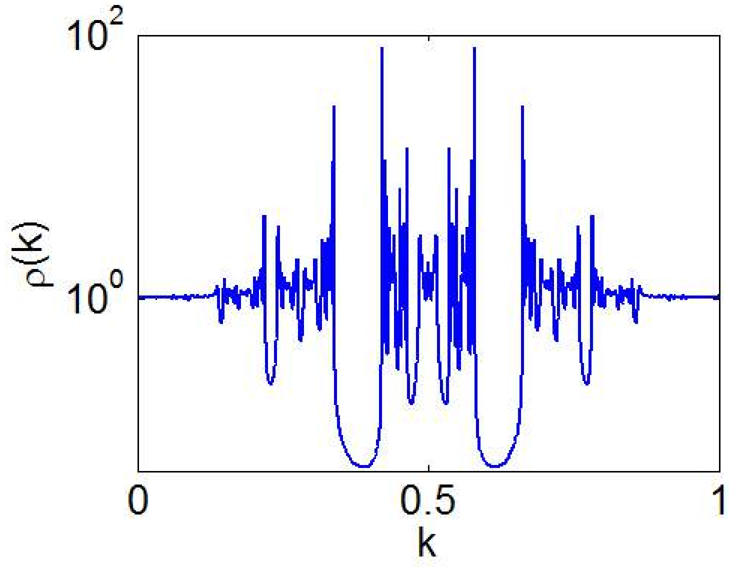

DMD front view

A diffraction measurement of winding numbers

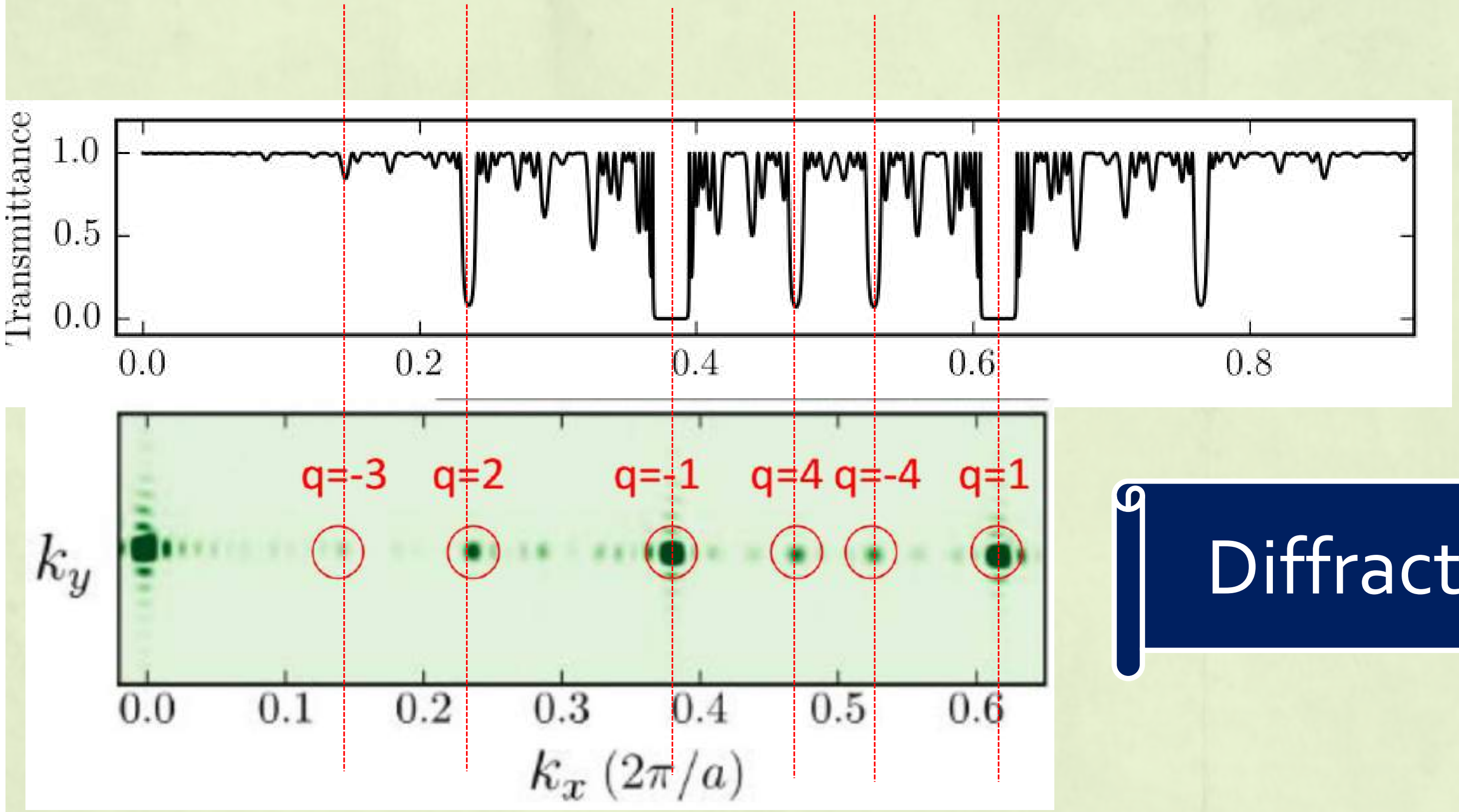
Fibonacci finite string



Related to the transmission spectrum ?



A. Dareau, E. Levy, E.A, F. Gerbier and J. Beugnon, 2016



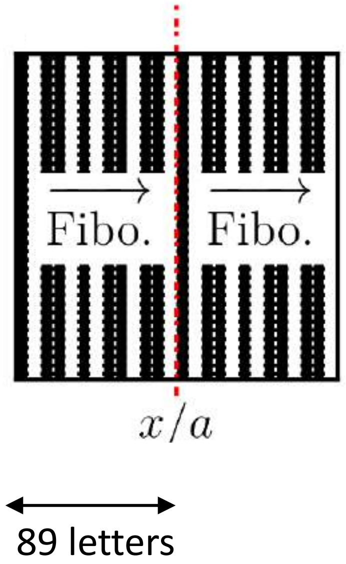
Transmission spectrum

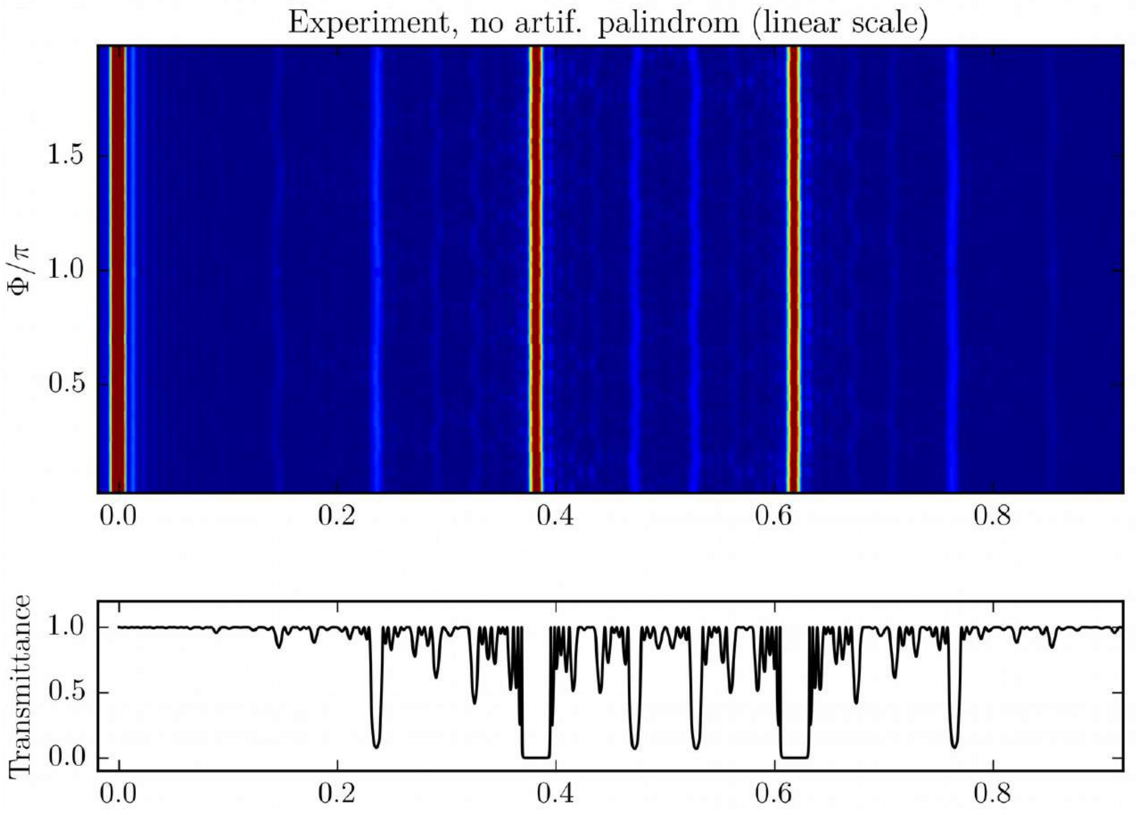
Diffraction pattern

A (diffraction) Young-slit measurement of (topological) winding numbers

Juxtaposition of 2 identical Fibonacci chains

DMD Pattern



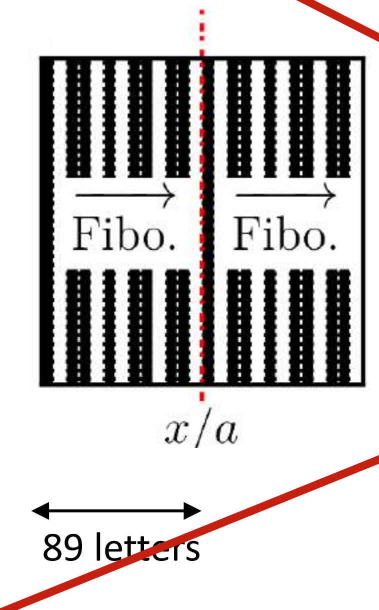


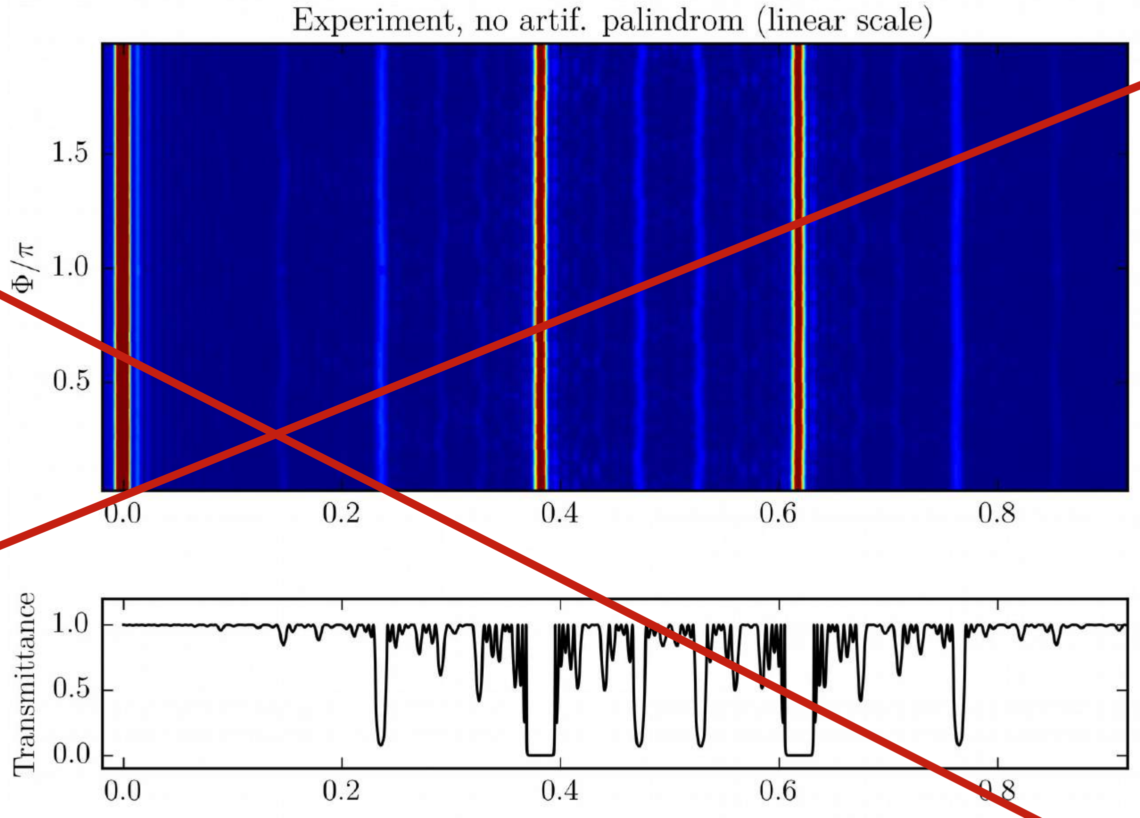
No effect of ϕ

Need to create edge states !

