Waves and quantum physics on fractals :

From continuous to discrete scaling symmetry

Eric Akkermans
Physics-Technion

Four lectures

- General introduction - Photons and Quantum Electrodynamics on fractals
- Interplay between topology and discrete scaling symmetry : Quasi-crystals
- Critical behaviour on fractals : BEC and superfluidity
- Efimov physics from geometric and spectral perspectives

Benefitted from discussions and collaborations with:

Technion:

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Dor Gittelman
Ariane Soret (ENS Cachan)
Or Raz
Omrie Ovdat
Ohad Shpielberg
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Rafael:

Eli Levy
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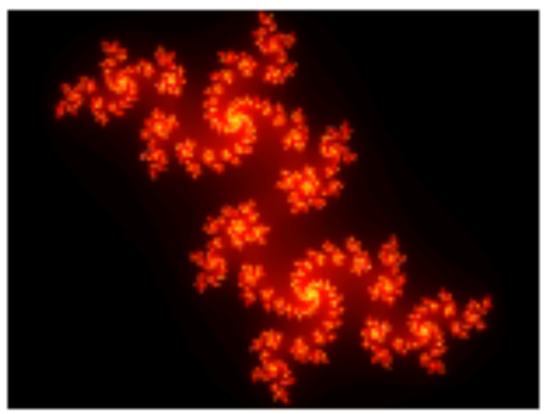
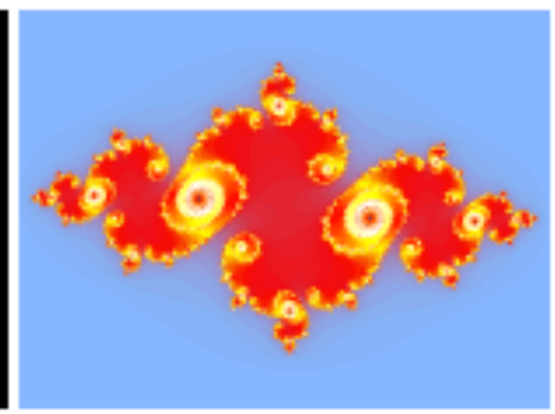
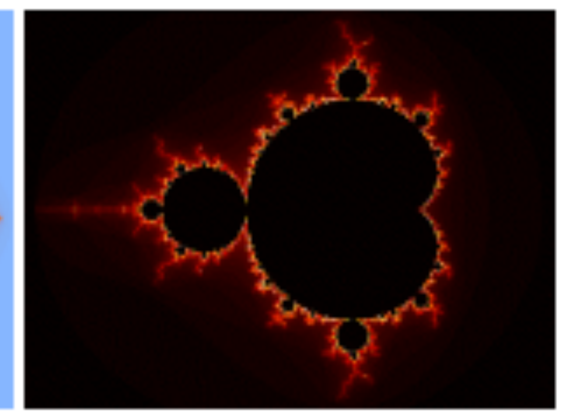
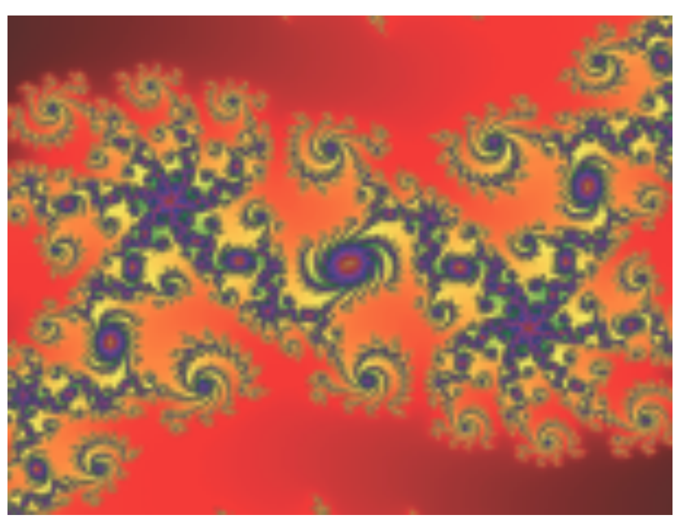
Gerald Dunne (UConn.)
Alexander Teplyaev (UConn.)
Raphael Voituriez (LPTMC, Jussieu)
Olivier Benichou (LPTMC, Jussieu)
Jacqueline Bloch (LPN, Marcoussis)
Dimitri Tanese (LPN, Marcoussis)
Florent Baboux (LPN, Marcoussis)
Alberto Amo (LPN, Marcoussis)
Julien Gabelli (LPS, Orsay)

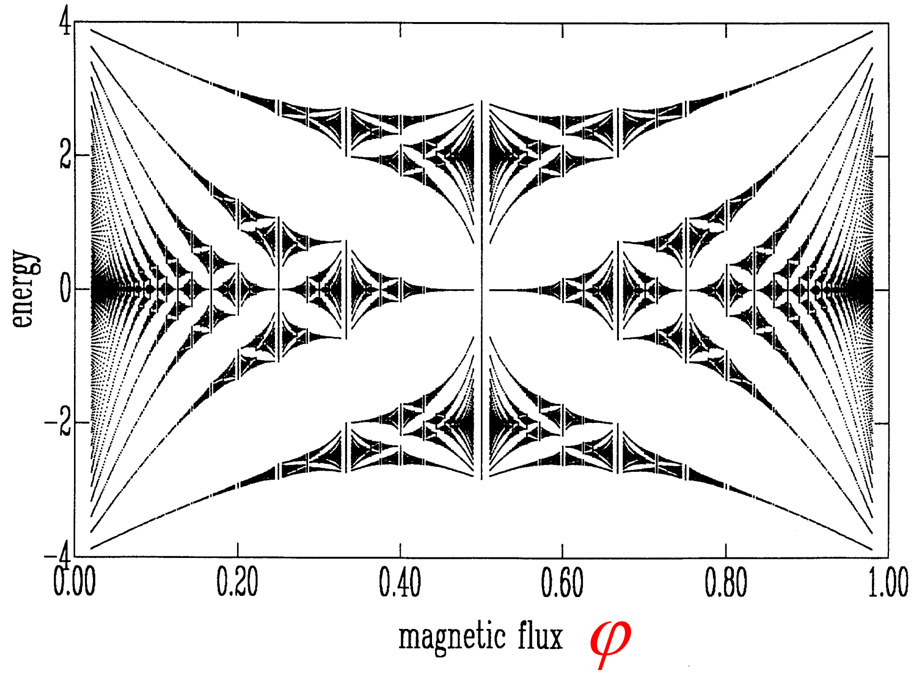
Part 1: General introduction to fractals

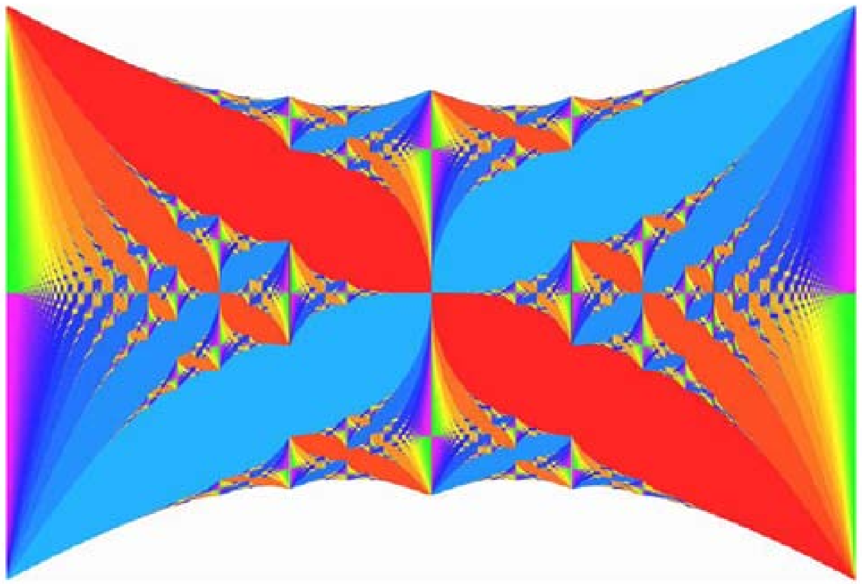
- attractive objects - Bear exotic names

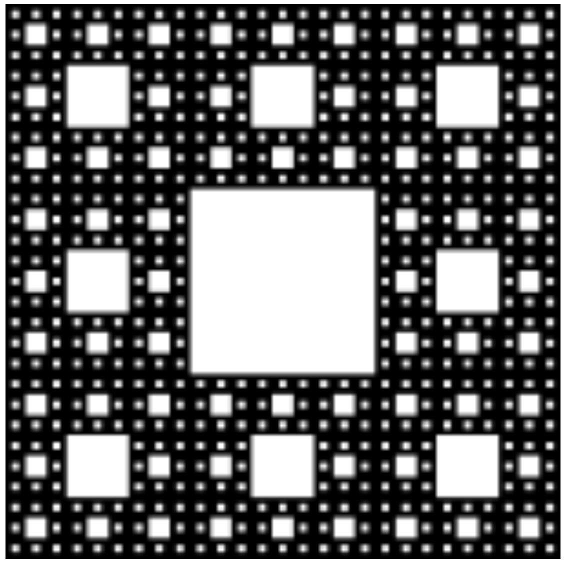


Julia sets

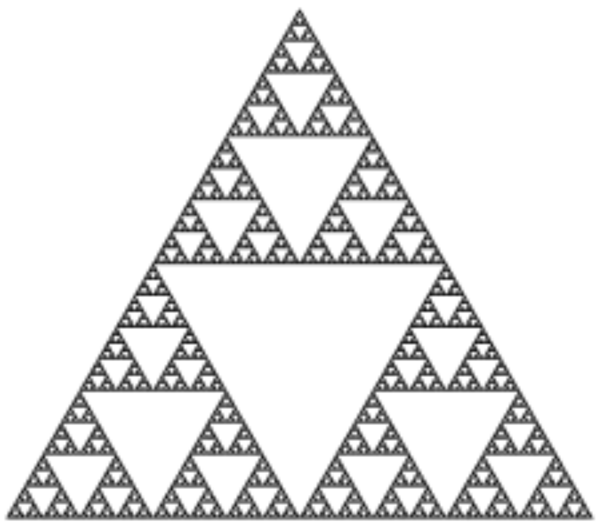




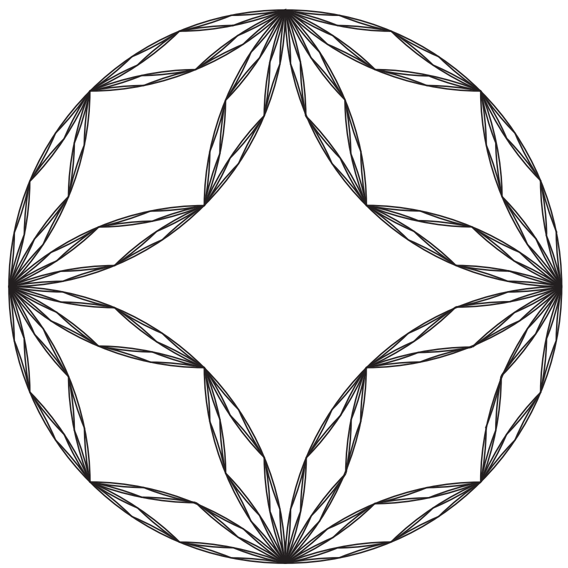
Hofstadter butterfly



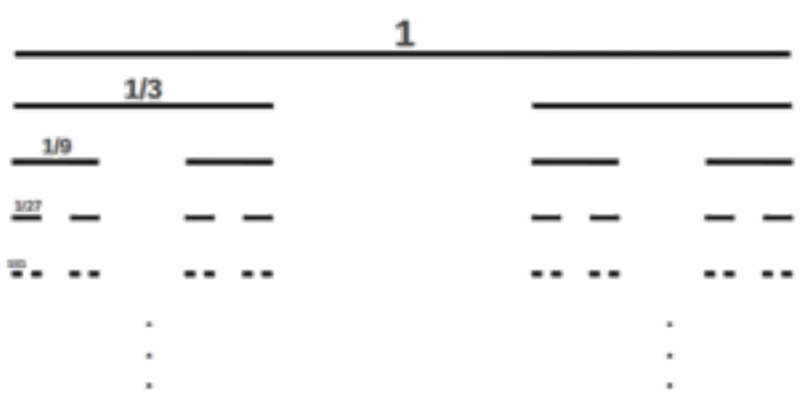
Sierpinski carpet



Sierpinski gasket



Diamond fractals



Triadic Cantor set

Convey the idea of highly symmetric objects yet with an unusual type of symmetry and a notion of extreme subdivision

Fractal : Iterative graph structure

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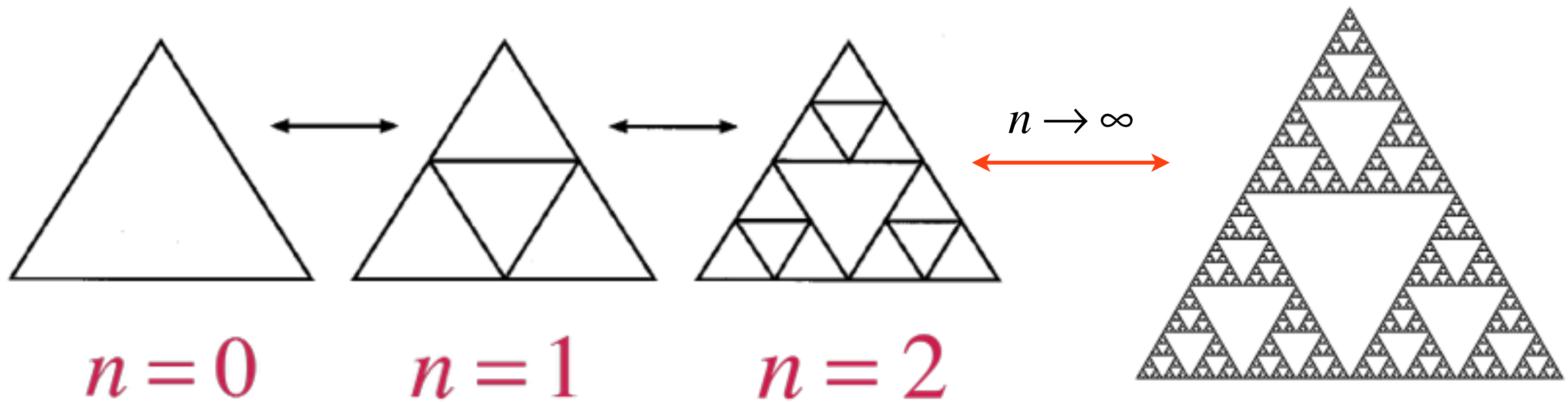

