Interplay between topology and discrete scaling symmetry : Fibonacci quasi-crystals

A topological system without magnetic field

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Enseigner la recherche en train de se faire

הקרן הלאומית למדע
ISRAEL SCIENCE FOUNDATION

A *spectral* rather than *geometric* perspective of fractals as in the first lecture

Today's program

- Spontaneous emission from a vacuum with a discrete scaling symmetry (fractal)
- Experimental study of the Fibonacci spectrum (polaritons)
- Some wanderings

Benefitted from discussions and collaborations with:

Technion:

Evgeni Gurevich (KLA-Tencor)
Dor Gittelman
Ariane Soret (ENS Cachan)
Or Raz
Omrie Ovdat
Ohad Shpielberg
Alex Leibenzon

Rafael:

Eli Levy
Assaf Barak
Amnon Fisher

Elsewhere:

Gerald Dunne (UConn.)
Alexander Teplyaev (UConn.)
Raphael Voituriez (LPTMC, Jussieu)
Olivier Benichou (LPTMC, Jussieu)
Jacqueline Bloch (LPN, Marcoussis)
Dimitri Tanese (LPN, Marcoussis)
Florent Baboux (LPN, Marcoussis)
Alberto Amo (LPN, Marcoussis)
Julien Gabelli (LPS, Orsay)

A large variety of problems are conveniently described using the existing classification in spectral classes

absolutely continuous

singular-continuous

point spectrum

A large variety of problems are conveniently described in terms of spectral classes

(absolutely continuous / singular-continuous / point spectrum):

- Anderson localisation
- Quantum and classical wave diffusion
- Random magnetism
- …

A large variety of problems are conveniently described in terms of spectral classes

absolutely continuous

What about a fractal spectrum ?

pectrum

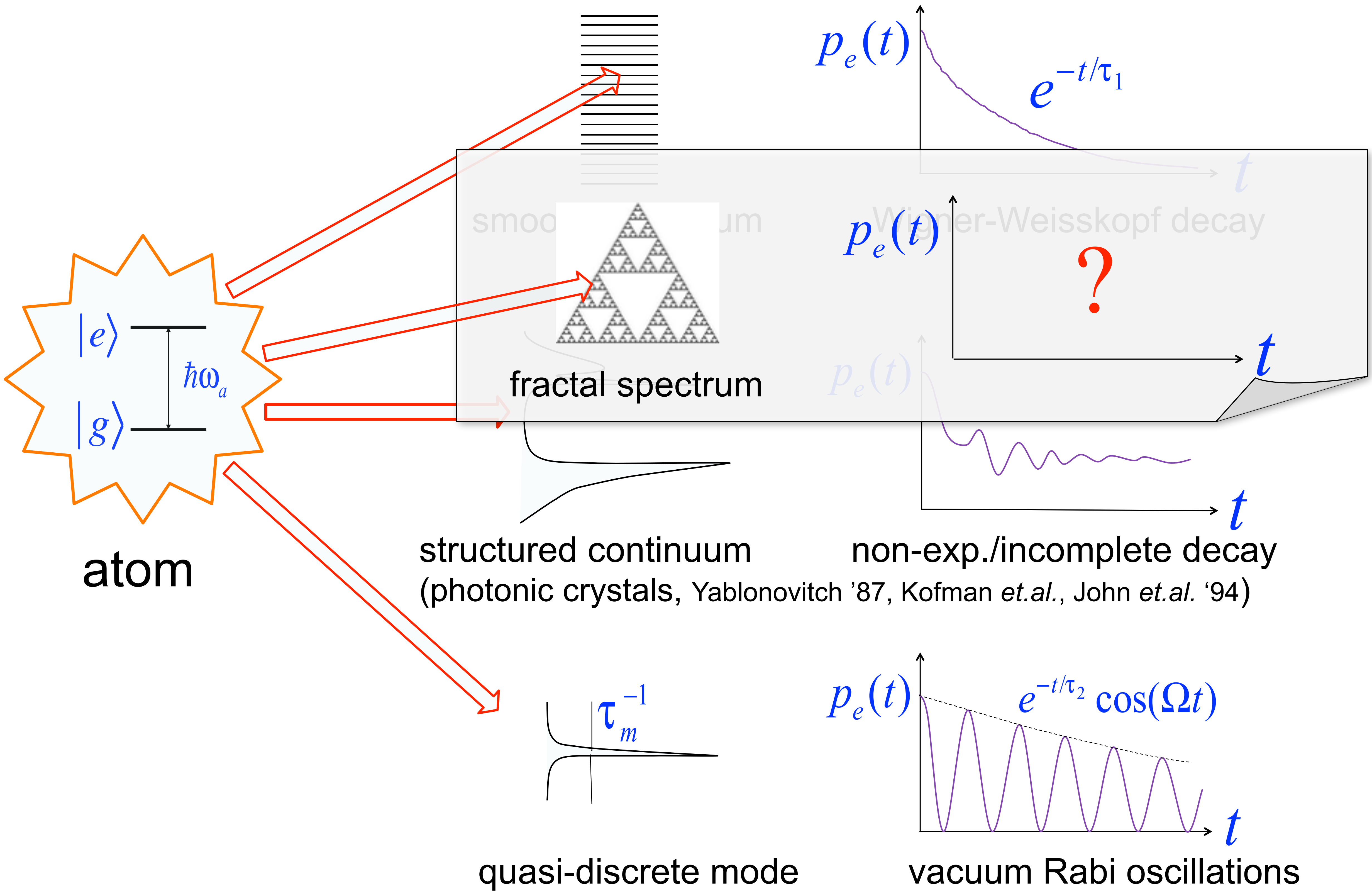
Part 1

An interesting problem to warm up…

Spontaneous emission from a fractal QED cavity/spectrum

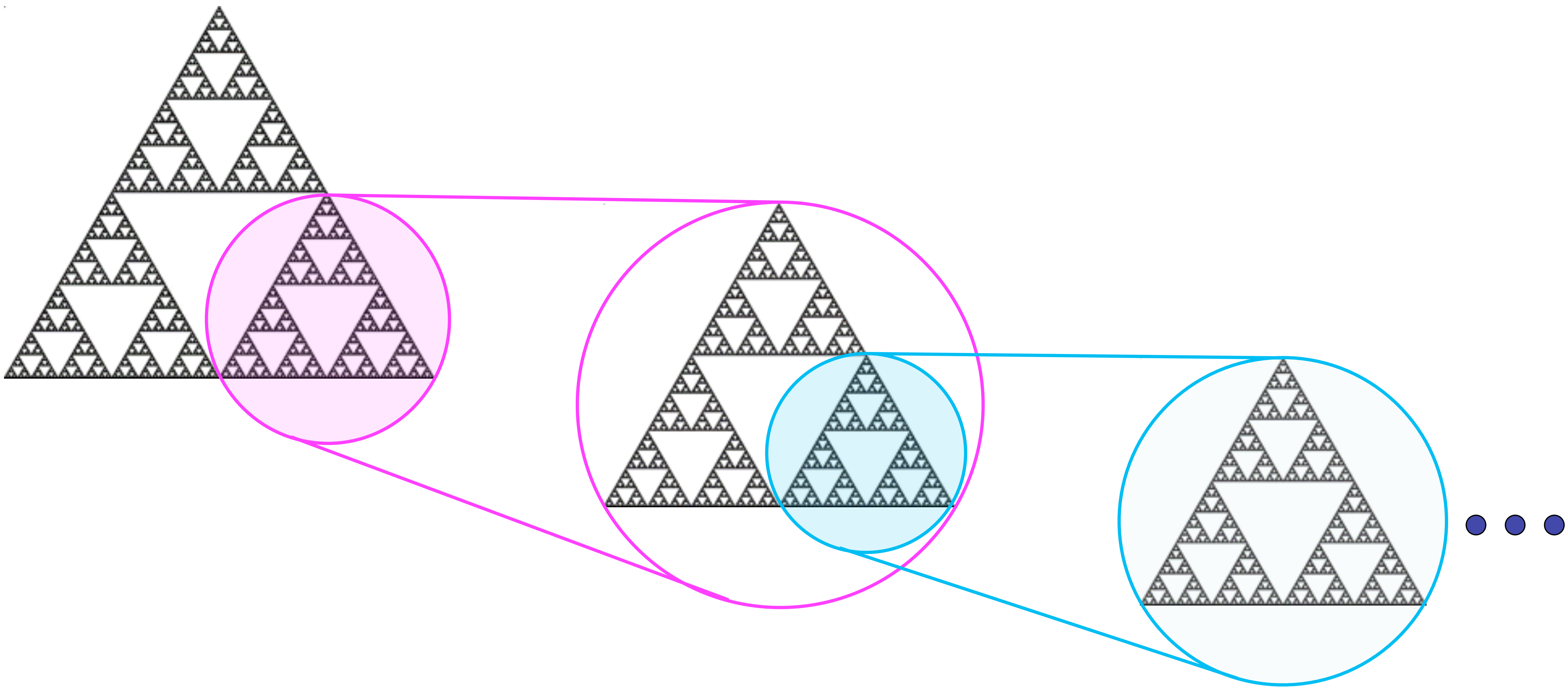
(courtesy of J. Gabelli)

Spontaneous emission for different QED vacua



Fractal spectrum ?

Fractal ↔ Self-similar



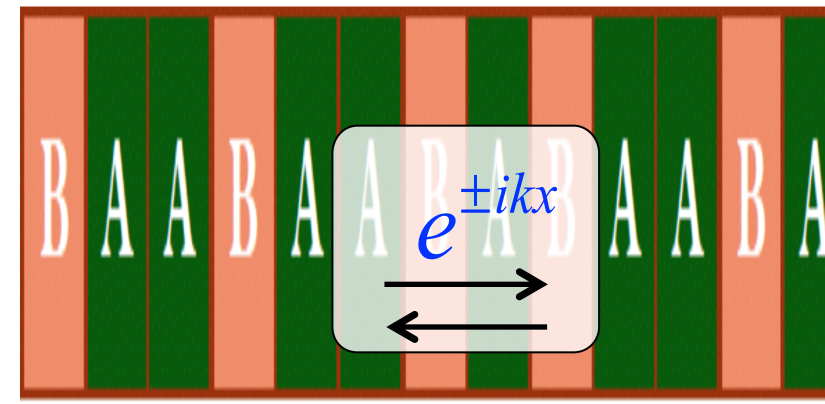
Discrete scaling symmetry

Fractal spectrum - an example

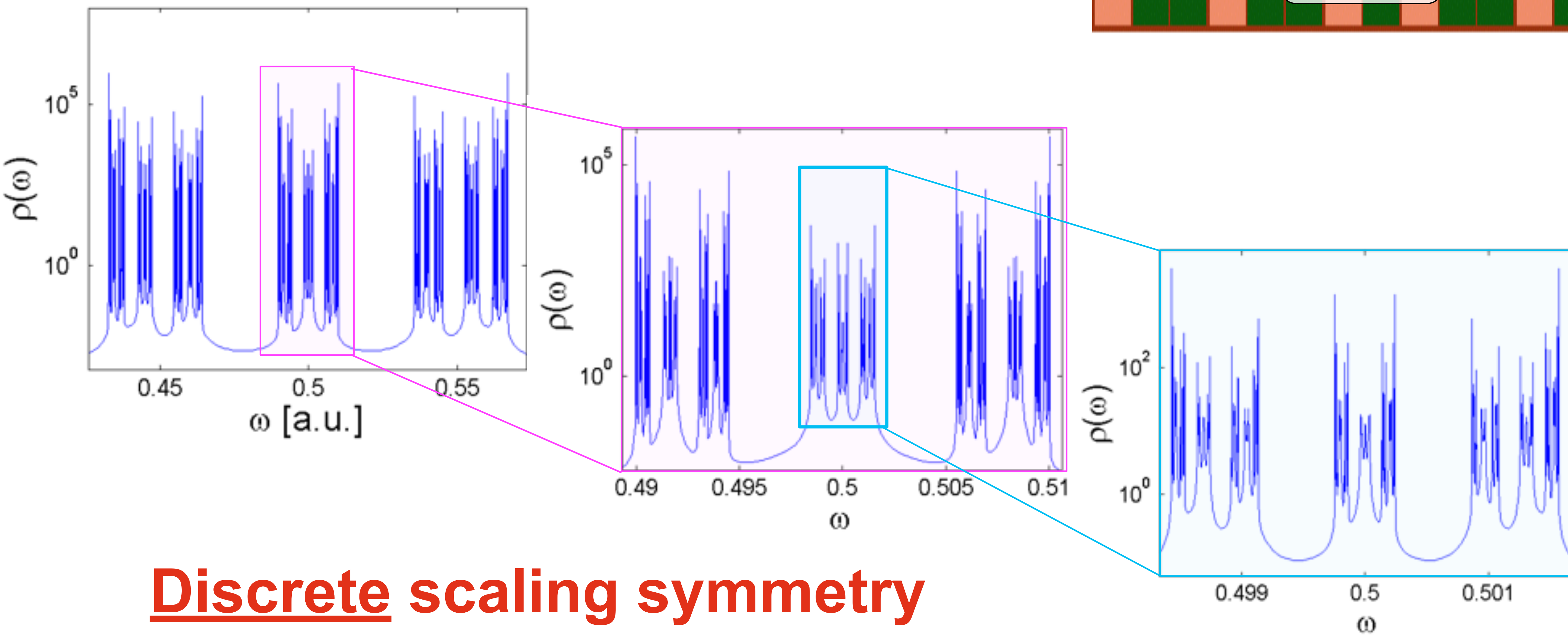
A quasi-periodic stack of dielectric layers of two types (n_A, n_B)

Fibonacci sequence: $S_{j\geq 2} = \left[S_{j-1} S_{j-2} \right], \quad S_0 = B, \quad S_1 = A$

A→AB→ABA→ABAAB→ABAABABA→…

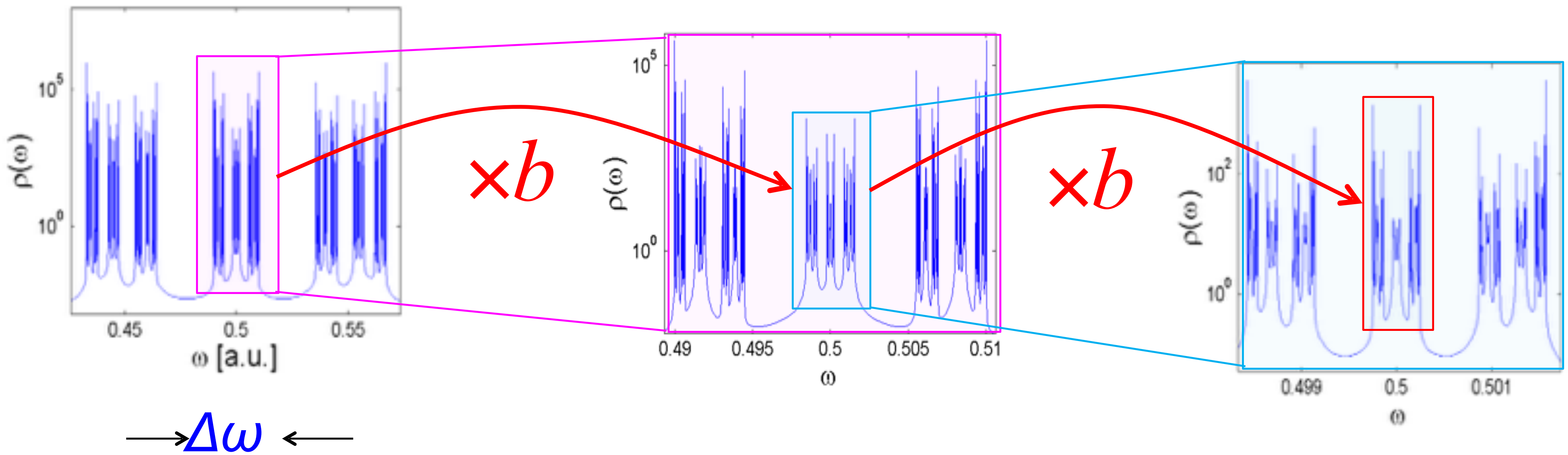


The density of modes $\rho(\omega)$:



Discrete scaling symmetry

Discrete scaling symmetry: formal description



Counting function: $N_\omega(\Delta\omega) \equiv \int_{\omega}^{\omega+\Delta\omega} \rho(\omega')d\omega'$ = (# of states in $[\omega, \omega+\Delta\omega]$)

$$N_\omega(b^p\Delta\omega) = a^p N_\omega(\Delta\omega), \quad p \in \mathbf{Z}$$

b, a - fixed scaling factors

⇒ **Discrete scaling symmetry**

Testing the discrete scaling symmetry

Scaling equation

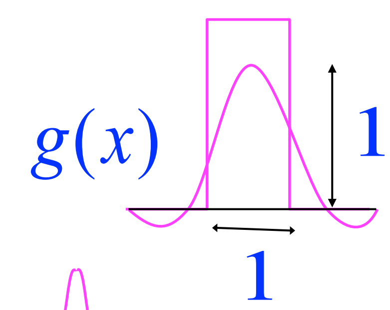
$$N_\omega(b^p \Delta\omega) = a^p N_\omega(\Delta\omega), \qquad N_\omega(\Delta\omega) \equiv \int_\omega^{\omega+\Delta\omega} \rho(\omega')d\omega'$$

has the following general solution (dimensionless ω):

$$N_\omega(\Delta\omega) = (\Delta\omega)^\alpha \times F\left(\frac{\ln|\Delta\omega|}{\ln b}\right), \qquad \alpha = \frac{\ln a}{\ln b}, \quad F(x+1) = F(x)$$