A Young-slit interference experiment

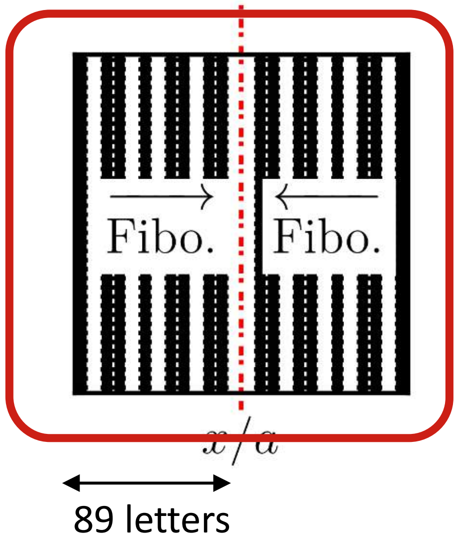
~~DMD Pattern~~

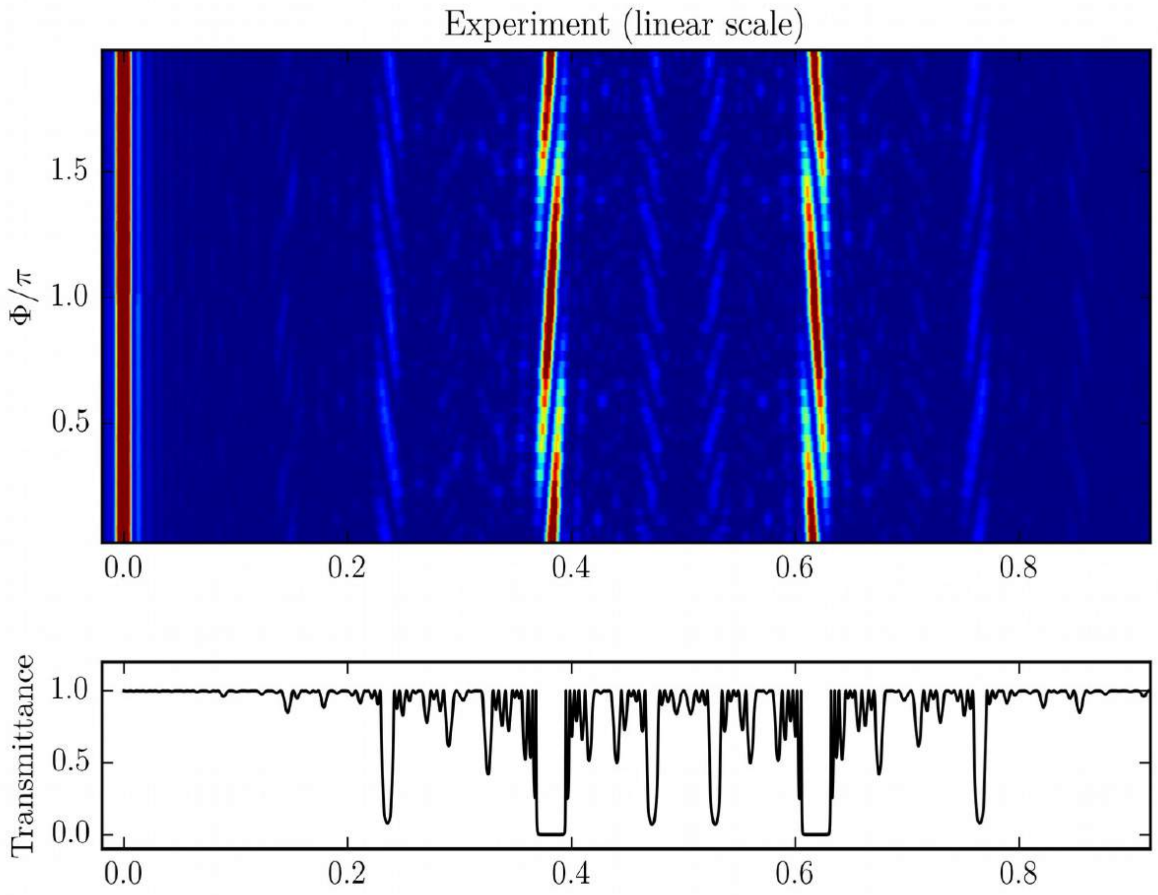


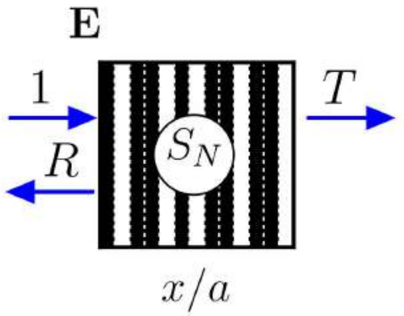


No effect of ϕ

DMD Pattern



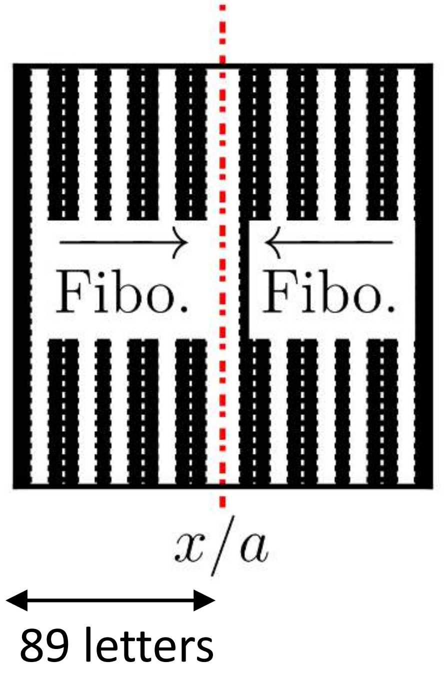




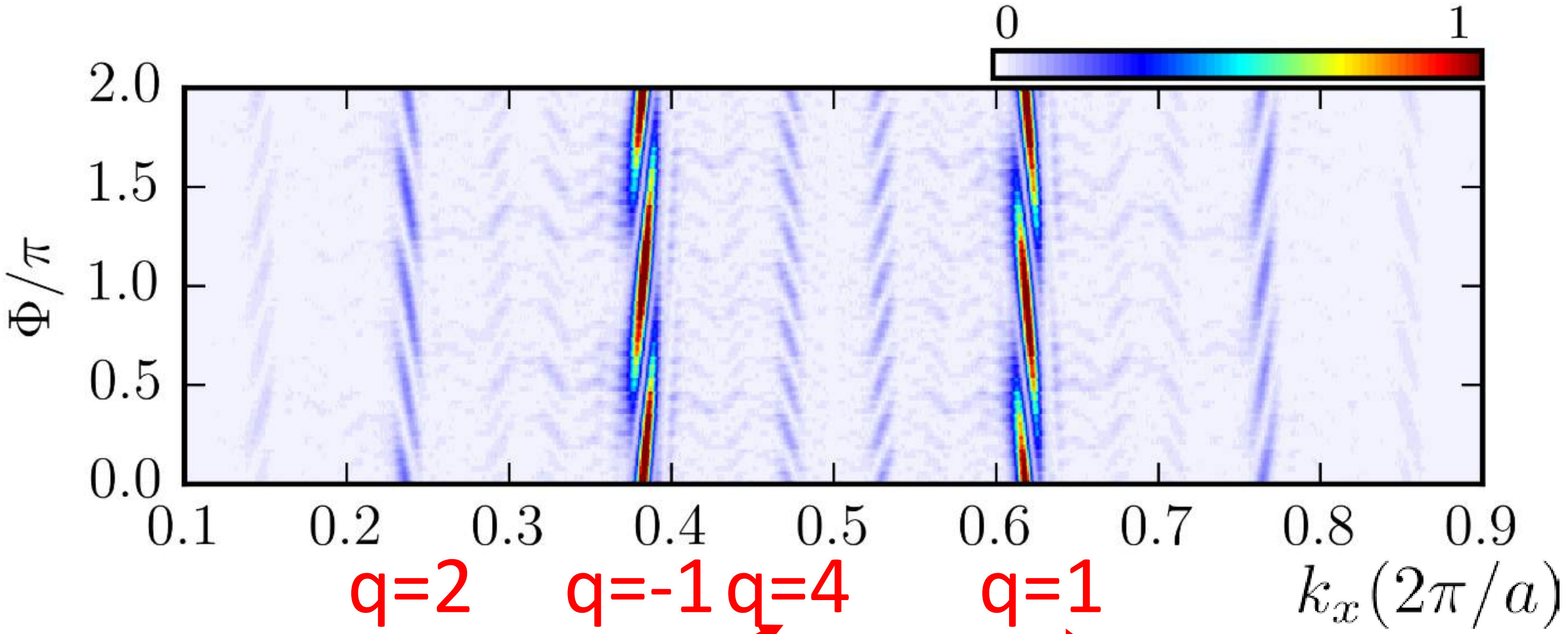
There is an effect of the phason ϕ

A Young-slit measurement of windings

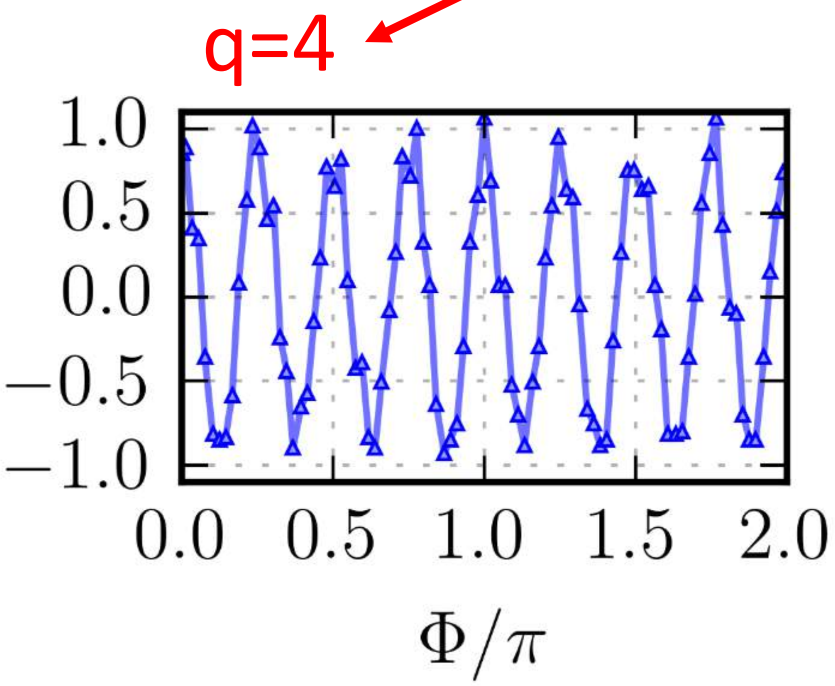
DMD Pattern

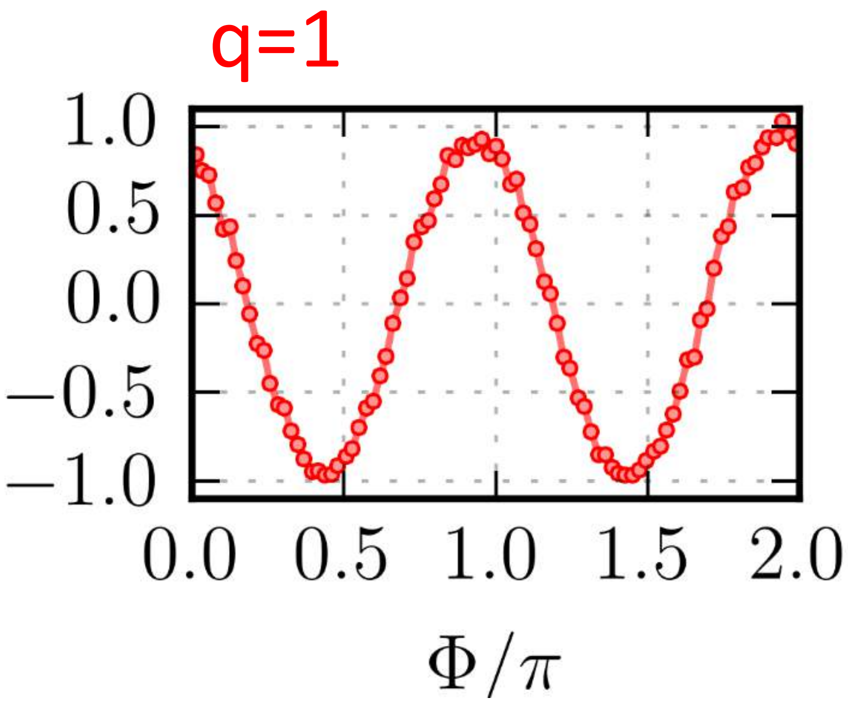


Diffraction pattern



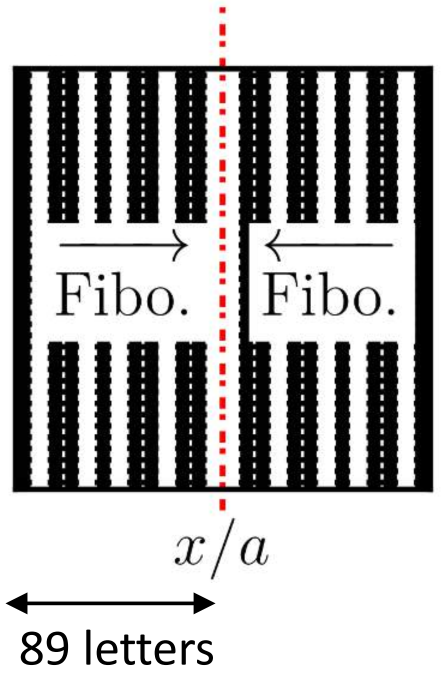
Cuts at a given k_x





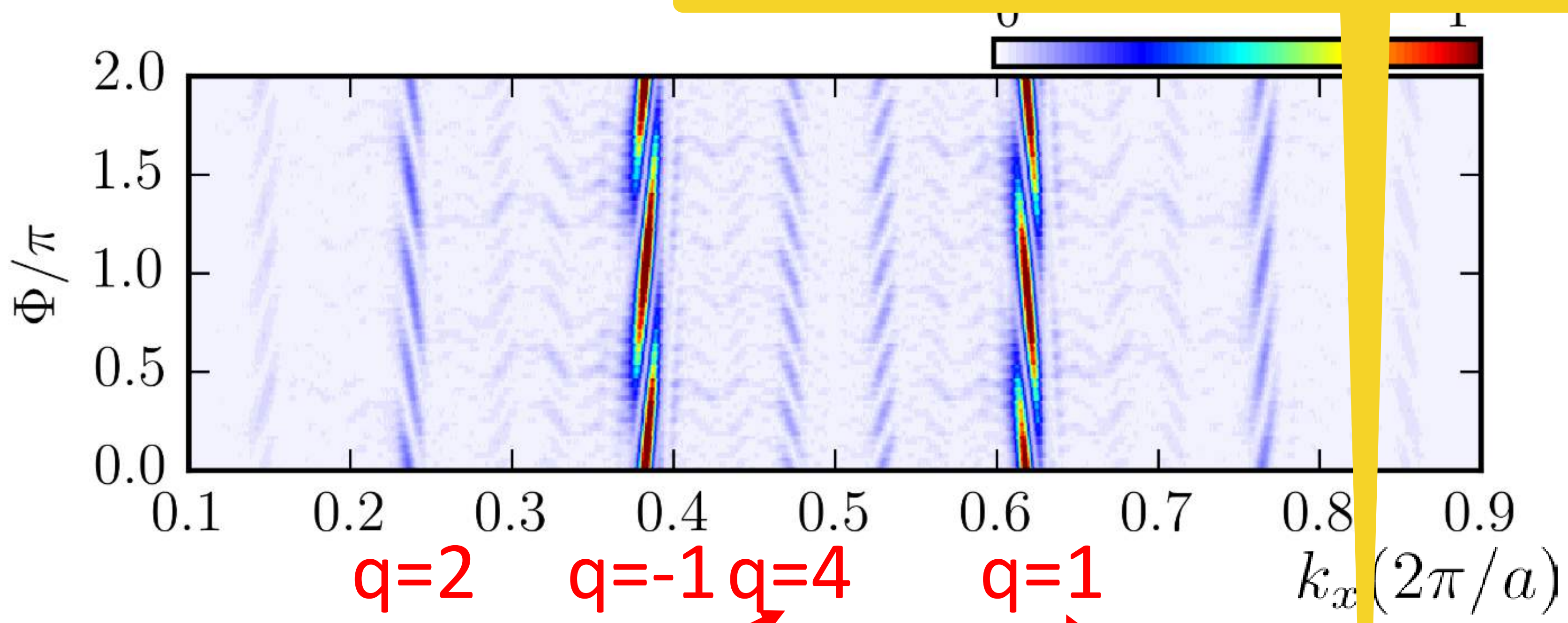
A Young-slit measurement of windings

DMD Pattern

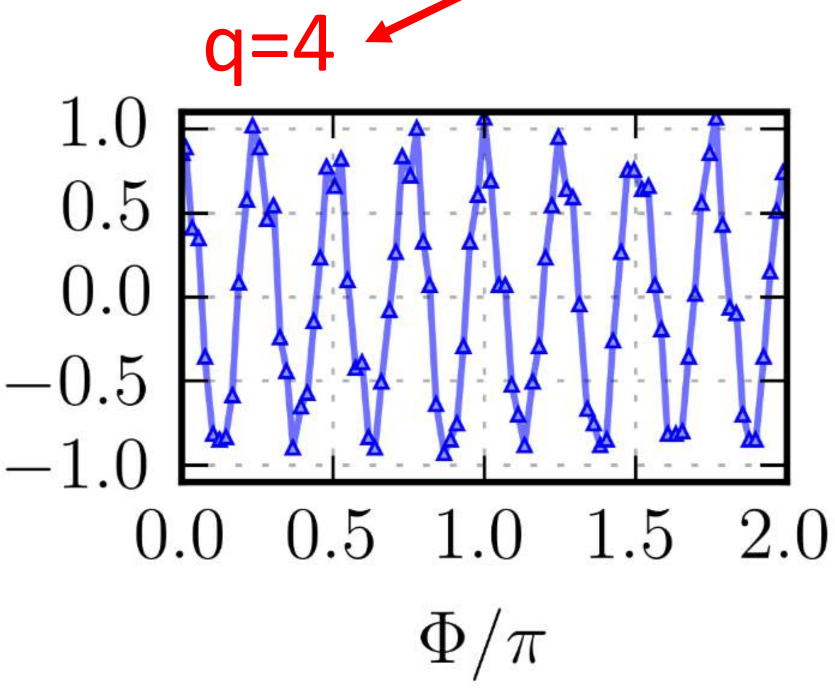


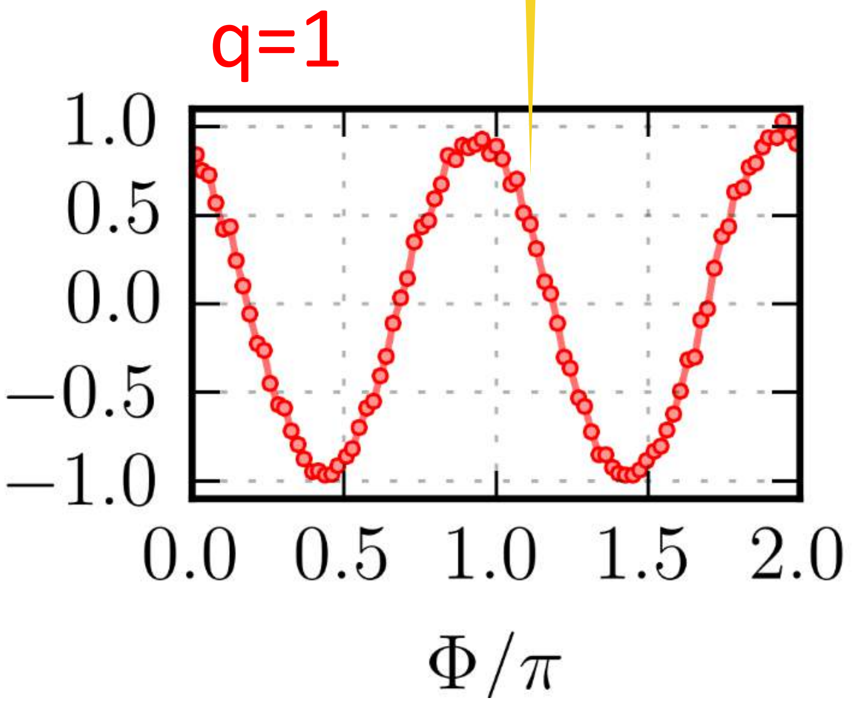
Diffra

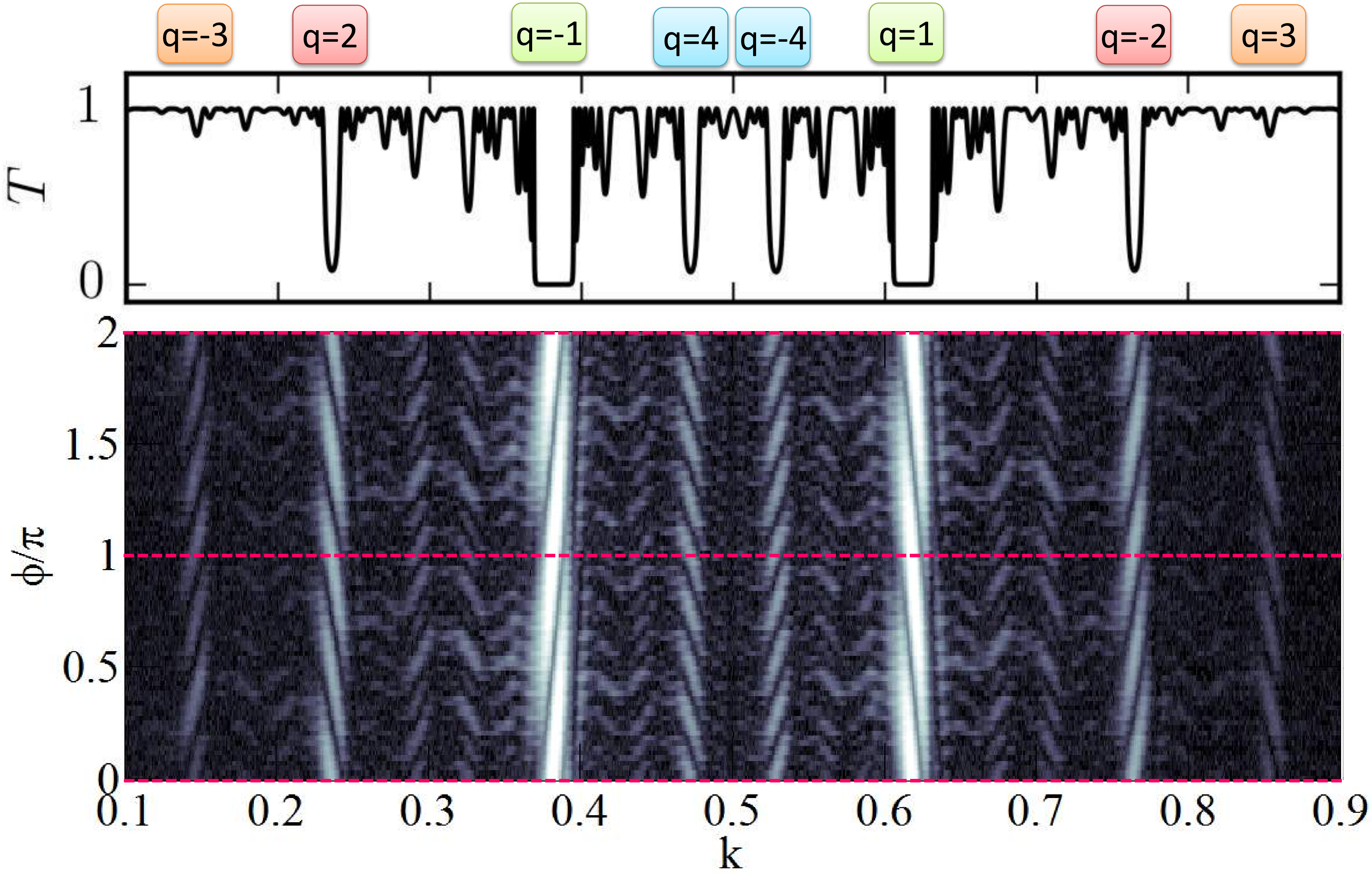
Split of the diffraction peak oscillates with a period $\pi/|q|$