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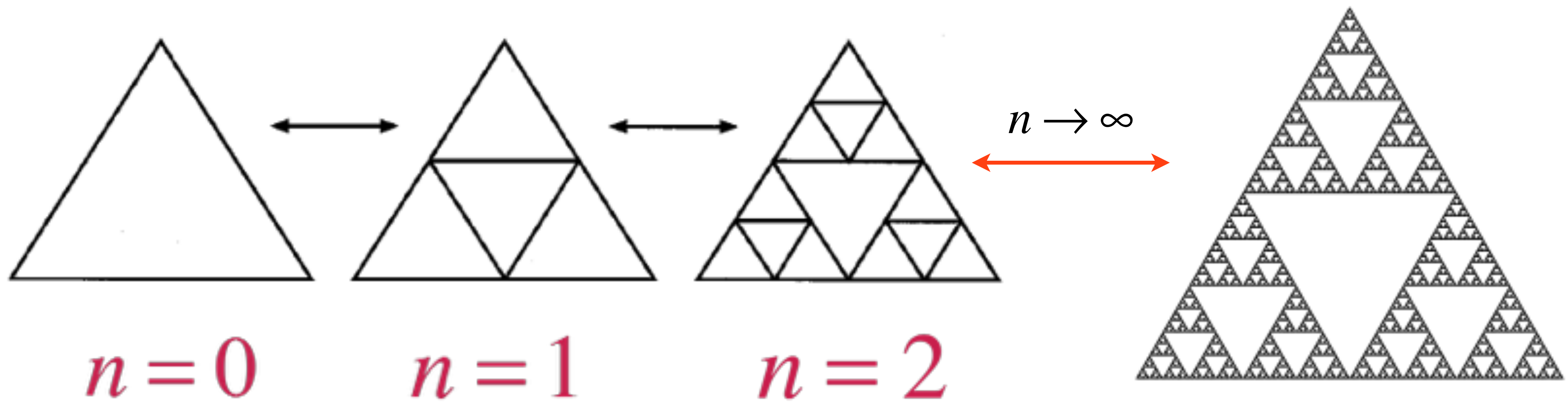

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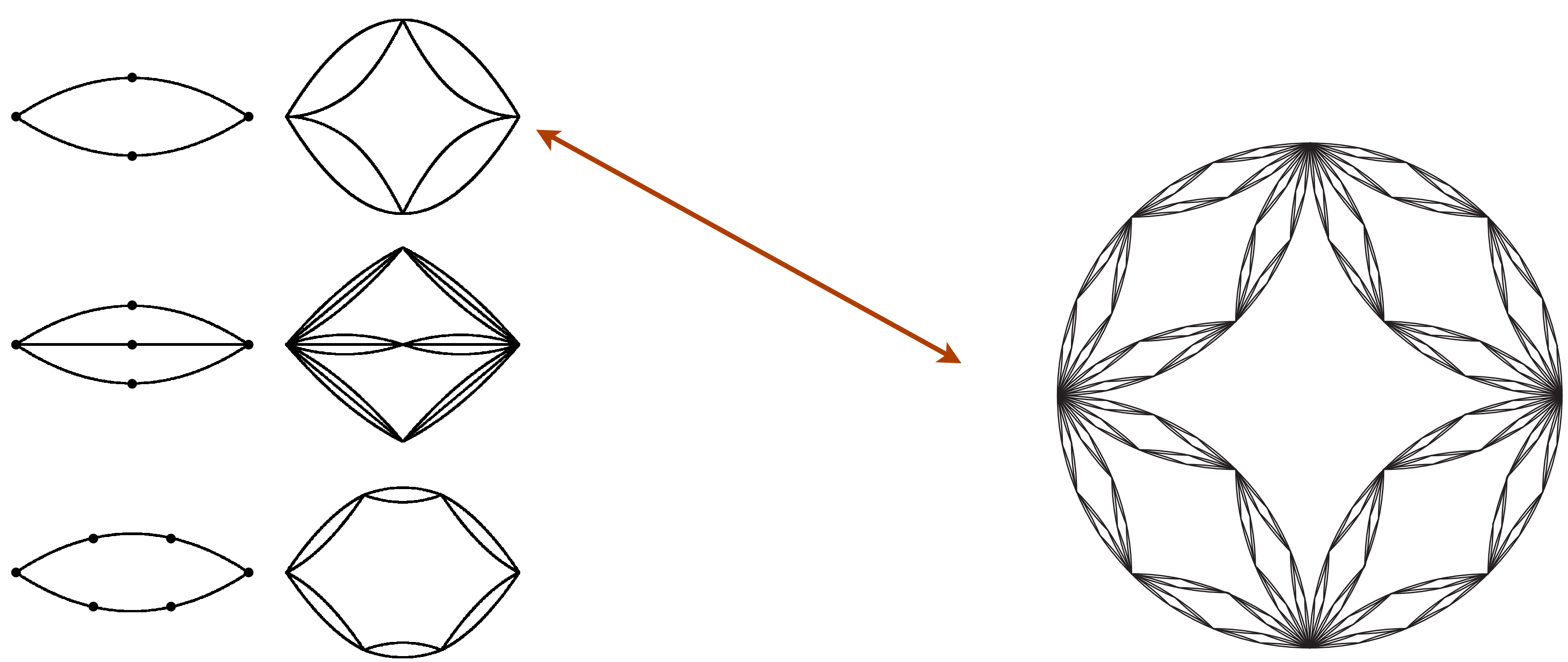
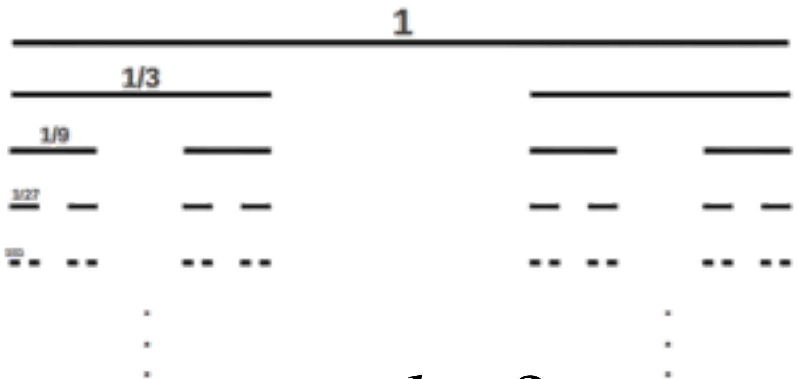


Diamond fractals

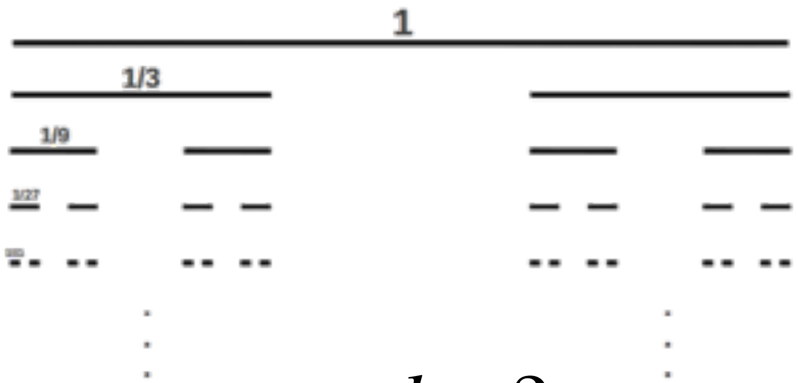
Fractal graph and Euclidean equivalent



$$d = ?$$

$$d = 1$$

Fractal graph and Euclidean equivalent



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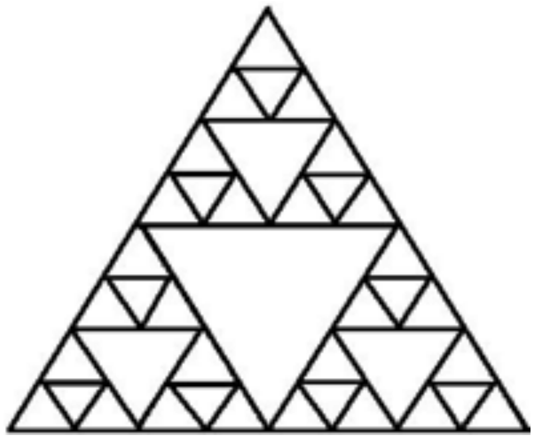
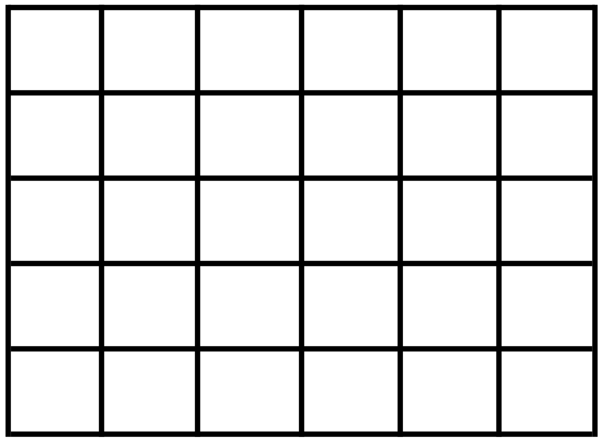




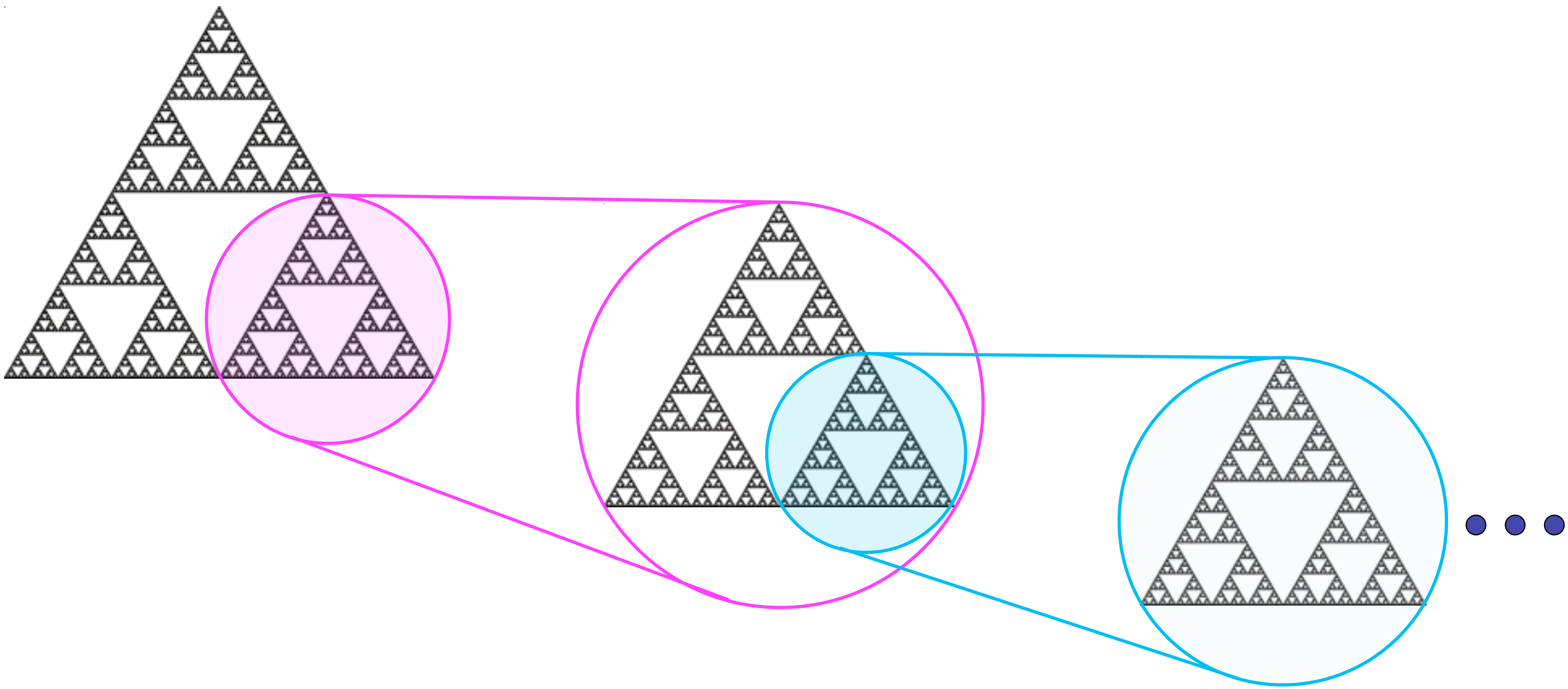
$d = 2$

Similar graphs but have very distinct properties

As opposed to Euclidean spaces characterised by translation symmetry, fractals possess a dilatation symmetry.