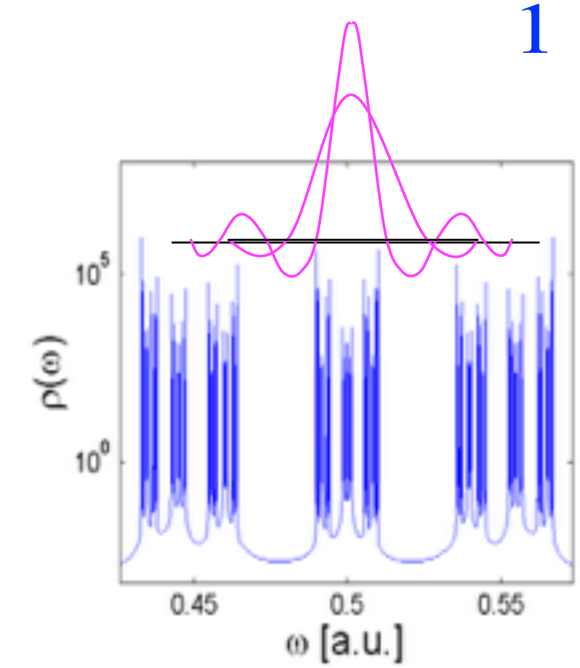
$0 \le \alpha \le 1$ - fractal exponent (absolutely continuous : $\alpha = 1$, pure-point : $\alpha = 0$)

Similarly for the convolution of $\rho(\omega)$ with a window function



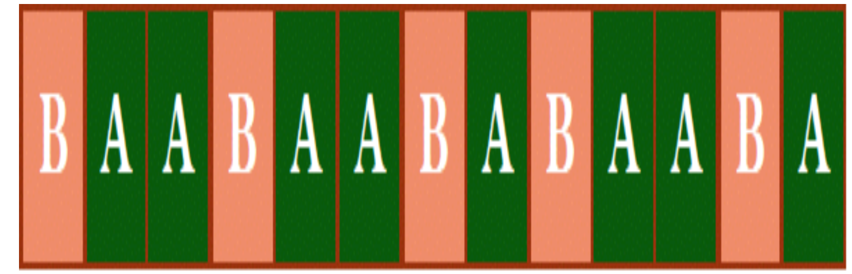
$$N_\omega^{(g)}(\Delta\omega) \equiv \int g\left(\frac{\omega'-\omega}{\Delta\omega}\right)\rho(\omega')d\omega' = (\Delta\omega)^\alpha \times F_g\left(\frac{\ln|\Delta\omega|}{\ln b}\right),$$

(Ghez and Vaienti, ’89: the wavelet transform of fractal measures)



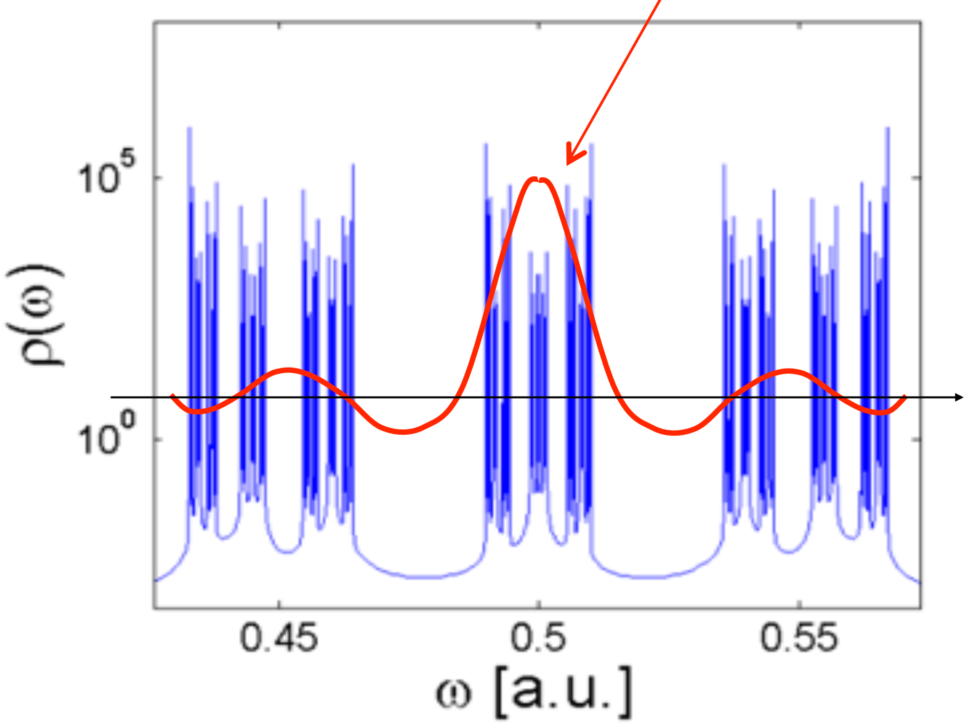
Testing the discrete scaling symmetry - an example

A quasi-periodic dielectric stack

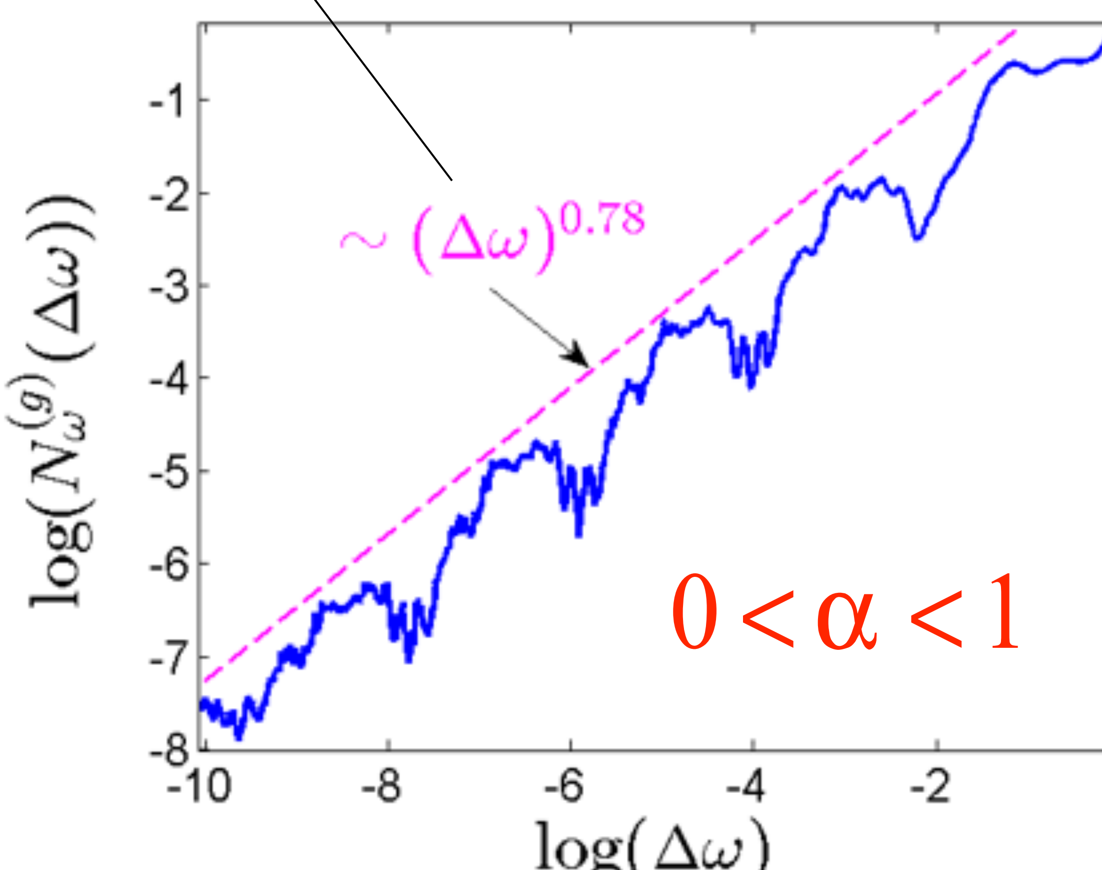


$$N_\omega^{(g)}(\Delta\omega) \equiv \int g\left(\frac{\omega' - \omega}{\Delta\omega}\right)\rho(\omega')d\omega' \overset{?}{=} (\Delta\omega)^\alpha \times F_g\left(\frac{\ln|\Delta\omega|}{\ln b}\right),$$

$$g(x) = \frac{\sin(x)}{\pi x}$$



numerics



$$\alpha = \alpha(n_A, n_B) = 0.777$$

Summarise

A quasi-periodic dielectric stack

does not have a geometric fractal structure, but…

Summarise

A quasi-periodic dielectric stack

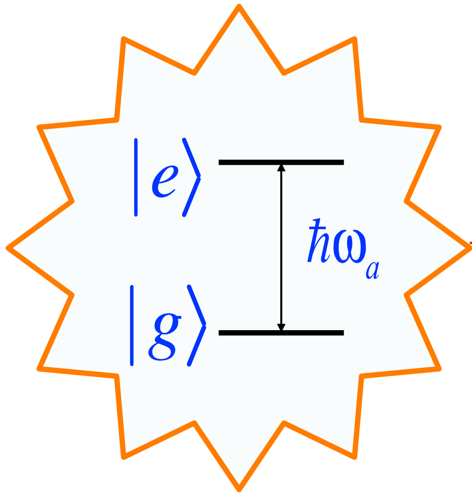
does not have a geometric fractal structure, but…

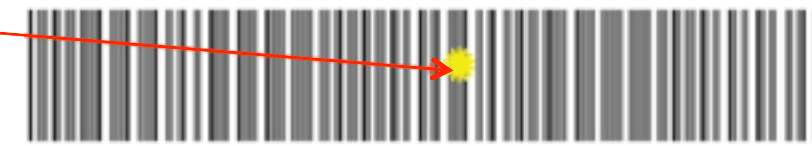
its spectrum has a fractal structure :

$$N_{\omega}(\Delta\omega) = (\Delta\omega)^{\alpha} \times F\left(\frac{\ln|\Delta\omega|}{\ln b}\right), \qquad \alpha = \frac{\ln a}{\ln b}, \quad F(x+1) = F(x)$$

Spectral fractal dimension d_s

Two-level atom coupled to a continuum of states





$$H_{Int} = \sum_k V_k a_k^\dagger |g\rangle\langle e| + h.c.$$

$$V_k \sim E_k(r_a)$$

$$H_{Atom} = \hbar\omega_a |e\rangle\langle e|$$

$$H_{Field} = \hbar\sum_k \omega_k a_k^\dagger a_k$$

We solve the time-dependent problem: $\left|\Psi(t=0)\right\rangle = |e,0_k\rangle$

$$|\Psi(t)\rangle = \alpha(t)e^{-i\omega_a t}|e,0_k\rangle + \int dk\,\rho(k)\beta_k(t)|g,1_k\rangle$$

density of photonic modes

$p_e(t) = |\alpha(t)|^2$ - the excited state probability

Two-level atom coupled to a continuum of states - basics

Probability amplitude

state after a time t :

$$U_e(t) = \langle e, 0_k | \hat{U}(t,0) | e, 0_k \rangle$$

$\hat{U}(t,0)$ evolution operator for the total Hamiltonian $H_{Atom} + H_{Int} + H_{Field}$

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A

$$U_e(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathrm{d}s \frac{e^{(s-i\omega_e)t}}{s + \tilde{\Phi}_e(s - i\omega_e)}.$$

$\tilde{\Phi}_e(s)$ is the Laplace transform of time correlation function of the field

$$\Phi_e(t) = \hbar^{-2} |d_{ge}|^2 \langle 0_k | \hat{E}_z(\mathbf{r}, t) \hat{E}_z^\dagger(\mathbf{r}, 0) | 0_k \rangle$$

Two-level atom coupled to a continuum of states - basics

Probability amplitude

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Note : local quantity

Two relevant energy scales for the pb. of spontaneous emission:

1. Strength $\Gamma_e(\omega_e)$ of the coupling between emitter and vacuum.

2. Spectral width Δ of $\Gamma_e(\omega_e)$

- Dimensionless coupling parameter :

$$g = \Gamma_e(\omega_e)/\Delta.$$

Strong vs. weak coupling

- Weak coupling limit $g \ll 1$,

Probability amplitude $U_e(t) = \langle e, 0_k | \hat{U}(t,0) | e, 0_k \rangle$ for spont. emission

is determined by the pole in $$U_e(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathrm{d}s \frac{e^{(s-i\omega_e)t}}{s + \tilde{\Phi}_e(s - i\omega_e)}.$$

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At long time, $t \gg \Gamma_e^{-1}(\omega_e)$
pole approx. breaks down (even in free space),

For a d-dimensional scalar QED vacuum, $U_e(t) \sim 1/t^{d+1}$.

Driven by the singularity at the edge $\omega = 0$ of the spectrum

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For a d-dimensional scalar QED vacuum, $U_e(t) \sim 1/t^{d+1}$.

holds also for structured photonics crystals
but not achievable for reasonably measurable
times !

Spectral dimension

- For a fractal vacuum, we have always $g \gg 1$ (strong coupling regime), even for a small

$$H_{int} = \sum_k (V_k^* a_k^\dagger |g\rangle\langle e| + \text{h.c.})$$

But the short time limit remains applicable !

Short time limit – the Fermi golden rule revisited

Short-time limit

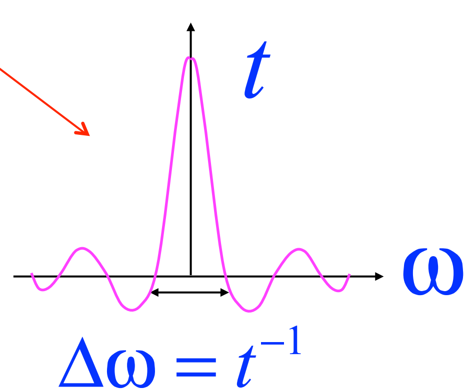
A standard perturbative treatment:

For short times, such that $\alpha(t) \approx \alpha(0) = 1$

the excited state probability is $|U_e(t)|^2 \simeq 1 - \int_0^t \mathrm{d}t'\, \Gamma_e(t'),$

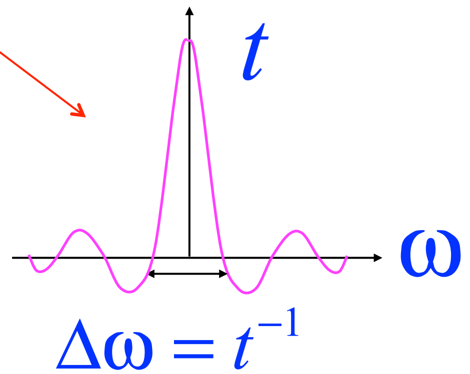
where the differential decay rate $\Gamma_e(t)$ is given by the well known expression:

$$\Gamma_e(t) = \frac{2}{\hbar^2} \int dk\, \rho(k)\, |V_k|^2 \frac{\sin(\omega_k - \omega_a)t}{(\omega_k - \omega_a)}$$



Fermi golden rule

$$\Gamma_e(t) = \frac{2}{\hbar^2}\int dk\,\rho(k)\,|V_k|^2 \frac{\sin(\omega_k-\omega_a)t}{(\omega_k-\omega_a)}$$



Valid for smooth spectrum + long times

$$\Gamma_e(t) = \frac{2}{\hbar^2}\int dk\,\rho(k)\,|V_k|^2 \pi\delta\,(\omega_k-\omega_a) = const = \Gamma_e$$

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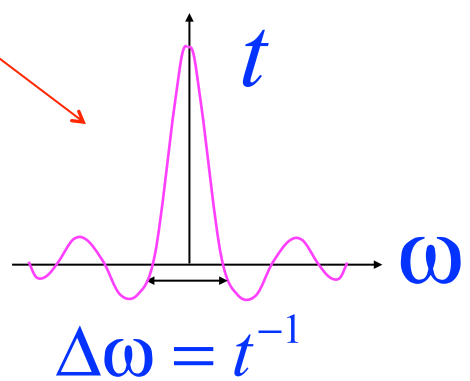


Valid for smooth spectrum + long times

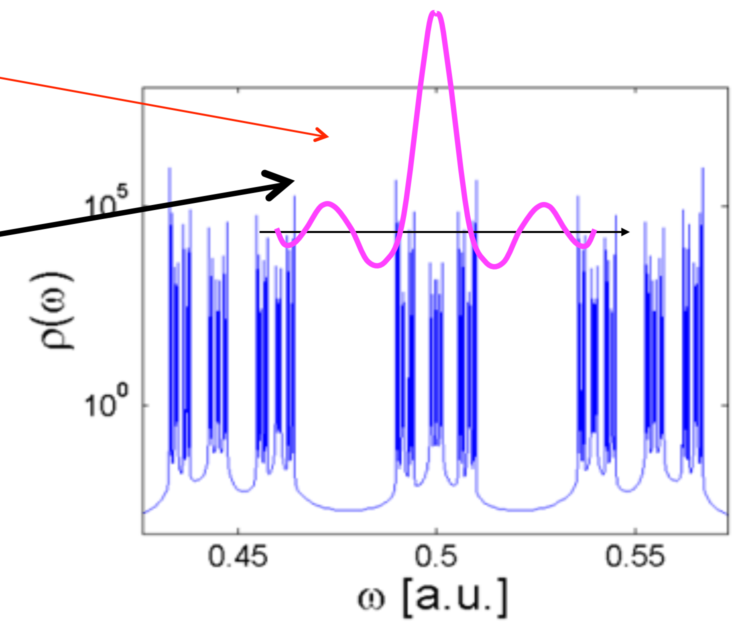
$$\Gamma_e(t) = \frac{2}{\hbar^2}\int dk\,\rho(k)\,|V_k|^2 \pi\delta(\omega_k - \omega_a) = const = \Gamma_e$$

This Γ_e coincides with the exponential decay rate (Wigner-Weisskopf):

$$|U_e(t)|^2 \approx 1 - \Gamma_e t \quad \longleftrightarrow \quad |U_e(t)|^2 = e^{-\Gamma_e t}$$

Short time limit - fractal spectrum

$$\Gamma_e(t) = \frac{2}{\hbar^2}\int dk\,\rho(k)\,|V_k|^2 \frac{\sin(\omega_k-\omega_a)t}{(\omega_k-\omega_a)}$$



Recall that the counting function satisfies

$$N_\omega^{(g)}(\Delta\omega) \equiv \int g\left(\frac{\omega'-\omega}{\Delta\omega}\right)\rho(\omega')d\omega' = (\Delta\omega)^\alpha \times F\left(\frac{\ln|\Delta\omega|}{\ln b}\right),$$