Cuts at a given k_x

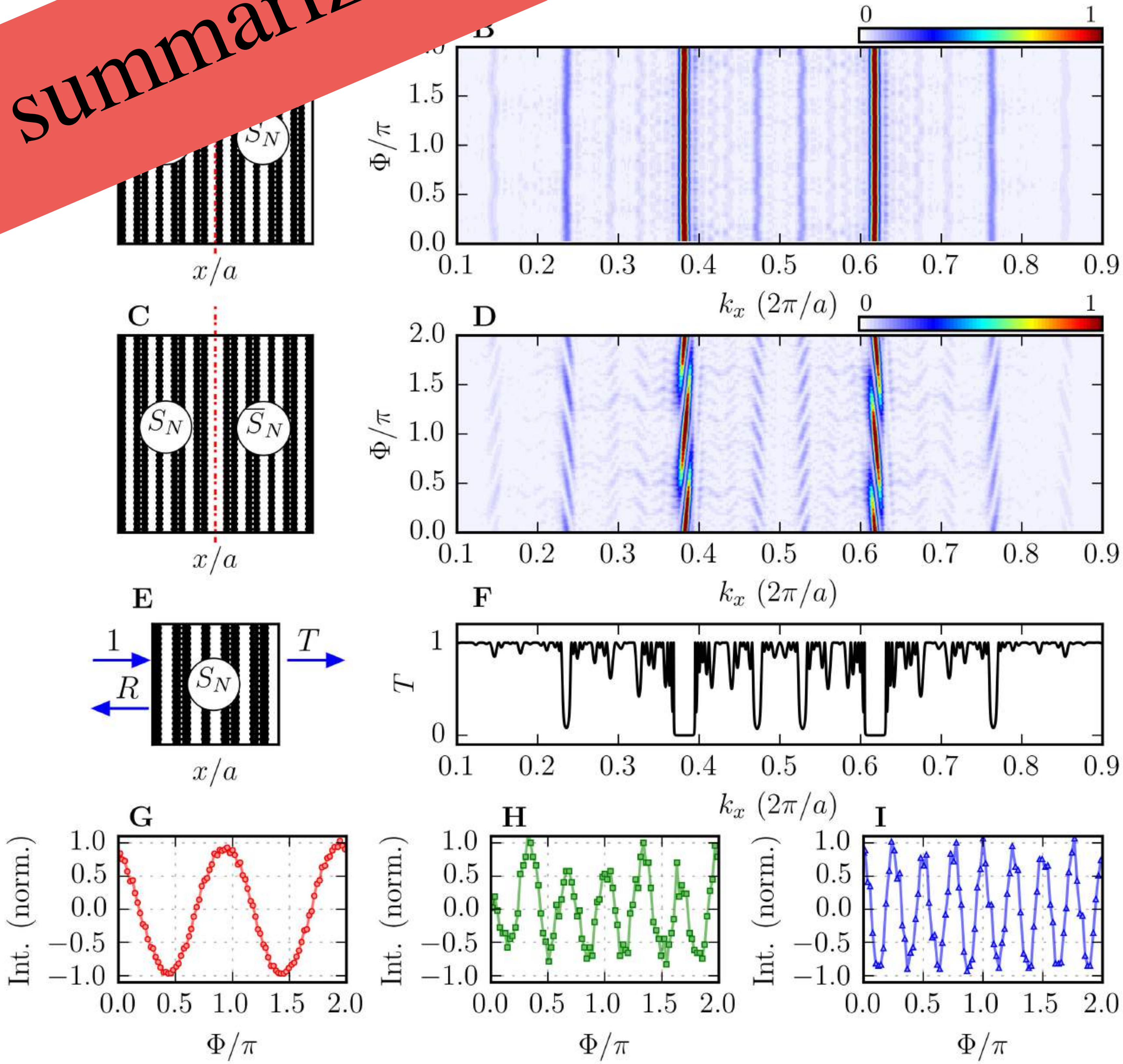




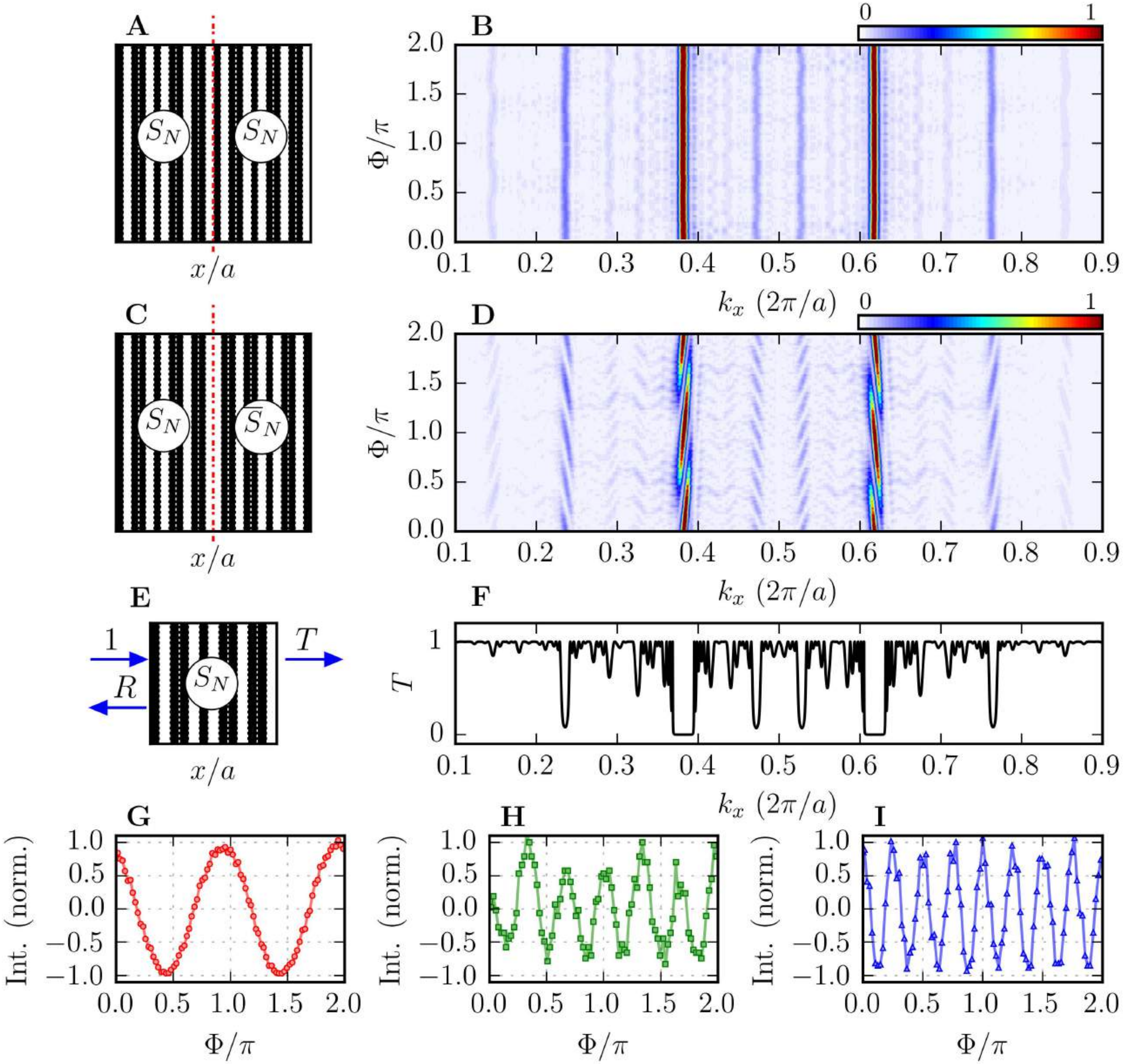

q=-3
q=2
q=-1
q=4
q=-4
q=1
q=-2
q=3
T
1
0
φ/π
2
1.5
1
0.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
k

A Young-sli asurement of windings

To summarize



A Young-slit measurement of windings



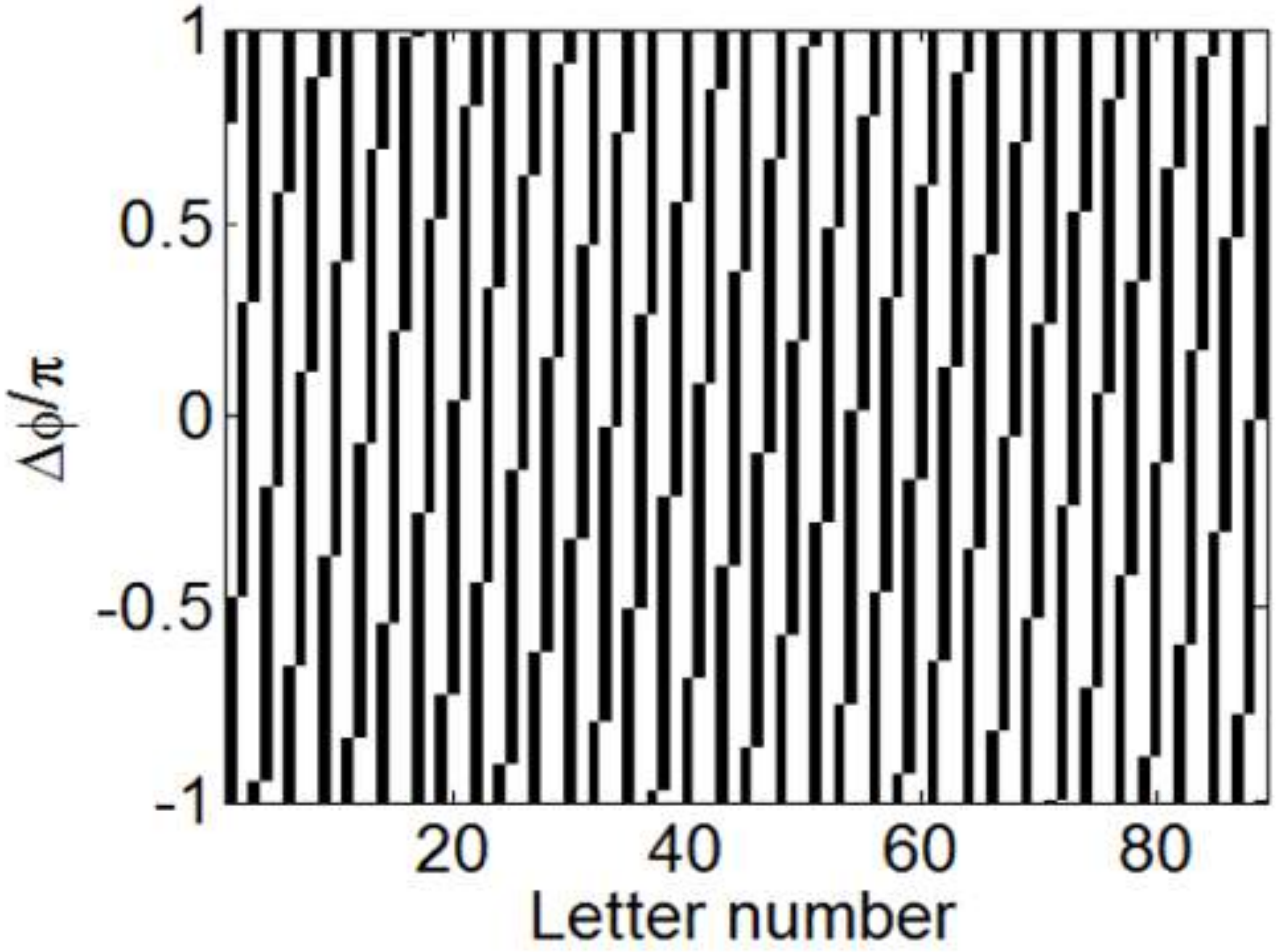
We can do better !

Single shot experiment for all winding numbers

2D diffraction experiment

Instead of

Consider all the realisations for all values of $\Delta\phi$ in a period



2D diffraction experiment

Instead of 

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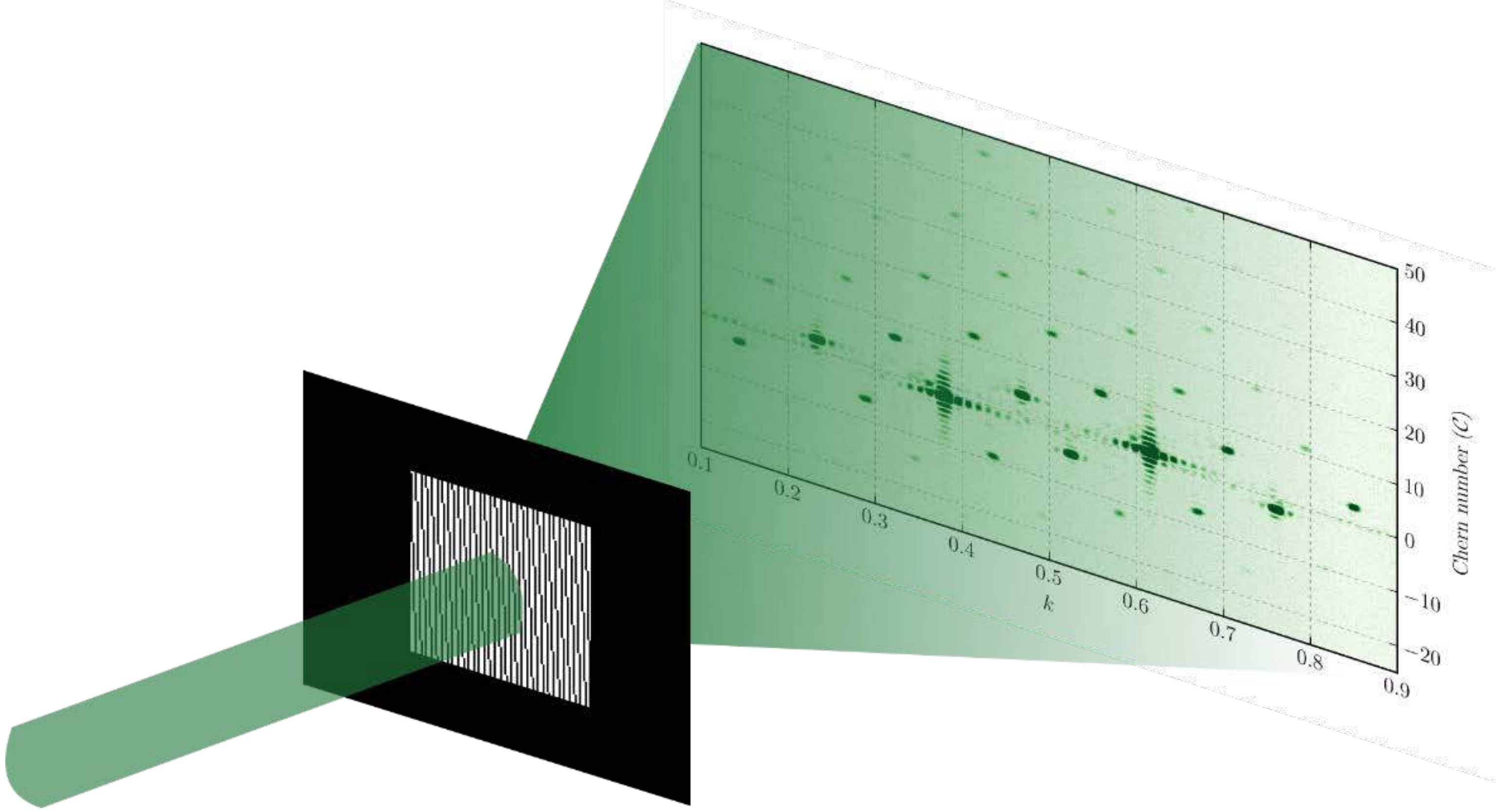
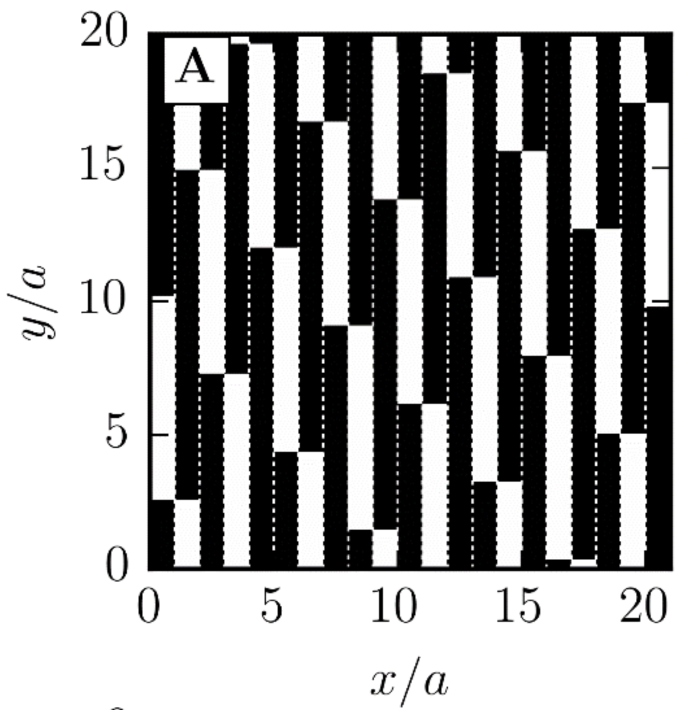
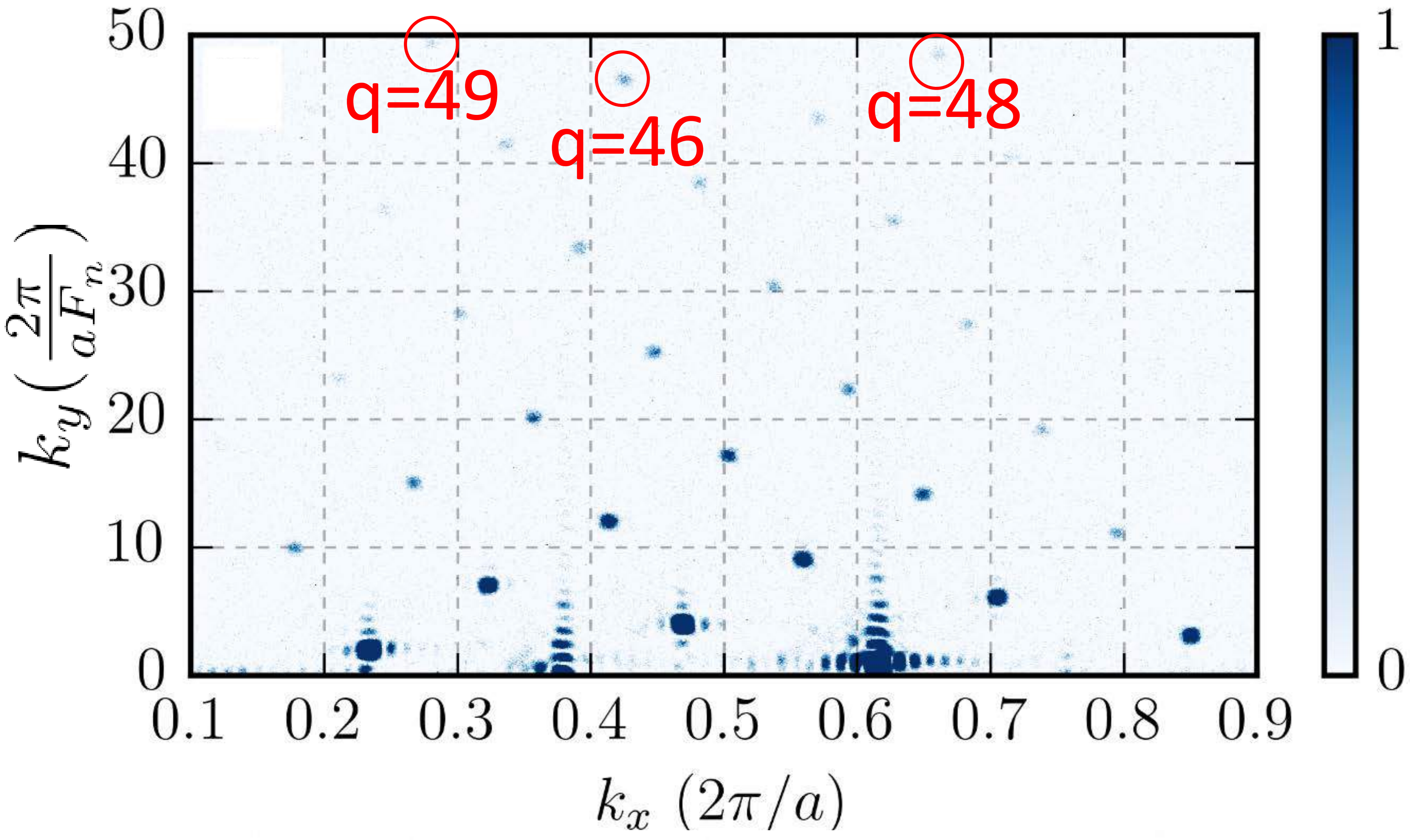