Fractals are self-similar objects

Fractal $\leftrightarrow$ Self-similar



Discrete scaling symmetry

Notion of dimension

- Euclidean systems are characterised by a single integer dimension : d =1,2,3,4,…10

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Yet, there is a single parameter d which enters into all physical quantities

On fractal manifolds, the notion of dimension depends on the measured physical quantity.

Dimensions are not necessarily integers

To summarise

- Euclidean manifolds : d is a fixed (boring ?) parameter.
- Fields (Stat. Mech., QFT) on Euclidean manifolds : d may be a variable (ε -expansion) quantity
- Underlying essential idea : spontaneous (continuous) symmetry breaking.
- Fractal manifolds : genuine non integer dimension
- Fields on fractals : d is not variable, but distinct physical quantities are characterised by different dimensions.
- Underlying essential idea : <u>discrete scaling symmetry</u>.

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    - Sometimes, the fractal structure is not geometrical but it is hidden at a more abstract level.

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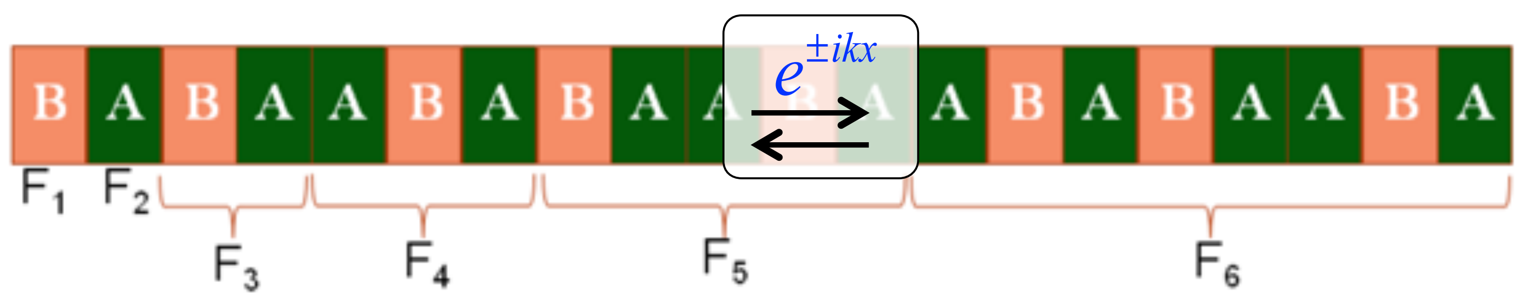
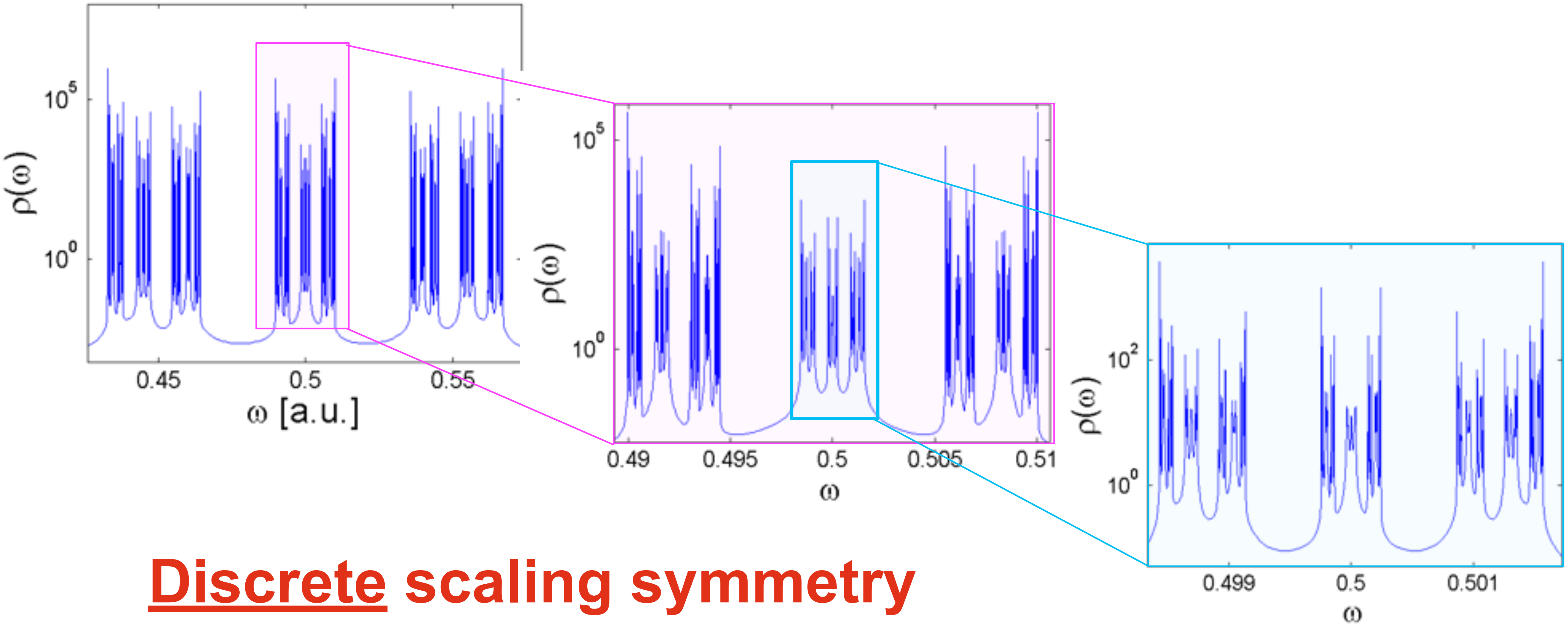
<u>Exemple</u> : Quasi-periodic stack of dielectric layers of 2 types A, B

Fibonacci sequence : $F_1 = B;\; F_2 = A;\; F_{j\geq 3} = \left[F_{j-2}F_{j-1}\right]$



Defines a cavity whose <u>mode spectrum</u> is fractal.

Density of modes $\rho(\omega)$:



Discrete scaling symmetry

Existence of a fractal behaviour (discrete scaling symmetry) may be the expression of a genuine specific symmetry (next lecture).

Why studying fractals in physics ?

Fractals or the skill of playing with dimensions

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Fractals or the skill of playing with dimensions

Fractals define a very useful testing ground for dimensionality dependent physical problems since distinct physical properties are characterised by different (usually non integer) dimensions.

Some examples :

- Anderson localization phase transition : exists for $d > 2$
- Bose-Einstein condensation $d > 2$
- Mermin-Wagner theorem (Superfluidity $d \geq 2$)
- Levy flights-Percolation (quantum and classical)
- Recurrence properties of random walks $d \geq 2$
- Renormalisability $d = 2$ and $d = 4$ special
- Quantum and classical phase transitions-Existence of topological defects...

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Is it a geometric, spectral, transport,… feature ?

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In this context, the fractal paradigm is interesting since it removes the degenerate role of dimension by assigning a different dimension to distinct physical properties.

Topological aspects

- We have discussed geometrical aspects : dimensions of manifolds, spectra,…
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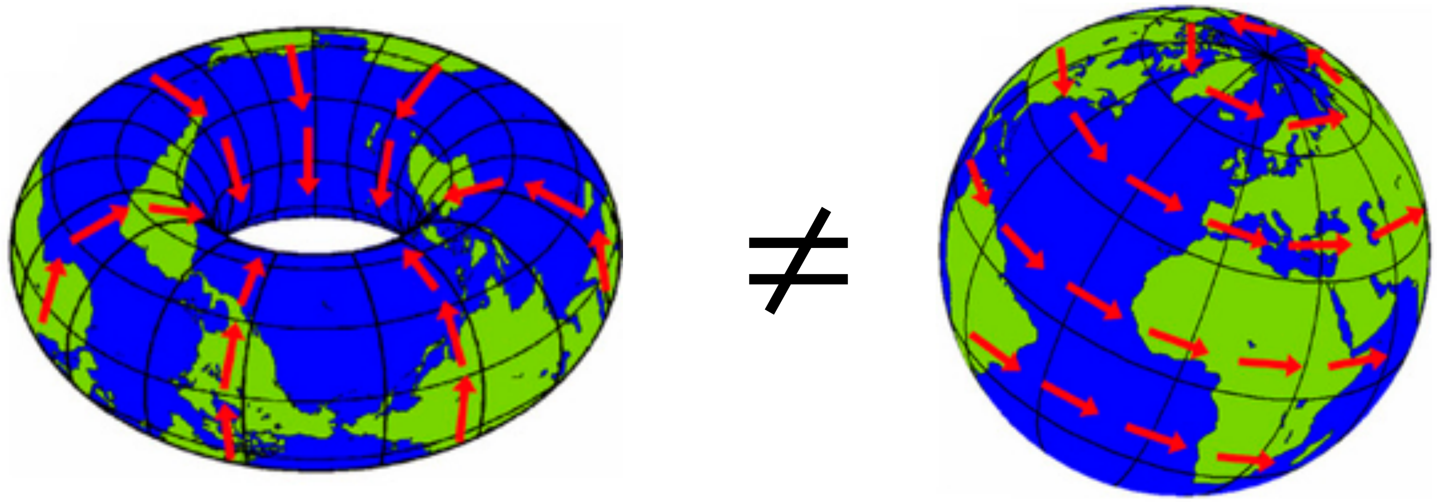
Example: Some d=2 surfaces cannot transform one into another by means of continuous transformations. This is expressed by a constraint a.k.a a topological invariant.



Euler-Poincare characteristics

$$\boxed{\chi(S) = 2(1-h)}$$

h : number of holes

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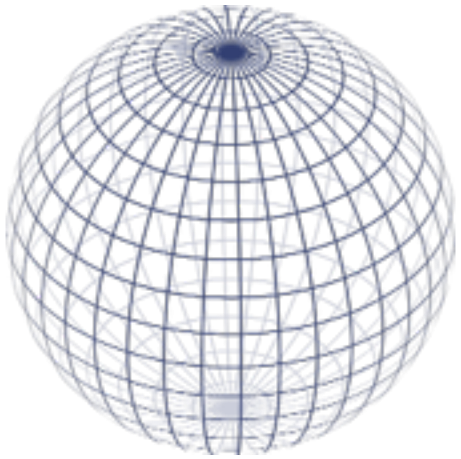
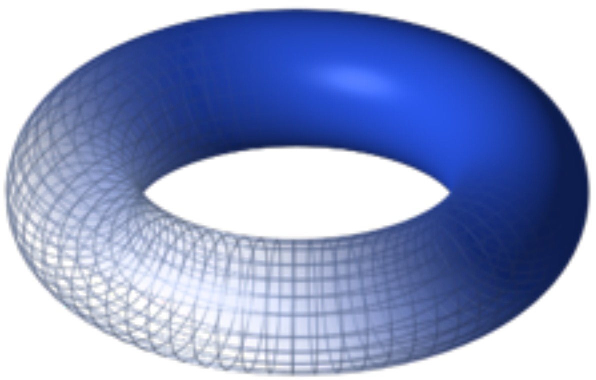


$$\chi(S_2) = 2$$



$$\chi(T_2) = 0$$

Euler-Poincare characteristics

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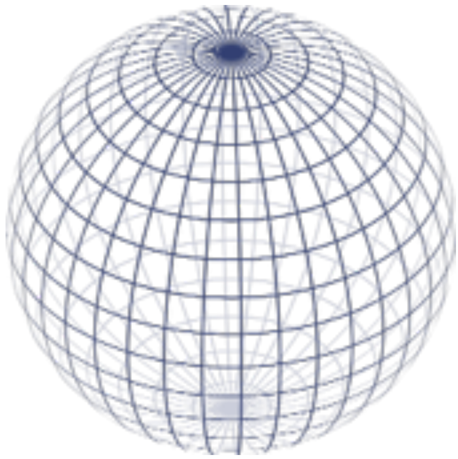

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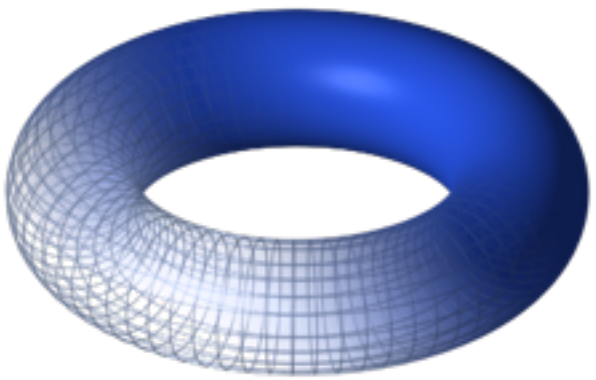
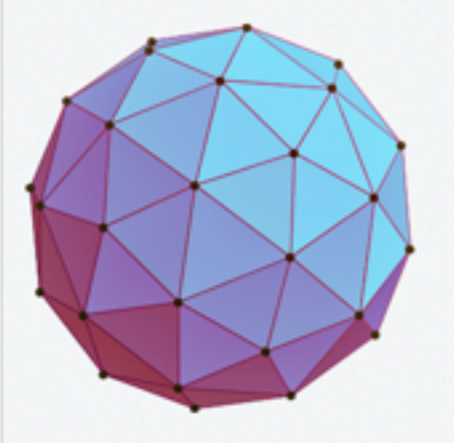


$\chi(T_2) = 0$

Euler : $\chi(S) = V - E + F$

V=# of vertices ; E = # of edges and F = # of faces



- Although the sphere and the torus have the same dimension, these two manifolds have distinguishable properties when it comes to topology namely defining fields and operators (*e.g.*, the Laplace operator $-\Delta$ which measure the energy cost to adapt a field to a specific manifold).