~~$\frac{\sin(\omega_k-\omega_a)t}{(\omega_k-\omega_a)} \to \pi\delta(\omega_k-\omega_a)$~~

We immediately conclude that the general form of $\Gamma_e(t)$ is:

$$\Gamma_e(t) = \tau^{-1} \times \left(\frac{t}{\tau}\right)^{1-\alpha} \times F\left(\frac{\ln(t/t_0)}{\ln b}\right), \qquad F(x+1) = F(x),$$

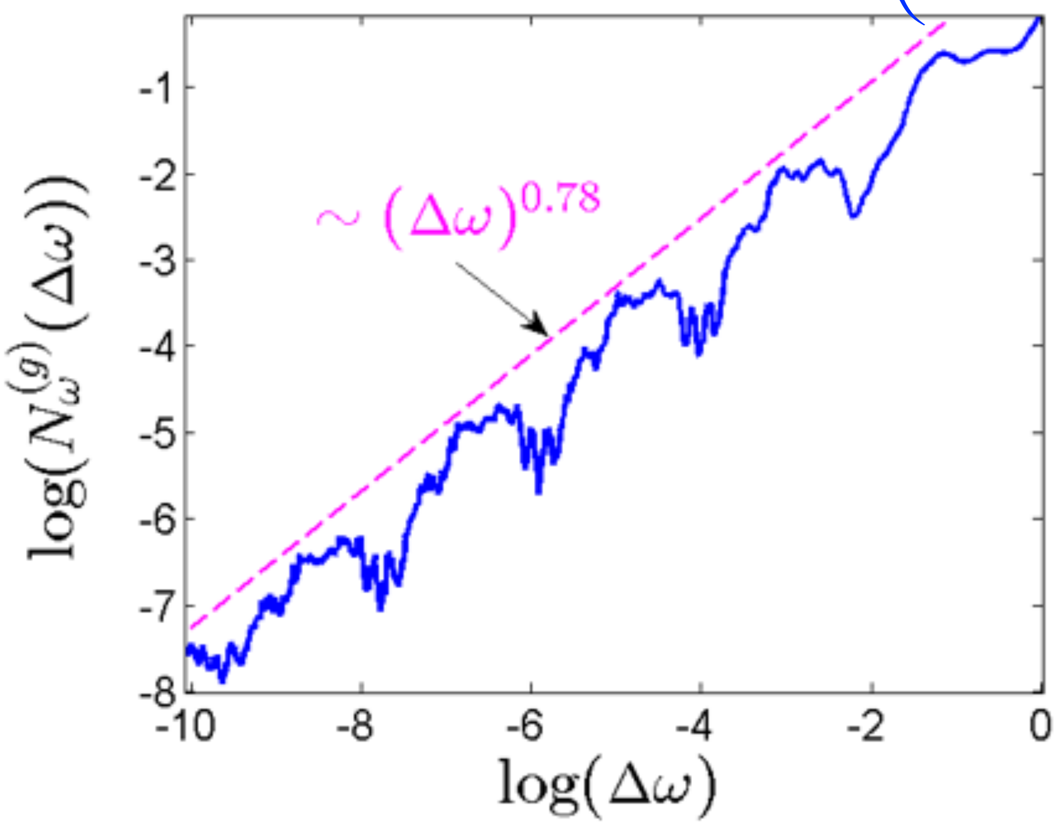
where

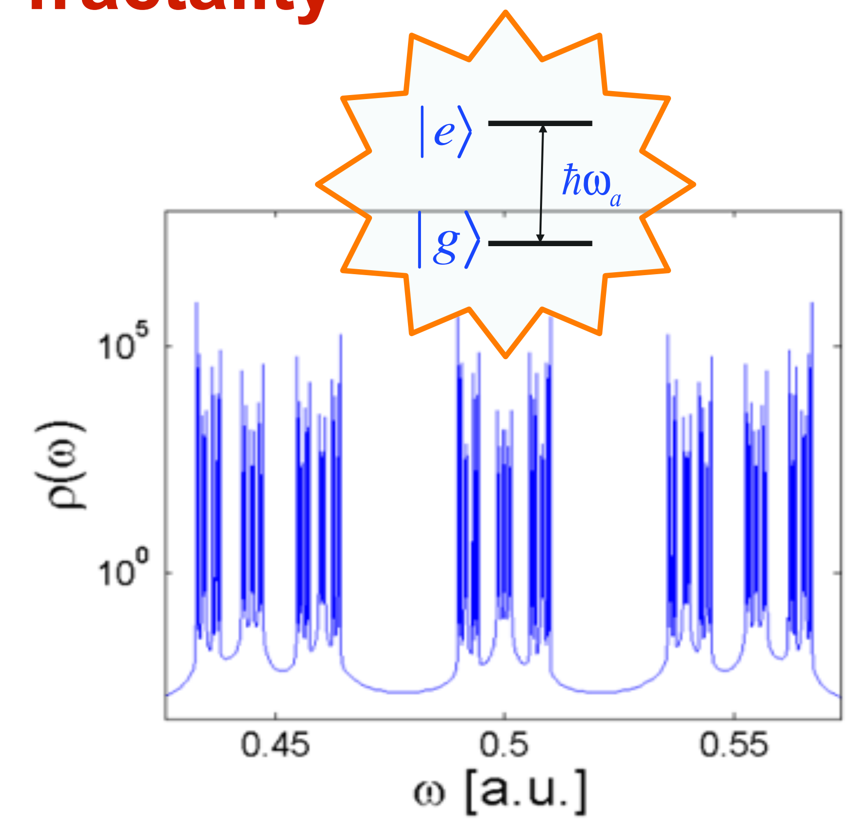
$0 \le \alpha \le 1, \quad b$ - fractal exponent and scaling factor of the spectrum

τ, t_0 - time scales, specific to the considered problem.

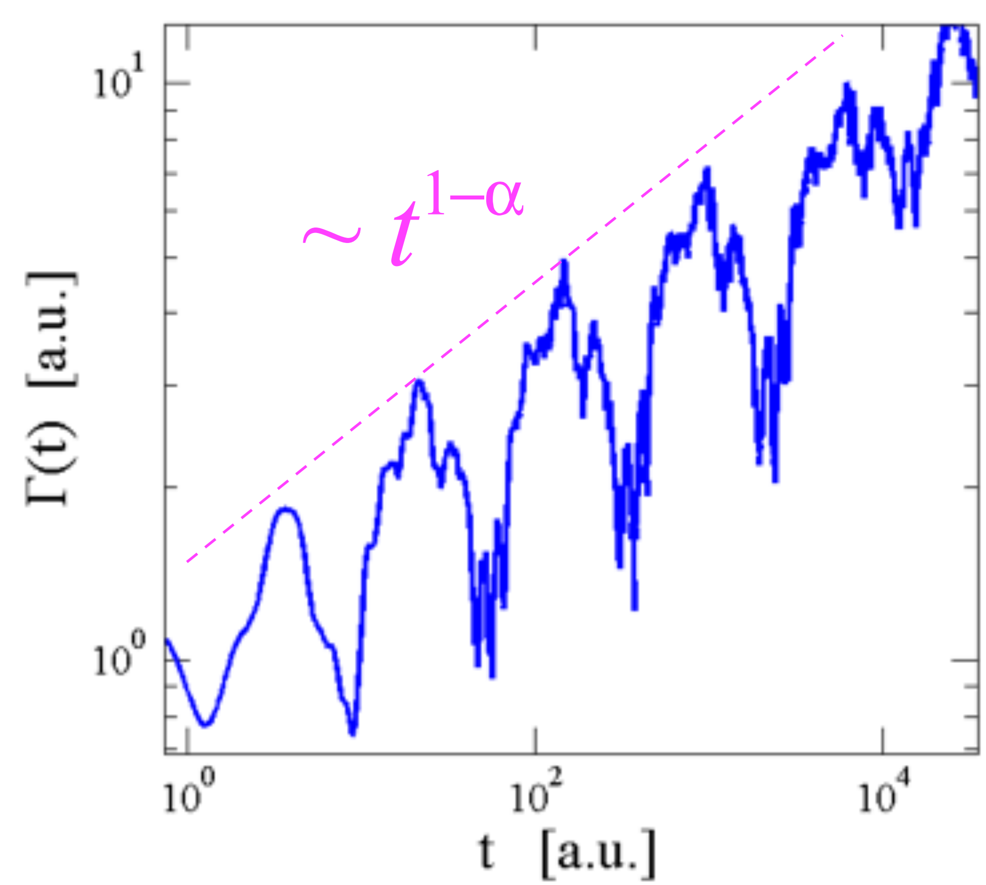
Spontaneous emission and vacuum fractality

$$N_{\omega}^{(g)}(\Delta\omega) = (\Delta\omega)^{\alpha} \times F_g\left(\frac{\ln|\Delta\omega|}{\ln b}\right),$$





$$\Delta\omega \sim t^{-1}$$



Differential decay rate

(at small times) $\Gamma(t) = \frac{dp_e(t)}{dt}$

To summarise

Spontaneous emission from a fractal vacuum

to the ________________________

Wigner-Weisskopf exponential decay.

The decay probability $|U_e(t)|^2$ is given by an algebraic time decrease modulated by a log-periodic function characteristic of the discrete scaling symmetry (fractal) of the vacuum,

$$|U_e(t)|^2 = t^{-2\gamma}\,\mathcal{G}\left(\frac{\ln t}{\lambda}\right)$$

The exponent γ is related to the spectral dimension.

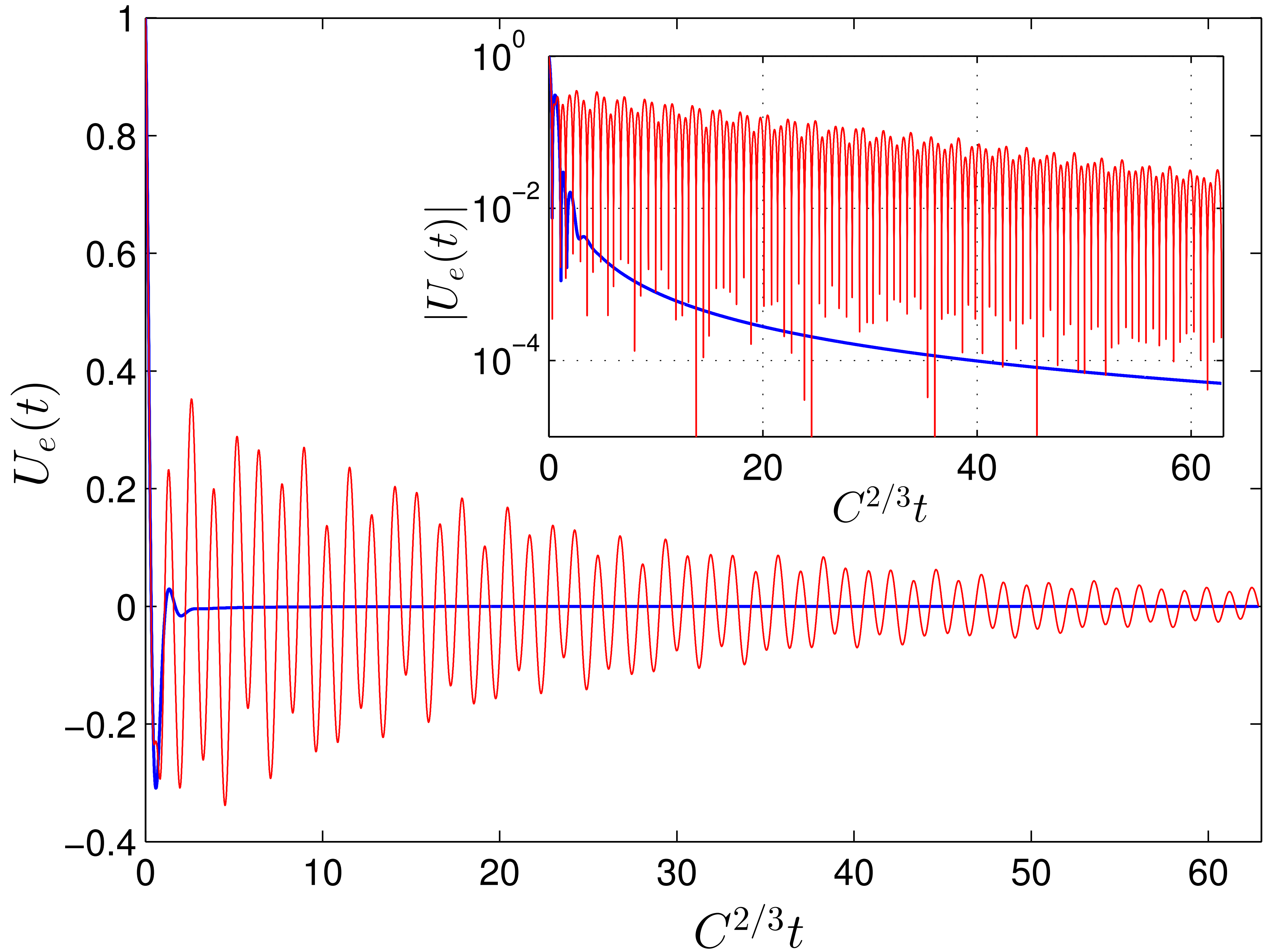
Beyond the short time regime- Strong coupling and Inhibition of spontaneous emission

A toy model

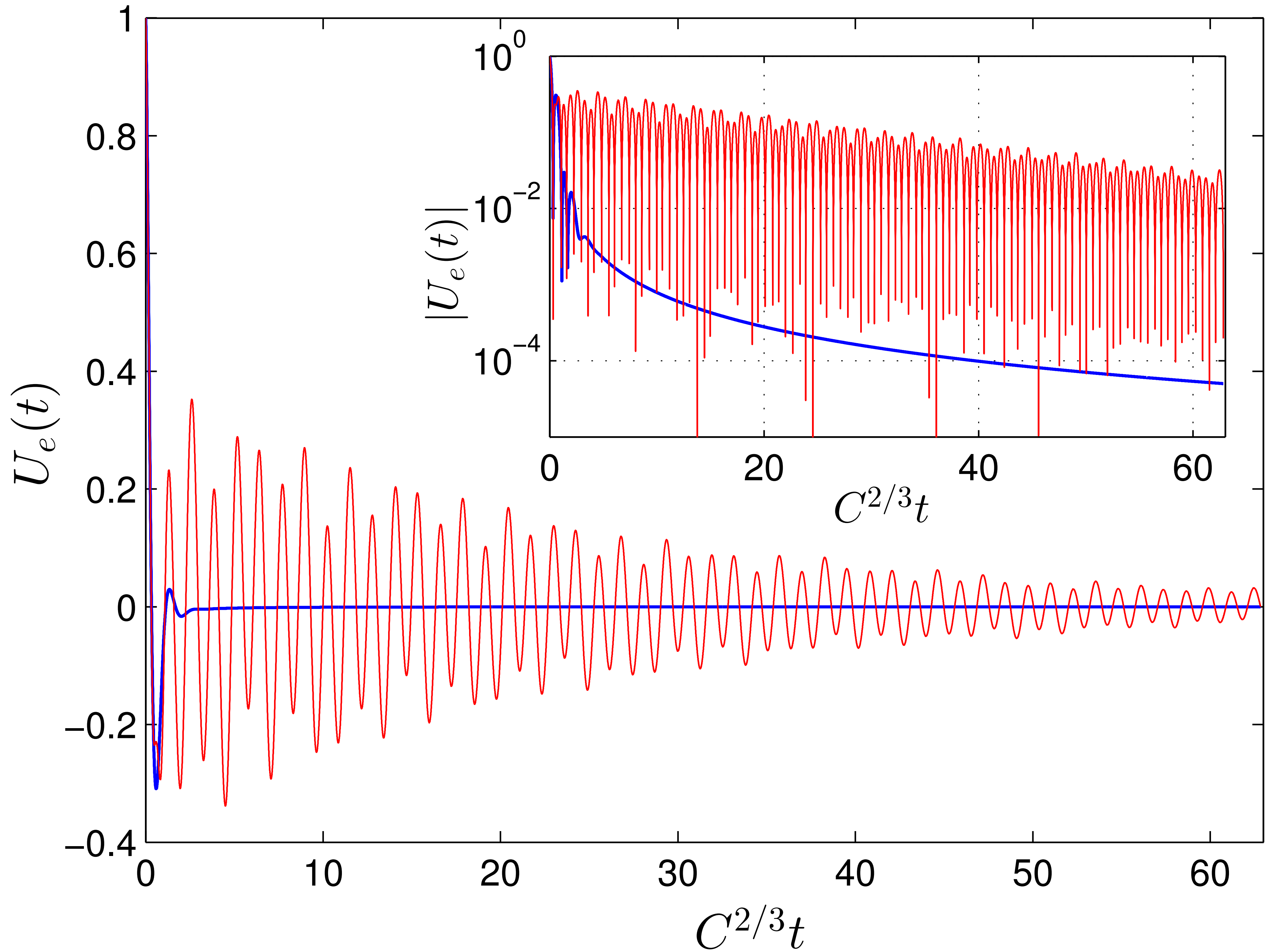
$$\rho_L(k,r_a) \sim \overline{|V_k|^2}\rho(\omega) = \frac{C}{|\omega-\omega_u|^{1-\alpha}}\left[1 + A\cdot\cos\left(\frac{\ln|\omega-\omega_u|}{\ln b}\right)\right],$$

incorporates basic ingredients :

- A singularity in the spectrum (power law decrease)
- Mimics the fractal properties
- Reproduces the scaling in the short time limit
- Can be treated analytically at all time scales



Strong coupling - non perturbative solution



STRO … UTION

$$\rho_L(k,r_a) \sim \overline{|V_k|^2}\rho(\omega) = \frac{C}{|\omega-\omega_u|^{1-\alpha}}\left[1 + A\cdot\cos\left(\frac{\ln|\omega-\omega_u|}{\ln b}\right)\right],$$