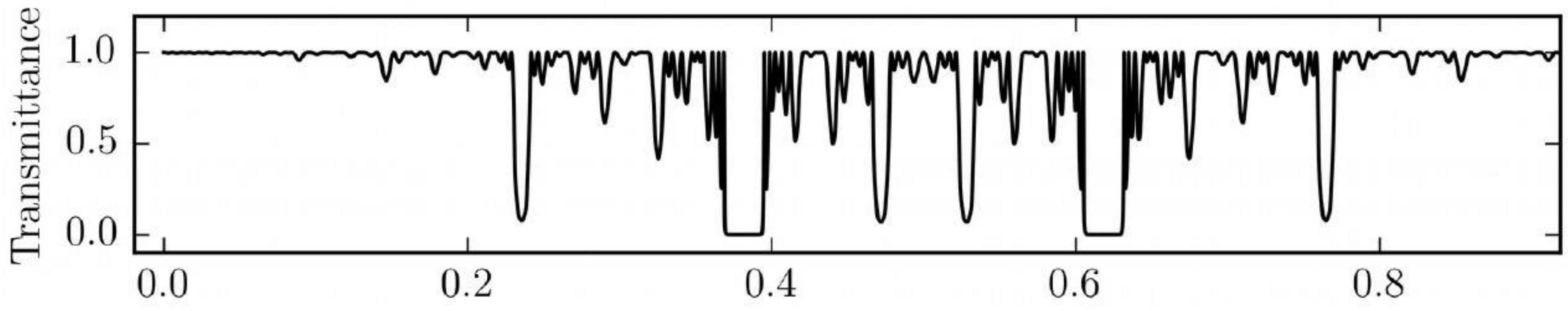
DMD Pattern

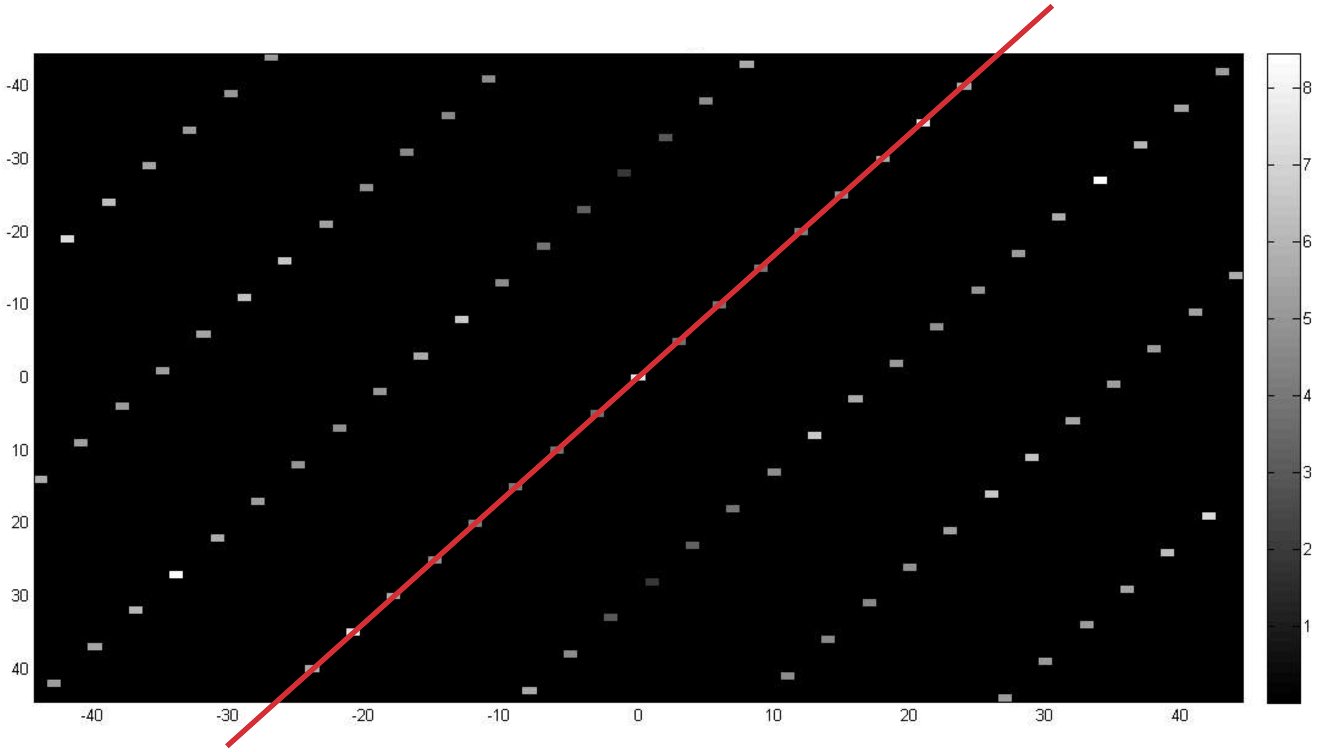


y axis is associated with Φ

Diffraction pattern






-40
-30
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-10
0
10
20
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-30
-20
-10
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20
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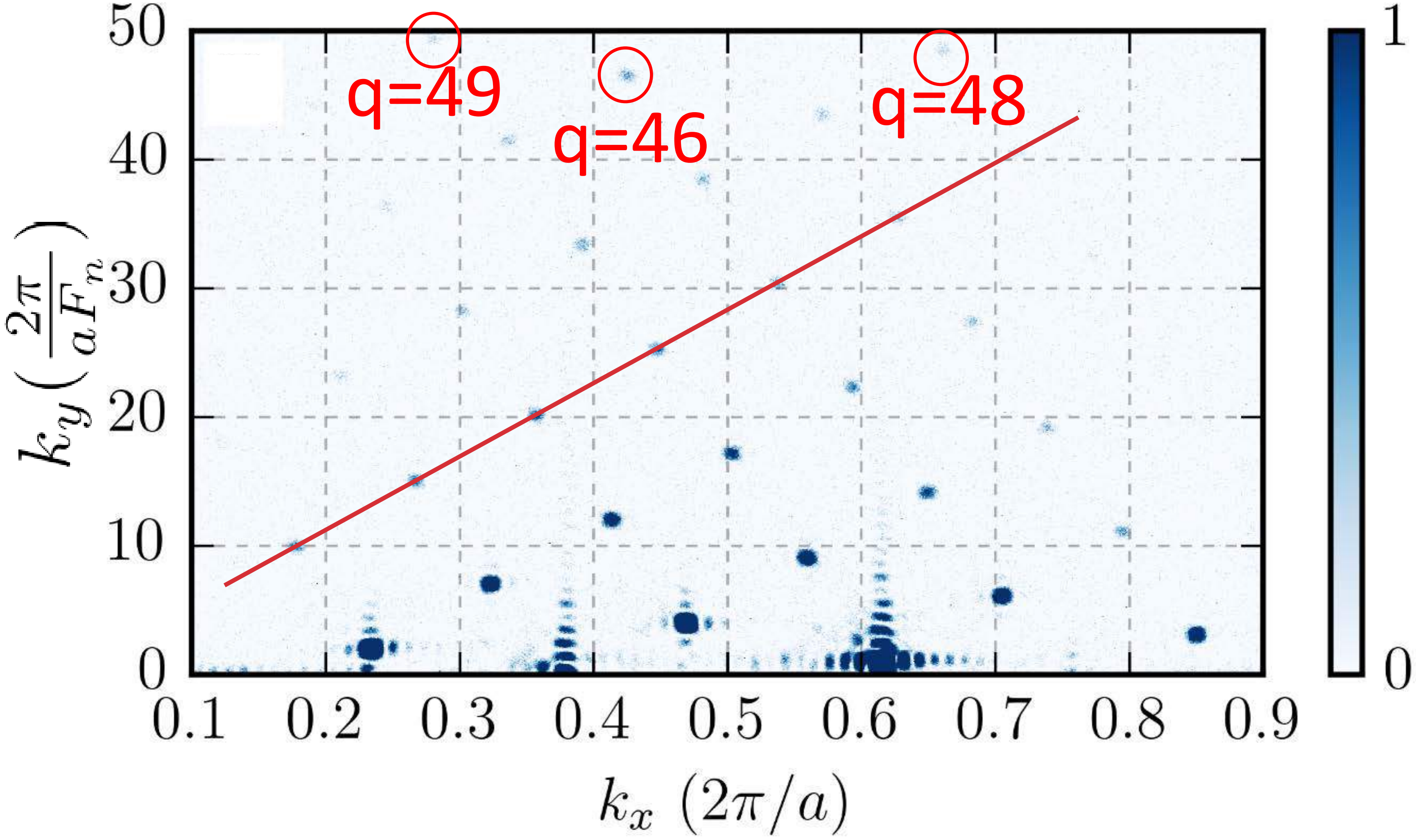
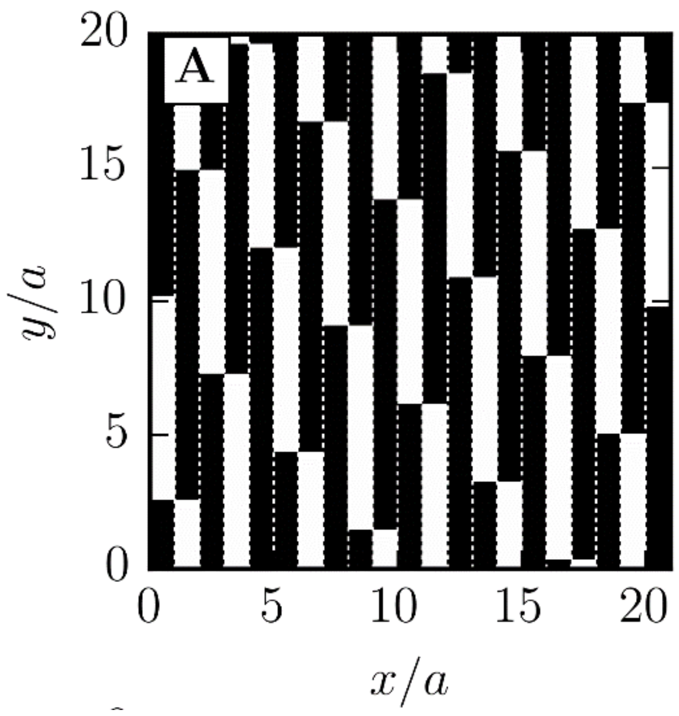
6
5
4
3
2
1


q=49
q=46
q=48
50
40
30
20
10
0
$k_y\left(\frac{2\pi}{aF_n}\right)$
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
$k_x\ (2\pi/a)$
1
0

DMD Pattern



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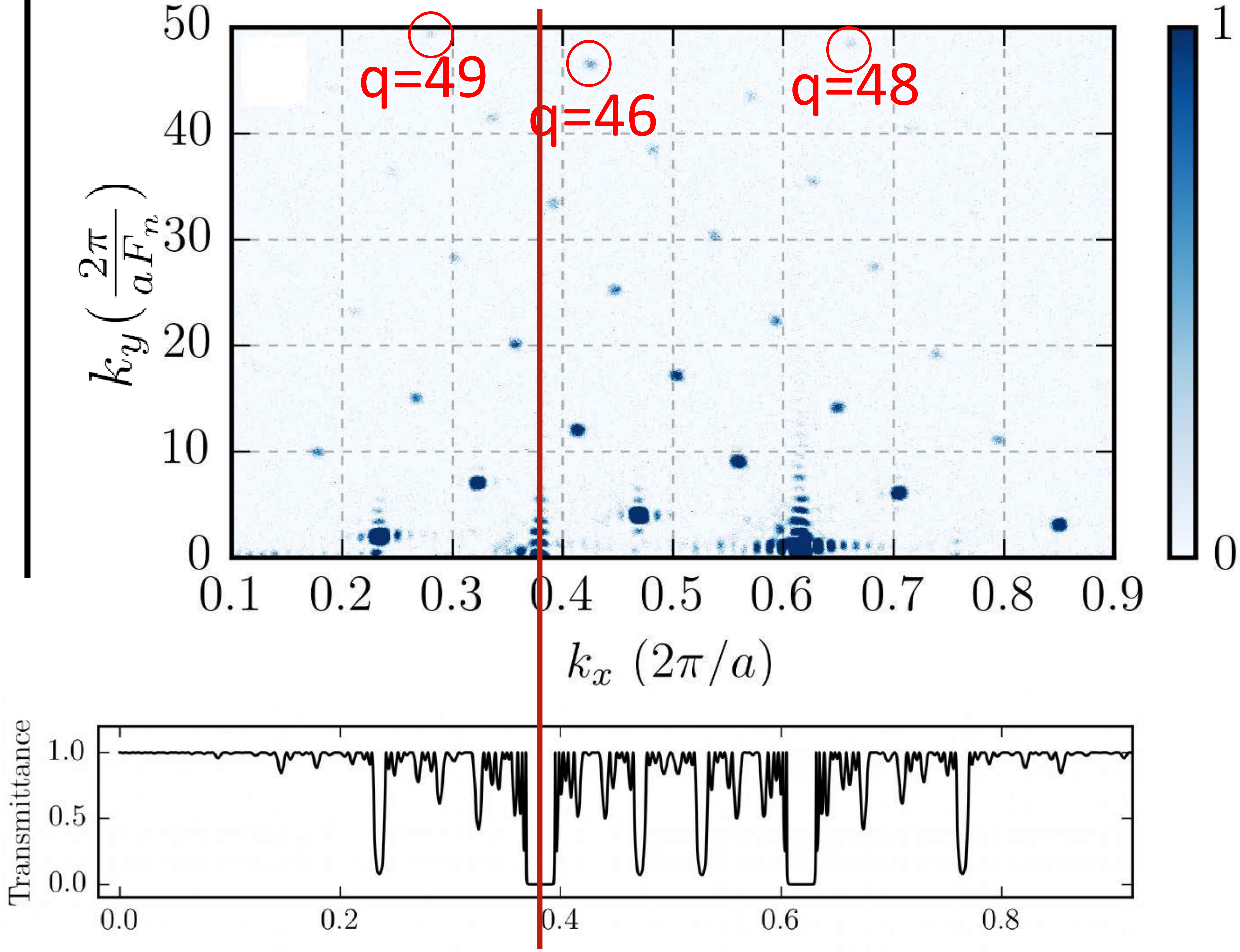
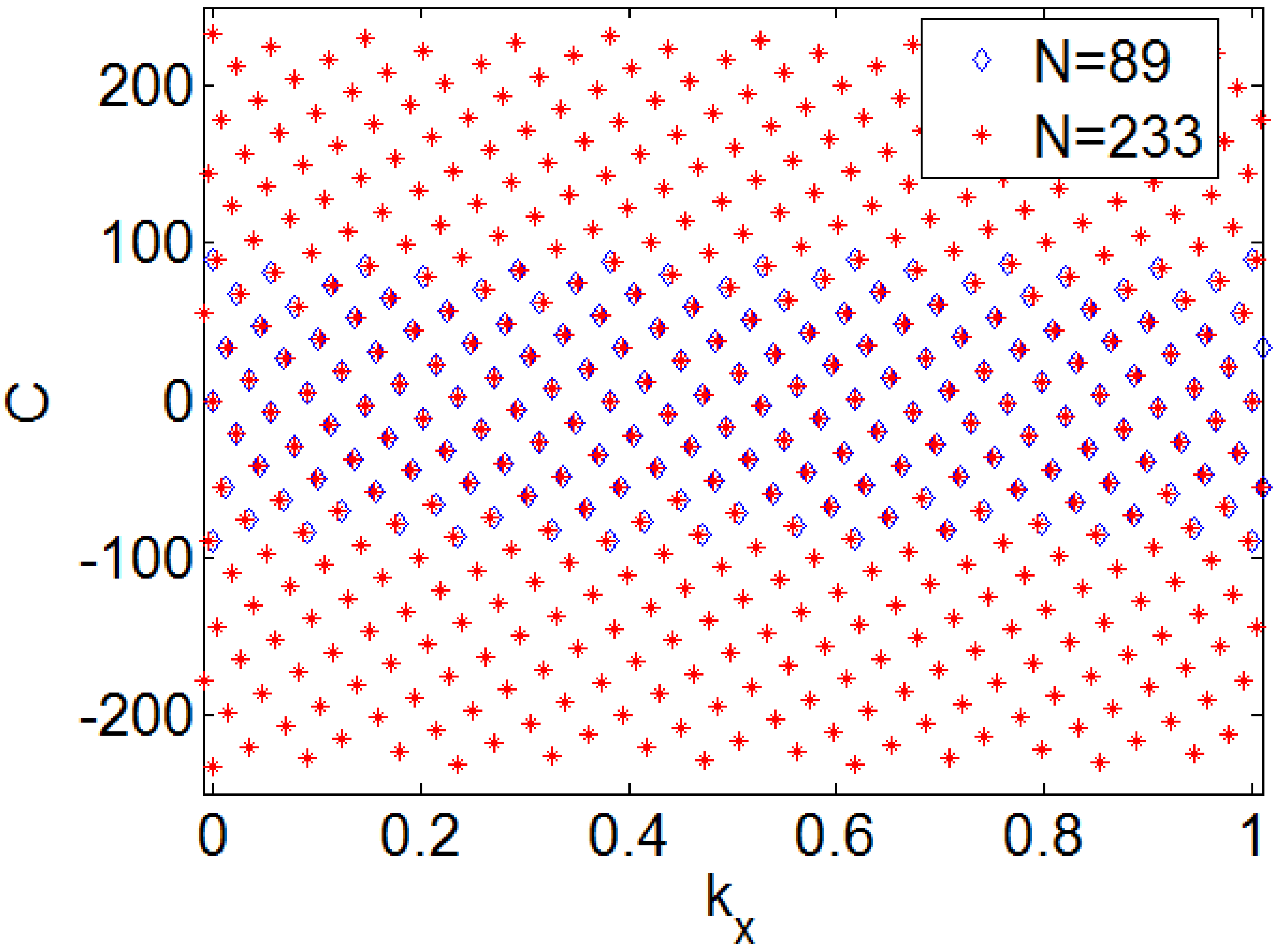


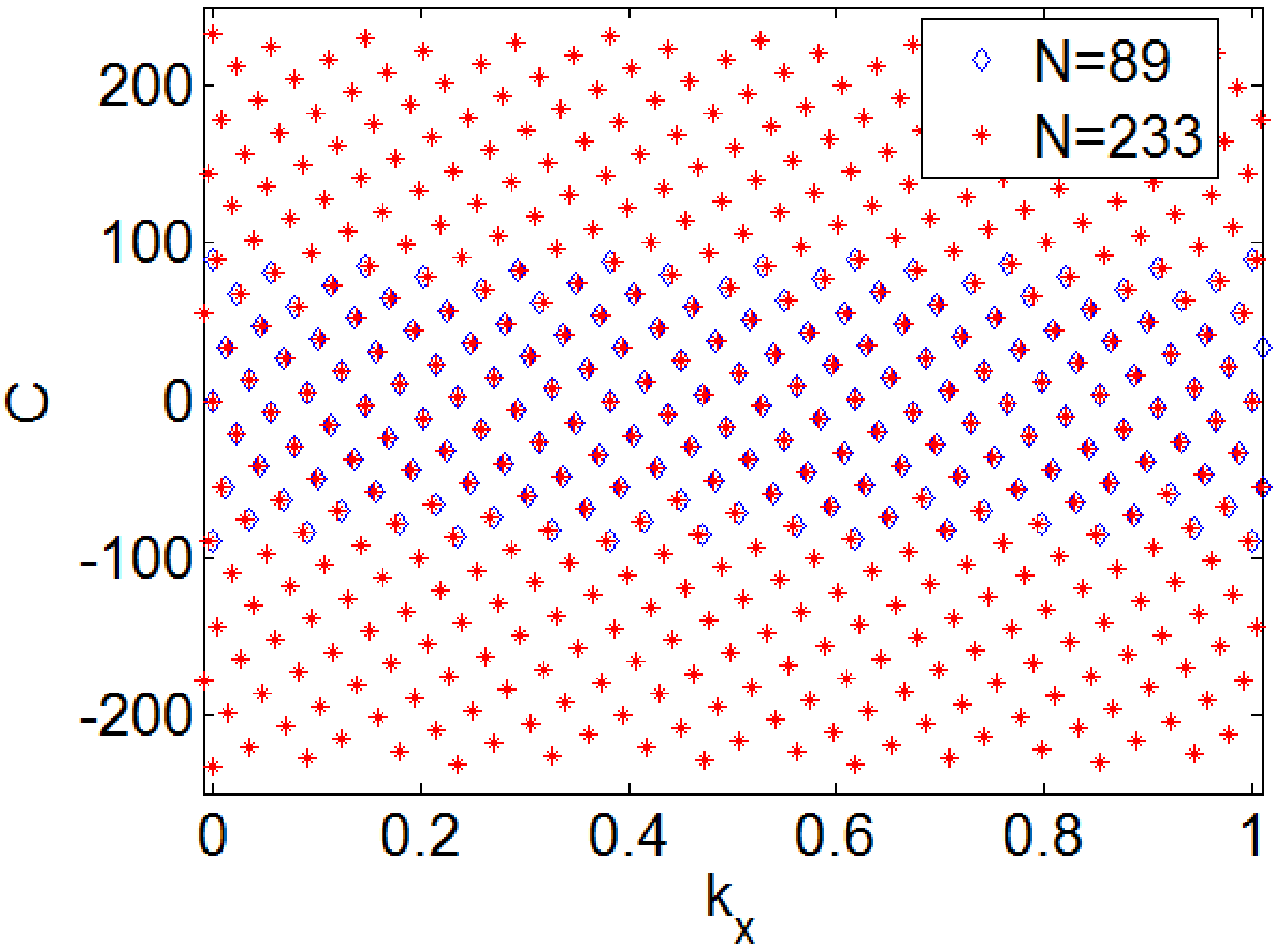
Structural origin of Chern numbers

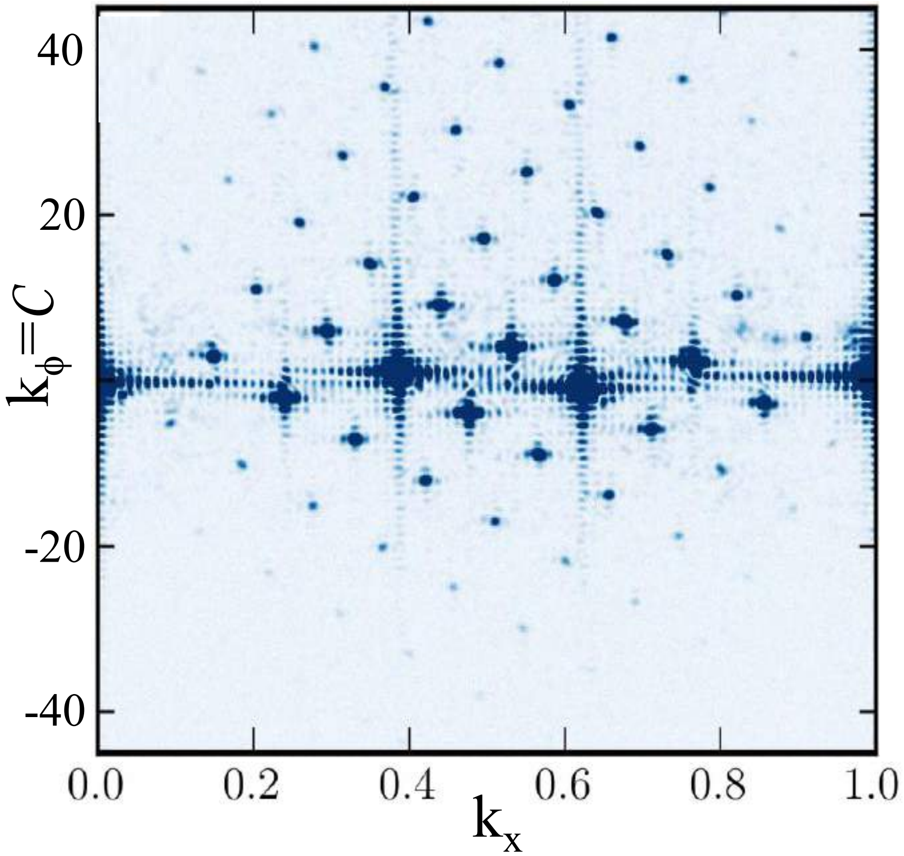
For a chain of size N, there exist N values of the Chern numbers