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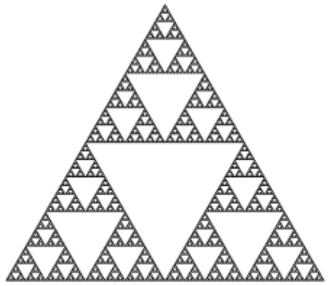
- Topology of fractals is at a much earlier stage : difficulty in defining operators on fractals.



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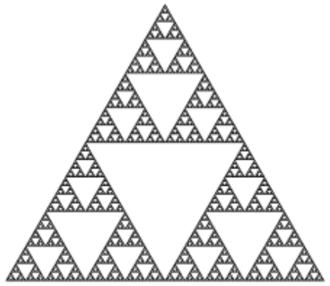
- Important progresses : gap labeling theorem (Bellissard '82), aspects of the QHE and Quasi-crystals (Lectures 2+3).



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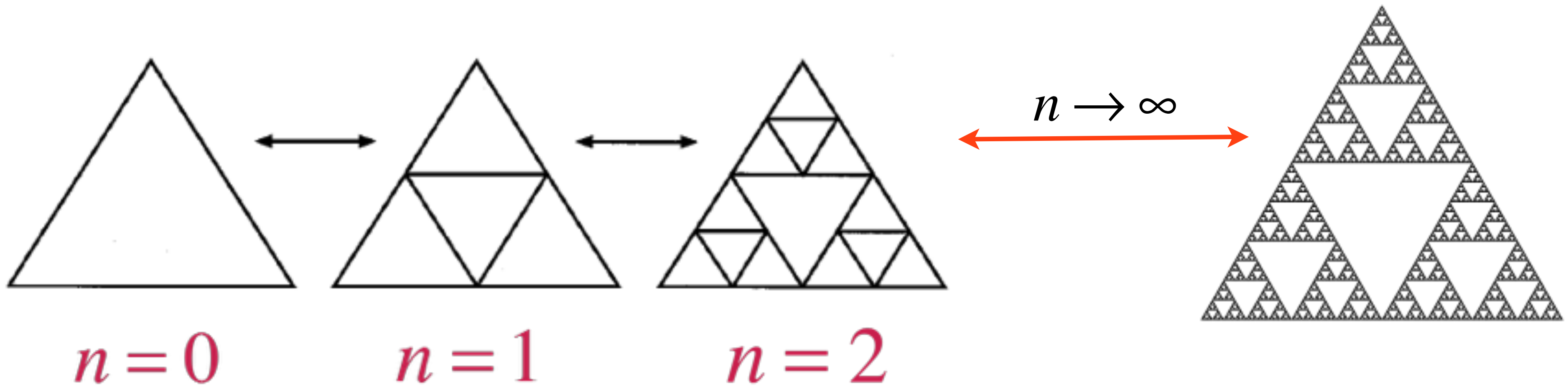
Part 2 :Fractal dimensions

A working definition of a fractal

Hausdorff geometric dimension

An iterative structure : Sierpinski gasket



Hausdorff geometric dimension

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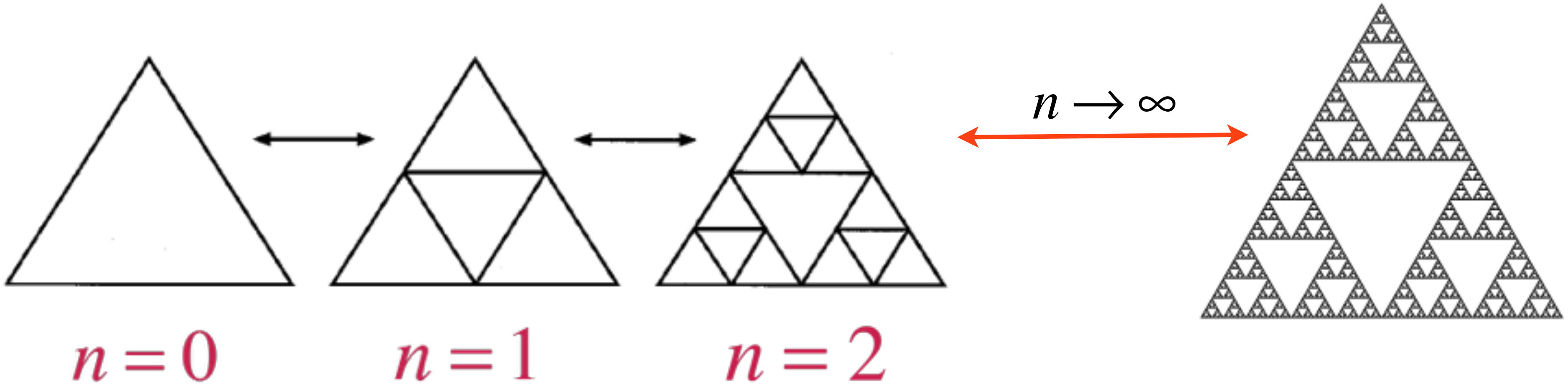

At each step n of the iteration, the fractal graph is characterised by its length scale $L_n = 2^n L$ and the number of bonds (mass) $M_n = 3^n M$.

Hausdorff geometric dimension

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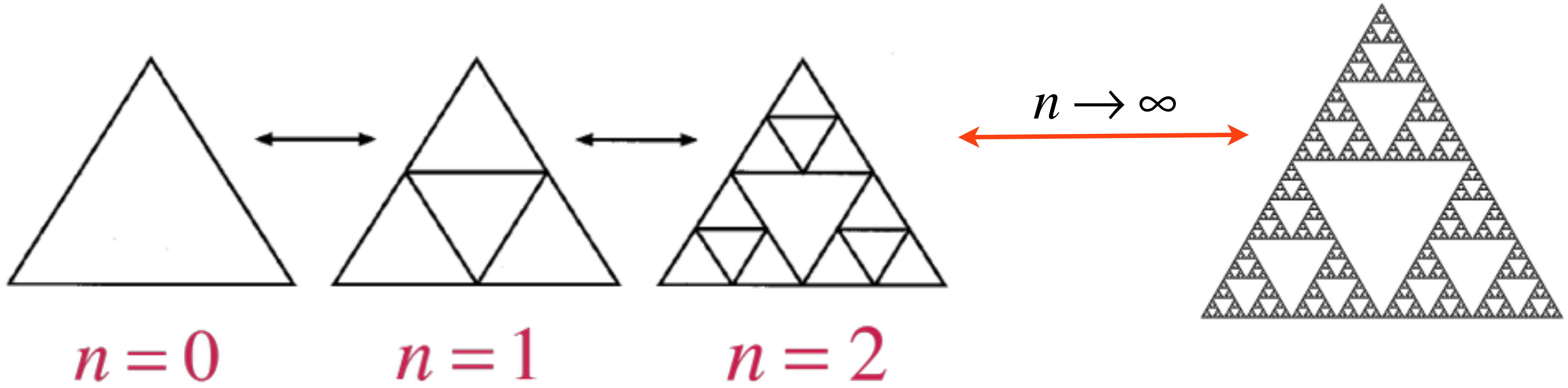

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Scaling of these dimensionless quantities allows to define a (mass) fractal dimension :

Hausdorff geometric dimension $\quad \dfrac{\ln M_n}{\ln L_n} \xrightarrow[n\to\infty]{} d_h$

$$d_h = \frac{\ln 3}{\ln 2} \sim 1.585$$

Hausdorff geometric dimension

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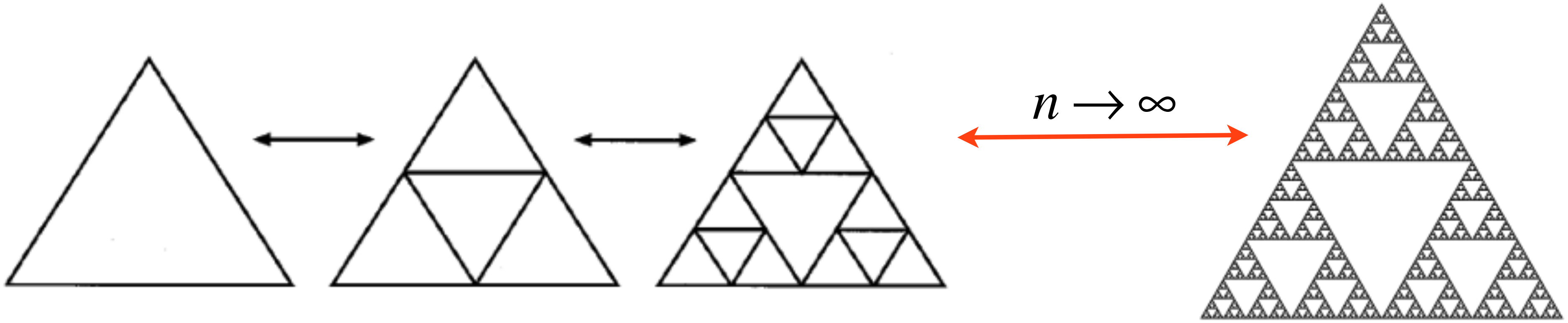

$n=0$ $n=1$ $n=2$

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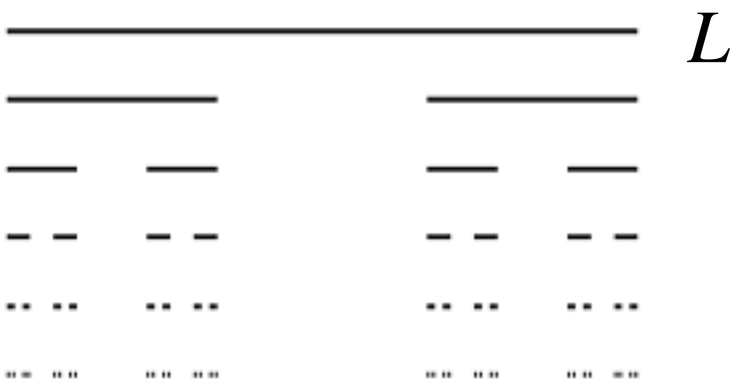
The Cantor set (disconnected zero measure)



$$M_n = 2^n M$$

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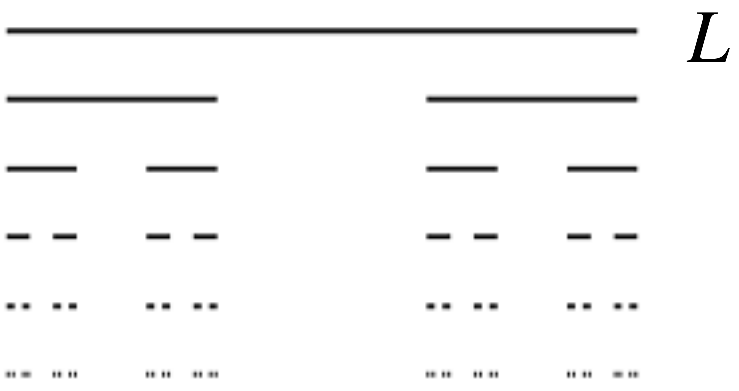

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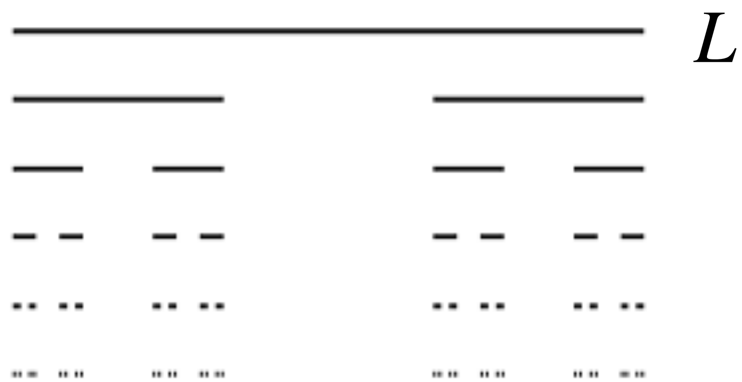

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The mass density observed after a magnification by a factor 3 is

$$2\,m\left(L/3\right) = m\left(L\right)$$

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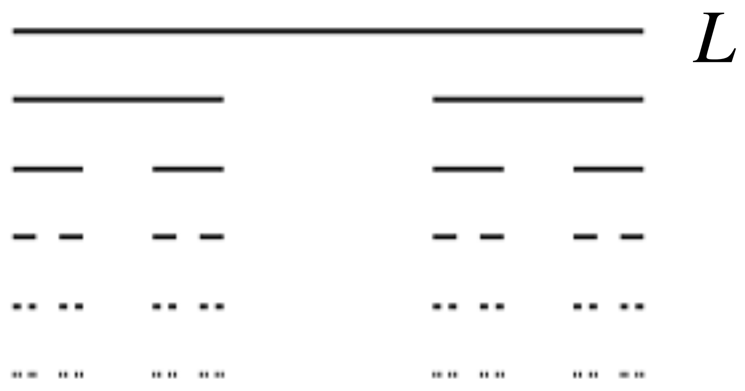