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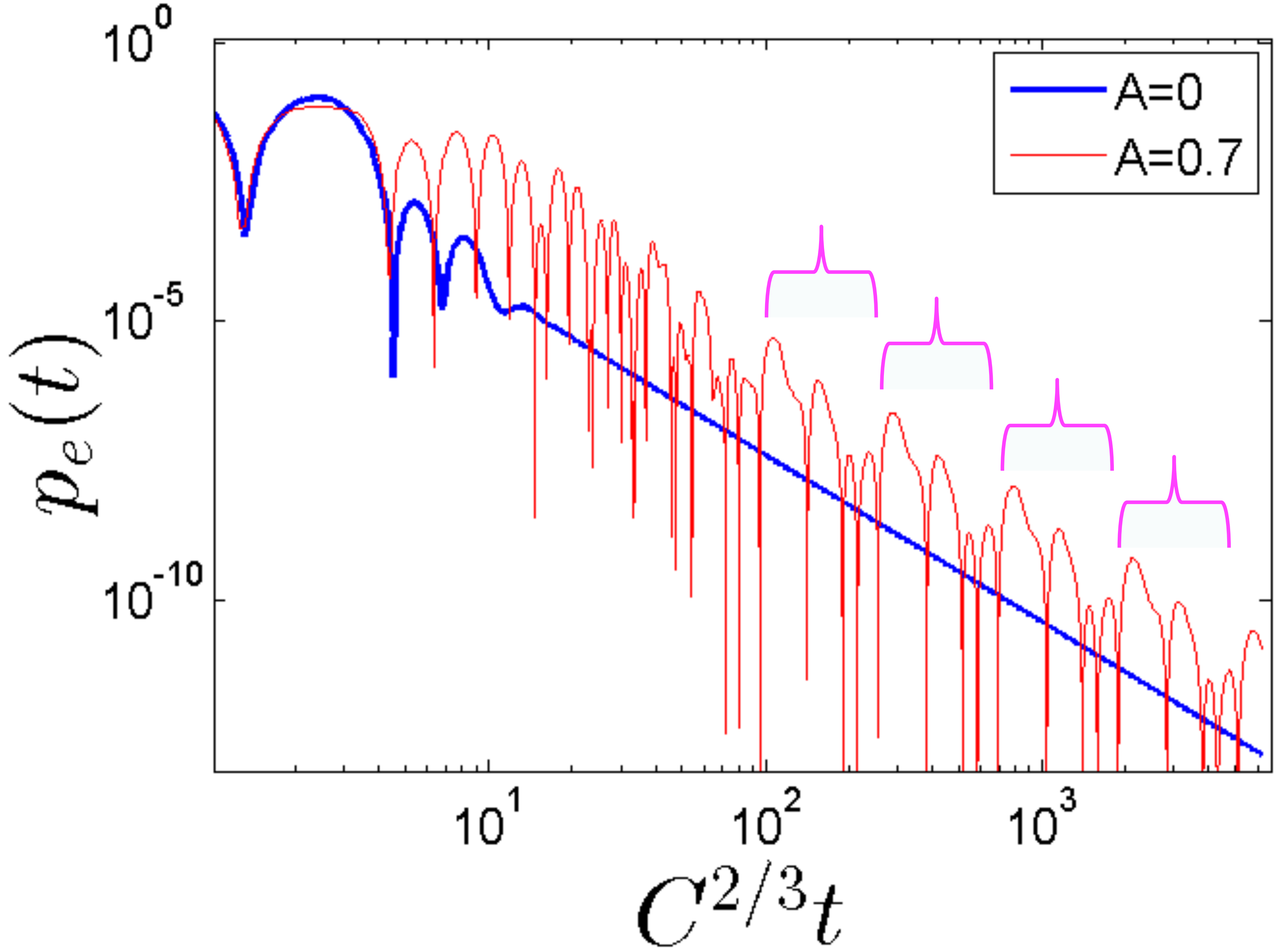

$$\alpha = \frac{1}{2}$$

$$\rho_L(k, r_a) \sim \overline{|V_k|^2}\rho(\omega) = \frac{C}{|\omega - \omega_u|^{1-\alpha}}\left[1 + A \cdot \cos\left(\frac{\ln|\omega - \omega_u|}{\ln b}\right)\right],$$

Part 2

Experimental study of a fractal energy spectrum :

Cavity polaritons in a Fibonacci quasi-periodic potential

The Fibonacci problem has a long and rich (theoretical and experimental) history.

(Kohmoto,Luck, Gellerman, Damanik, Bellissard,Simon,...)

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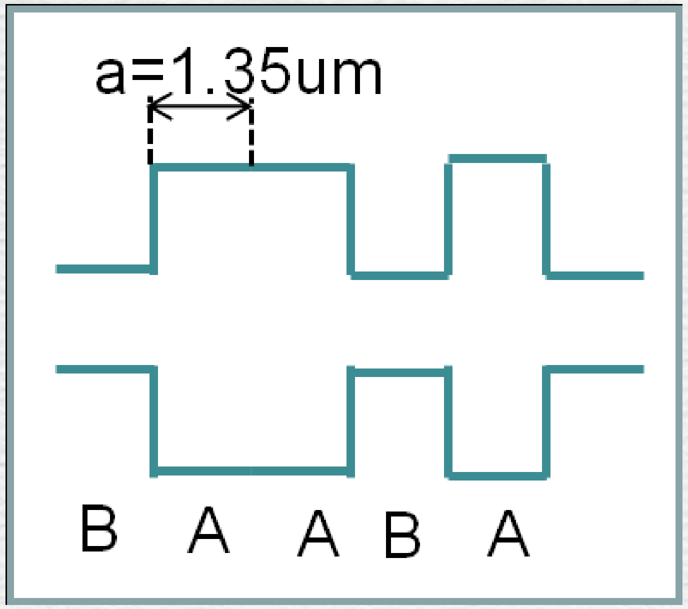
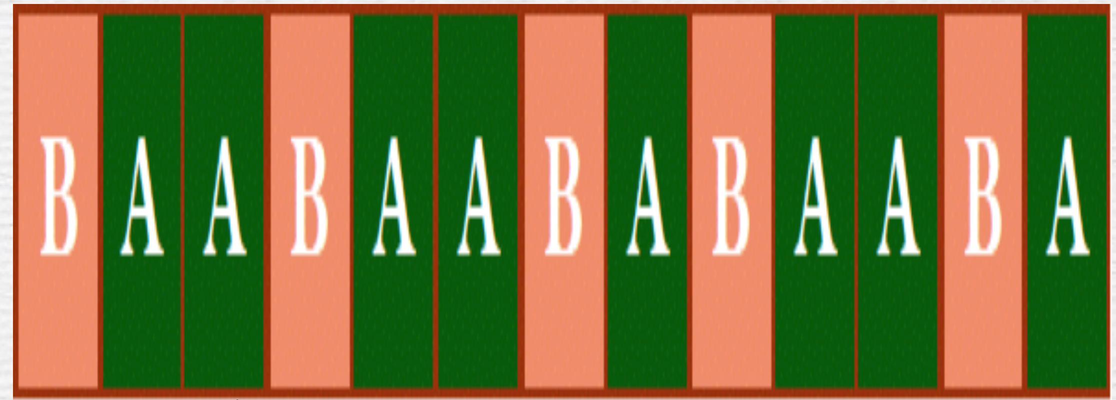
But still much to be done...

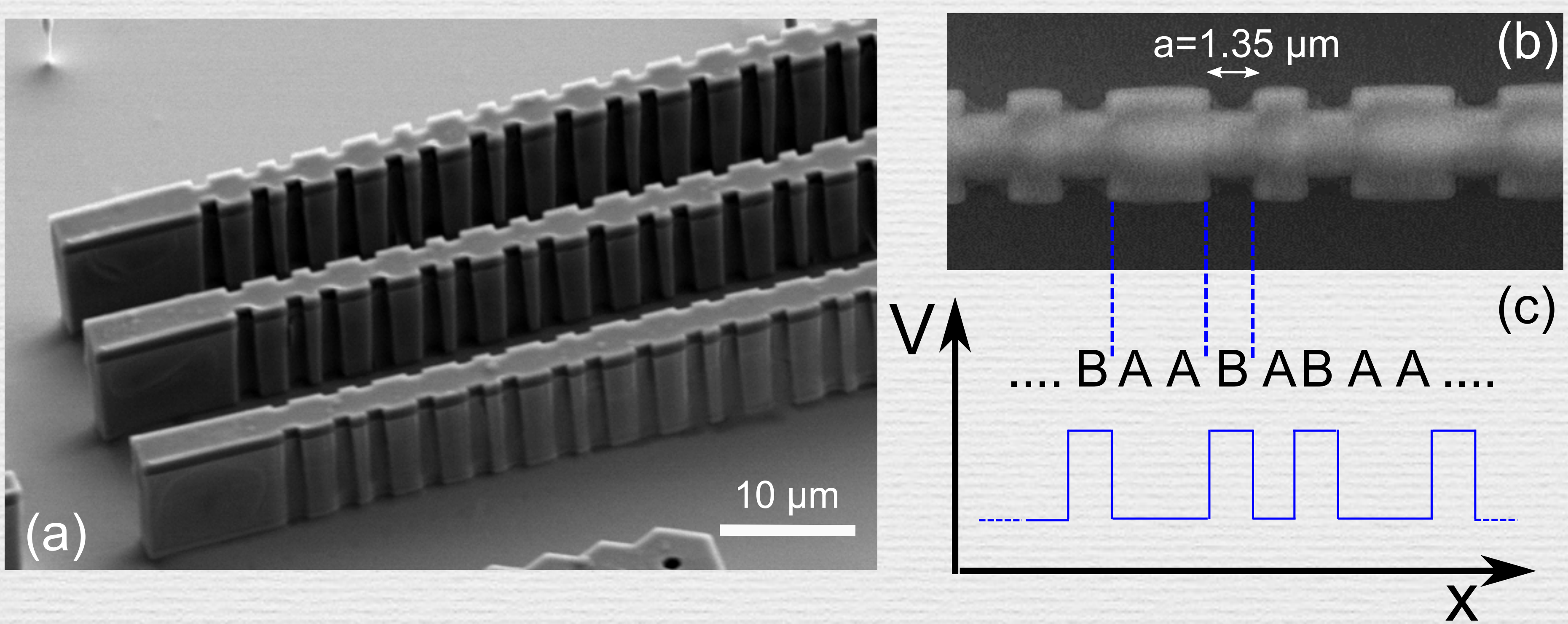




Fibonacci sequence: $S_{j\geq 2} = [S_{j-1}S_{j-2}], \quad S_0 = B, \ \ S_1 = A$

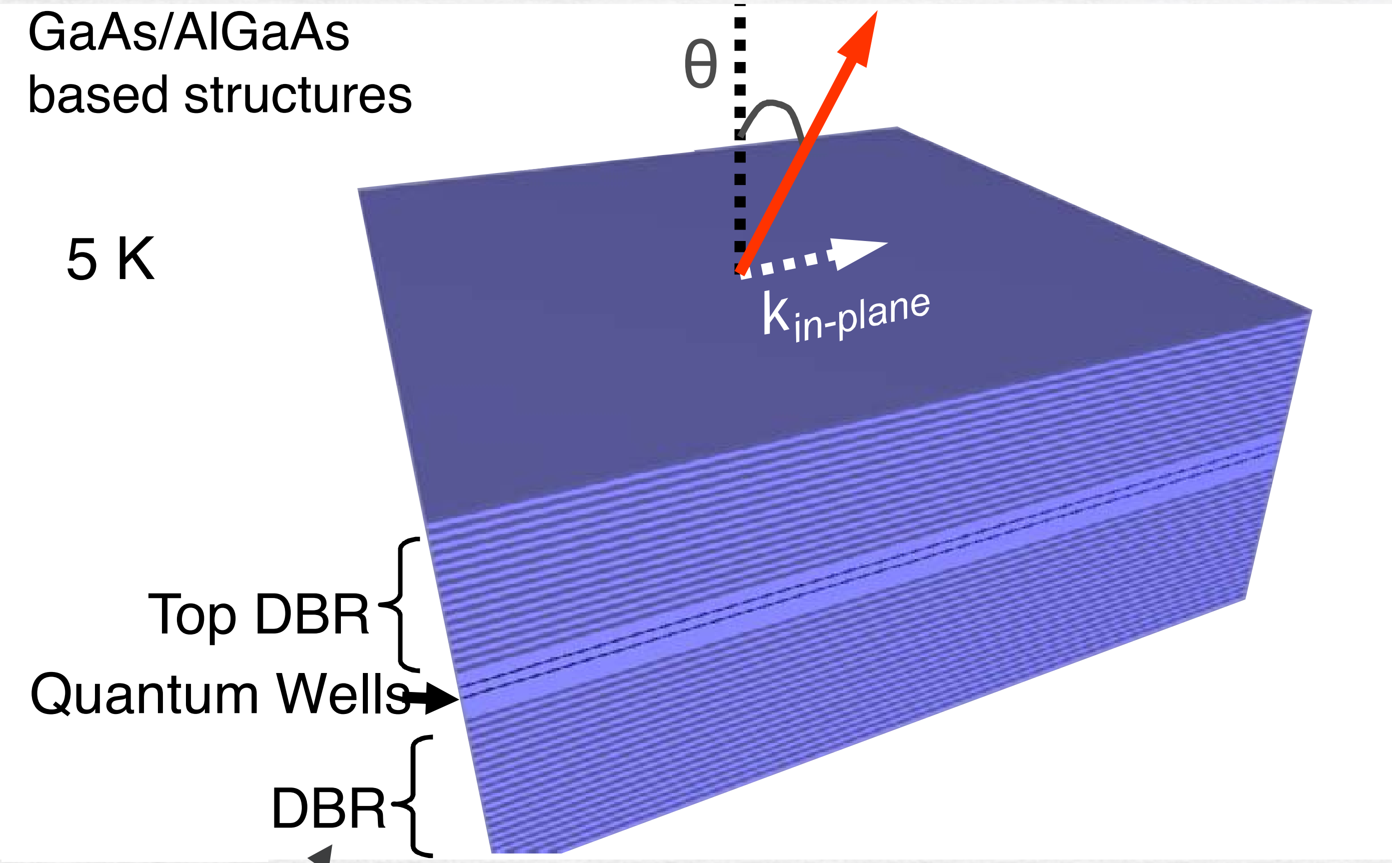
A→AB→ABA→ABAAB→ABAABABA→…

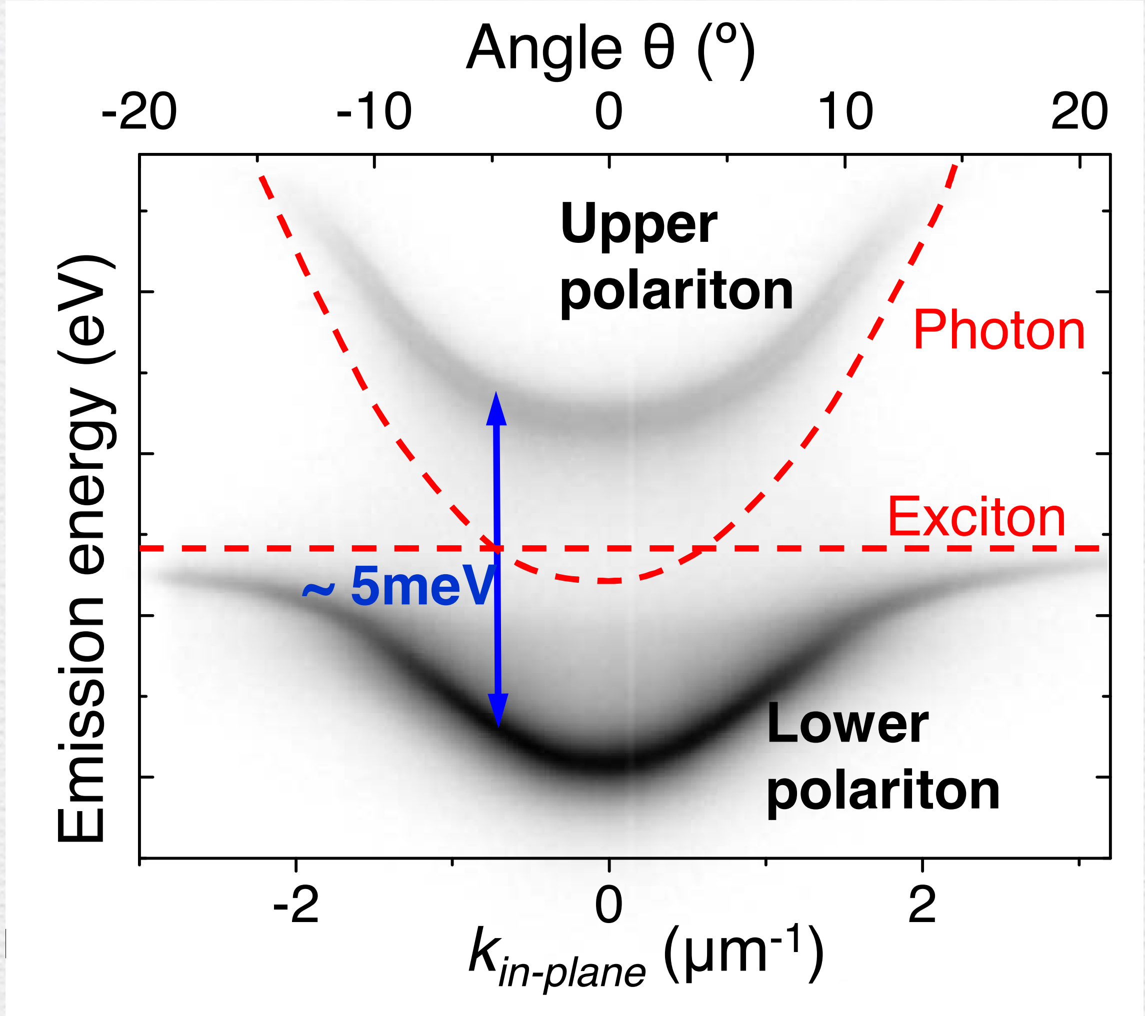
Number of letters of a sequence S_j is the Fibonacci number F_j so that $F_j = F_{j-1} + F_{j-2}$



(233 letters)