Structural origin of Chern numbers

For a chain of size N, there exist N values of the Chern numbers - all visible in the diffraction experiment

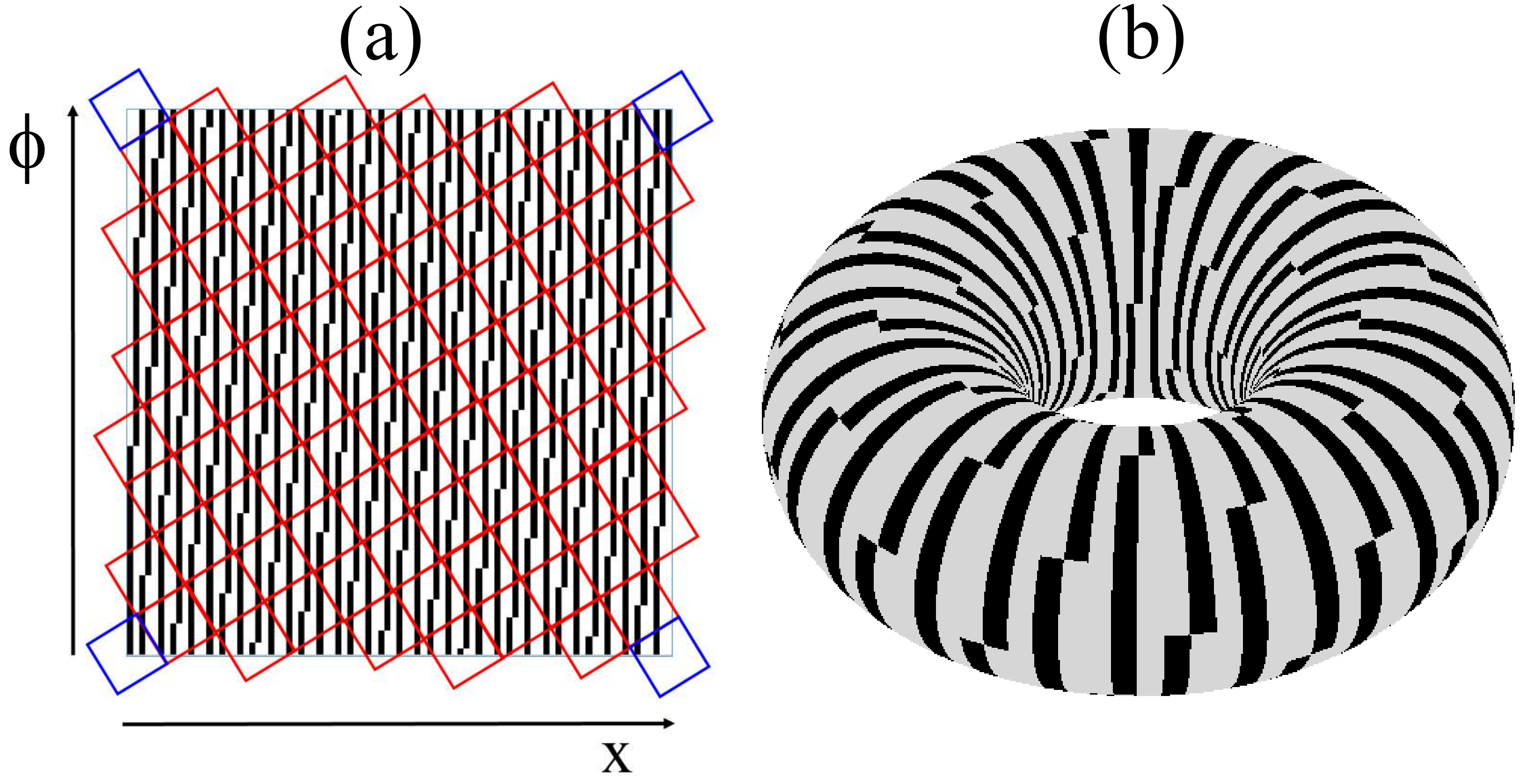




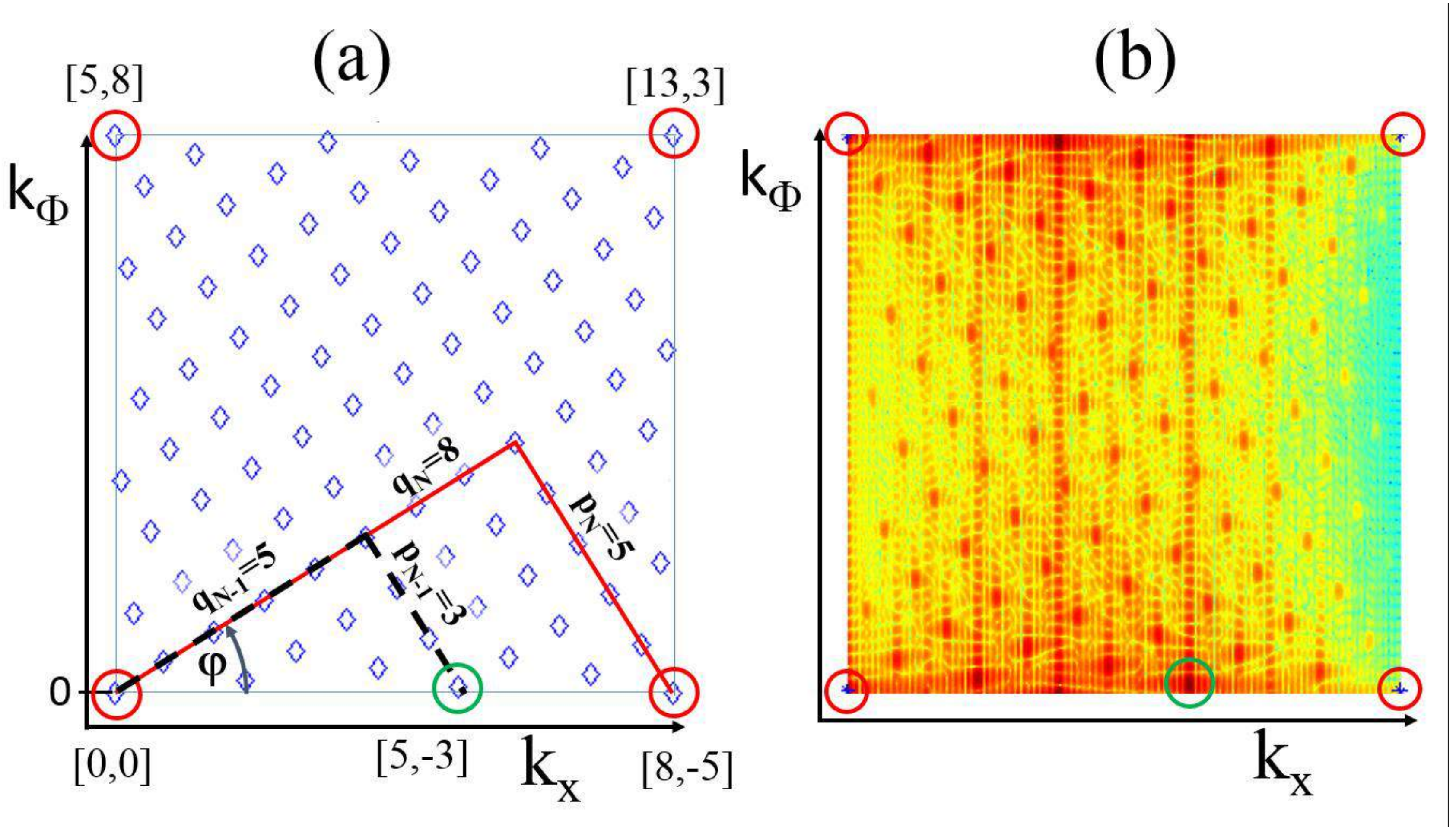
Chern numbers appear in pairs (-q,q)

Quasi-Brillouin zone

Real space



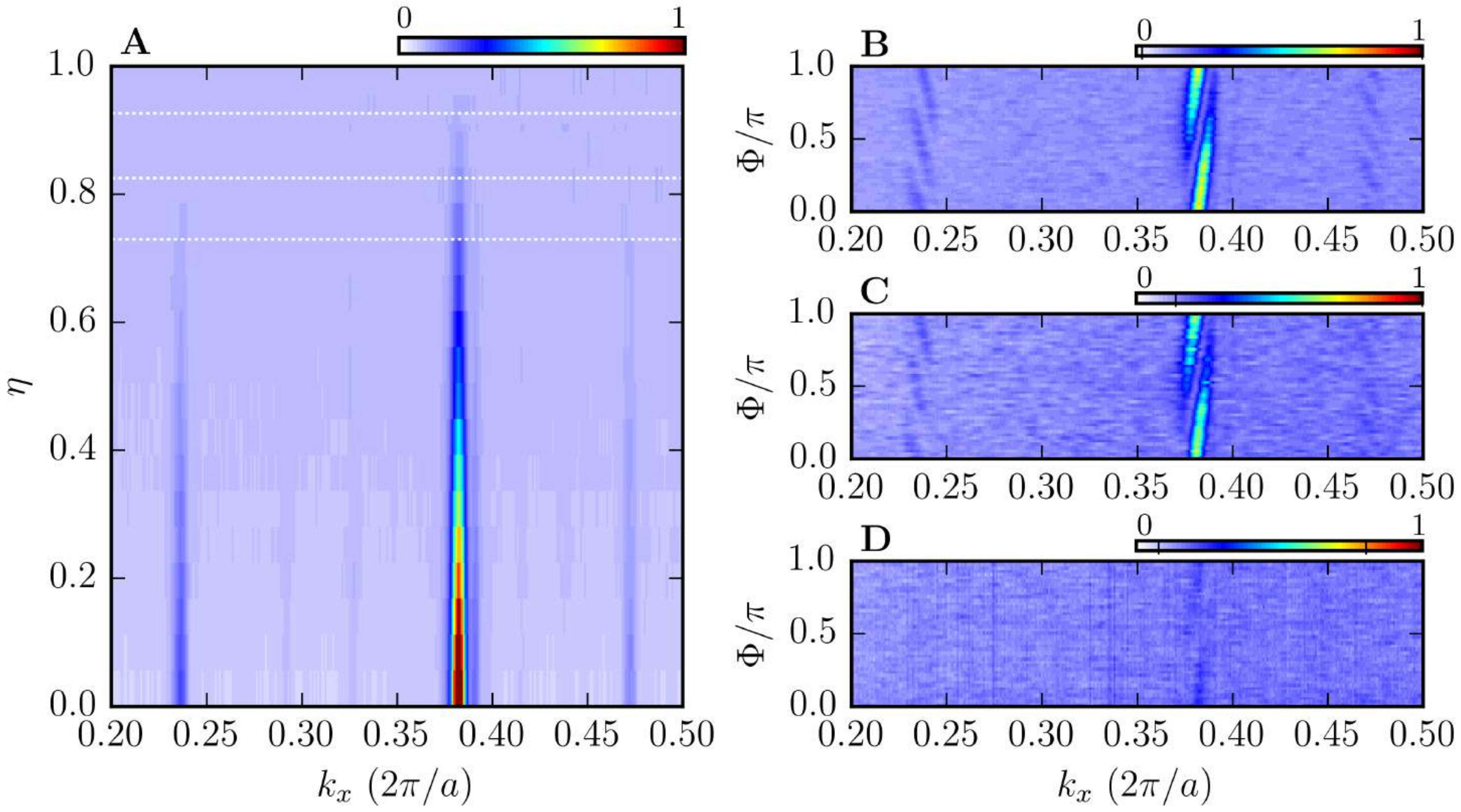
Reciprocal space



Robustness of topological features against noise.

Chern numbers are advertised as topological invariants, i.e. rather insensitive against disorder.

Make it more quantitative

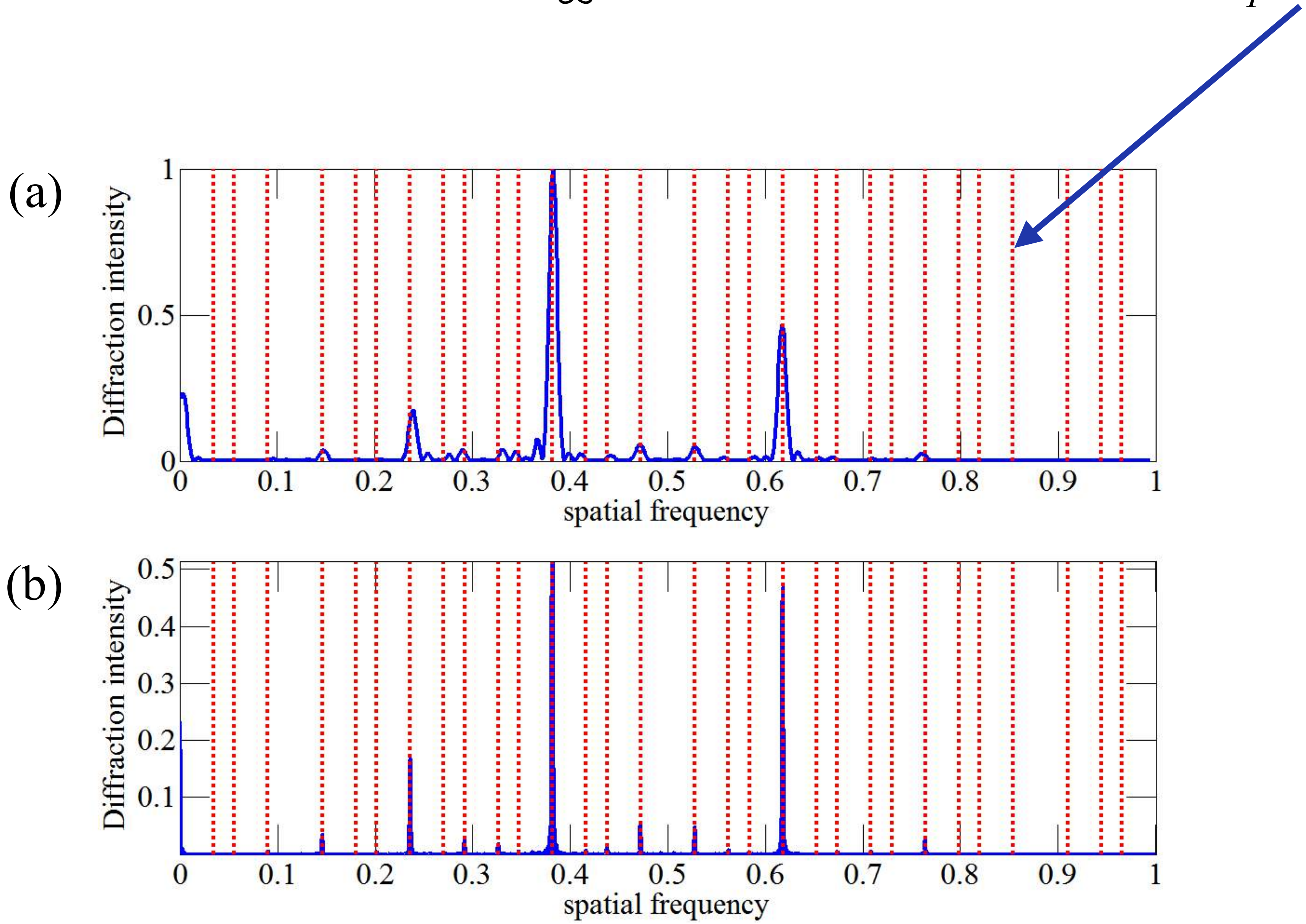


Diffraction peaks-Finite chains

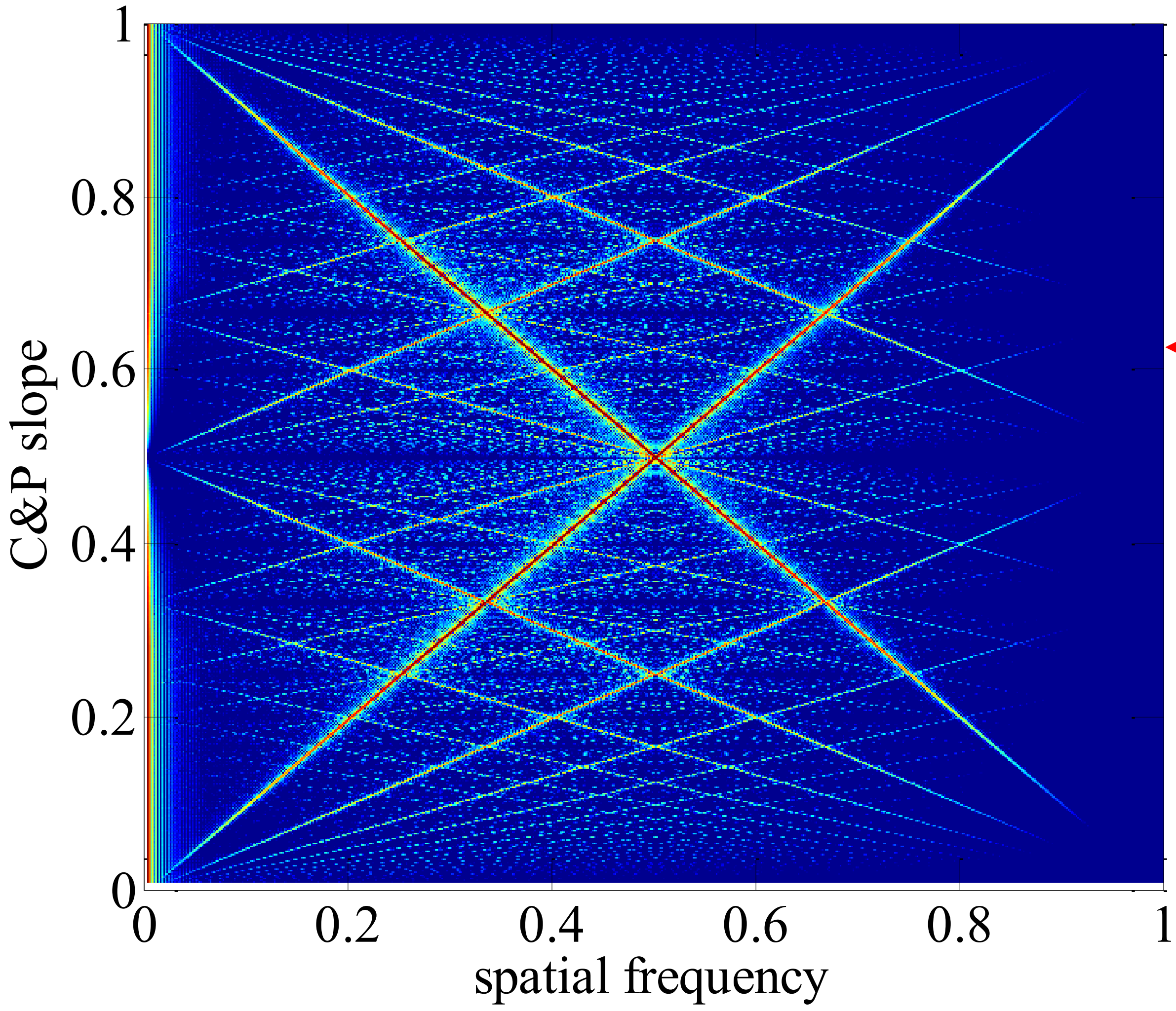
For the infinite Fibonacci chain S_∞ Bragg peaks are at $k_q = p + q\tau^{-1}$