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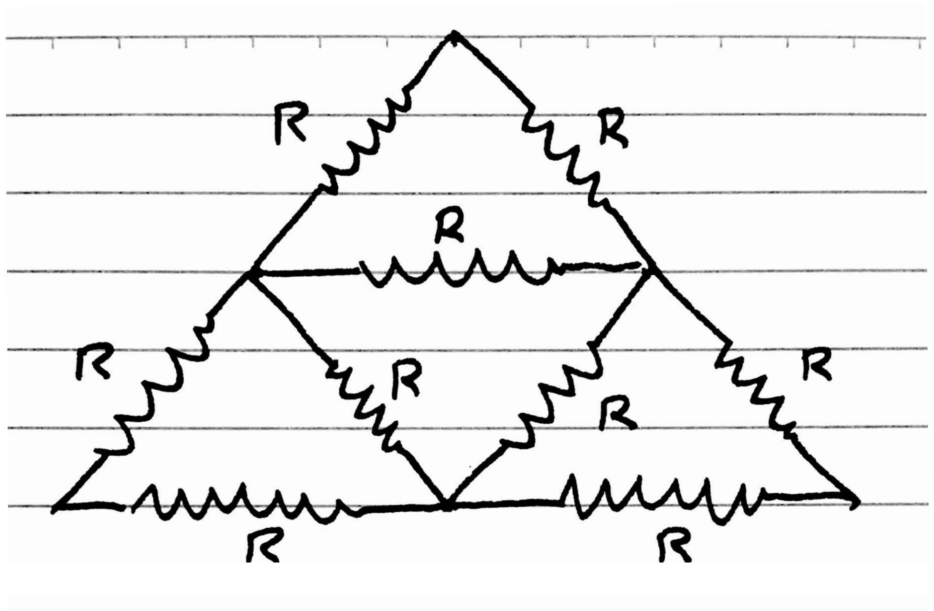
$$2m\left(L/3\right) = m(L) \Leftrightarrow 2m(L) = m(3L)$$

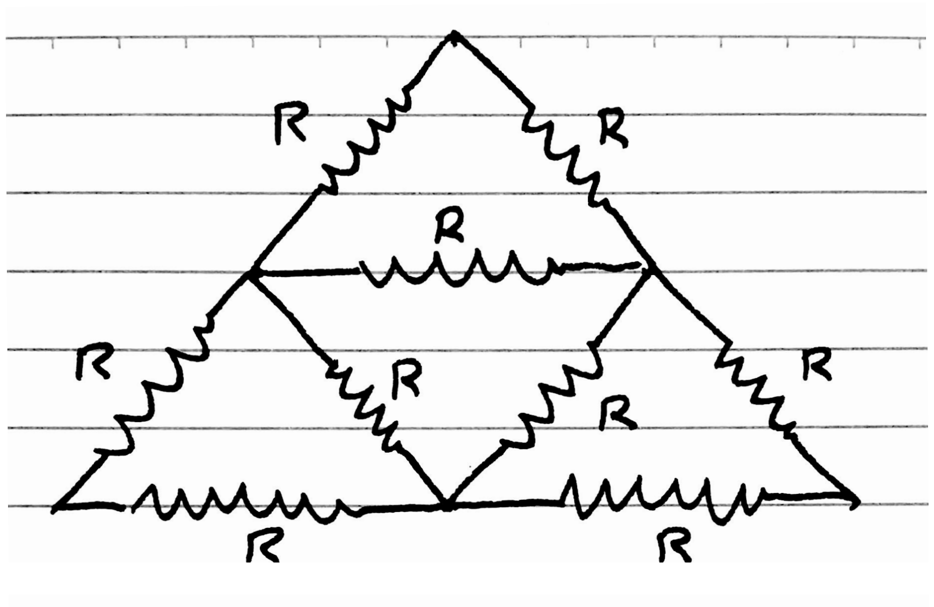
Electric dimension : ζ



Electric Sierpinski network

Scaling of the equivalent electric resistance R(L)

Electric dimension : ζ



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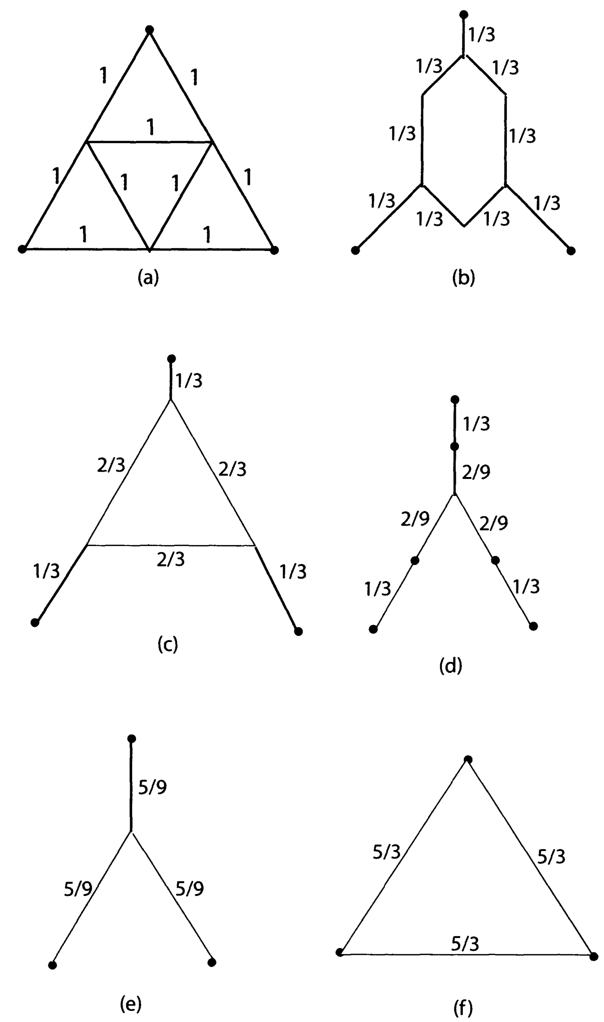

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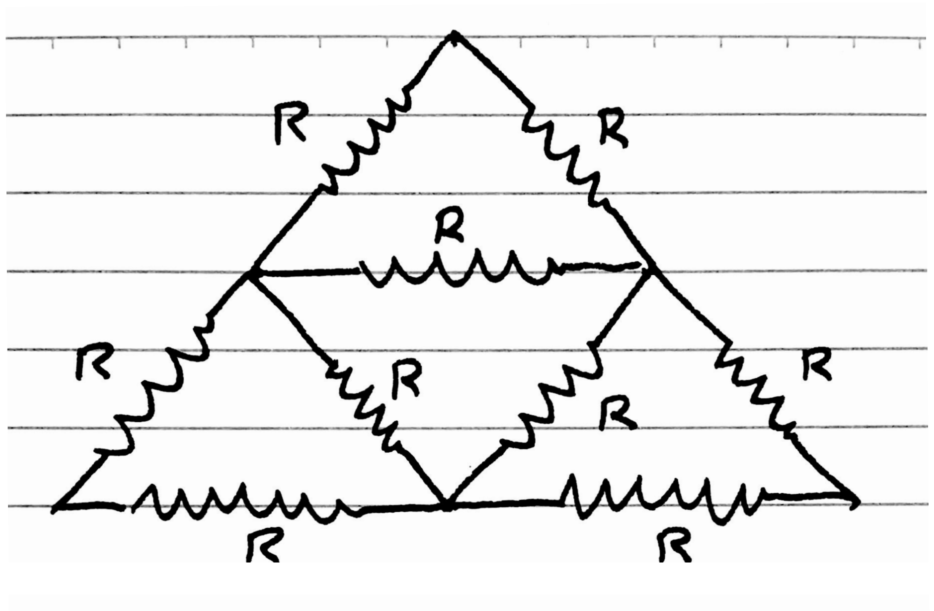

Electric Sierpinski network

$$R(L) \sim L^{\varsigma}$$

$$\varsigma = \frac{\ln {5}/{3}}{\ln 2}$$

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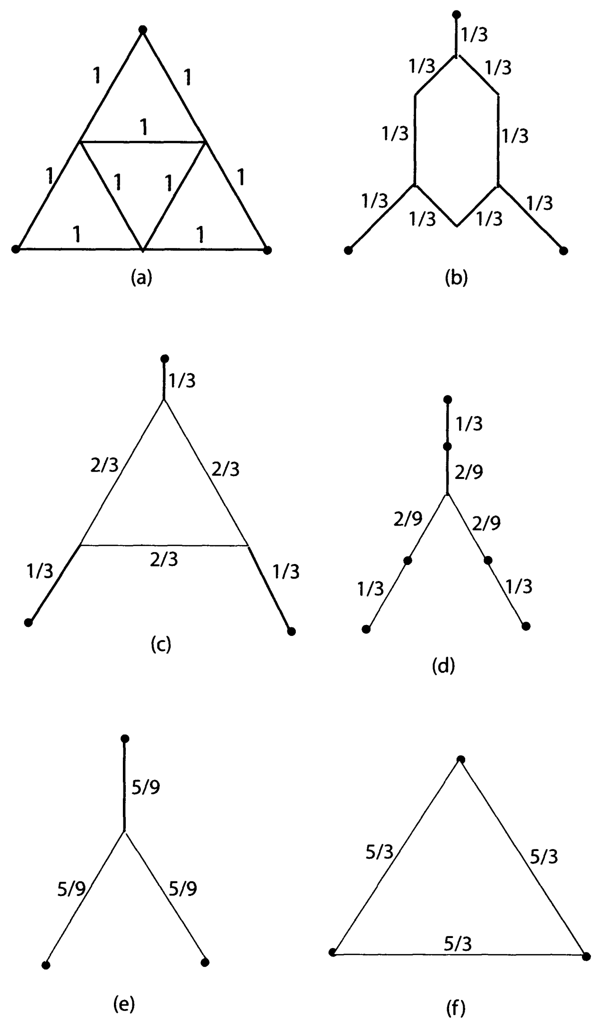

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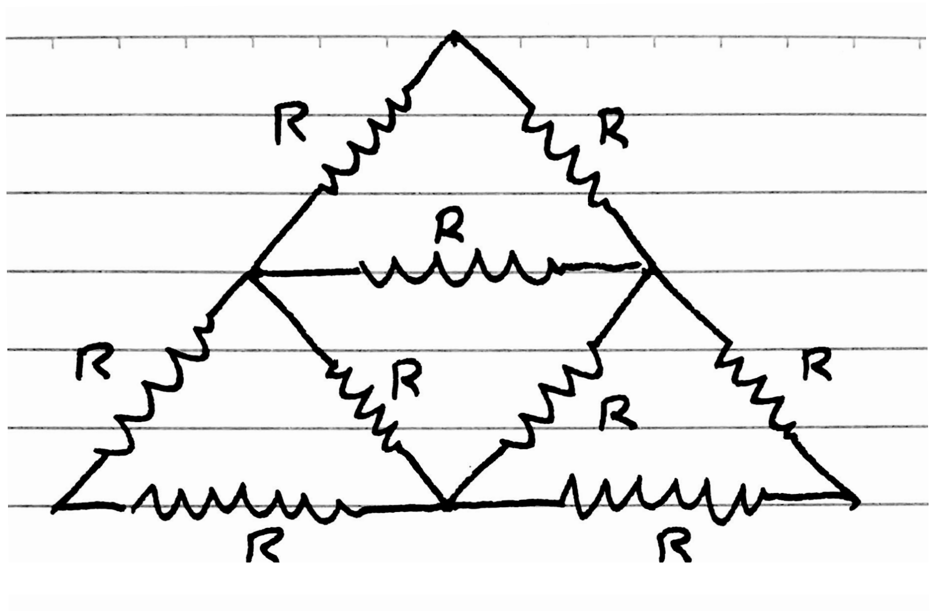

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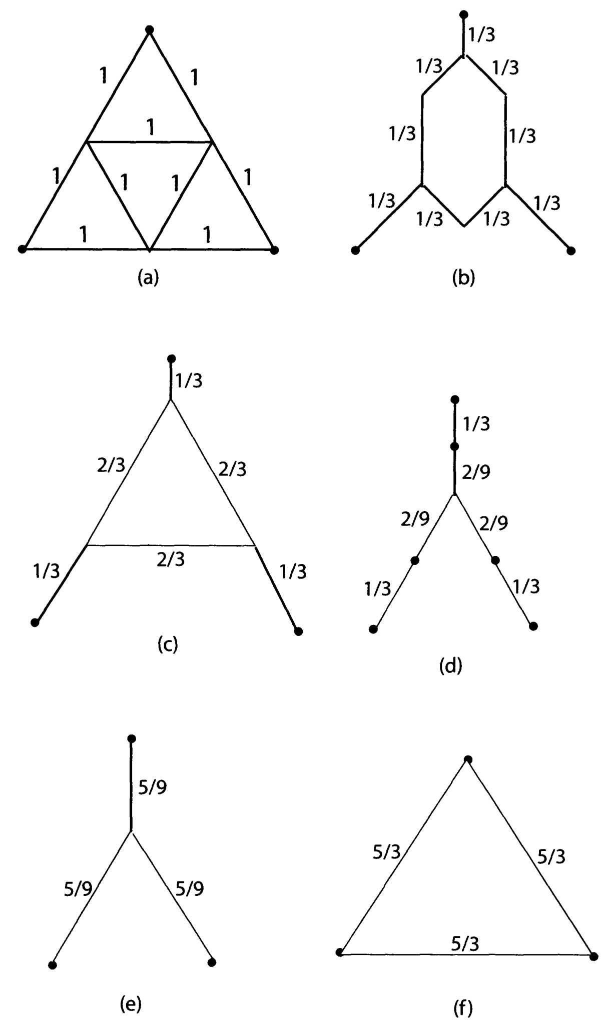

$$\boxed{R(L) \sim L^{\varsigma}}$$

$$\varsigma = \frac{\ln 5/3}{\ln 2}$$

The electric dimension ς is different from $d_h = \ln 3 / \ln 2$

Classical diffusion - walk dimension d_w

On an Euclidean manifold, we write the mean square displacement

$$\langle r^2(t) \rangle = Dt$$

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Related through the Einstein relation

Continuous *vs*. discrete scale invariance

In all previous cases, we have found that exponents are determined by a scaling relation:

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$$f(a\,x) = b\,f(x)$$

If this relation is satisfied $\forall\, b(a) \in \mathbb{R}$, the system has a continuous scale invariance

General solution (by direct inspection) $f(x) = C\,x^{\alpha}$

with $\alpha = \dfrac{\ln b}{\ln a}$ (does not need to be an integer)