Basics on cavity polaritons





(Distributed) Bragg reflectors

Cavity polaritons :

an optical cavity mode and confined excitons (quantum wells)

C. Weisbuch et al. PRL,

Cavity polaritons are described using a **d=2** *Schrödinger* eq.

$$E\psi(x,y) = -\frac{\hbar^2}{2m_{ph}}\Delta_\perp \psi(x,y)$$

with the effective photon mass $m_{ph} = {n^2 E_c}/{c^2}$

n = effective refractive index, $\Delta_\perp \equiv \partial_x^2 + \partial_y^2$

$E_c = \frac{\hbar c}{n} k_z$ = energy of the fundamental mode of the cavity

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Eigenmodes of the d=2 problem ⟶ numerics

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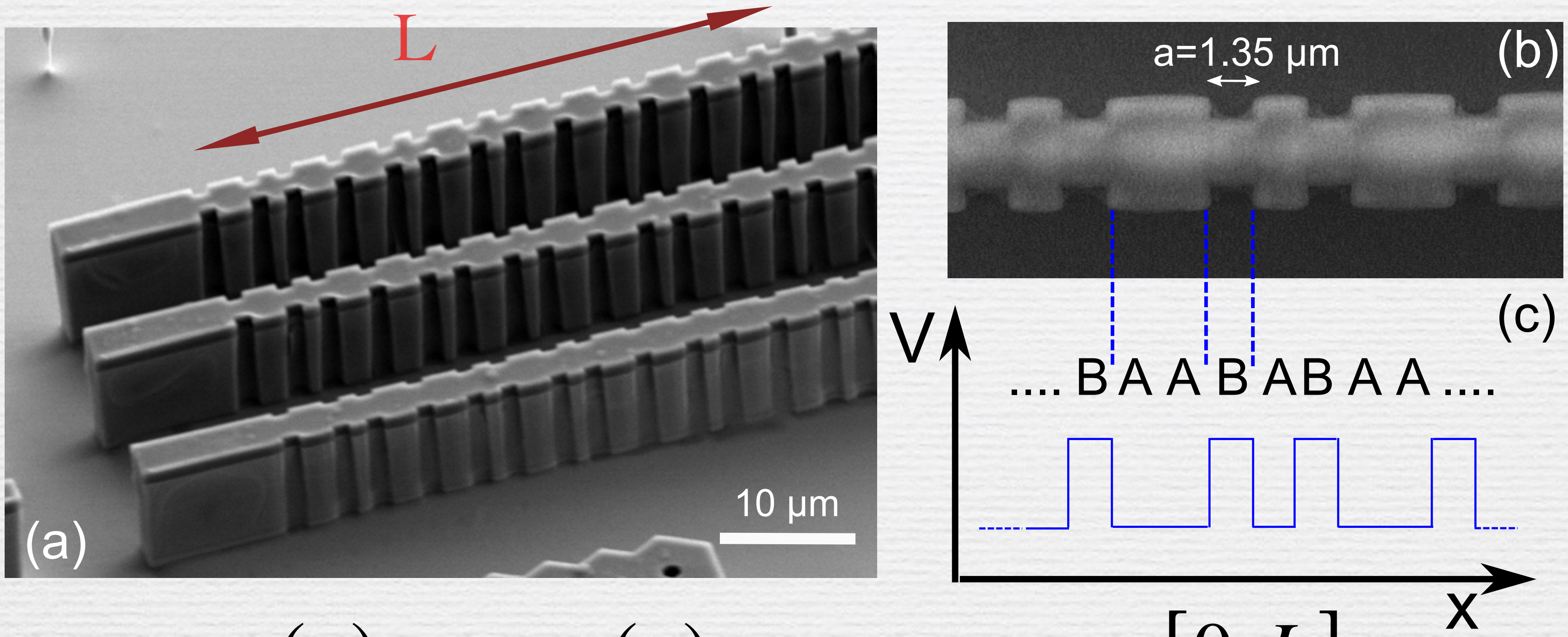
Eigenmodes of the d=2 problem ⟶ numerics

Well controlled d=1 effective model is preferable !

$$E\varphi(x) = \frac{\hbar^2}{2m_{ph}}\left[-\frac{d^2}{dx^2} + V(x)\right]\varphi(x)$$

$V(x)$?



$w(x)$

$$-\frac{w(x)}{2} \leq y \leq \frac{w(x)}{2}$$

$$x \in [0, L]$$

$$E\varphi(x) = \frac{\hbar^2}{2m_{ph}}\left[-\frac{d^2}{dx^2} + V(x)\right]\varphi(x)$$

$$V(x) = \frac{\pi^2}{w^2(x)} + \frac{\pi^2 + 3}{12}\left(\frac{w'(x)}{w(x)}\right)^2$$

Adiabatic approx.

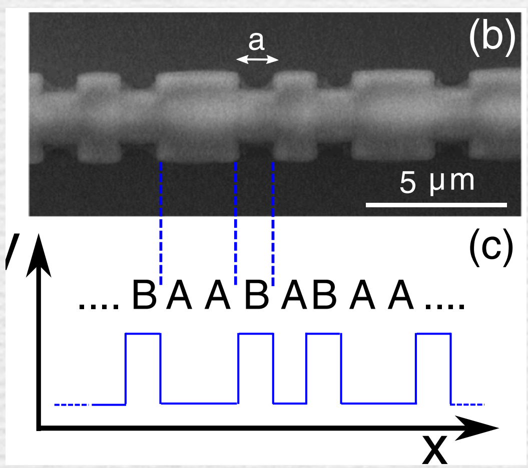
Non perturbative correction - unusual !
Steps sharpness

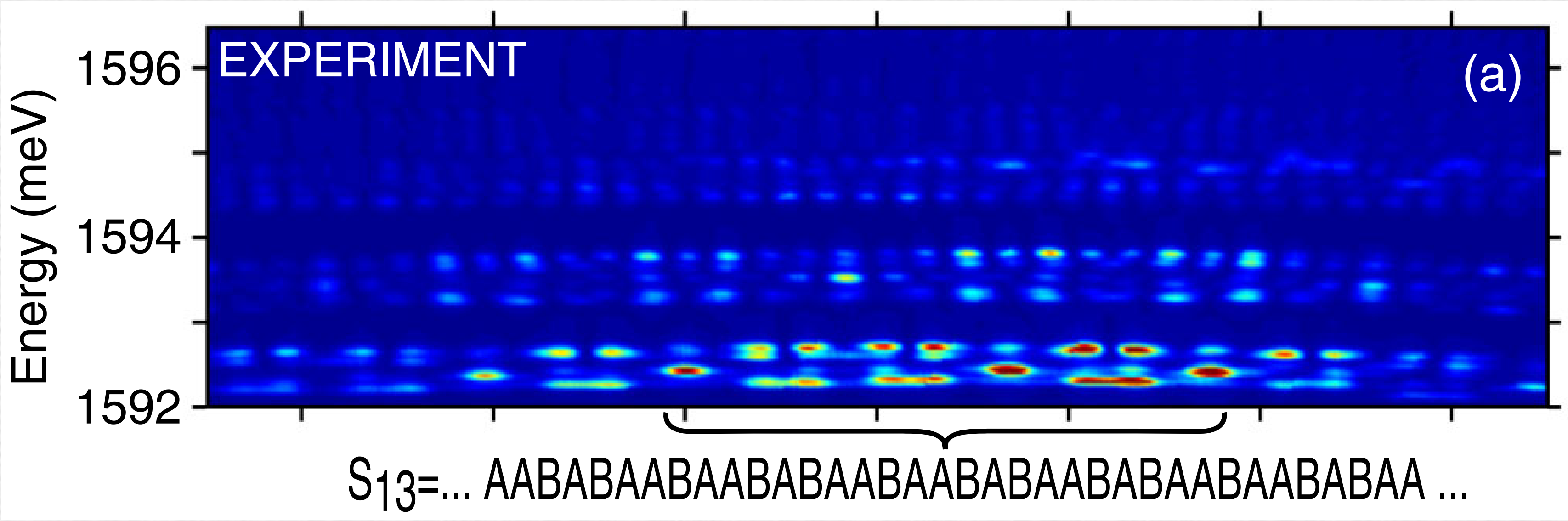
D. Tanese, J. Bloch, E. Gurevich, E.A. PRL, 2014.

Advantages of cavity polaritons :

allow for a
excitations both in real and momentum spaces.

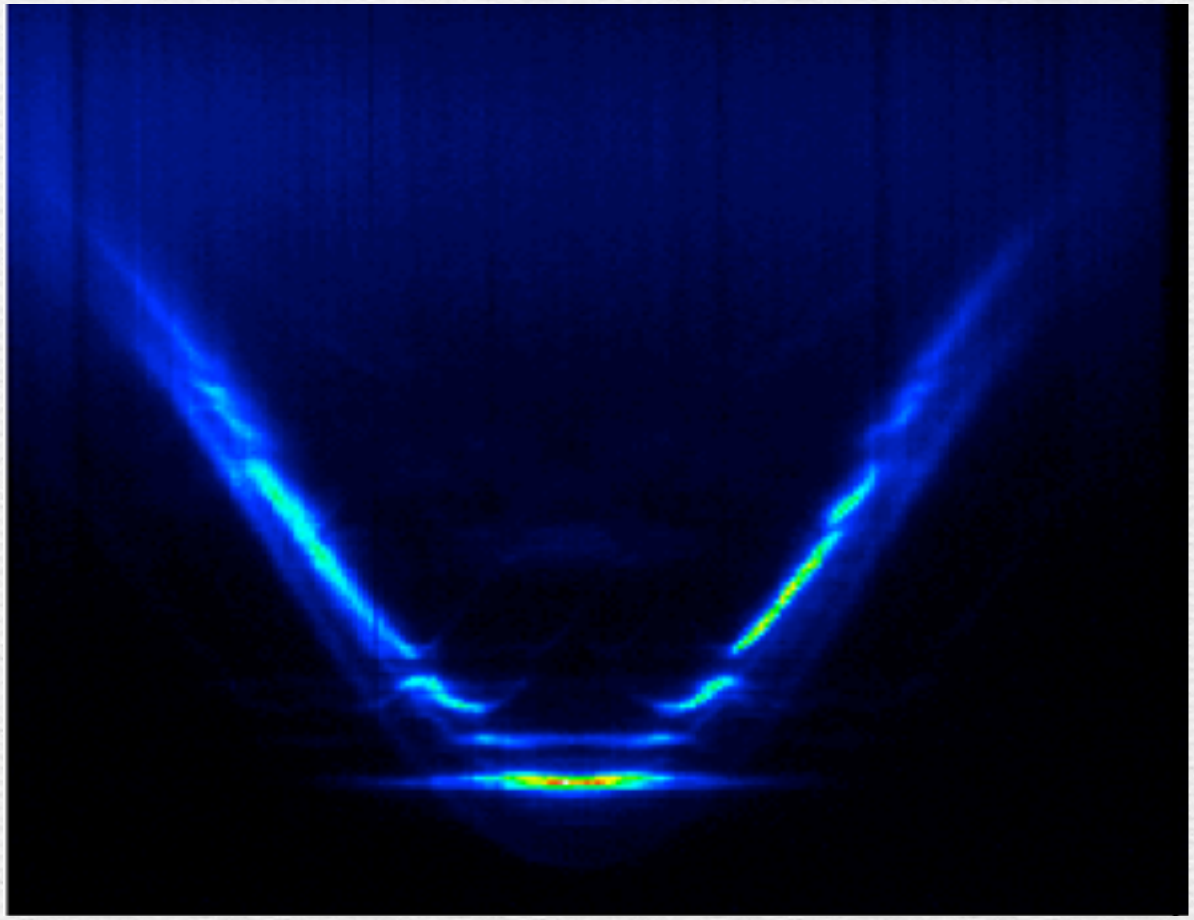
→ Visualisation/imaging of individual eigenmodes





→ Wave packet dynamics (under study)

Measure of spectral function $E(k)$ intensity maps



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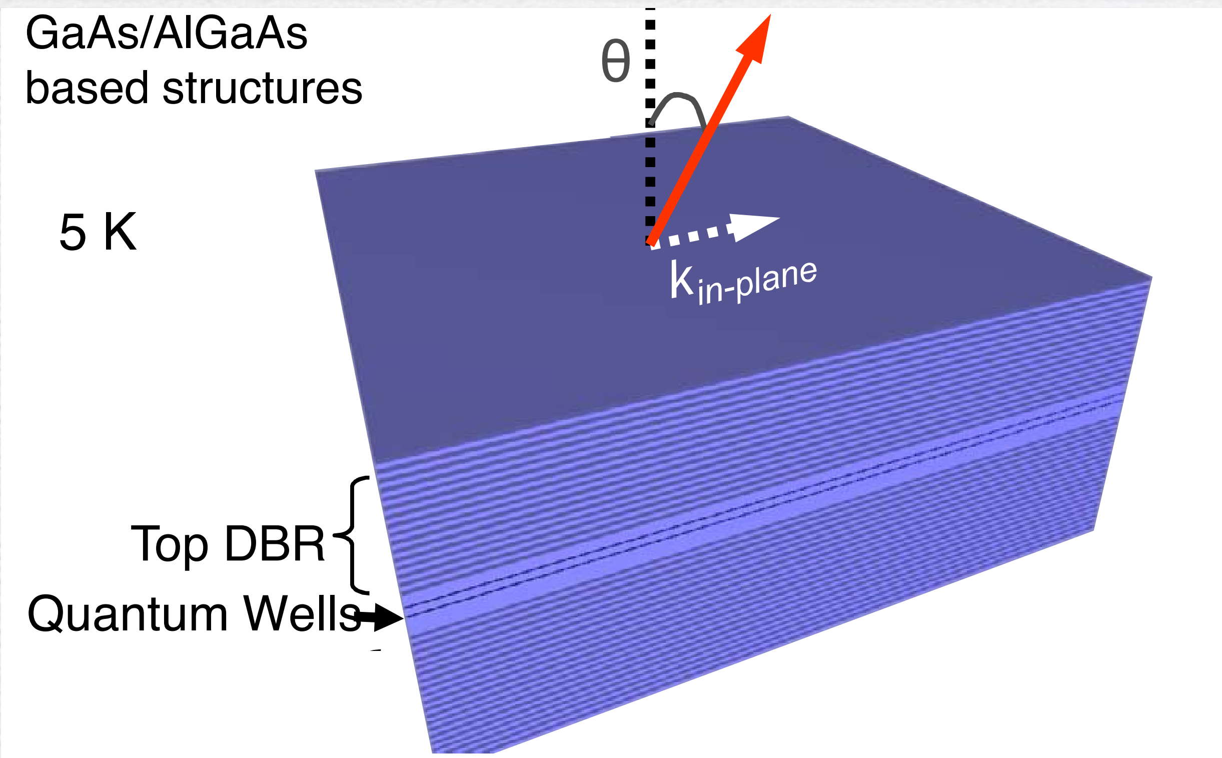