S_{10}

S_{14}



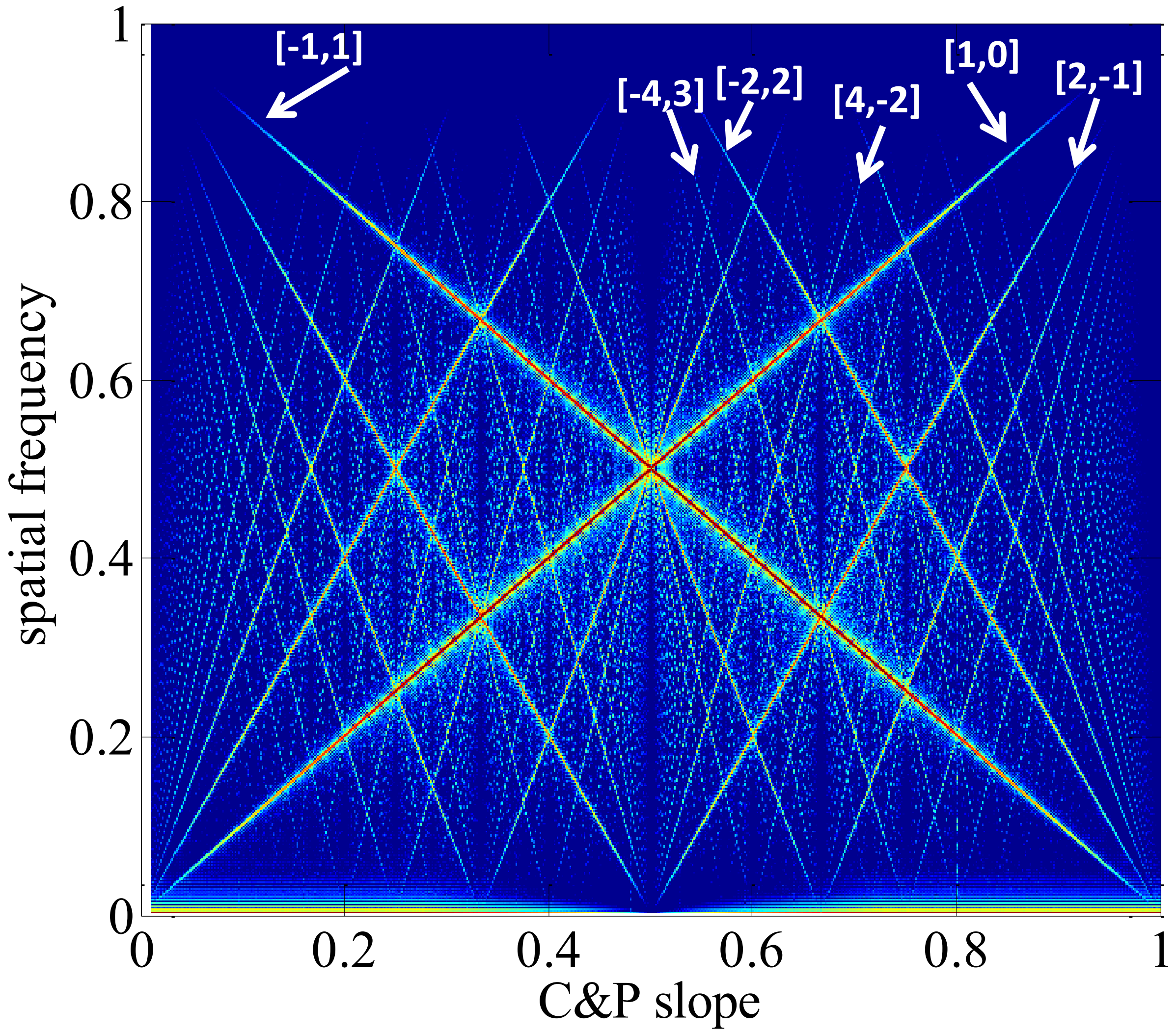
Wannier diagram



Fibonacci chain slope τ^{-1}

$$y = qx + p$$

Spectrum and Wannier diagram



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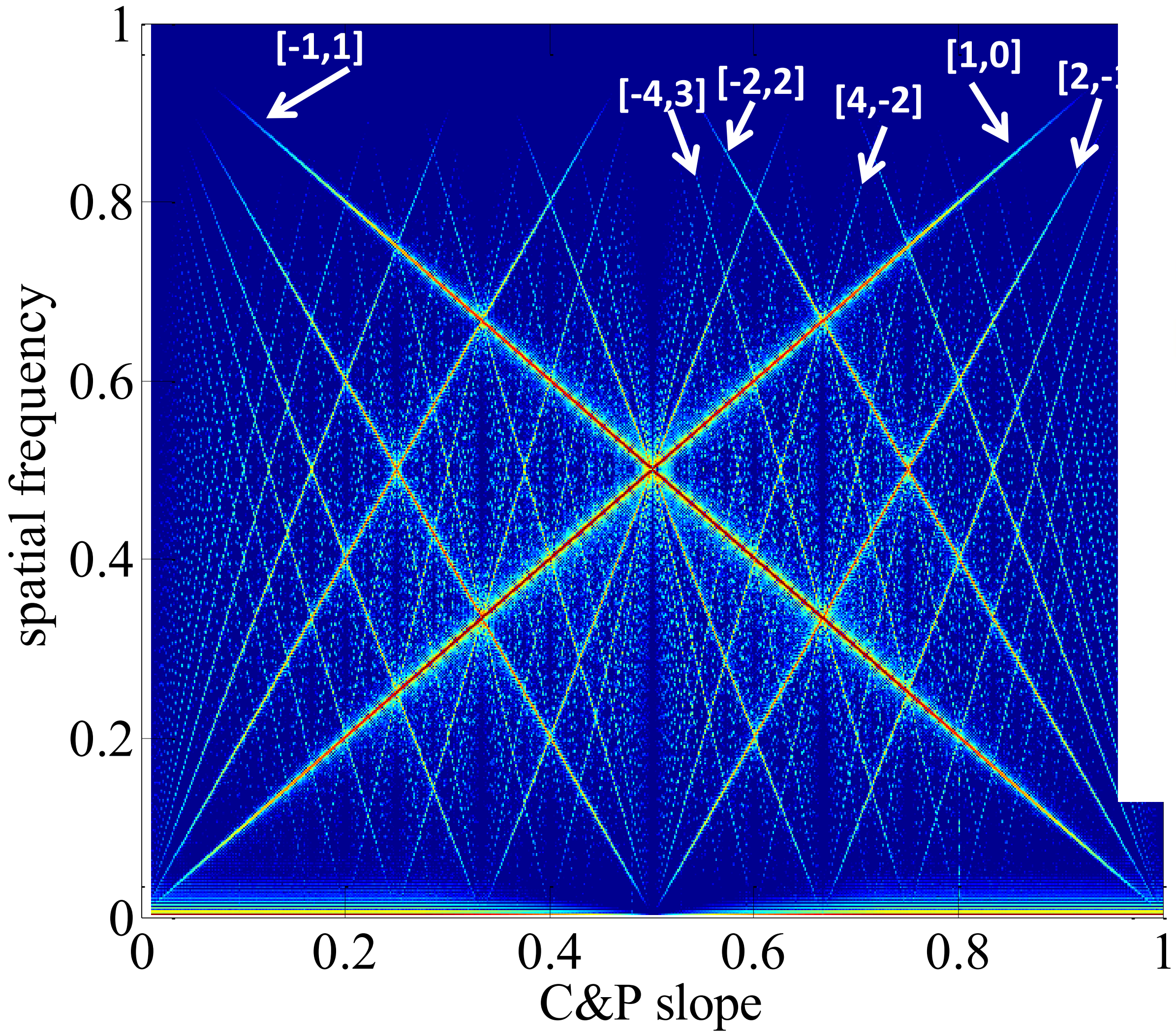

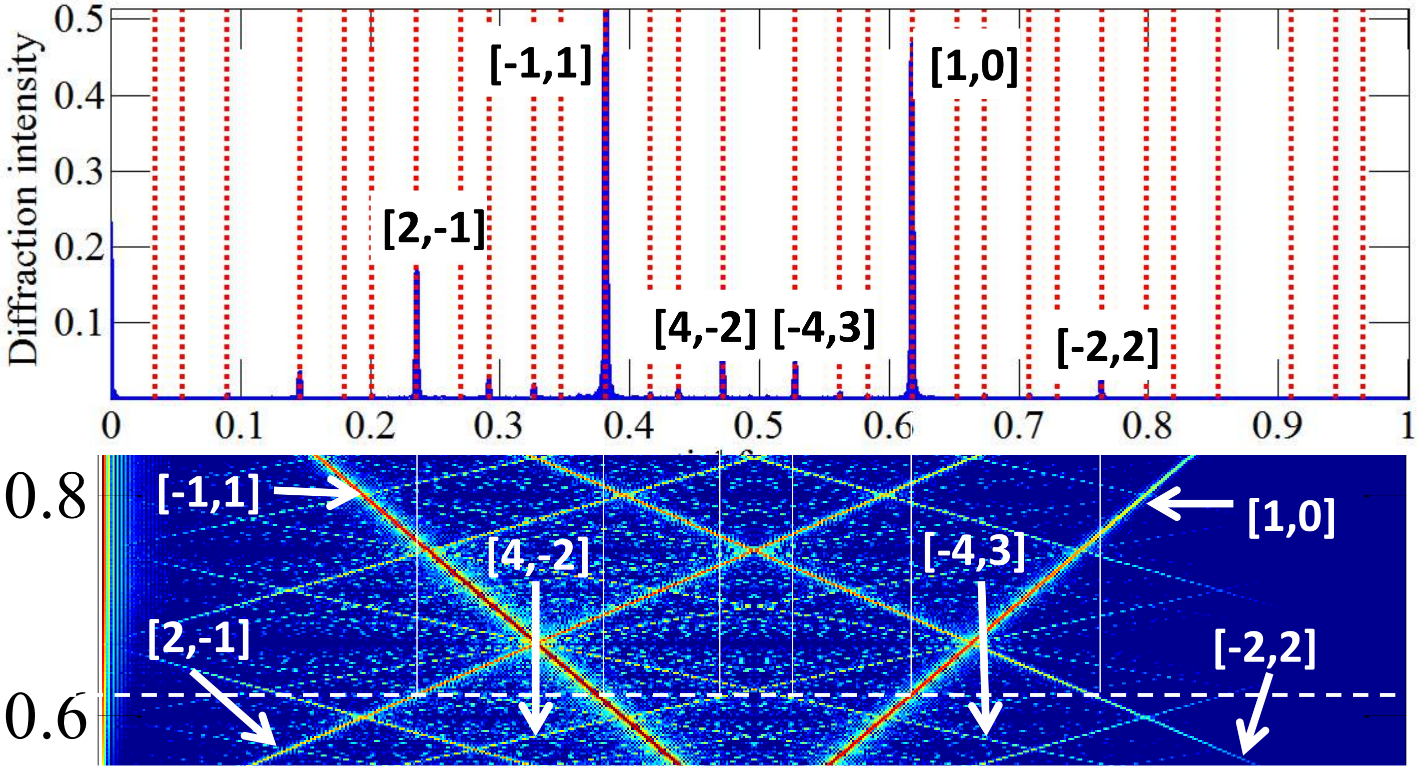

Summary-Further directions

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- Scattering theory gives a simple way to calculate/measure Chern numbers.
- Never underestimate the information contained in 2x2 matrices.

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- Gap traversing of edge states is completely determined by corresponding gap Chern numbers.
- Scattering theory gives a simple way to calculate/measure Chern numbers.
- Never underestimate the information contained in 2x2 matrices.
- Topological Chern numbers are also contained in structural data of the quasi-crystals. They can be retrieved from a Young-slit diffraction experiment. This approach allows for a measurement of high numbers even for short chains.

Summary

- Complete characterisation of the fractal spectrum of quasi-periodic systems.

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Summary

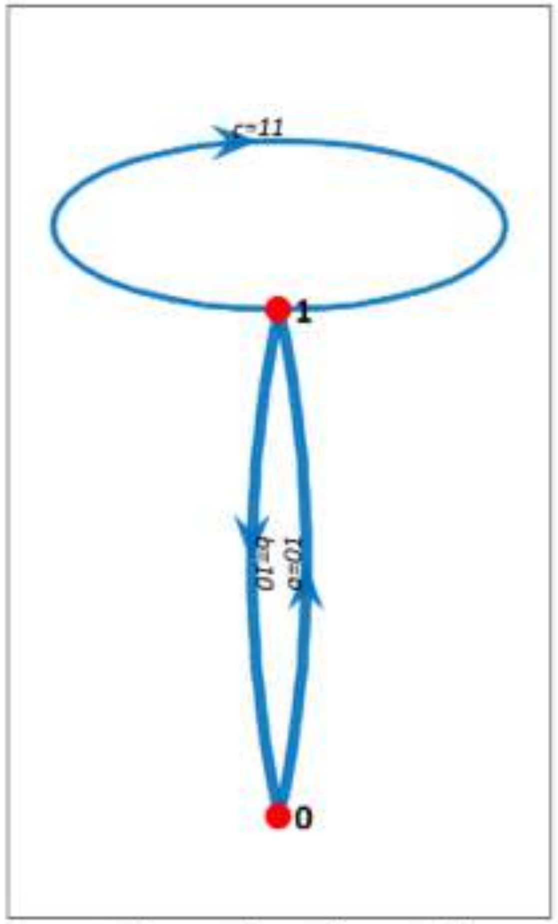
- Complete characterisation of the fractal spectrum of quasi-periodic systems.
- Given a topological meaning to the integers labelling the gaps of the fractal spectrum.
- Proposed a complete algebraic structure to account for the topological integers (Abelian group structure isomorphic to $\mathbb{Z}/F_N\mathbb{Z}$
- This Abelian group is isomorphic to the cohomology group $H^{(1)}$ defined on (Bratelli) graphs associated to the quasi periodic structures.