Continuous *vs.* discrete scale invariance

In all previous cases, we have found that exponents are determined by a scaling relation:

$$f(a\,x) = b\,f(x)$$

If this relation is satisfied $\forall\, b(a) \in \mathbb{R}$, the system has a continuous scale invariance

General solution (by direct inspection) $f(x) = C\,x^{\alpha}$

with $\alpha = \dfrac{\ln b}{\ln a}$ (does not need to be an integer)

Power laws are signature of scale invariance

Discrete scale invariance

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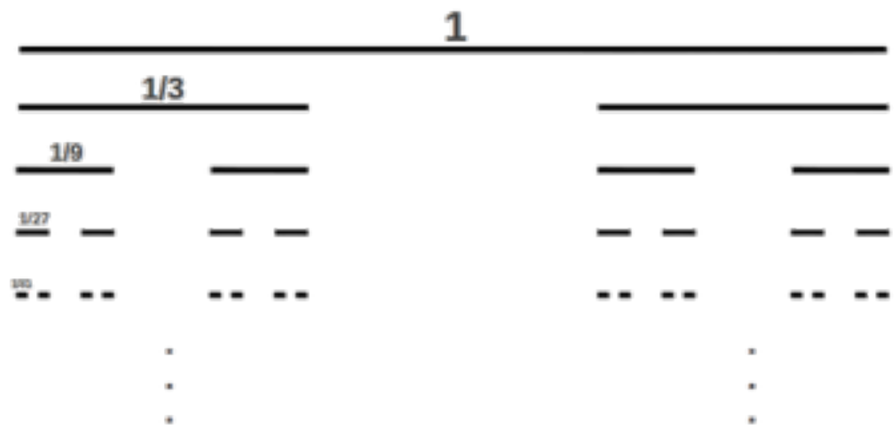
Generalizes to scaling equations :

$$f(ax) = b\,f(x) + g(x), \quad \textit{with fixed}\ (a,b)$$

Relation between the two cases : discrete *vs*. continuous



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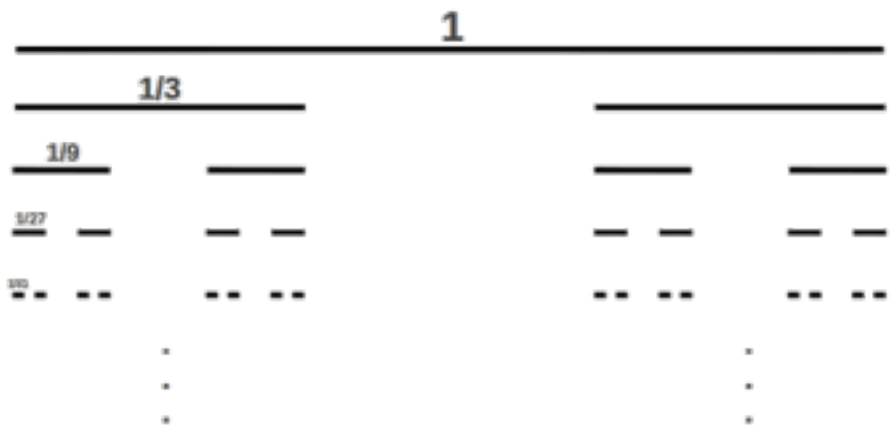

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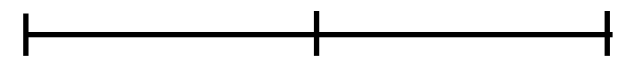

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$$\forall\, b(a) \in \mathbb{R}$$

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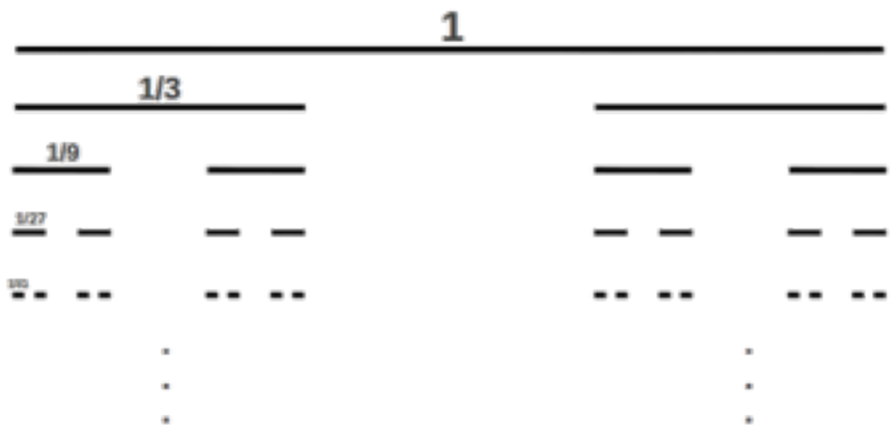

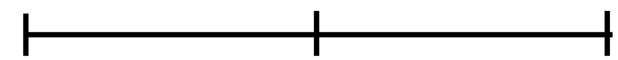

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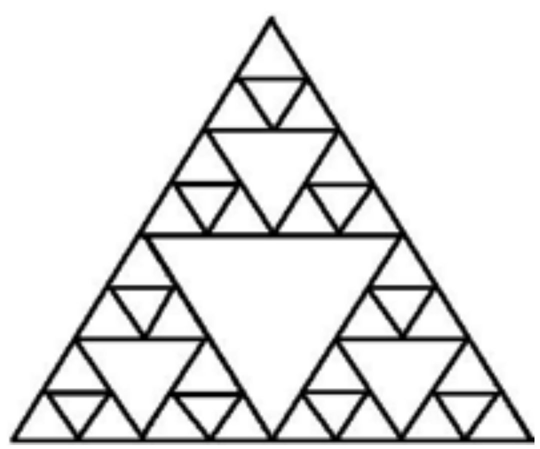

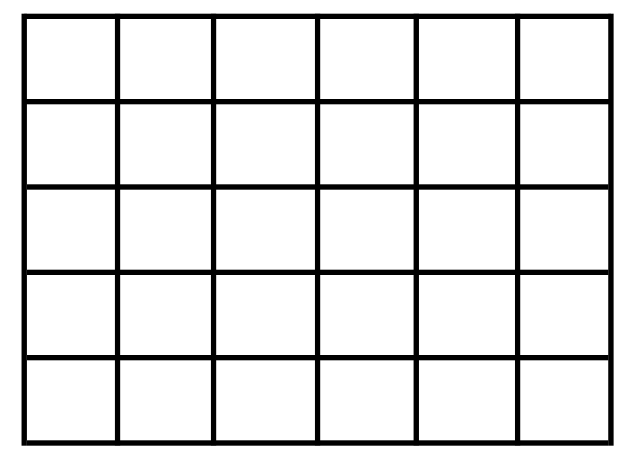

$d = 2$

$m(2L) = 3m(L)$ $\quad (a,b) = (2,3)$

Both satisfy $f(a\,x) = b\,f(x)$ but with fixed (a,b) for the fractals.

Complex fractal exponents and oscillations

For a discrete scale invariance, $f(x) = x^{\alpha} G\left(\frac{\ln x}{\ln a}\right)$

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<u>The existence of such an infinite set is sometimes taken as a definition of an underlying fractal structure.</u>

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Mellin or ζ-transform :

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Inverse Mellin transform gives immediately:

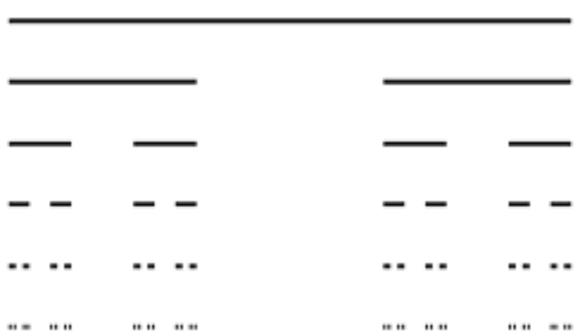
$$f(x) = x^{\frac{\ln b}{\ln a}} G\left(\frac{\ln x}{\ln a}\right) \qquad \text{with} \qquad G(u+1) = G(u)$$

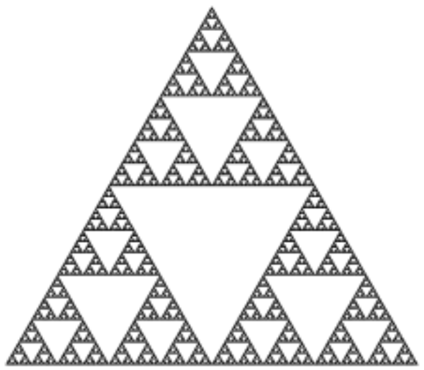
Part 3: Operators and fields on fractal manifolds

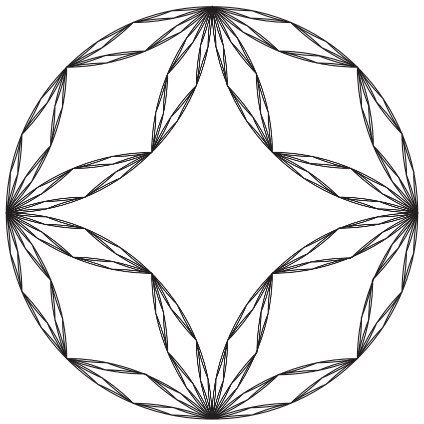
Operators are often expressed by local differential equations relating the space-time behaviour of a field

Ex. Wave equation $$\frac{\partial^2 u}{\partial t^2} = \Delta u$$

Such local equations cannot be defined on a fractal







But operators are essential quantities for physics!

- Quantum transport in fractal structures :

  e.g., networks, waveguides, ...

  electrons, photons

  dependence on temperature, on external fields (E, B)

- Density of states
- Scattering matrix (transmission/reflection)

But operators are essential quantities for physics!

- Quantum fields on fractals, *e.g.*, fermions (spin 1/2), photons (spin 1) - canonical quantisation (Fourier modes) - path integral quantisation : path integrals, Brownian motion.

- "curved space QFT" or quantum gravity

- Scaling symmetry (renormalisation group) - critical behaviour.

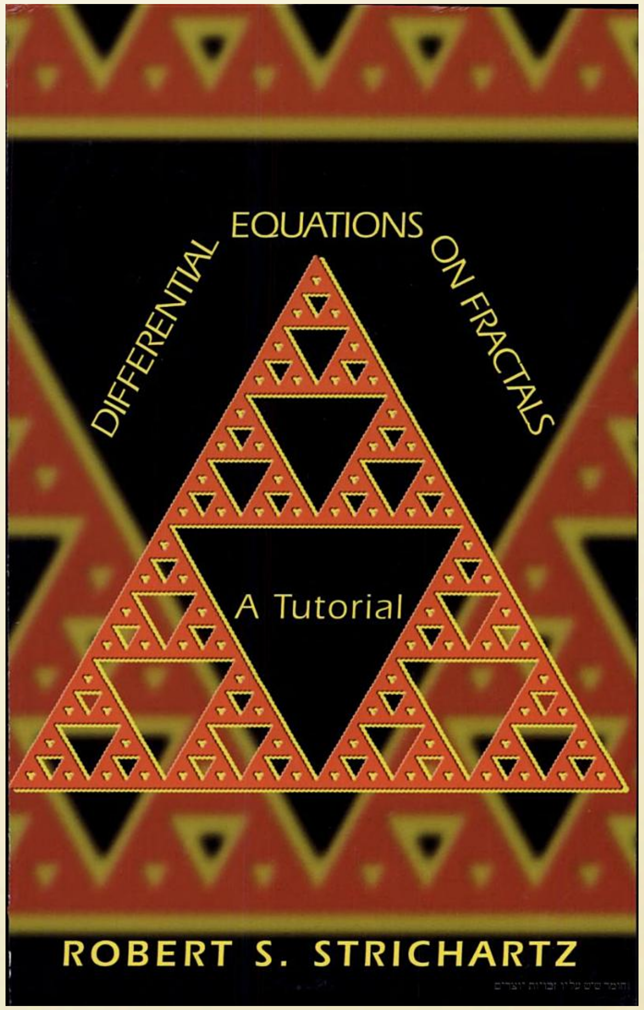
Recent new ideas

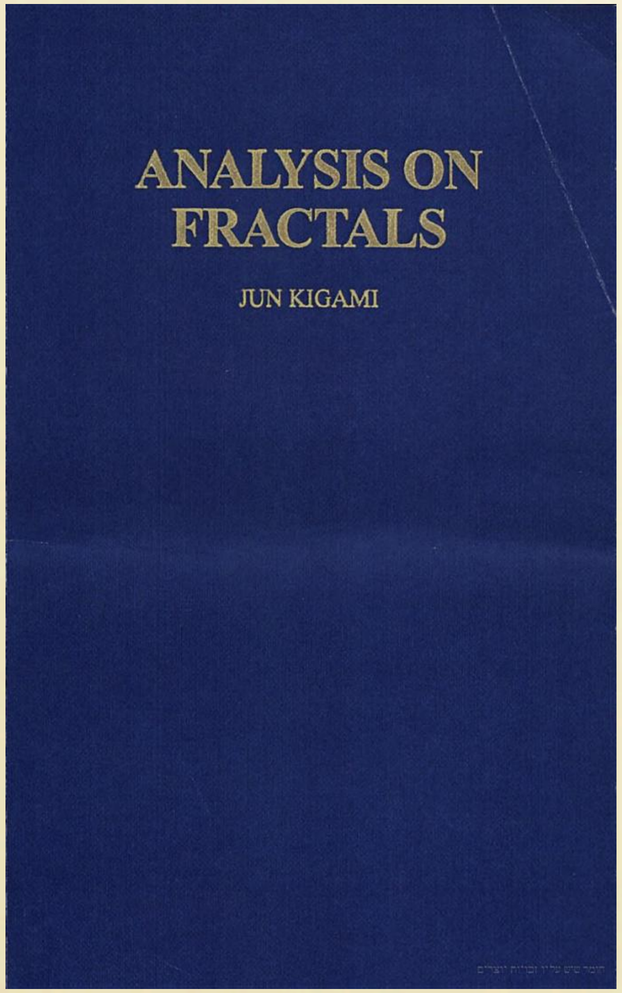
>2000

Maths.