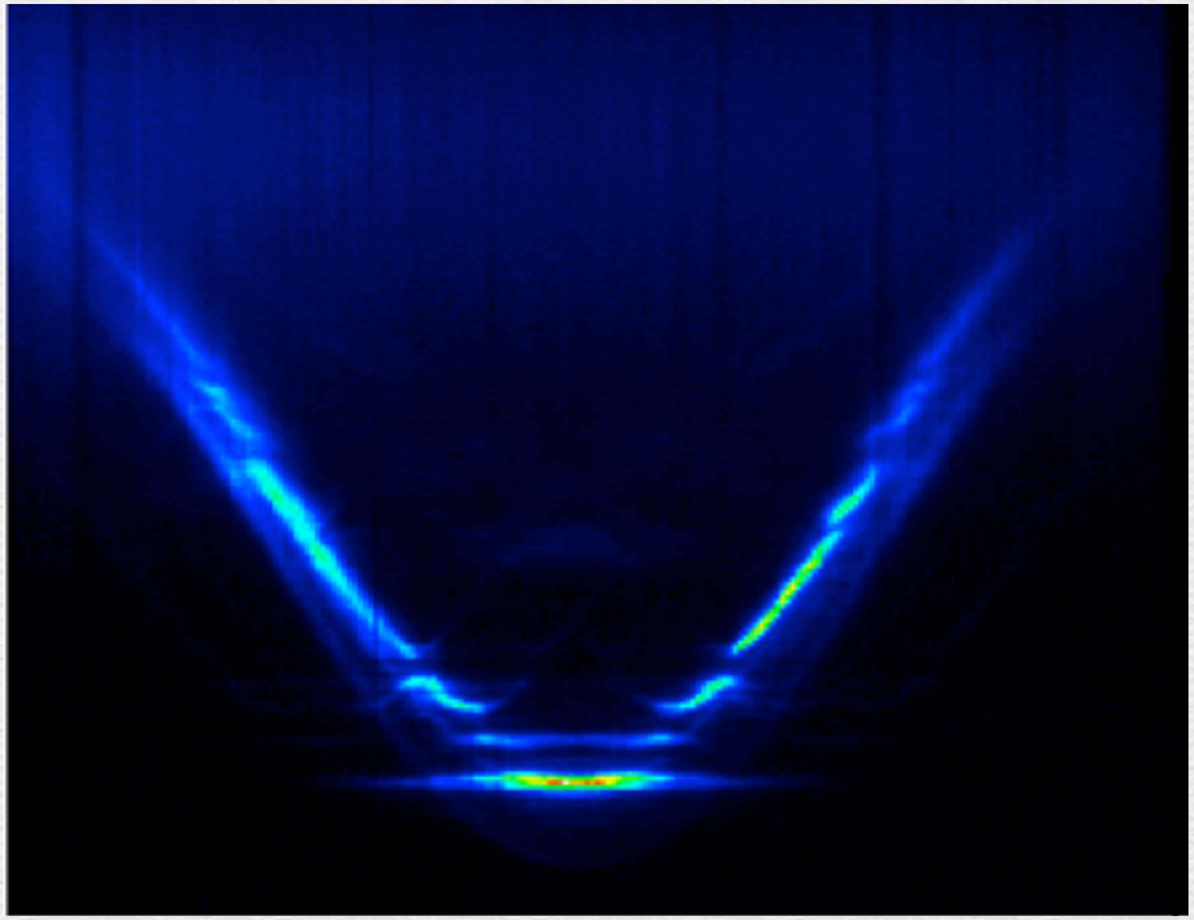

k

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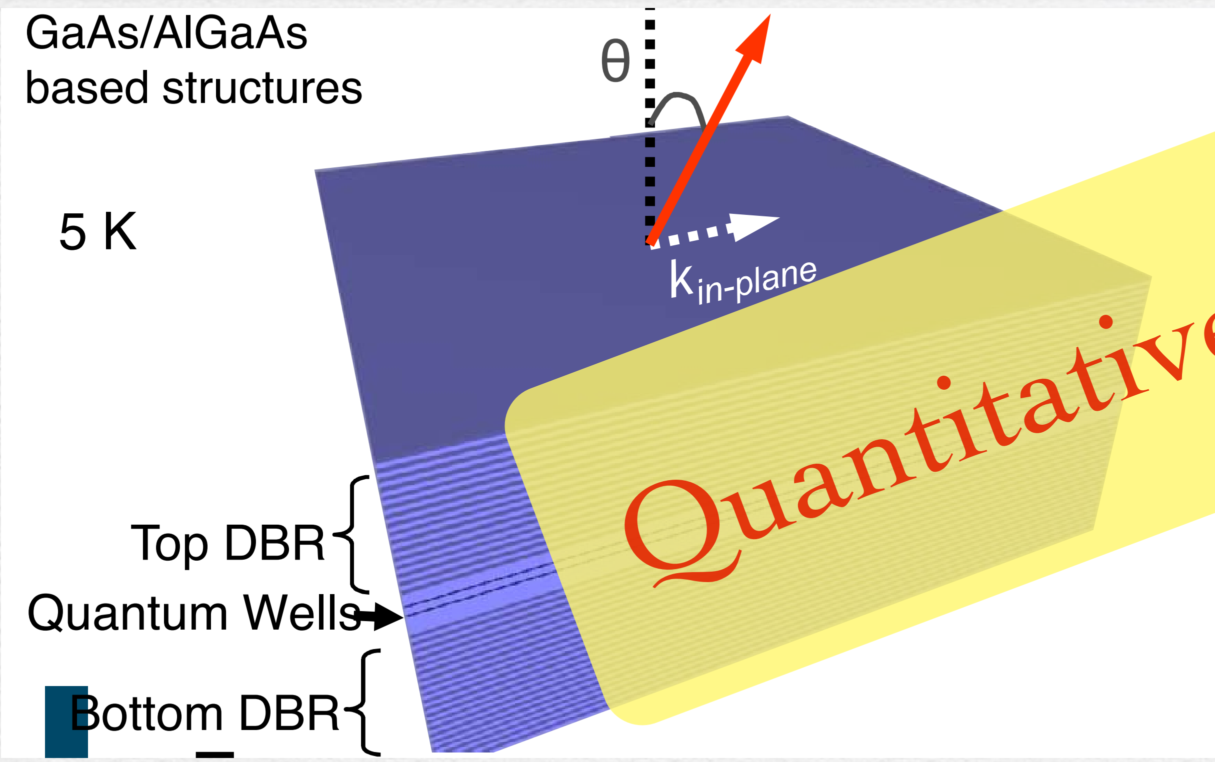
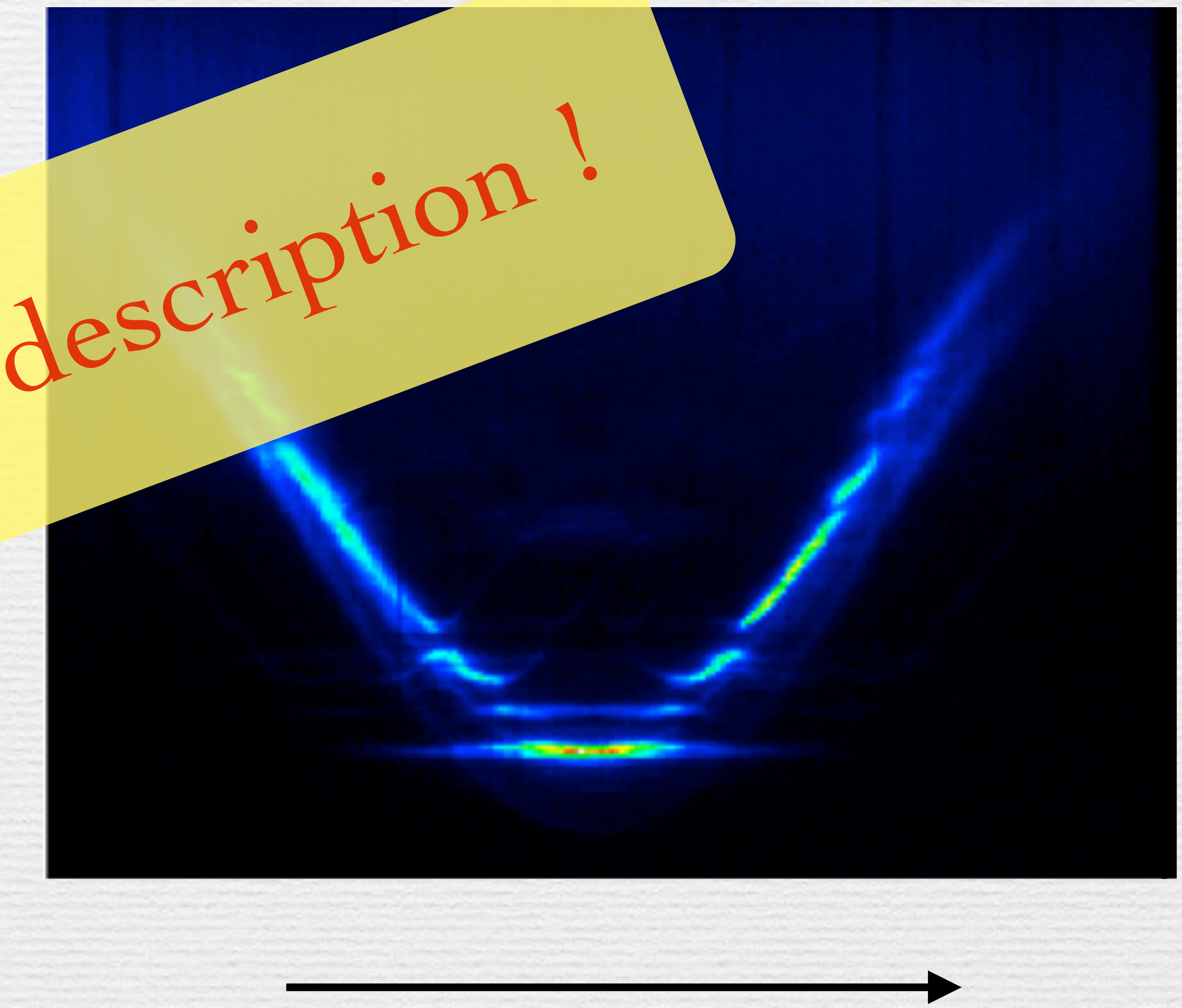




k

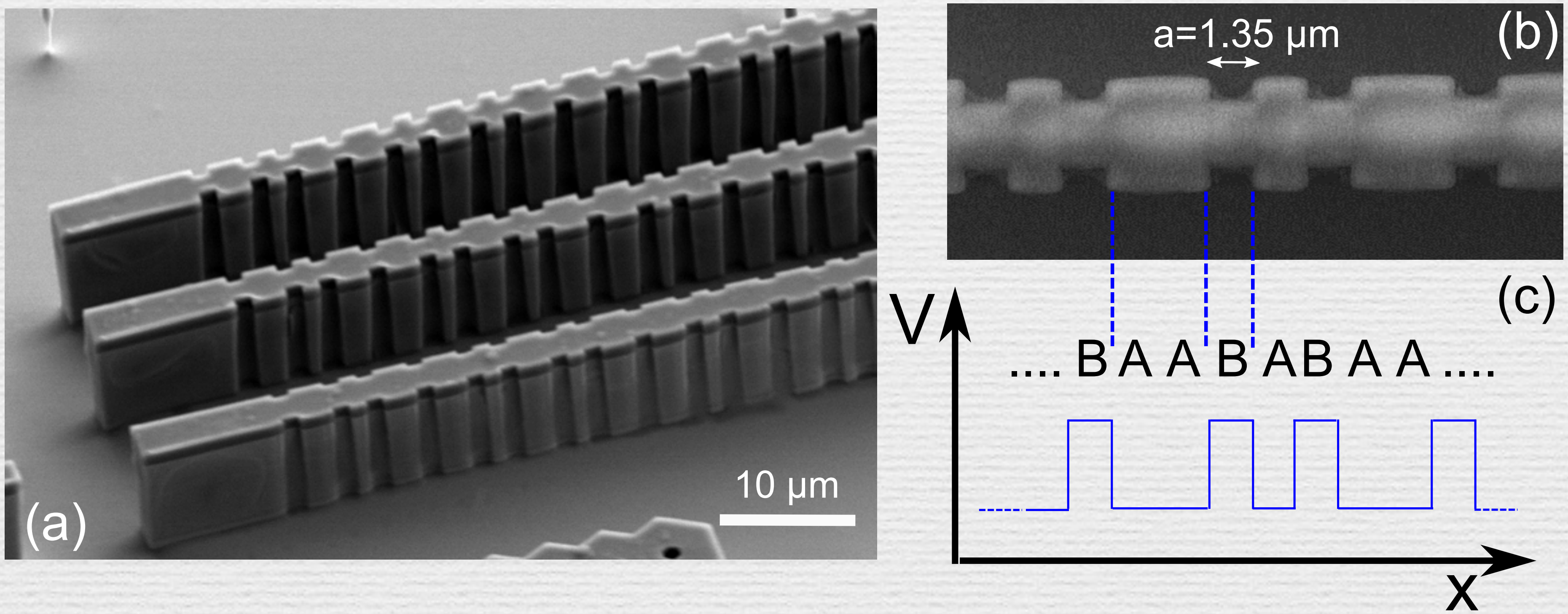
Quantitative description !

Effective 1D model

$$\left[-\frac{\hbar^2}{2M}\frac{d^2}{dx^2}+V(x)\right]\psi(x)=E\psi(x)$$

where

$$V(x)=\sum_n \chi\left(\sigma^{-1}n\right)u_b\left(x-an\right)$$



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$$\chi(x)=\begin{cases}1, & 0<x<2-\sigma\\ 0, & 2-\sigma<x<1\end{cases}$$

Characteristic function

$\sigma = \sqrt{5}+1\big/2 \approx 1.62$ is the golden mean

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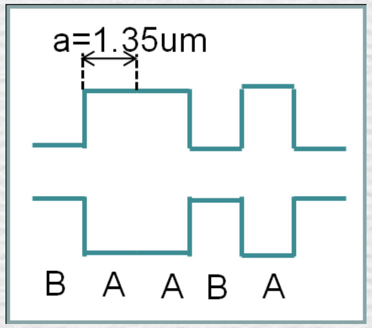
Characteristic function

Shape of each letter



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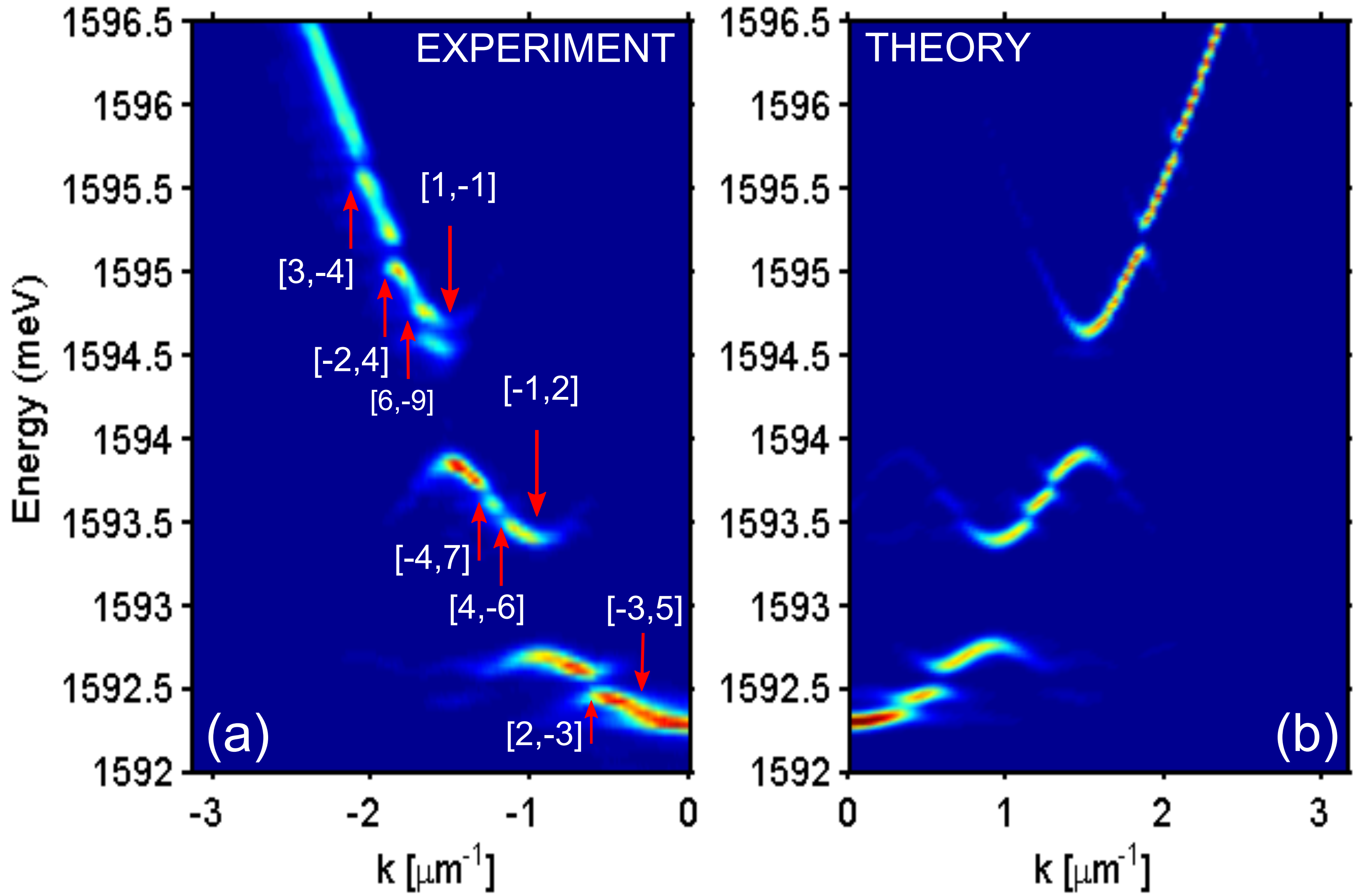
EXPERIMENT
THEORY
Energy (meV)
1596.5
1596
1595.5
1595
1594.5
1594
1593.5
1593
1592.5
1592
[1,-1]
[3,-4]
[-2,4]
[6,-9]
[-1,2]
[-4,7]
[4,-6]
[-3,5]
[2,-3]
(a)
(b)
-3
-2
-1
0
1
2
3
k [μm^{-1}]
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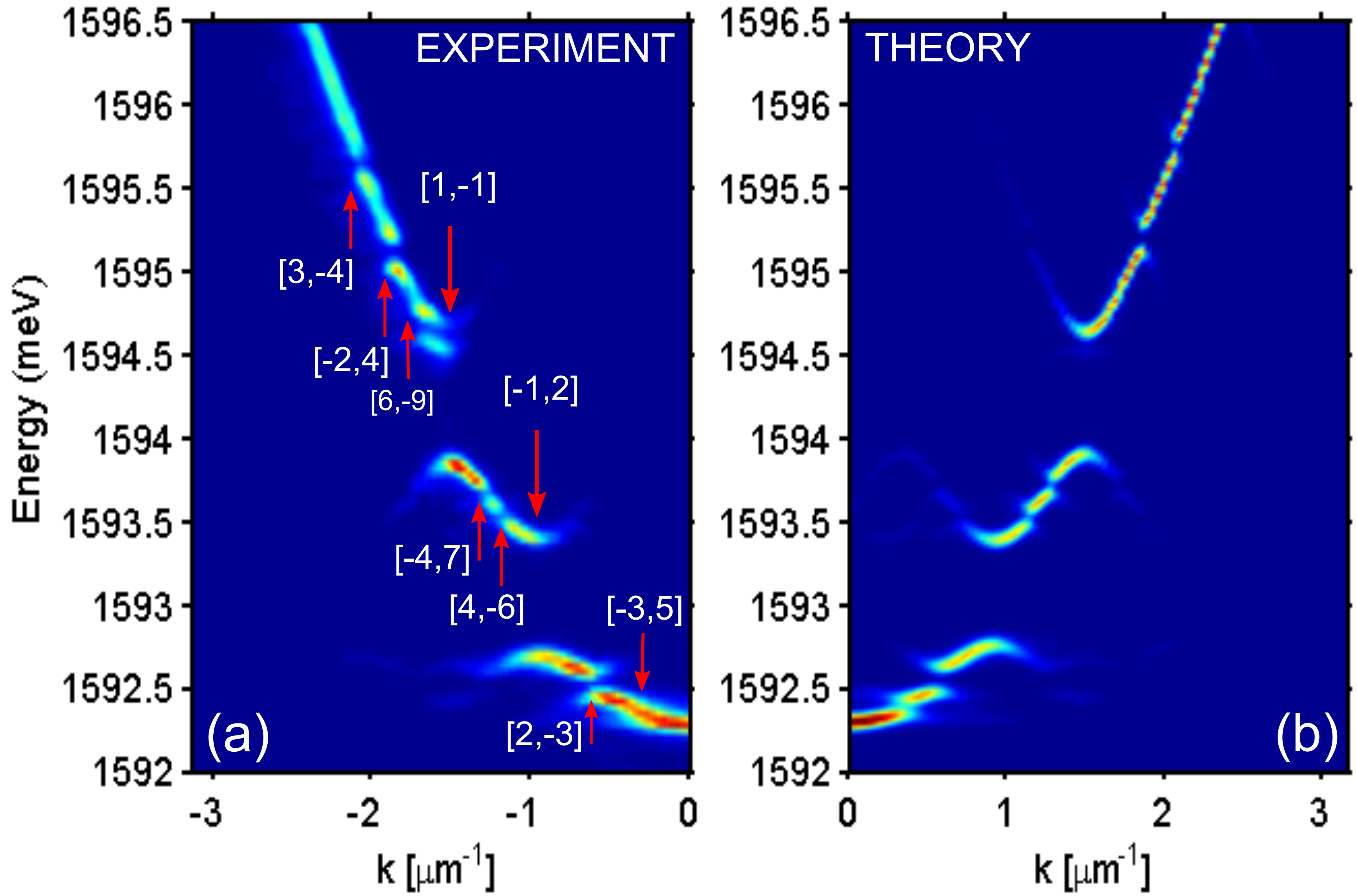