Name	**Substitution**		**Substitution on Doublets**		**Self Properties**		**Cohomology**		**Zeta Function**	**Gap Labeling Theorem**		**Properties**	
	Rule σ_1	Occurrence M_1	Rule σ_2	Occurrence M_2	Eigenvalue	Char. Polynomial	$H^0(G)$	$H^1(G)$	$\zeta(z)$			Pisot char.	Periodicity
Fibonacci	$\begin{array}{l}0 \mapsto 01\\1 \mapsto 0\end{array}$	$\begin{pmatrix}1&1\\1&0\end{pmatrix}$	$\begin{array}{l}a \mapsto bc\\b \mapsto bc\\c \mapsto a\end{array}$	$\begin{pmatrix}0&1&1\\0&1&1\\1&0&0\end{pmatrix}$	τ	$\lambda^2-\lambda-1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\frac{1-z}{1-z-z^2}$	$p+q\cdot\tau$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Cantor Set	$\begin{array}{l}0 \mapsto 010\\1 \mapsto 111\end{array}$	$\begin{pmatrix}2&1\\0&3\end{pmatrix}$	$\begin{array}{l}a \mapsto aba\\b \mapsto ccb\\c \mapsto ccc\end{array}$	$\begin{pmatrix}2&1&0\\0&1&2\\0&0&3\end{pmatrix}$	3	$\lambda^2-5\lambda+6=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{(1-2z)(1-3z)}$	$\frac{k}{3^N}$	$k,N\in\mathbb{Z}$	not primitive	limit-quasiperiodic
Non-Pisot	$\begin{array}{l}0 \mapsto 0001\\1 \mapsto 011\end{array}$	$\begin{pmatrix}3&1\\1&2\end{pmatrix}$	$\begin{array}{l}a \mapsto aabc\\b \mapsto aabc\\c \mapsto bdc\\d \mapsto bdc\end{array}$	$\begin{pmatrix}2&1&1&0\\2&1&1&0\\0&1&1&1\\0&1&1&1\end{pmatrix}$	$\tau+2$	$\lambda^2-5\lambda+5=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{1-5z+5z^2}$	$\frac{p+q\cdot\tau}{5^N}$	$p,q,N\in\mathbb{Z}$	non-Pisot	limit-quasiperiodic
Periodic	$\begin{array}{l}0 \mapsto 01\\1 \mapsto 01\end{array}$	$\begin{pmatrix}1&1\\1&1\end{pmatrix}$	$\begin{array}{l}a \mapsto ab\\b \mapsto ab\end{array}$	$\begin{pmatrix}1&1\\1&1\end{pmatrix}$	2	$\lambda^2-2\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^1$	$\frac{1-z}{1-2z}$	$\frac{k}{2}$	$k\in\mathbb{Z}$	Pisot	periodic
Thue-Morse	$\begin{array}{l}0 \mapsto 01\\1 \mapsto 10\end{array}$	$\begin{pmatrix}1&1\\1&1\end{pmatrix}$	$\begin{array}{l}a \mapsto bc\\b \mapsto bd\\c \mapsto ca\\d \mapsto cb\end{array}$	$\begin{pmatrix}0&1&1&0\\0&1&0&1\\1&0&1&0\\0&1&1&0\end{pmatrix}$	2	$\lambda^2-2\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{(1-2z)(1+z)}$	$\frac{k}{3\cdot 2^N}$	$k,N\in\mathbb{Z}$	Pisot	aperiodic
Sierpiński	$\begin{array}{l}0 \mapsto 01010\\1 \mapsto 11\end{array}$	$\begin{pmatrix}3&2\\0&2\end{pmatrix}$	$\begin{array}{l}a \mapsto ababa\\b \mapsto cb\\c \mapsto cc\end{array}$	$\begin{pmatrix}3&2&0\\0&1&1\\0&0&2\end{pmatrix}$	3	$\lambda^2-5\lambda+6=0$	$\mathbb{Z}^1$	$\mathbb{Z}^4$	$\frac{1-z}{(1-2z)(1-3z)}$	$\frac{k}{3^N}$	$k,N\in\mathbb{Z}$	not primitive	limit-quasiperiodic
Degen. Sierpiński	$\begin{array}{l}0 \mapsto 0001\\1 \mapsto 1112\\2 \mapsto 1112\end{array}$	$\begin{pmatrix}3&1&0\\0&3&1\\0&3&1\end{pmatrix}$	$\begin{array}{l}a \mapsto aabc\\b \mapsto aabd\\c \mapsto ddef\\d \mapsto ddeg\\e \mapsto ddeg\\f \mapsto ddef\\g \mapsto ddeg\end{array}$	$\begin{pmatrix}2&1&1&0&0&0&0\\2&1&0&1&0&0&0\\0&0&0&2&1&1&0\\0&0&0&2&1&0&1\\0&0&0&2&1&0&1\\0&0&0&2&1&1&0\\0&0&0&2&1&0&1\end{pmatrix}$	4	$\lambda^3-7\lambda^2+12\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{(1-3z)(1-4z)}$	$\frac{k}{4^N}$	$k,N\in\mathbb{Z}$	not primitive	aperiodic
Period Doubling	$\begin{array}{l}0 \mapsto 01\\1 \mapsto 00\end{array}$	$\begin{pmatrix}1&1\\2&0\end{pmatrix}$	$\begin{array}{l}a \mapsto bc\\b \mapsto bc\\c \mapsto aa\end{array}$	$\begin{pmatrix}0&1&1\\0&1&1\\2&0&0\end{pmatrix}$	2	$\lambda^2-\lambda-2=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{(1-2z)(1+z)}$	$\frac{k}{3\cdot 2^N}$	$k,N\in\mathbb{Z}$	non-Pisot	limit-quasiperiodic
Circle Sequence	$\begin{array}{l}0 \mapsto 202\\1 \mapsto 02202\\2 \mapsto 01202\end{array}$	$\begin{pmatrix}1&0&2\\2&0&3\\2&1&2\end{pmatrix}$	$\begin{array}{l}a \mapsto dbd\\b \mapsto dbd\\c \mapsto bedbd\\d \mapsto acdbe\\e \mapsto acdbd\end{array}$	$\begin{pmatrix}0&1&0&2&0\\0&1&0&2&0\\0&2&0&2&1\\1&1&1&1&1\\1&1&1&2&0\end{pmatrix}$	τ^3	$\lambda^3-3\lambda^2-5\lambda-1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{(1+z)(1-4z-z^2)}$	$\frac{1}{2}(p+q\cdot\tau)$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Rudin-Shapiro	$\begin{array}{l}0 \mapsto 02\\1 \mapsto 32\\2 \mapsto 01\\3 \mapsto 31\end{array}$	$\begin{pmatrix}1&0&1&0\\0&0&1&1\\1&1&0&0\\0&1&0&1\end{pmatrix}$	$\begin{array}{l}a \mapsto bf\\b \mapsto be\\c \mapsto he\\d \mapsto hf\\e \mapsto ac\\f \mapsto ad\\g \mapsto gd\\h \mapsto gc\end{array}$	$\begin{pmatrix}0&1&0&0&0&1&0&0\\0&1&0&0&1&0&0&0\\0&0&0&0&1&0&0&1\\0&0&0&0&0&1&0&1\\1&0&1&0&0&0&0&0\\1&0&0&1&0&0&0&0\\0&0&0&1&0&0&1&0\\0&0&1&0&0&0&1&0\end{pmatrix}$	2	$\lambda^4-2\lambda^3-2\lambda^2+4\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^9$	$\frac{1-z}{(1-2z)(1-2z^2)(1+z)}$	$\frac{k}{2^N}$	$k,N\in\mathbb{Z}$	non-Pisot	aperiodic
Skau Example #1	$\begin{array}{l}0 \mapsto 001\\1 \mapsto 0101\end{array}$	$\begin{pmatrix}2&1\\2&2\end{pmatrix}$	$\begin{array}{l}a \mapsto abc\\b \mapsto abc\\c \mapsto bcbc\end{array}$	$\begin{pmatrix}1&1&1\\1&1&1\\0&2&2\end{pmatrix}$	$\sqrt{2}+2$	$\lambda^2-4\lambda+2=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{1-4z+2z^2}$	$\frac{p+q\sqrt{2}}{2^N}$	$p,q,N\in\mathbb{Z}$	Pisot	limit-quasiperiodic
Skau Example #2	$\begin{array}{l}0 \mapsto 010\\1 \mapsto 01\end{array}$	$\begin{pmatrix}2&1\\1&1\end{pmatrix}$	$\begin{array}{l}a \mapsto bca\\b \mapsto bca\\c \mapsto bc\end{array}$	$\begin{pmatrix}1&1&1\\1&1&1\\0&1&1\end{pmatrix}$	$\tau+1$	$\lambda^2-3\lambda+1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\frac{1-z}{1-3z+z^2}$	$p+q\cdot\tau$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Skau Example #3	$\begin{array}{l}0 \mapsto 001\\1 \mapsto 10\end{array}$	$\begin{pmatrix}2&1\\1&1\end{pmatrix}$	$\begin{array}{l}a \mapsto abc\\b \mapsto abd\\c \mapsto ca\\d \mapsto cb\end{array}$	$\begin{pmatrix}1&1&1&0\\1&1&0&1\\1&0&1&0\\0&1&1&0\end{pmatrix}$	$\tau+1$	$\lambda^2-3\lambda+1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^5$	$\frac{1-z}{(1+z)(1-3z+z^2)}$	$\frac{p+q\cdot\tau}{5}$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Skau Example #4	$\begin{array}{l}0 \mapsto 010\\1 \mapsto 1001\end{array}$	$\begin{pmatrix}2&1\\2&2\end{pmatrix}$	$\begin{array}{l}a \mapsto bca\\b \mapsto bcb\\c \mapsto cabc\end{array}$	$\begin{pmatrix}1&1&1\\0&2&1\\1&1&2\end{pmatrix}$	$\sqrt{2}+2$	$\lambda^2-4\lambda+2=0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\frac{1-z}{1-4z+2z^2}$	$\frac{p+q\sqrt{2}}{2^N}$	$p,q,N\in\mathbb{Z}$	Pisot	limit-quasiperiodic
Chacon	$\begin{array}{l}0 \mapsto 0010\\1 \mapsto 1\end{array}$	$\begin{pmatrix}3&1\\0&1\end{pmatrix}$	$\begin{array}{l}a \mapsto abca\\b \mapsto abcb\\c \mapsto c\end{array}$	$\begin{pmatrix}2&1&1\\1&2&1\\0&0&1\end{pmatrix}$	3	$\lambda^2-4\lambda+3=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1}{1-3z}$	$\frac{k}{3^N}$	$k,N\in\mathbb{Z}$	not primitive	limit-quasiperiodic
Golden Mean Squared	$\begin{array}{l}0 \mapsto 100\\1 \mapsto 10\end{array}$	$\begin{pmatrix}2&1\\1&1\end{pmatrix}$	$\begin{array}{l}a \mapsto cab\\b \mapsto cab\\c \mapsto cb\end{array}$	$\begin{pmatrix}1&1&1\\1&1&1\\0&1&1\end{pmatrix}$	$\tau+1$	$\lambda^2-3\lambda+1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\frac{1-z}{1-3z+z^2}$	$p+q\cdot\lambda_1$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Silver Mean Squared	$\begin{array}{l}0 \mapsto 1001000\\1 \mapsto 100\end{array}$	$\begin{pmatrix}5&2\\2&1\end{pmatrix}$	$\begin{array}{l}a \mapsto cabcaab\\b \mapsto cabcaab\\c \mapsto cab\end{array}$	$\begin{pmatrix}3&2&2\\3&2&2\\1&1&1\end{pmatrix}$	$2\sqrt{2}+3$	$\lambda^2-6\lambda+1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\frac{1-z}{1-6z+z^2}$	$p+q\cdot\lambda_1$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Copper Mean Squared	$\begin{array}{l}0 \mapsto 1000100010000\\1 \mapsto 1000\end{array}$	$\begin{pmatrix}10&3\\3&1\end{pmatrix}$	$\begin{array}{l}a \mapsto caabcaabcaaab\\b \mapsto caabcaabcaaab\\c \mapsto caab\end{array}$	$\begin{pmatrix}7&3&3\\7&3&3\\2&1&1\end{pmatrix}$	$\frac{3\sqrt{13}}{2}+\frac{11}{2}$	$\lambda^2-11\lambda+1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\frac{1-z}{1-11z+z^2}$	$p+q\cdot\lambda_1$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Luck Ternary #1	$\begin{array}{l}0 \mapsto 01\\1 \mapsto 02\\2 \mapsto 012\end{array}$	$\begin{pmatrix}1&1&0\\1&0&1\\1&1&1\end{pmatrix}$	$\begin{array}{l}a \mapsto ac\\b \mapsto ac\\c \mapsto be\\d \mapsto be\\e \mapsto ade\end{array}$	$\begin{pmatrix}1&0&1&0&0\\1&0&1&0&0\\0&1&0&0&1\\0&1&0&0&1\\1&0&0&1&1\end{pmatrix}$	2.247	$\lambda^3-2\lambda^2-\lambda+1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{1-2z-z^2+z^3}$	$p+q\cdot\lambda_1+r\cdot\lambda_1^2$	$p,q,r\in\mathbb{Z}$	Pisot	quasiperiodic
Luck Ternary #2	$\begin{array}{l}0 \mapsto 2\\1 \mapsto 0\\2 \mapsto 12\end{array}$	$\begin{pmatrix}0&0&1\\1&0&0\\0&1&1\end{pmatrix}$	$\begin{array}{l}a \mapsto c\\b \mapsto a\\c \mapsto be\\d \mapsto bc\\e \mapsto bd\end{array}$	$\begin{pmatrix}0&0&1&0&0\\1&0&0&0&0\\0&1&0&0&1\\0&1&1&0&0\\0&1&0&1&0\end{pmatrix}$	1.4656	$\lambda^3-\lambda^2-1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{1-z-z^3}$	$p+q\cdot\lambda_1+r\cdot\lambda_1^2$	$p,q,r\in\mathbb{Z}$	Pisot	quasiperiodic
Periodic 1-2	$\begin{array}{l}0 \mapsto 011\\1 \mapsto 011\end{array}$	$\begin{pmatrix}1&2\\1&2\end{pmatrix}$	$\begin{array}{l}a \mapsto acb\\b \mapsto acb\\c \mapsto acb\end{array}$	$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$	3	$\lambda^2-3\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^1$	$\frac{1-z}{1-3z}$	$\frac{k}{3}$	$k\in\mathbb{Z}$	Pisot	periodic
Periodic 1-3	$\begin{array}{l}0 \mapsto 0111\\1 \mapsto 0111\end{array}$	$\begin{pmatrix}1&3\\1&3\end{pmatrix}$	$\begin{array}{l}a \mapsto accb\\b \mapsto accb\\c \mapsto accb\end{array}$	$\begin{pmatrix}1&1&2\\1&1&2\\1&1&2\end{pmatrix}$	4	$\lambda^2-4\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^1$	$\frac{1-z}{1-4z}$	$\frac{k}{4}$	$k\in\mathbb{Z}$	Pisot	periodic
Periodic 1-4	$\begin{array}{l}0 \mapsto 01111\\1 \mapsto 01111\end{array}$	$\begin{pmatrix}1&4\\1&4\end{pmatrix}$	$\begin{array}{l}a \mapsto acccb\\b \mapsto acccb\\c \mapsto acccb\end{array}$	$\begin{pmatrix}1&1&3\\1&1&3\\1&1&3\end{pmatrix}$	5	$\lambda^2-5\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^1$	$\frac{1-z}{1-5z}$	$\frac{k}{5}$	$k\in\mathbb{Z}$	Pisot	periodic
Periodic 2-3	$\begin{array}{l}0 \mapsto 00111\\1 \mapsto 00111\end{array}$	$\begin{pmatrix}2&3\\2&3\end{pmatrix}$	$\begin{array}{l}a \mapsto abddc\\b \mapsto abddc\\c \mapsto abddc\\d \mapsto abddc\end{array}$	$\begin{pmatrix}1&1&1&2\\1&1&1&2\\1&1&1&2\\1&1&1&2\end{pmatrix}$	5	$\lambda^2-5\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^1$	$\frac{1-z}{1-5z}$	$\frac{k}{5}$	$k\in\mathbb{Z}$	Pisot	periodic
Periodic 2-5	$\begin{array}{l}0 \mapsto 0011111\\1 \mapsto 0011111\end{array}$	$\begin{pmatrix}2&5\\2&5\end{pmatrix}$	$\begin{array}{l}a \mapsto abddddc\\b \mapsto abddddc\\c \mapsto abddddc\\d \mapsto abddddc\end{array}$	$\begin{pmatrix}1&1&1&4\\1&1&1&4\\1&1&1&4\\1&1&1&4\end{pmatrix}$	7	$\lambda^2-7\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^1$	$\frac{1-z}{1-7z}$	$\frac{k}{7}$	$k\in\mathbb{Z}$	Pisot	periodic
Golden Mean	$\begin{array}{l}0 \mapsto 10\\1 \mapsto 0\end{array}$	$\begin{pmatrix}1&1\\1&0\end{pmatrix}$	$\begin{array}{l}a \mapsto cb\\b \mapsto ca\\c \mapsto b\end{array}$	$\begin{pmatrix}0&1&1\\1&0&1\\0&1&0\end{pmatrix}$	τ	$\lambda^2-\lambda-1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\frac{1-z}{1-z-z^2}$	$p+q\cdot\lambda_1$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Silver Mean	$\begin{array}{l}0 \mapsto 100\\1 \mapsto 0\end{array}$	$\begin{pmatrix}2&1\\1&0\end{pmatrix}$	$\begin{array}{l}a \mapsto cab\\b \mapsto caa\\c \mapsto b\end{array}$	$\begin{pmatrix}1&1&1\\2&0&1\\0&1&0\end{pmatrix}$	$\sqrt{2}+1$	$\lambda^2-2\lambda-1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\frac{1-z}{1-2z-z^2}$	$p+q\cdot\lambda_1$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Copper Mean	$\begin{array}{l}0 \mapsto 1000\\1 \mapsto 0\end{array}$	$\begin{pmatrix}3&1\\1&0\end{pmatrix}$	$\begin{array}{l}a \mapsto caab\\b \mapsto caaa\\c \mapsto b\end{array}$	$\begin{pmatrix}2&1&1\\3&0&1\\0&1&0\end{pmatrix}$	$\frac{\sqrt{13}}{2}+\frac{3}{2}$	$\lambda^2-3\lambda-1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\frac{1-z}{1-3z-z^2}$	$p+q\cdot\lambda_1$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Marginal	$\begin{array}{l}0 \mapsto 001\\1 \mapsto 011\end{array}$	$\begin{pmatrix}2&1\\1&2\end{pmatrix}$	$\begin{array}{l}a \mapsto abc\\b \mapsto abc\\c \mapsto bdc\\d \mapsto bdc\end{array}$	$\begin{pmatrix}1&1&1&0\\1&1&1&0\\0&1&1&1\\0&1&1&1\end{pmatrix}$	3	$\lambda^2-4\lambda+3=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1}{1-3z}$	$\frac{k}{2\cdot 3^N}$	$k,N\in\mathbb{Z}$	non-Pisot	limit-quasiperiodic
Luck non-Pisot	$\begin{array}{l}0 \mapsto 0001\\1 \mapsto 110\end{array}$	$\begin{pmatrix}3&1\\1&2\end{pmatrix}$	$\begin{array}{l}a \mapsto aabc\\b \mapsto aabd\\c \mapsto dca\\d \mapsto dcb\end{array}$	$\begin{pmatrix}2&1&1&0\\2&1&0&1\\1&0&1&1\\0&1&1&1\end{pmatrix}$	$\tau+2$	$\lambda^2-5\lambda+5=0$	$\mathbb{Z}^1$	$\mathbb{Z}^5$	$\frac{1-z}{(z+1)(1-5z+5z^2)}$	$\frac{p+q\cdot\tau}{11\cdot 5^N}$	$p,q,N\in\mathbb{Z}$	non-Pisot	limit-quasiperiodic
Binary non-Pisot	$\begin{array}{l}0 \mapsto 01\\1 \mapsto 000\end{array}$	$\begin{pmatrix}1&1\\3&0\end{pmatrix}$	$\begin{array}{l}a \mapsto bc\\b \mapsto bc\\c \mapsto aaa\end{array}$	$\begin{pmatrix}0&1&1\\0&1&1\\3&0&0\end{pmatrix}$	$\frac{\sqrt{13}}{2}+\frac{1}{2}$	$\lambda^2-\lambda-3=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{1-z-3z^2}$	$\frac{p+q\cdot\lambda_1}{3^N}$	$p,q,N\in\mathbb{Z}$	non-Pisot	limit-quasiperiodic
Ternary non-Pisot	$\begin{array}{l}0 \mapsto 2\\1 \mapsto 0\\2 \mapsto 101\end{array}$	$\begin{pmatrix}0&0&1\\1&0&0\\1&2&0\end{pmatrix}$	$\begin{array}{l}a \mapsto e\\b \mapsto f\\c \mapsto b\\d \mapsto a\\e \mapsto cad\\f \mapsto cac\end{array}$	$\begin{pmatrix}0&0&0&0&1&0\\0&0&0&0&0&1\\0&1&0&0&0&0\\1&0&0&0&0&0\\1&0&1&1&0&0\\1&0&2&0&0&0\end{pmatrix}$	1.5214	$\lambda^3-\lambda-2=0$	$\mathbb{Z}^1$	$\mathbb{Z}^6$	$\frac{1}{1-z^2-2z^3}$	$\frac{p+q\cdot\lambda_1+r\cdot\lambda_1^2}{2^N}$	$p,q,r,N\in\mathbb{Z}$	non-Pisot	limit-quasiperiodic