Michel Lapidus

Bob Strichartz





Jun Kigami

Intermezzo : heat and waves

From classical diffusion to wave propagation

There is an important relation between classical diffusion and wave propagation on a manifold.

It expresses this profound idea that it is possible to measure and characterise a manifold using waves, more precisely with the eigenvalue spectrum of the Laplacian operator.

Differential operator
“propagating probe”

geometry
curvature
volume
dimension

$\longleftrightarrow$

Spectral data
Heat kernel
Zeta function

physically:
Laplacian

Mathematical physics

Use propagating physical waves/particles to probe geometry

- spectral information: density of states, transport, heat kernel, ...
- geometric information: dimension, volume, boundaries, shape, ...

1910 Lorentz: why is the Jeans radiation law only dependent on the volume ?

1911 Weyl : relation between asymptotic eigenvalues and dimension/ volume.

1966 Kac : can one hear the shape of a drum ?

Important examples

- Heat equation $\dfrac{\partial u}{\partial t} = \Delta u$
- Wave equation $\dfrac{\partial^2 u}{\partial t^2} = \Delta u$

Schr. equation. $i\dfrac{\partial u}{\partial t} = \Delta u$

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$$u(x,t) = \int d\mu(y) P_t(x,y) u(y,0)$$

Schr. equation. $$i\frac{\partial u}{\partial t} = \Delta u$$

$$P_t(x,y) = \int_{x(0)=x,\,x(t)=y} \mathcal{D}x\, e^{-(i)\int_0^t \dot{x}^2 d\tau}$$

Brownian motion

$$P_t(x,y) \sim \frac{1}{t^{d/2}} \sum_n a_n(x,y) t^n$$

Heat kernel expansion

$$P_t(x,y) \sim \sum_{geodesics} (\#) e^{-(i)S_{classical}(x,y,t)}$$

Gutzwiller - instantons

Spectral functions

$$P_t(x,y) = \langle y | e^{-\Delta t} | x \rangle = \sum_{\lambda} \psi_{\lambda}^{*}(y)\psi_{\lambda}(x)e^{-\lambda t}$$

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Return probability

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Mellin transform

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Weyl expansion

The heat kernel is related to the density of states of the Laplacian

There are "Laplace transform" of each other:

$$Z(t) = \int_0^\infty d\omega \, \rho(\omega) \, e^{-t\omega}$$

From the Weyl expansion, it is thus possible to obtain the density of states.

How does it work ?

$$\frac{\partial \phi}{\partial t} = D \frac{\partial^2 \phi}{\partial x^2}$$

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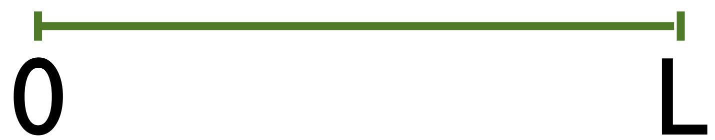
We can characterise the "spatial geometry" by watching how the heat flows.
The heat kernel $Z_d(t)$ is

$$Z_d(t) = \int_{Vol.} d^d x P_t(x,x) = \frac{Volume}{(4\pi D t)^{d/2}}$$ $\longrightarrow$ access the volume of the manifold

Boundary terms- Hearing the shape of a drum

Mark Kac (1966)



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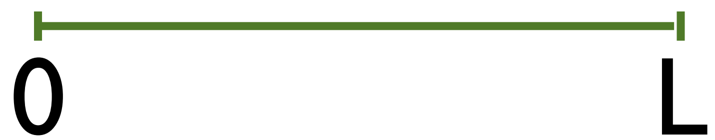

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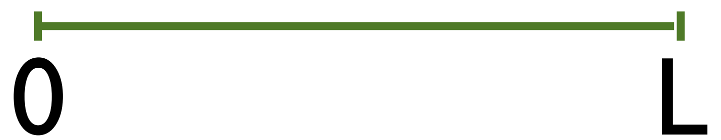

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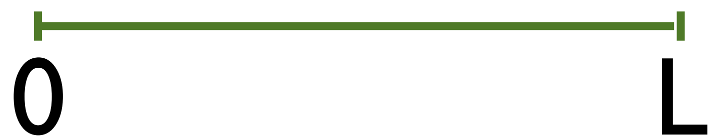

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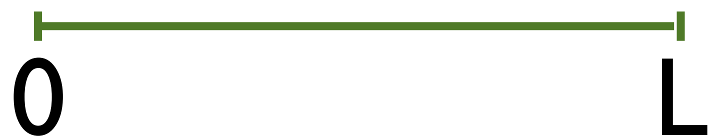

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Weyl expansion (2d) :

$$\boxed{Z_{d=2}(t) \sim \frac{Vol.}{4\pi t} - \frac{L}{4}\frac{1}{\sqrt{4\pi t}} + \frac{1}{6} + \ldots}$$

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sensitive to boundary

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$$Z_{d=2}(t) \sim \frac{Vol.}{4\pi t} - \frac{L}{4}\frac{1}{\sqrt{4\pi t}} + \frac{1}{6} + \dots$$

bulk

sensitive to boundary

integral of bound. curvature

ζ-function $\quad \zeta_Z(s) = Tr\frac{1}{\Delta^s} = \sum_\lambda \frac{1}{\lambda^s}$

$$Dirichlet : \lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 1,2,\ldots$$

$$\zeta(s) = \sum_{n=0}^{\infty}\left(\frac{L^2}{n^2\pi^2}\right)^s = \frac{L^{2s}}{\pi^{2s}}\sum_{n=0}^{\infty}\frac{1}{n^{2s}} \equiv \frac{L^{2s}}{\pi^{2s}}\zeta_R(2s)$$

$\zeta_R(2s)$ has a simple pole at $s = \frac{1}{2}$ $\left(s = \frac{d}{2}\right)$ so that,

$$Z(t) = \frac{1}{2i\pi}\int_{a-i\infty}^{a+i\infty} ds\, t^{-s}\,\Gamma(s)\zeta(s) \sim \frac{L}{2\pi}t^{-1/2}\,\Gamma(1/2) + \ldots$$

$$= \frac{L}{\sqrt{4\pi t}} + \ldots$$

How does it work on a fractal ?

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Differently…

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Differently…

No access to the eigenvalue spectrum but we know how to calculate the Heat Kernel.

$$Z(t) = Tr\, e^{-\Delta t} = \int dx \langle x | e^{-\Delta t} | x \rangle = \sum_{\lambda} e^{-\lambda t}$$

and thus, the density of states,

$$Z(t) = \int_0^\infty d\omega\, \rho(\omega)\, e^{-t\omega}$$

More precisely, we have,

$$Z(t)=\sum_{k=1}^{\infty} e^{-k^2\pi^2 t}+B\sum_{n=0}^{\infty} L_n^{d_h}\sum_{k=1}^{\infty} e^{-k^2\pi^2 t\, L_n^{d_w}}$$

where $L_n = a^n$ is the total length upon iteration of the elementary step

$$\zeta(s)=\frac{\zeta_R(2s)}{\pi^{2s}}+\sum_{n=0}^{\infty} L_n^{d_h}\sum_{k=1}^{\infty}\left(\frac{1}{k^2\pi^2 L_n^{d_w}}\right)^s$$

$$=\frac{\zeta_R(2s)}{\pi^{2s}}\left(1+\sum_{n=0}^{\infty} L_n^{d_h-d_w s}\right)$$

$$=\frac{\zeta_R(2s)}{\pi^{2s}}\left(\frac{2-a^{d_h-d_w s}}{1-a^{d_h-d_w s}}\right)$$

which has poles at

$$a^{d_h-d_w s}=1$$

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The whole calculation !

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Infinite number of complex poles : complex fractal dimensions.

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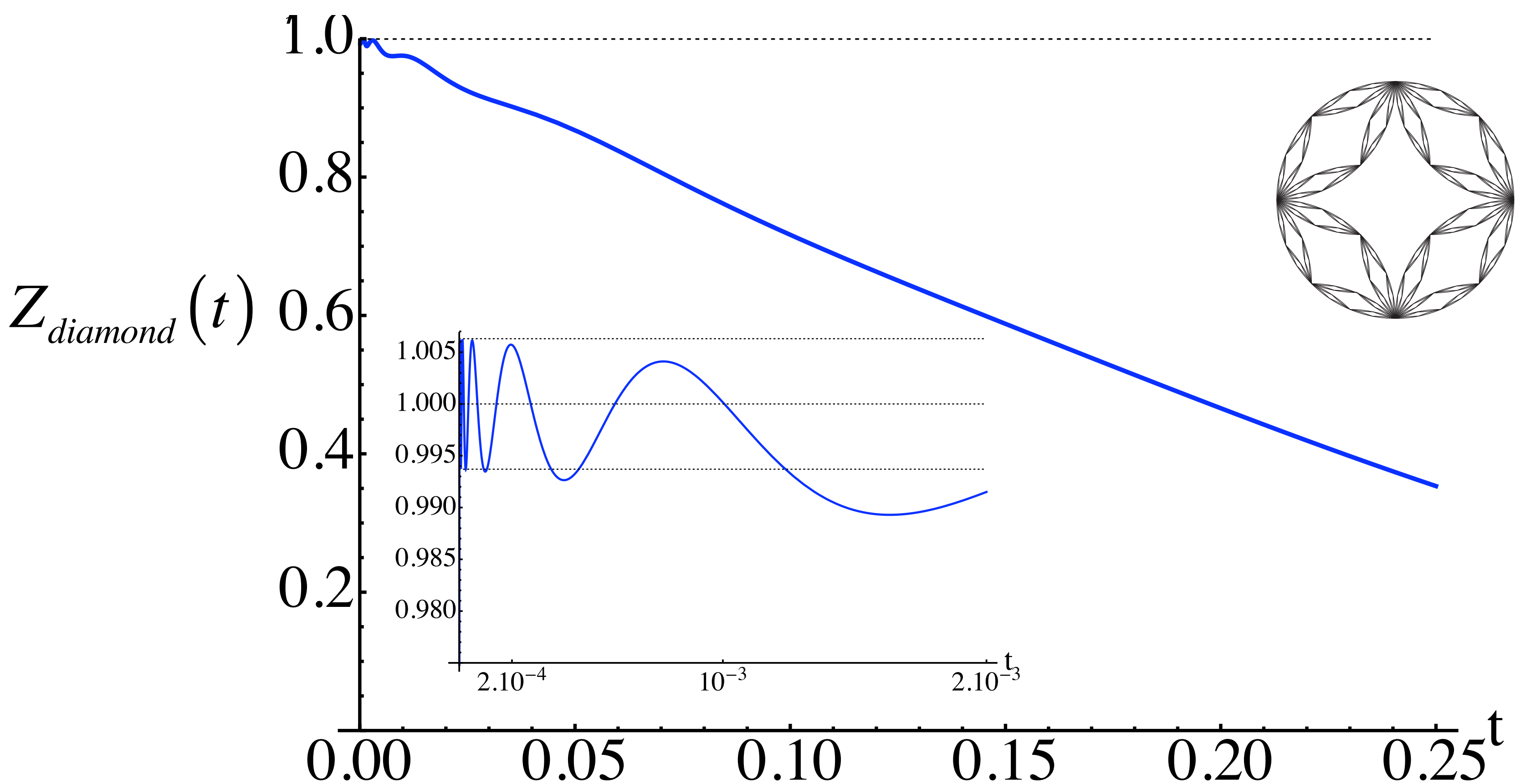
Infinite number of complex poles : complex fractal dimensions.

They control the behaviour of the heat kernel which exhibits oscillations!



A new fractal dimension : **spectral dimension** d_s

Another surprise :

Notion of spectral volume

From the previous expression we obtain $Z(t)$

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Consider for simplicity $n=1$, namely $s_1 = \frac{d_s}{2} + \frac{2i\pi}{d_w \ln a} \equiv \frac{d_s}{2} + i\delta$

$$Z(t) = \mathrm{Re}\left(\frac{V_S}{t^{\frac{d_s}{2}+i\delta}} \right)$$

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$$\boxed{Z(t) \sim \frac{V_s}{t^{d_s/2}} \cos\left(\frac{2\pi}{d_w \ln a} \ln t\right)}$$

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to compare with

$$Z_d(t) = \int_{Vol.} d^d x P_t(x,x) = \frac{Volume}{(4\pi D t)^{d/2}}$$

Spectral volume ?