Exact numerical 2D calculation or 1D with the non perturbative term

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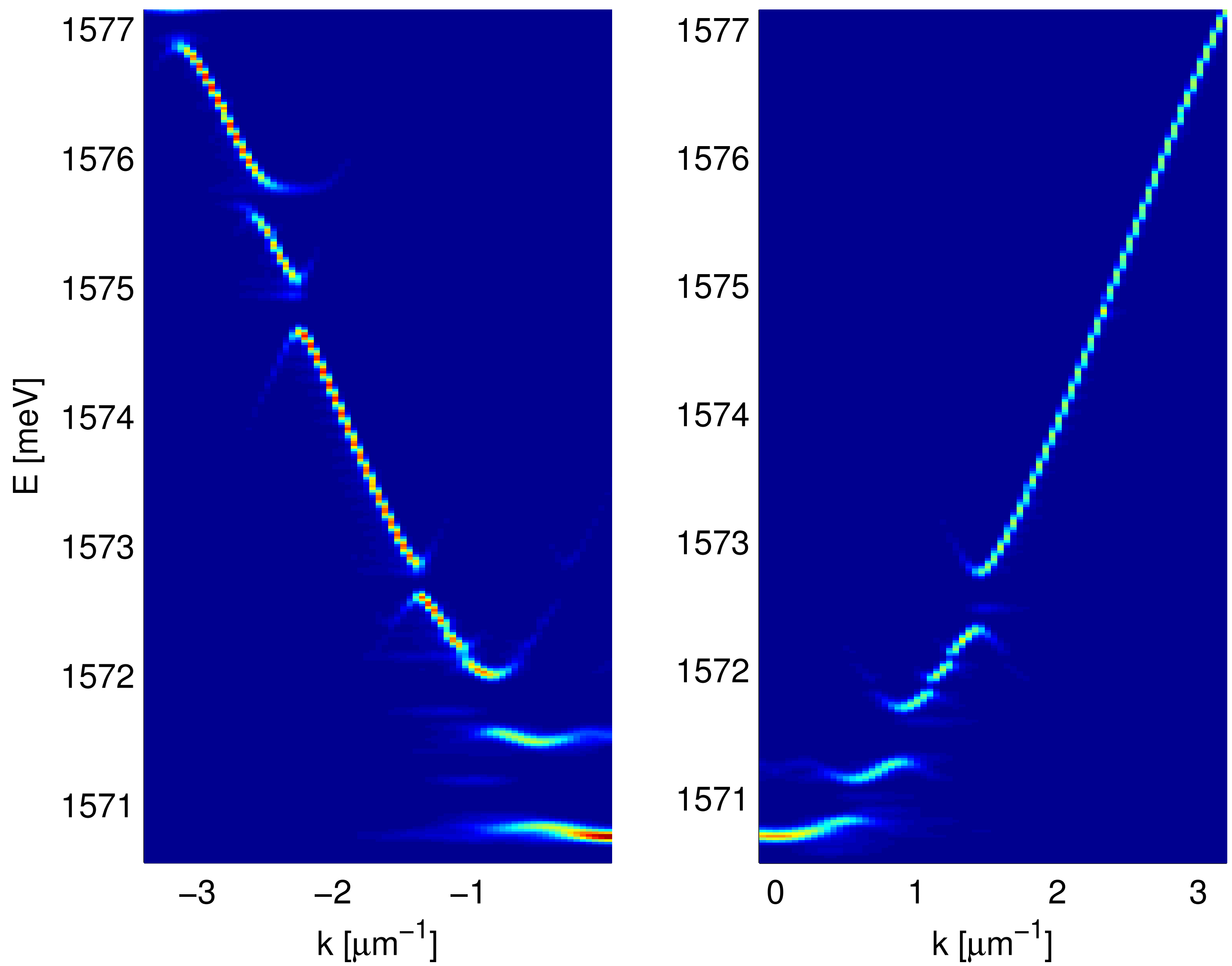
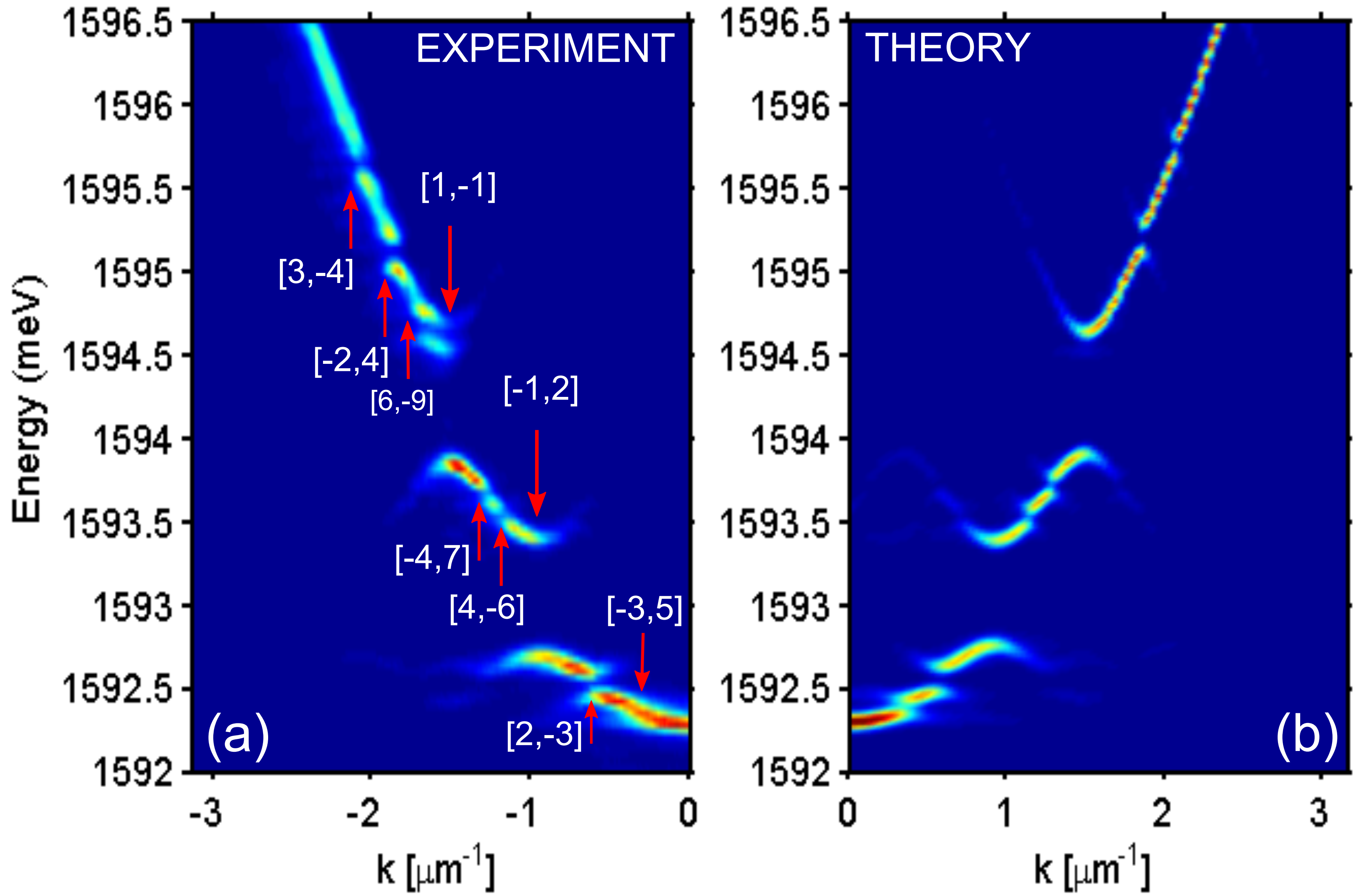
1D calculation without the non perturbative term



Labeling the gaps...



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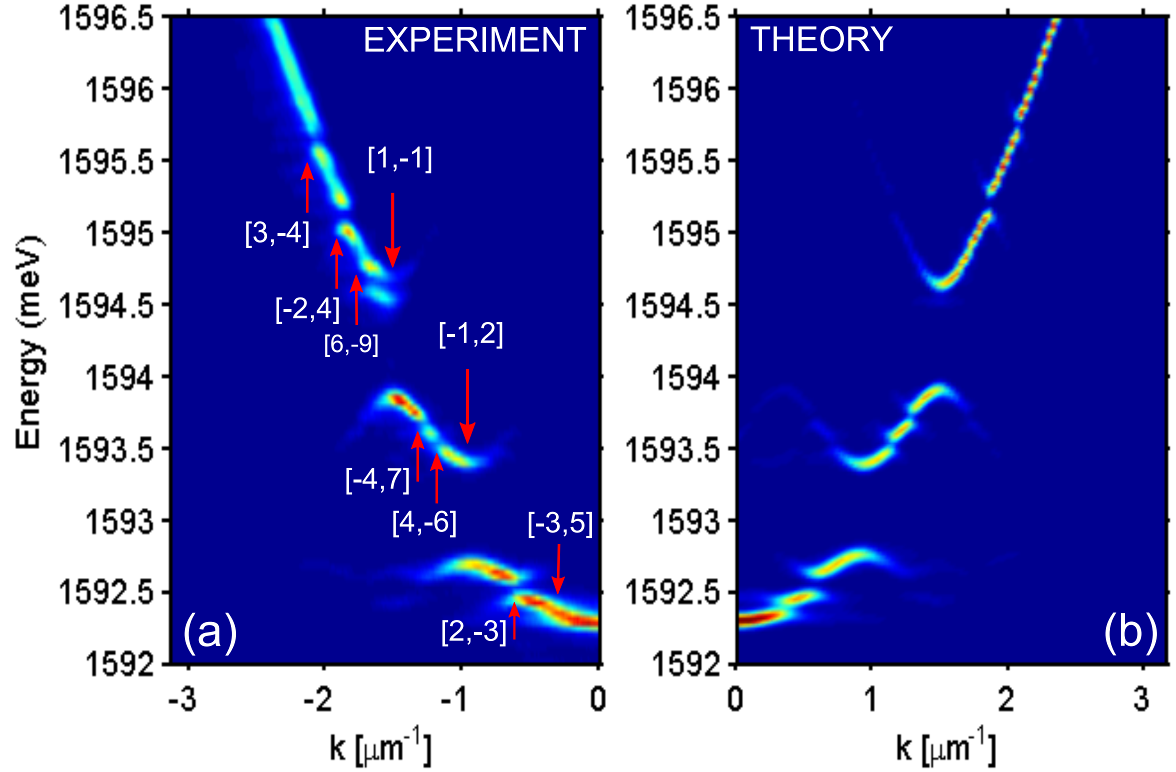
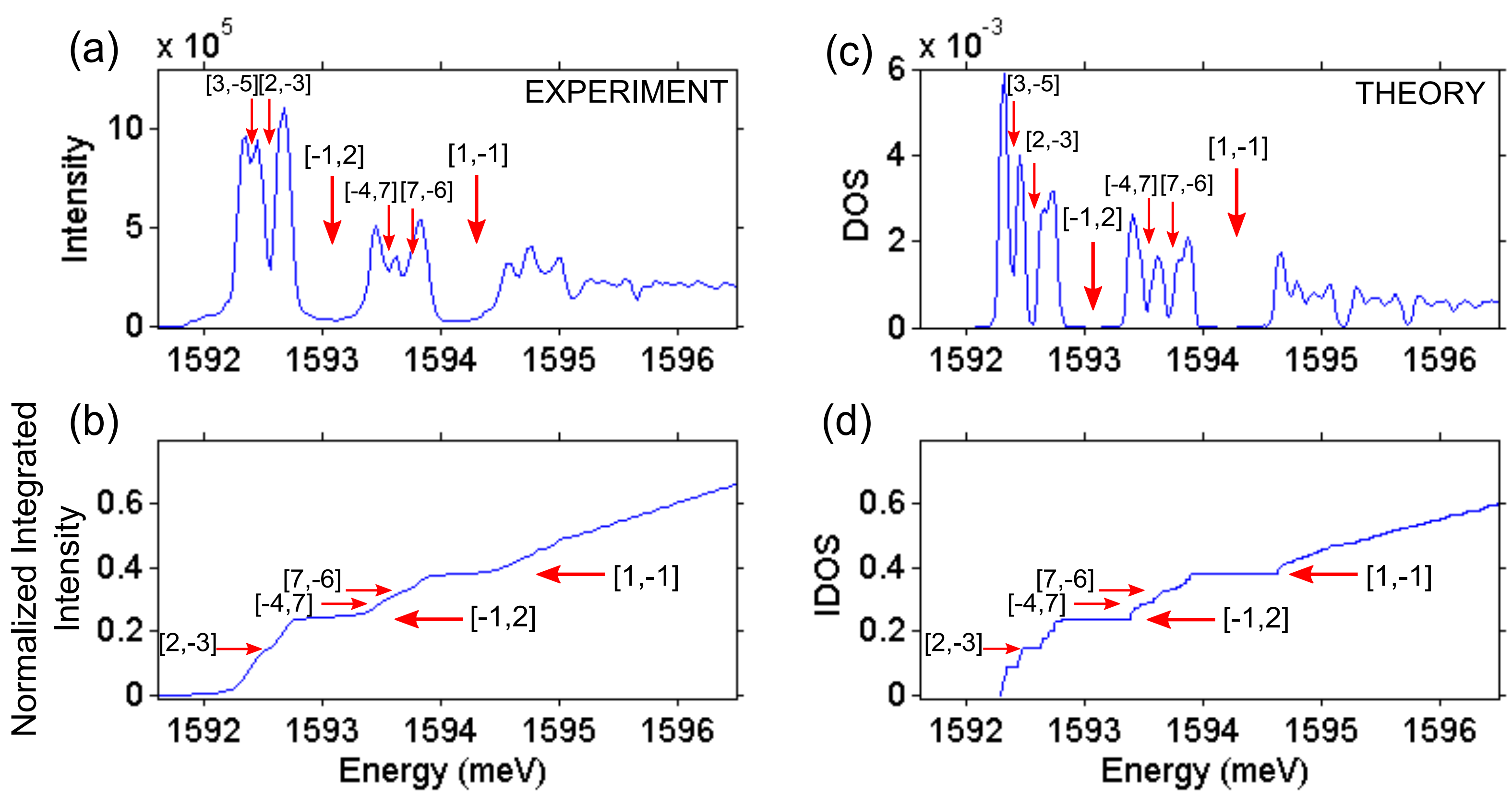
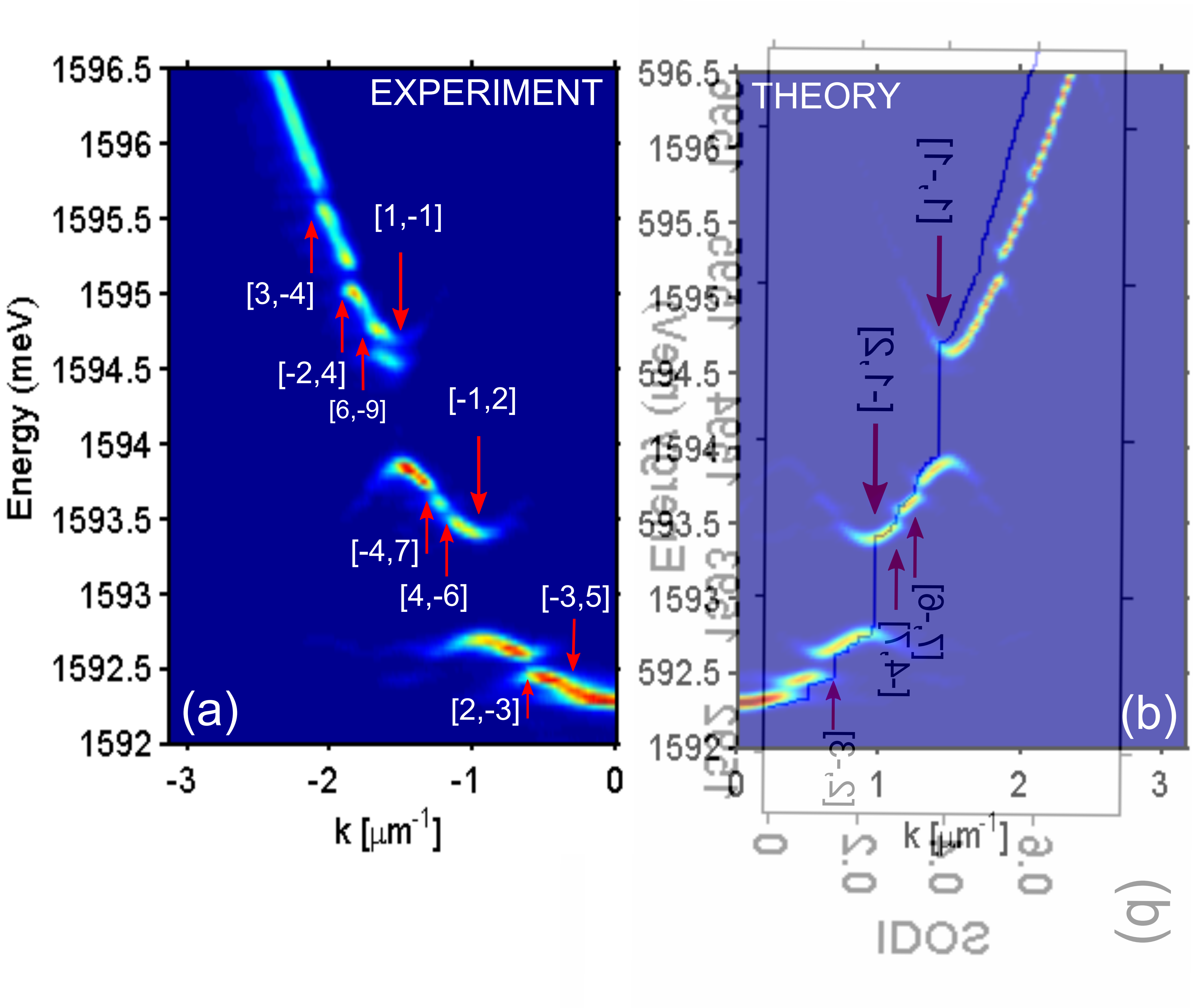


Calculating the integrated density of states (IDOS)





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Integrated density of states (IDOS)-Gap labeling

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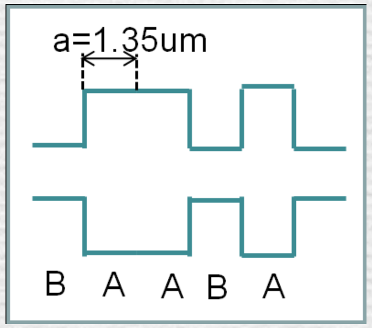

$\sigma = \sqrt{5}+1/2 \approx 1.62$ is the golden mean

$$V(k) = u_b(k) \times \sum_{p,q} \chi_q \, \delta\left(ka - 2\pi[p + \sigma q]\right)$$

Each pair $\{p,q\}$ of integers defines a unique Bragg peak (σ is irrational).