Name	Substitution		Substitution on Doublets		Self Properties		Cohomology		Zeta Function	Gap Labeling Theorem		Properties	
	Rule σ_1	Occurrence M_1	Rule σ_2	Occurrence M_2	Eigenvalue	Char. Polynomial	$H^0(G)$	$H^1(G)$	$\zeta(z)$			Pisot char.	Periodicity
Fibonacci	$0 \mapsto 01$ $1 \mapsto 0$	$\begin{pmatrix}1&1\\1&0\end{pmatrix}$	$a \mapsto bc$ $b \mapsto bc$ $c \mapsto a$	$\begin{pmatrix}0&1&1\\0&1&1\\1&0&0\end{pmatrix}$	τ	$\lambda^2-\lambda-1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\frac{1-z}{1-z-z^2}$	$p+q\cdot\tau$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Cantor Set	$0 \mapsto 010$ $1 \mapsto 111$	$\begin{pmatrix}2&1\\0&3\end{pmatrix}$	$a \mapsto aba$ $b \mapsto ccb$ $c \mapsto ccc$	$\begin{pmatrix}2&1&0\\0&1&2\\0&0&3\end{pmatrix}$	3	$\lambda^2-5\lambda+6=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{(1-2z)\,(1-3z)}$	$\frac{k}{3^N}$	$k,N\in\mathbb{Z}$	not primitive	limit-quasiperiodic
Non-Pisot	$0 \mapsto 0001$ $1 \mapsto 011$	$\begin{pmatrix}3&1\\1&2\end{pmatrix}$	$a \mapsto aabc$ $b \mapsto aabc$ $c \mapsto bdc$ $d \mapsto bdc$	$\begin{pmatrix}2&1&1&0\\2&1&1&0\\0&1&1&1\\0&1&1&1\end{pmatrix}$	$\tau+2$	$\lambda^2-5\lambda+5=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{1-5z+5z^2}$	$\frac{p+q\cdot\tau}{5^N}$	$p,q,N\in\mathbb{Z}$	non-Pisot	limit-quasiperiodic
Periodic	$0 \mapsto 01$ $1 \mapsto 01$	$\begin{pmatrix}1&1\\1&1\end{pmatrix}$	$a \mapsto ab$ $b \mapsto ab$	$\begin{pmatrix}1&1\\1&1\end{pmatrix}$	2	$\lambda^2-2\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^1$	$\frac{1-z}{1-2z}$	$\frac{k}{2}$	$k\in\mathbb{Z}$	Pisot	periodic
Thue-Morse	$0 \mapsto 01$ $1 \mapsto 10$	$\begin{pmatrix}1&1\\1&1\end{pmatrix}$	$a \mapsto bc$ $b \mapsto bd$ $c \mapsto ca$ $d \mapsto cb$	$\begin{pmatrix}0&1&1&0\\0&1&0&1\\1&0&1&0\\0&1&1&0\end{pmatrix}$	2	$\lambda^2-2\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{(1-2z)\,(1+z)}$	$\frac{k}{3\cdot 2^N}$	$k,N\in\mathbb{Z}$	Pisot	aperiodic
Sierpiński	$0 \mapsto 01010$ $1 \mapsto 11$	$\begin{pmatrix}3&2\\0&2\end{pmatrix}$	$a \mapsto ababa$ $b \mapsto cb$ $c \mapsto cc$	$\begin{pmatrix}3&2&0\\0&1&1\\0&0&2\end{pmatrix}$	3	$\lambda^2-5\lambda+6=0$	$\mathbb{Z}^1$	$\mathbb{Z}^4$	$\frac{1-z}{(1-2z)\,(1-3z)}$	$\frac{k}{3^N}$	$k,N\in\mathbb{Z}$	not primitive	limit-quasiperiodic
Degen. Sierpiński	$0 \mapsto 0001$ $1 \mapsto 1112$ $2 \mapsto 1112$	$\begin{pmatrix}3&1&0\\0&3&1\\0&3&1\end{pmatrix}$	[redacted] $b \mapsto aabd$ $c \mapsto ddef$ $d \mapsto ddeg$ $e \mapsto ddeg$ $f \mapsto ddef$ $g \mapsto ddeg$	$\begin{pmatrix}2&1&1&0&0&0&0\\2&1&0&1&0&0&0\\0&0&0&2&1&1&0\\0&0&0&2&1&0&1\\0&0&0&2&1&0&1\\0&0&0&2&1&1&0\\0&0&0&2&1&0&1\end{pmatrix}$	4	$\lambda^3-7\lambda^2+12\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{(1-3z)\,(1-4z)}$	$\frac{k}{4^N}$	$k,N\in\mathbb{Z}$	not primitive	aperiodic
Period Doubling	$0 \mapsto 01$ $1 \mapsto 00$	$\begin{pmatrix}1&1\\2&0\end{pmatrix}$	$a \mapsto bc$ $b \mapsto bc$ $c \mapsto aa$	$\begin{pmatrix}0&1&1\\0&1&1\\2&0&0\end{pmatrix}$	2	$\lambda^2-\lambda-2=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{(1-2z)\,(1+z)}$	$\frac{k}{3\cdot 2^N}$	$k,N\in\mathbb{Z}$	non-Pisot	limit-quasiperiodic
Circle Sequence	$0 \mapsto 202$ $1 \mapsto 02202$ $2 \mapsto 01202$	$\begin{pmatrix}1&0&2\\2&0&3\\2&1&2\end{pmatrix}$	$a \mapsto dbd$ $b \mapsto dbd$ $c \mapsto bedbd$ $d \mapsto acdbe$ $e \mapsto acdbd$	$\begin{pmatrix}0&1&0&2&0\\0&1&0&2&0\\0&2&0&2&1\\1&1&1&1&1\\1&1&1&2&0\end{pmatrix}$	τ^3	$\lambda^3-3\lambda^2-5\lambda-1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{(1+z)\,(1-4z-z^2)}$	$\frac{1}{2}\,(p+q\cdot\tau)$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Rudin-Shapiro	$0 \mapsto 02$ $1 \mapsto 32$ $2 \mapsto 01$ $3 \mapsto 31$	$\begin{pmatrix}1&0&1&0\\0&0&1&1\\1&1&0&0\\0&1&0&1\end{pmatrix}$	$a \mapsto bf$ $b \mapsto be$ $c \mapsto he$ $d \mapsto hf$ $e \mapsto ac$ $f \mapsto ad$ $g \mapsto gd$ $h \mapsto gc$	$\begin{pmatrix}0&1&0&0&0&1&0&0\\0&1&0&0&1&0&0&0\\0&0&0&0&1&0&0&1\\0&0&0&0&0&1&0&1\\1&0&1&0&0&0&0&0\\1&0&0&1&0&0&0&0\\0&0&0&1&0&0&1&0\\0&0&1&0&0&0&1&0\end{pmatrix}$	2	$\lambda^4-2\lambda^3-2\lambda^2+4\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^9$	$\frac{1-z}{(1-2z)\,(1-2z^2)\,(1+z)}$	$\frac{k}{2^N}$	$k,N\in\mathbb{Z}$	non-Pisot	aperiodic
Skau Example #1	$0 \mapsto 001$ $1 \mapsto 0101$	$\begin{pmatrix}2&1\\2&2\end{pmatrix}$	$a \mapsto abc$ $b \mapsto abc$ $c \mapsto bcbc$	$\begin{pmatrix}1&1&1\\1&1&1\\0&2&2\end{pmatrix}$	$\sqrt{2}+2$	$\lambda^2-4\lambda+2=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{1-4z+2z^2}$	$\frac{p+q\sqrt{2}}{2^N}$	$p,q,N\in\mathbb{Z}$	Pisot	limit-quasiperiodic
Skau Example #2	$0 \mapsto 010$ $1 \mapsto 01$	$\begin{pmatrix}2&1\\1&1\end{pmatrix}$	$a \mapsto bca$ $b \mapsto bca$ $c \mapsto bc$	$\begin{pmatrix}1&1&1\\1&1&1\\0&1&1\end{pmatrix}$	$\tau+1$	$\lambda^2-3\lambda+1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\frac{1-z}{1-3z+z^2}$	$p+q\cdot\tau$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Skau Example #3	$0 \mapsto 001$ $1 \mapsto 10$	$\begin{pmatrix}2&1\\1&1\end{pmatrix}$	$a \mapsto abc$ $b \mapsto abd$ $c \mapsto ca$ $d \mapsto cb$	$\begin{pmatrix}1&1&1&0\\1&1&0&1\\1&0&1&0\\0&1&1&0\end{pmatrix}$	$\tau+1$	$\lambda^2-3\lambda+1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^5$	$\frac{1-z}{(1+z)\,(1-3z+z^2)}$	$\frac{p+q\cdot\tau}{5}$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Skau Example #4	$0 \mapsto 010$ $1 \mapsto 1001$	$\begin{pmatrix}2&1\\2&2\end{pmatrix}$	$a \mapsto bca$ $b \mapsto bcb$ $c \mapsto cabc$	$\begin{pmatrix}1&1&1\\0&2&1\\1&1&2\end{pmatrix}$	$\sqrt{2}+2$	$\lambda^2-4\lambda+2=0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\frac{1-z}{1-4z+2z^2}$	$\frac{p+q\sqrt{2}}{2^N}$	$p,q,N\in\mathbb{Z}$	Pisot	limit-quasiperiodic
Chacon	$0 \mapsto 0010$ $1 \mapsto 1$	$\begin{pmatrix}3&1\\0&1\end{pmatrix}$	$a \mapsto abca$ $b \mapsto abcb$ $c \mapsto c$	$\begin{pmatrix}2&1&1\\1&2&1\\0&0&1\end{pmatrix}$	3	$\lambda^2-4\lambda+3=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1}{1-3z}$	$\frac{k}{3^N}$	$k,N\in\mathbb{Z}$	not primitive	limit-quasiperiodic
Golden Mean Squared	$0 \mapsto 100$ $1 \mapsto 10$	$\begin{pmatrix}2&1\\1&1\end{pmatrix}$	$a \mapsto cab$ $b \mapsto cab$ $c \mapsto cb$	$\begin{pmatrix}1&1&1\\1&1&1\\0&1&1\end{pmatrix}$	$\tau+1$	$\lambda^2-3\lambda+1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\frac{1-z}{1-3z+z^2}$	$p+q\cdot\lambda_1$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Silver Mean Squared	$0 \mapsto 1001000$ $1 \mapsto 100$	$\begin{pmatrix}5&2\\2&1\end{pmatrix}$	$a \mapsto cabcaab$ $b \mapsto cabcaab$ $c \mapsto cab$	$\begin{pmatrix}3&2&2\\3&2&2\\1&1&1\end{pmatrix}$	$2\sqrt{2}+3$	$\lambda^2-6\lambda+1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\frac{1-z}{1-6z+z^2}$	$p+q\cdot\lambda_1$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Copper Mean Squared	$0 \mapsto 1000100010000$ $1 \mapsto 1000$	$\begin{pmatrix}10&3\\3&1\end{pmatrix}$	$a \mapsto caabcaabcaaab$ $b \mapsto caabcaabcaaab$ $c \mapsto caab$	$\begin{pmatrix}7&3&3\\7&3&3\\2&1&1\end{pmatrix}$	$\frac{3\sqrt{13}}{2}+\frac{11}{2}$	$\lambda^2-11\lambda+1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\frac{1-z}{1-11z+z^2}$	$p+q\cdot\lambda_1$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Luck Ternary #1	$0 \mapsto 01$ $1 \mapsto 02$ $2 \mapsto 012$	$\begin{pmatrix}1&1&0\\1&0&1\\1&1&1\end{pmatrix}$	$a \mapsto ac$ $b \mapsto ac$ $c \mapsto be$ $d \mapsto be$ $e \mapsto ade$	$\begin{pmatrix}1&0&1&0&0\\1&0&1&0&0\\0&1&0&0&1\\0&1&0&0&1\\1&0&0&1&1\end{pmatrix}$	2.247	$\lambda^3-2\lambda^2-\lambda+1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{1-2z-z^2+z^3}$	$p+q\cdot\lambda_1+r\cdot\lambda_1^2$	$p,q,r\in\mathbb{Z}$	Pisot	quasiperiodic
Luck Ternary #2	$0 \mapsto 2$ $1 \mapsto 0$ $2 \mapsto 12$	$\begin{pmatrix}0&0&1\\1&0&0\\0&1&1\end{pmatrix}$	$a \mapsto c$ $b \mapsto a$ $c \mapsto be$ $d \mapsto bc$ $e \mapsto bd$	$\begin{pmatrix}0&0&1&0&0\\1&0&0&0&0\\0&1&0&0&1\\0&1&1&0&0\\0&1&0&1&0\end{pmatrix}$	1.4656	$\lambda^3-\lambda^2-1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{1-z-z^3}$	$p+q\cdot\lambda_1+r\cdot\lambda_1^2$	$p,q,r\in\mathbb{Z}$	Pisot	quasiperiodic
Periodic 1-2	$0 \mapsto 011$ $1 \mapsto 011$	$\begin{pmatrix}1&2\\1&2\end{pmatrix}$	$a \mapsto acb$ $b \mapsto acb$ $c \mapsto acb$	$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$	3	$\lambda^2-3\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^1$	$\frac{1-z}{1-3z}$	$\frac{k}{3}$	$k\in\mathbb{Z}$	Pisot	periodic
Periodic 1-3	$0 \mapsto 0111$ $1 \mapsto 0111$	$\begin{pmatrix}1&3\\1&3\end{pmatrix}$	$a \mapsto accb$ $b \mapsto accb$ $c \mapsto accb$	$\begin{pmatrix}1&1&2\\1&1&2\\1&1&2\end{pmatrix}$	4	$\lambda^2-4\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^1$	$\frac{1-z}{1-4z}$	$\frac{k}{4}$	$k\in\mathbb{Z}$	Pisot	periodic
Periodic 1-4	$0 \mapsto 01111$ $1 \mapsto 01111$	$\begin{pmatrix}1&4\\1&4\end{pmatrix}$	$a \mapsto acccb$ $b \mapsto acccb$ $c \mapsto acccb$	$\begin{pmatrix}1&1&3\\1&1&3\\1&1&3\end{pmatrix}$	5	$\lambda^2-5\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^1$	$\frac{1-z}{1-5z}$	$\frac{k}{5}$	$k\in\mathbb{Z}$	Pisot	periodic
Periodic 2-3	$0 \mapsto 00111$ $1 \mapsto 00111$	$\begin{pmatrix}2&3\\2&3\end{pmatrix}$	$a \mapsto abddc$ $b \mapsto abddc$ $c \mapsto abddc$ $d \mapsto abddc$	$\begin{pmatrix}1&1&1&2\\1&1&1&2\\1&1&1&2\\1&1&1&2\end{pmatrix}$	5	$\lambda^2-5\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^1$	$\frac{1-z}{1-5z}$	$\frac{k}{5}$	$k\in\mathbb{Z}$	Pisot	periodic
Periodic 2-5	$0 \mapsto 0011111$ $1 \mapsto 0011111$	$\begin{pmatrix}2&5\\2&5\end{pmatrix}$	$a \mapsto abddddc$ $b \mapsto abddddc$ $c \mapsto abddddc$ $d \mapsto abddddc$	$\begin{pmatrix}1&1&1&4\\1&1&1&4\\1&1&1&4\\1&1&1&4\end{pmatrix}$	7	$\lambda^2-7\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^1$	$\frac{1-z}{1-7z}$	$\frac{k}{7}$	$k\in\mathbb{Z}$	Pisot	periodic
Golden Mean	$0 \mapsto 10$ $1 \mapsto 0$	$\begin{pmatrix}1&1\\1&0\end{pmatrix}$	$a \mapsto cb$ $b \mapsto ca$ $c \mapsto b$	$\begin{pmatrix}0&1&1\\1&0&1\\0&1&0\end{pmatrix}$	τ	$\lambda^2-\lambda-1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\frac{1-z}{1-z-z^2}$	$p+q\cdot\lambda_1$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Silver Mean	$0 \mapsto 100$ $1 \mapsto 0$	$\begin{pmatrix}2&1\\1&0\end{pmatrix}$	$a \mapsto cab$ $b \mapsto caa$ $c \mapsto b$	$\begin{pmatrix}1&1&1\\2&0&1\\0&1&0\end{pmatrix}$	$\sqrt{2}+1$	$\lambda^2-2\lambda-1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\frac{1-z}{1-2z-z^2}$	$p+q\cdot\lambda_1$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Copper Mean	$0 \mapsto 1000$ $1 \mapsto 0$	$\begin{pmatrix}3&1\\1&0\end{pmatrix}$	$a \mapsto caab$ $b \mapsto caaa$ $c \mapsto b$	$\begin{pmatrix}2&1&1\\3&0&1\\0&1&0\end{pmatrix}$	$\frac{\sqrt{13}}{2}+\frac{3}{2}$	$\lambda^2-3\lambda-1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\frac{1-z}{1-3z-z^2}$	$p+q\cdot\lambda_1$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Marginal	$0 \mapsto 001$ $1 \mapsto 011$	$\begin{pmatrix}2&1\\1&2\end{pmatrix}$	$a \mapsto abc$ $b \mapsto abc$ $c \mapsto bdc$ $d \mapsto bdc$	$\begin{pmatrix}1&1&1&0\\1&1&1&0\\0&1&1&1\\0&1&1&1\end{pmatrix}$	3	$\lambda^2-4\lambda+3=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1}{1-3z}$	$\frac{k}{2\cdot 3^N}$	$k,N\in\mathbb{Z}$	non-Pisot	limit-quasiperiodic
Luck non-Pisot	$0 \mapsto 0001$ $1 \mapsto 110$	$\begin{pmatrix}3&1\\1&2\end{pmatrix}$	$a \mapsto aabc$ $b \mapsto aabd$ $c \mapsto dca$ $d \mapsto dcb$	$\begin{pmatrix}2&1&1&0\\2&1&0&1\\1&0&1&1\\0&1&1&1\end{pmatrix}$	$\tau+2$	$\lambda^2-5\lambda+5=0$	$\mathbb{Z}^1$	$\mathbb{Z}^5$	$\frac{1-z}{(z+1)\,(1-5z+5z^2)}$	$\frac{p+q\cdot\tau}{11\cdot 5^N}$	$p,q,N\in\mathbb{Z}$	non-Pisot	limit-quasiperiodic
Binary non-Pisot	$0 \mapsto 01$ $1 \mapsto 000$	$\begin{pmatrix}1&1\\3&0\end{pmatrix}$	$a \mapsto bc$ $b \mapsto bc$ $c \mapsto aaa$	$\begin{pmatrix}0&1&1\\0&1&1\\3&0&0\end{pmatrix}$	$\frac{\sqrt{13}}{2}+\frac{1}{2}$	$\lambda^2-\lambda-3=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{1-z-3z^2}$	$\frac{p+q\cdot\lambda_1}{3^N}$	$p,q,N\in\mathbb{Z}$	non-Pisot	limit-quasiperiodic
Ternary non-Pisot	$0 \mapsto 2$ $1 \mapsto 0$ $2 \mapsto 101$	$\begin{pmatrix}0&0&1\\1&0&0\\1&2&0\end{pmatrix}$	$a \mapsto e$ $b \mapsto f$ $c \mapsto b$ $d \mapsto a$ $e \mapsto cad$ $f \mapsto cac$	$\begin{pmatrix}0&0&0&0&1&0\\0&0&0&0&0&1\\0&1&0&0&0&0\\1&0&0&0&0&0\\1&0&1&1&0&0\\1&0&2&0&0&0\end{pmatrix}$	1.5214	$\lambda^3-\lambda-2=0$	$\mathbb{Z}^1$	$\mathbb{Z}^6$	$\frac{1}{1-z^2-2z^3}$	$\frac{p+q\cdot\lambda_1+r\cdot\lambda_1^2}{2^N}$	$p,q,r,N\in\mathbb{Z}$	non-Pisot	limit-quasiperiodic

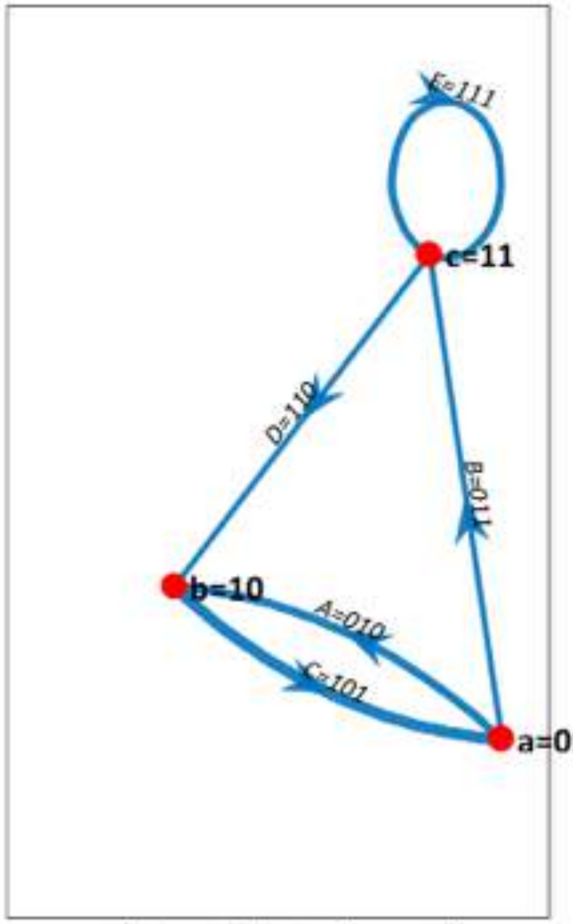
Sierpinski gasket

Sierpiński for Letters



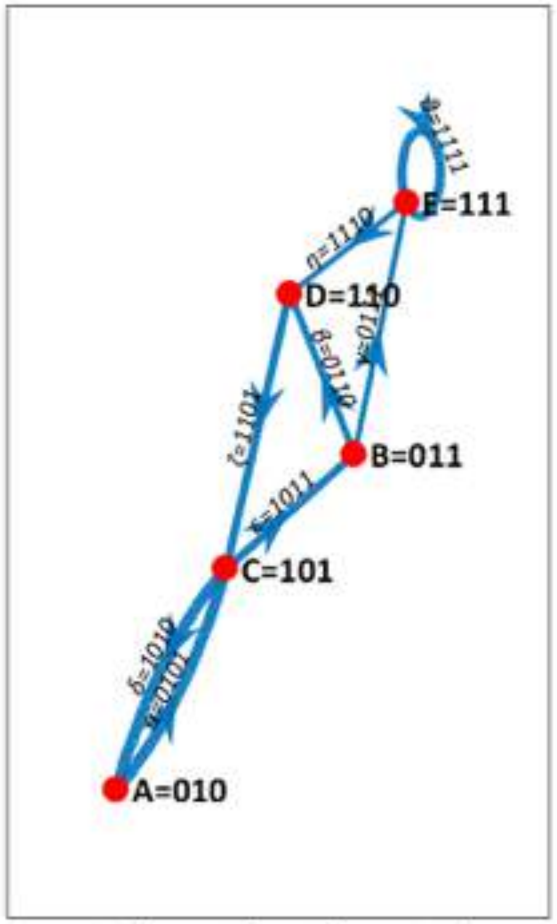
$H^0 \cong \mathbb{Z}^1,\ H^1 \cong \mathbb{Z}^2$

Sierpiński for Nodes



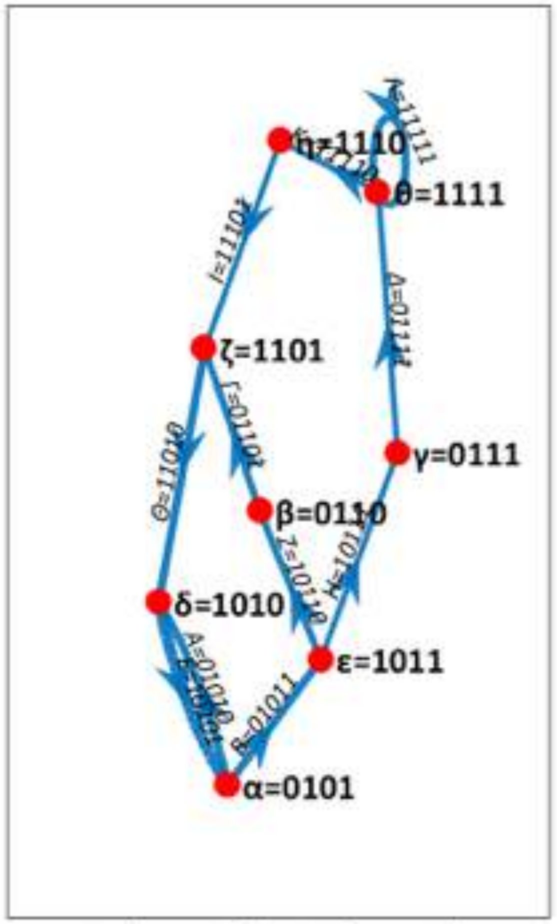
$H^0 \cong \mathbb{Z}^1,\ H^1 \cong \mathbb{Z}^3$

Sierpiński for Edges



$H^0 \cong \mathbb{Z}^1,\ H^1 \cong \mathbb{Z}^4$

Sierpiński for Frames



$H^0 \cong \mathbb{Z}^1,\ H^1 \cong \mathbb{Z}^4$

$M_1 = \begin{pmatrix} 3 & 2 \\ 0 & 2 \end{pmatrix}, \quad \begin{matrix} \sigma_1(0) = 01010 \\ \sigma_1(1) = 11 \end{matrix}$

$\lambda_1^{(1)} = 3, \quad v_1^{(1)} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \end{pmatrix}$

$M_2 = \begin{pmatrix} 3 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \quad \begin{matrix} \sigma_2(a) = ababa \\ \sigma_2(b) = cb \\ \sigma_2(c) = cc \end{matrix}$

$\lambda_1^{(2)} = 3, \quad v_1^{(2)} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$

Gaps: $\frac{k}{3^N} \cap [0,1)$ for $k, N \in \mathbb{Z}$

not primitive , limit-quasiperiodic

Inflation: $\zeta(z) = \frac{1 - z}{(2\,z - 1)\,(3\,z - 1)}$

Thank you for your attention.