Geometric volume described by the Hausdorff dimension is large (infinite)

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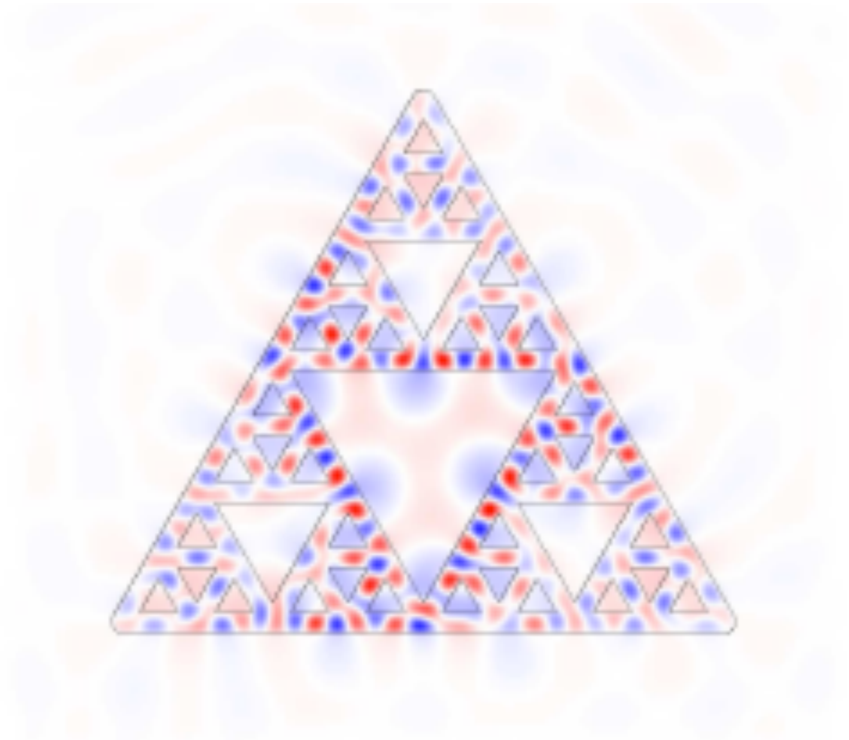
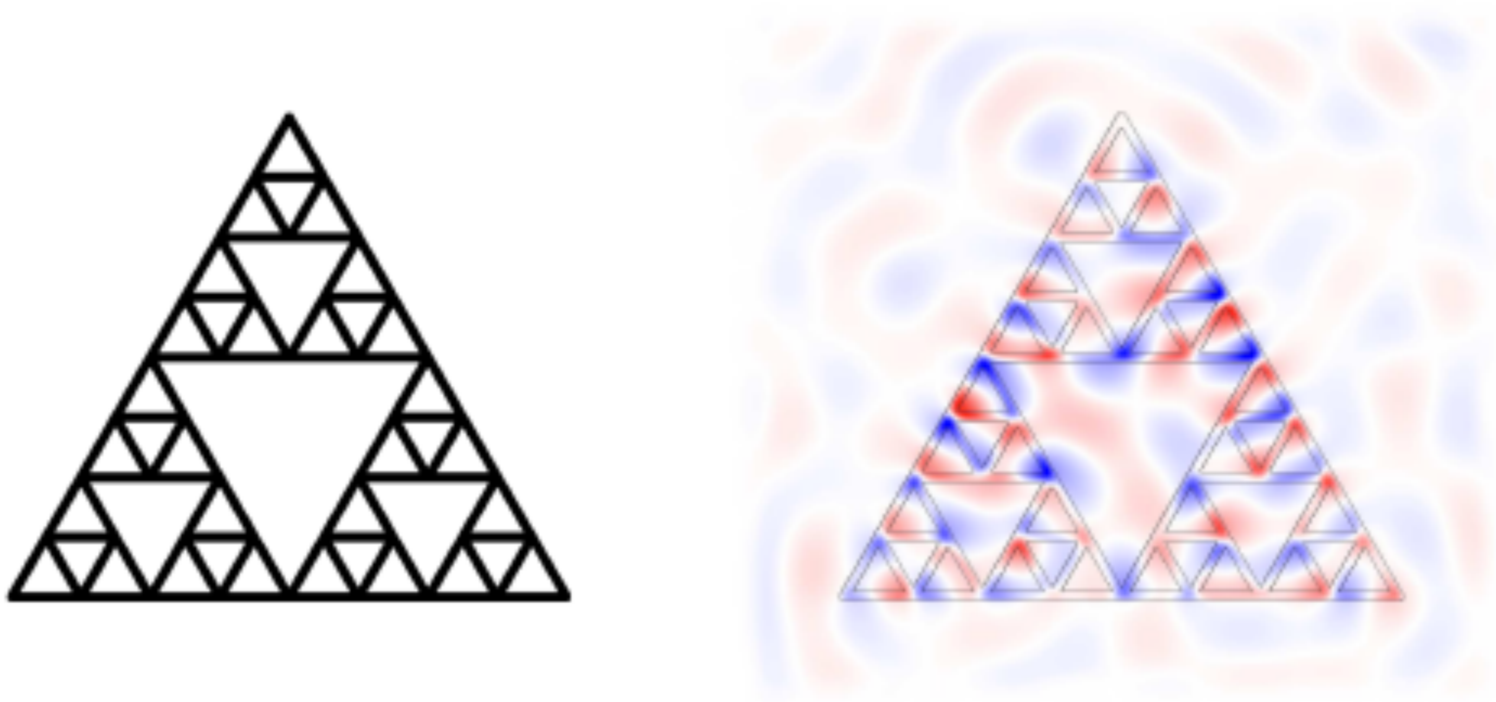


Spectral volume V_s
is the finite volume occupied by the modes

Numerical solution of Maxwell eqs. in the Sierpinski gasket
(courtesy of S.F. Liew and H. Cao, Yale)

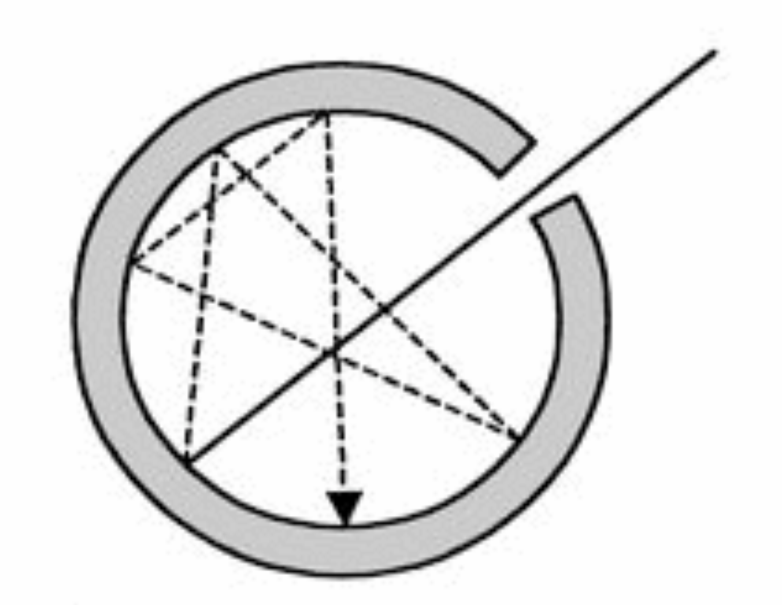
Part 4: Physical application.
Thermodynamics of photons on fractals



Quantisation of the electromagnetic field in a waveguide fractal structure.

How to measure the spectral volume

Blackbody radiation from a fractal or thermodynamics without phase space



Equation of state at thermodynamic equilibrium relating pressure, volume and internal energy:

$$PV = U/d$$

In an enclosure with a perfectly reflecting surface there can form standing electromagnetic waves analogous to tones of an organ pipe; we shall confine our attention to very high overtones. Jeans asks for the energy in the frequency interval $d\nu$ ... It is here that there arises the mathematical problem to prove that the number of sufficiently high overtones that lies in the interval ν to $\nu+d\nu$ is independent of the shape of the enclosure and is simply proportional to its volume.

H. Lorentz, 1910

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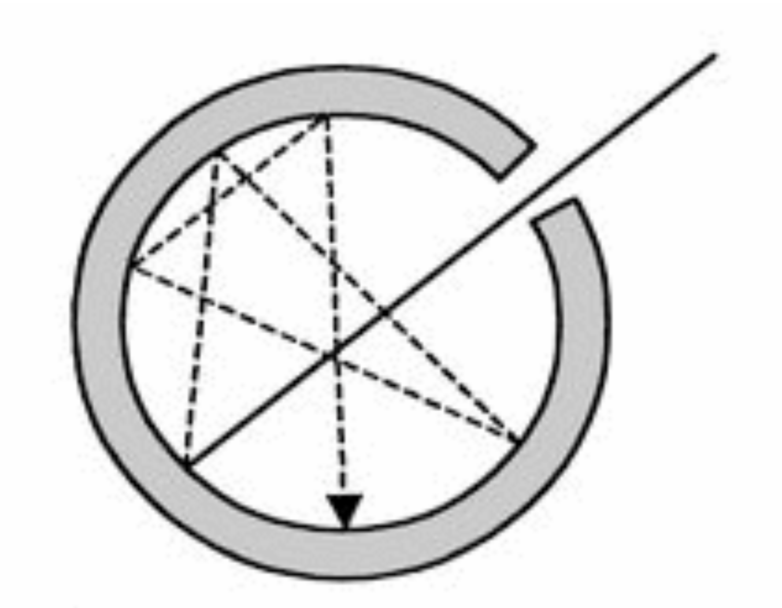

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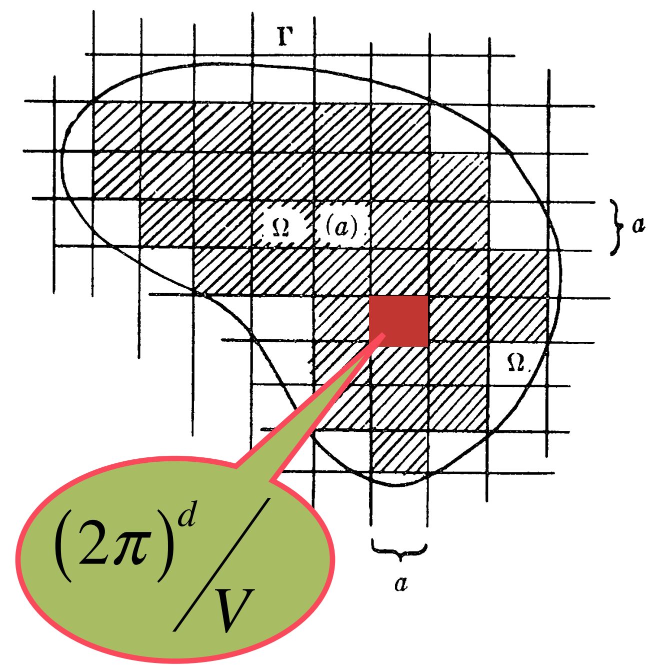
Spectral volume ?

Usual approach : count modes in momentum space

Calculate the partition (generating) function $z(T,V)$ for a blackbody of [large volume]() V in [dimension]() d



Mode decomposition of the field $\quad \omega = c\left|\vec{k}\right| = c\,V^{-1/d}\,2\pi\left|\vec{n}\right|$

$$\boxed{\ln z(T,V) = Q\left(L_\beta \big/ V^{1/d}\right)}$$

with $\quad L_\beta \equiv \beta\hbar c$

(photon thermal wavelength)

$\beta = 1/k_B T$

Thermodynamics :

$$U = -\frac{\partial}{\partial \beta} \ln z(T, V) = -\left(\frac{dQ}{dx}\right) \hbar c V^{-1/d}$$

$$P = \frac{1}{\beta}\left(\frac{\partial}{\partial V} \ln z\right)_T = -\left(\frac{dQ}{dx}\right) \frac{\hbar c V^{-1/d}}{V d}$$

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Stefan-Boltzmann $U \propto V T^{d+1}$ is a consequence of $\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P$

Adiabatic expansion $V T^d = Cte$

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Volume of a fractal is usually infinite.

Nevertheless,

$$\boxed{PV_s = U/d_s}$$

V_s is the “spectral volume”.

But we can re-phrase the thermodynamic problem in terms of heat kernel and zeta function !

Partition function of quantum radiation at equilibrium- General formulation - General geometric shape

$$\ln z(T,V) = -\frac{1}{2}\ln Det_{M\times V}\left(\frac{\partial^2}{\partial\tau^2} + c^2\Delta\right)$$

Looks (almost) like a bona fide wave equation but proper time.

This expression does not rely on mode decomposition, but results from the thermodynamic equilibrium (Keldysh-Schwinger).

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Rescale by $L_\beta \equiv \beta\hbar c$

Partition function of quantum radiation at equilibrium- General formulation - General geometric shape

$$\ln z(T,V) = -\frac{1}{2}\ln Det_{M\times V}\left(\frac{\partial^2}{\partial u^2} + L_\beta^2 \Delta\right)$$

M: circle of radius $L_\beta \equiv \beta\hbar c$

Matsubara frequencies

Spatial manifold (fractal)

Thermal equilibrium of photons on a spatial manifold V at temperature T is described by the (scaled) wave equation on $M \times V$

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$$Z\left(L_\beta^2 \tau\right)$$

Heat kernel

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Heat kernel

Large volume limit (a high temperature limit) $V \gg L_\beta^d \Leftrightarrow k_B T \gg \hbar c / V^{1/d}$

Weyl expansion:

$$Z\left(L_\beta^2\tau\right) \sim \frac{V}{\left(4\pi L_\beta^2 \tau\right)^{d/2}}$$

$$\ln z(T,V) = \frac{1}{2}\int_0^\infty \frac{d\tau}{\tau} f(\tau)\, Tr_V\, e^{-\tau L_\beta^2 \Delta}$$

+ Weyl expansion $\Longrightarrow$ $\boxed{\ln z(T,V) \sim \frac{V}{L_\beta^d}}$

$$\Downarrow$$

$$PV = U/d$$

Thermodynamics measures the spectral volume

On a fractal…

$$Z\left(L_\beta^2 \tau\right) \sim \frac{V_s}{\left(4\pi L_\beta^2 \tau\right)^{d_s/2}} f\left(\ln \tau\right)$$

On a fractal…

Spectral volume

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On a fractal…

Spectral volume

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Spectral dimension

On a fractal…

Spectral volume

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Spectral dimension

Thermodynamic equation of state for a fractal manifold

$$\boxed{PV_s = U/d_s}$$

Thermodynamics measures the spectral volume and the spectral dimension.

Something a bit weird…

Looks like, on a fractal, coordinate and momentum spaces involve different dimensions…

- Euclidean manifold : coordinate space has dimension d

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$$\Delta x \, \Delta k \sim V_m^{1/d_h} \, V_s^{-1/d_s} \sim ?$$

uncertainty principle ?

Thank you for your attention.