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Bragg peaks at values $k = Q \equiv \frac{1}{a}(F_{j+1} p + F_j q) \xrightarrow[j\to\infty]{} \frac{1}{a}(p + q\sigma)$

Perturbation theory

$$\left[-\frac{\hbar^2}{2M}\frac{d^2}{dx^2}+V(x)\right]\psi(x)=E\psi(x)$$

small

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small

Experimentally, it is not the case !

Perturbation theory (small V)

For the (quasi) crystal, a series of gaps open at each value of the (independent) Bragg peaks (Bloch thm.).

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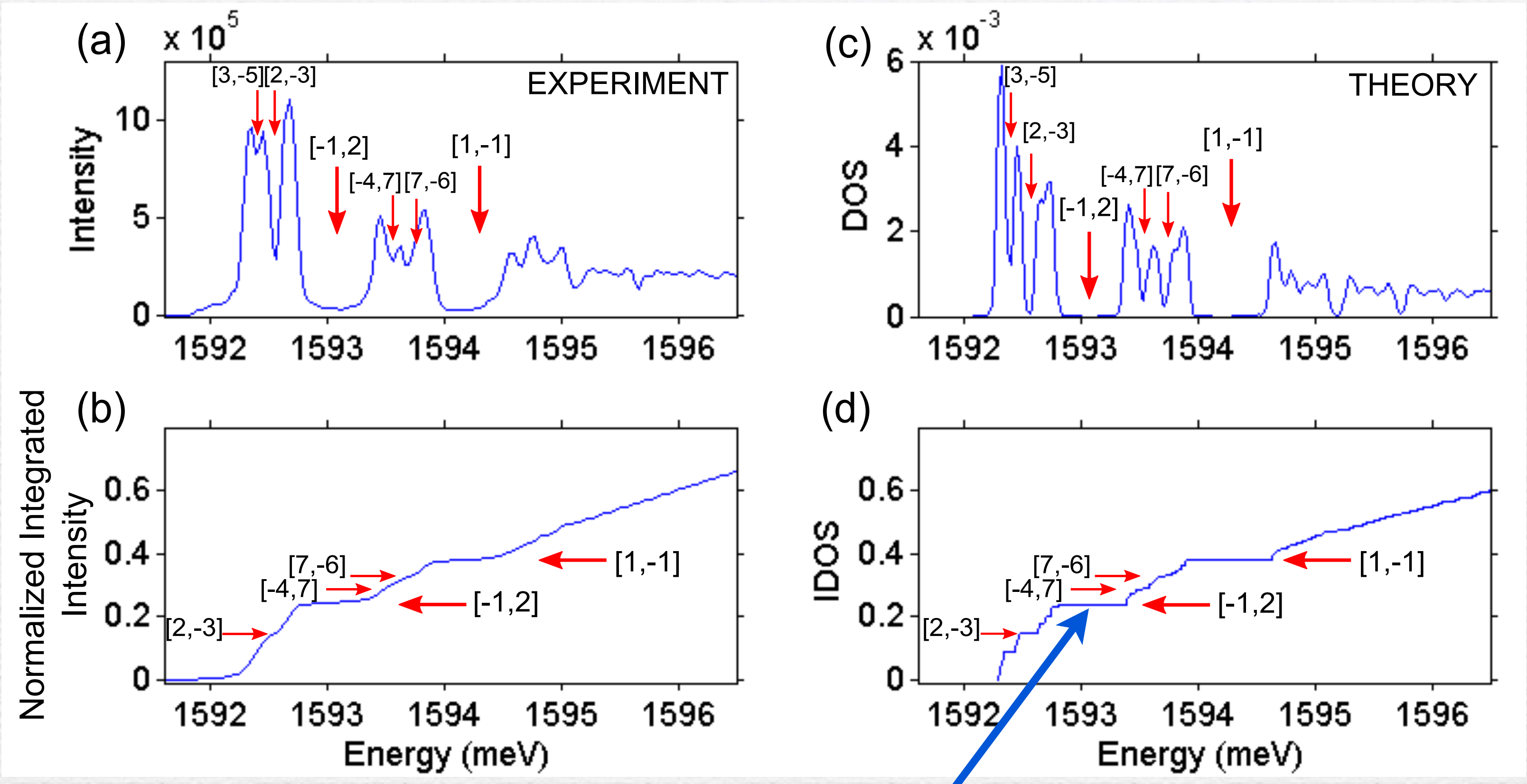
The (normalized) IDOS inside a gap labeled by $\{p,q\}$ is

$$N\left(\varepsilon = E_{Q_{p,q}/2}\right) = p + q\sigma$$

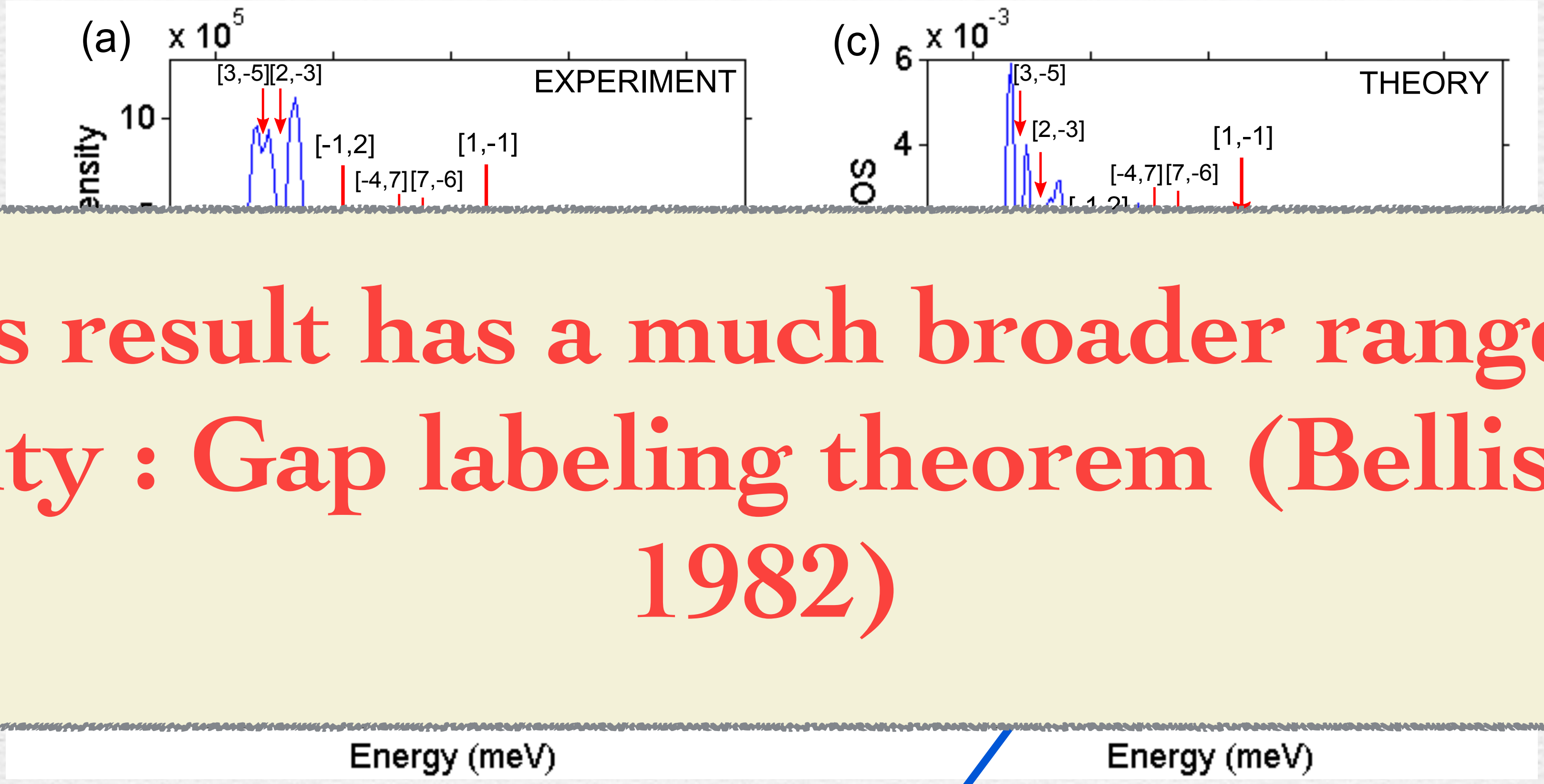
Integrated Density of States-Gap Labeling



$$N\left(\varepsilon = E_{Q_{p,q}/2}\right) = p + q\sigma$$

within a $\{p,q\}$ gap

Integrated Density of States-Gap Labeling



This result has a much broader range of validity : Gap labeling theorem (Bellissard, 1982)

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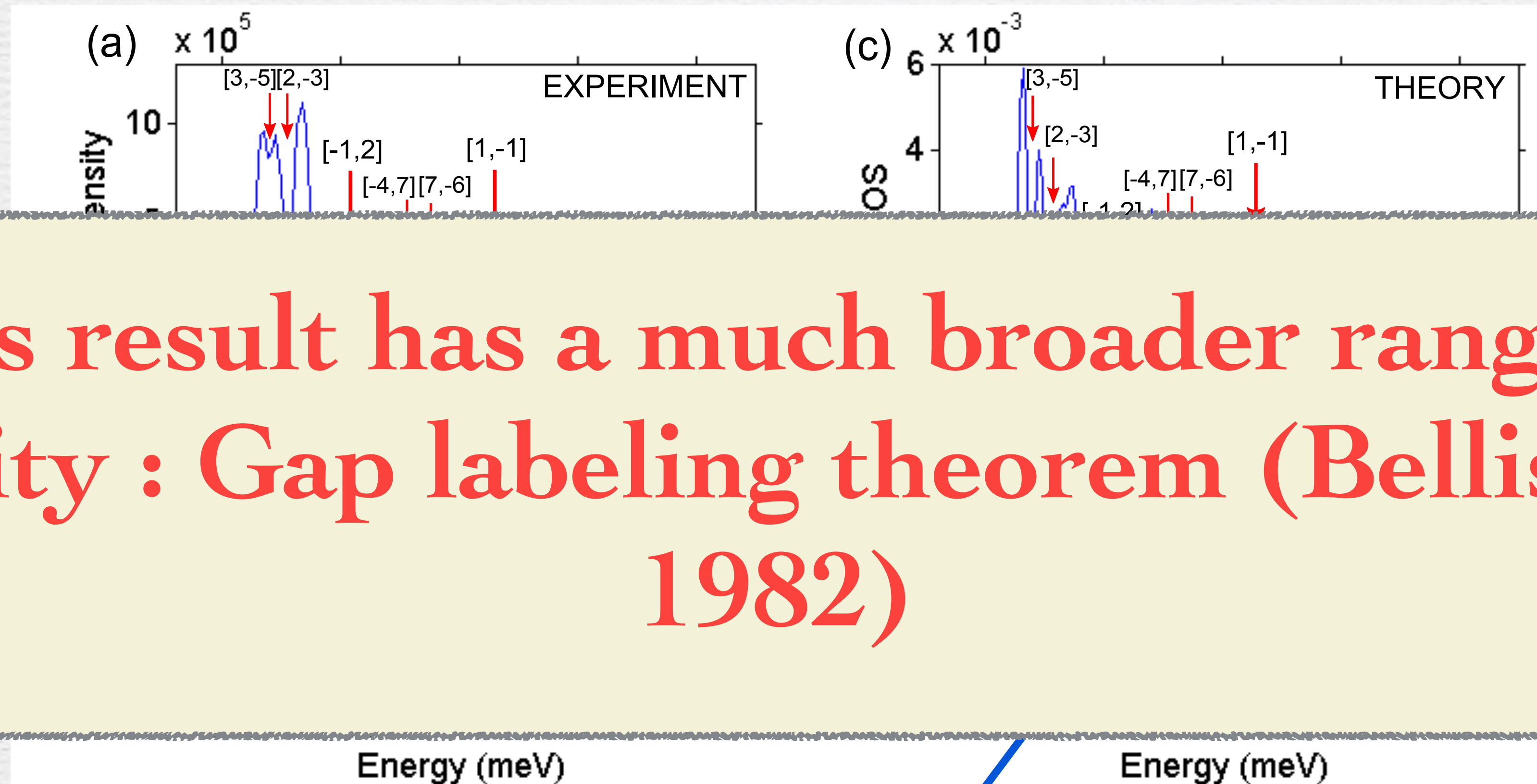

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Topological invariants (Chern numbers) independent of potential strength, inhomogeneity, ...

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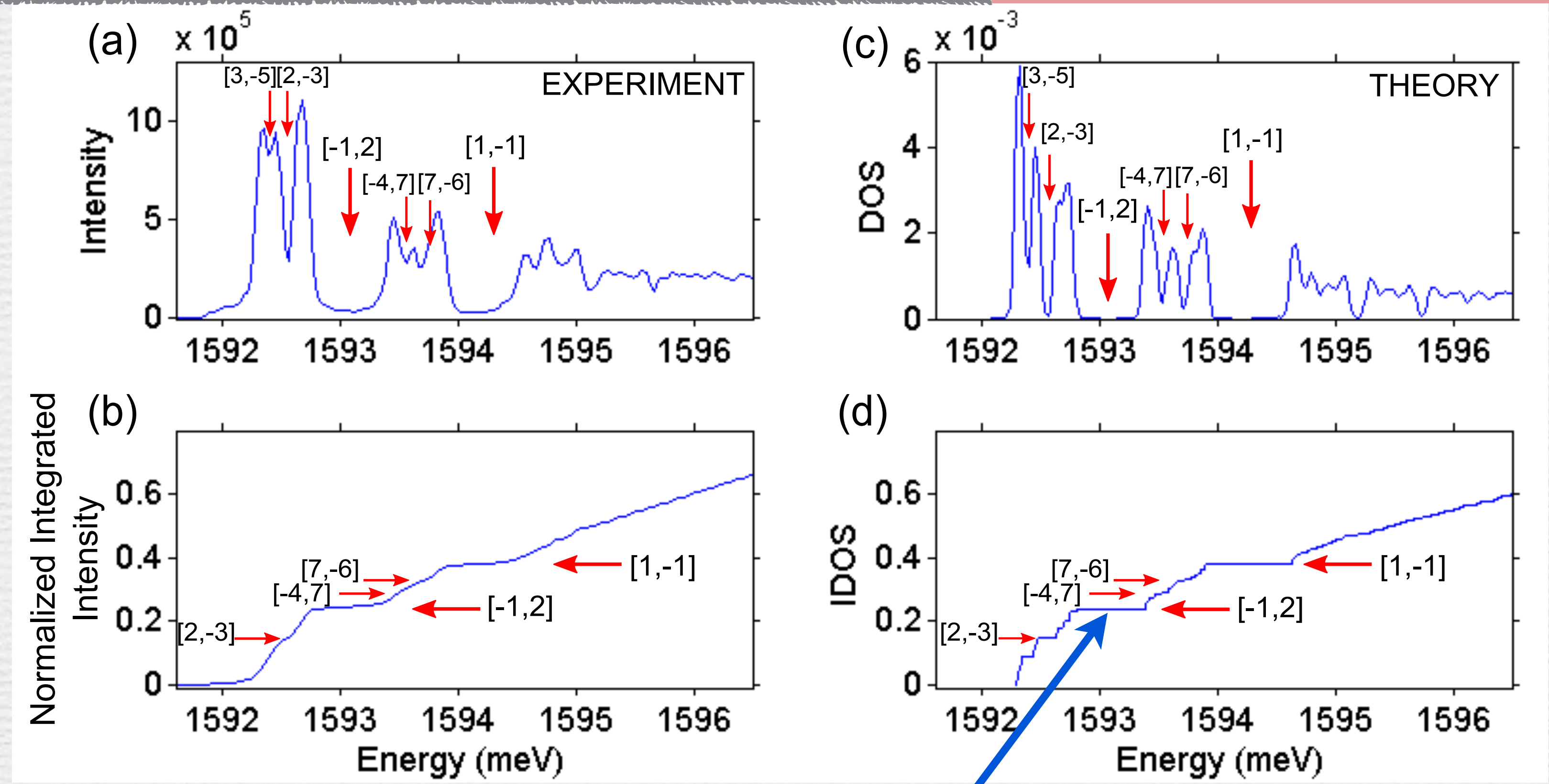

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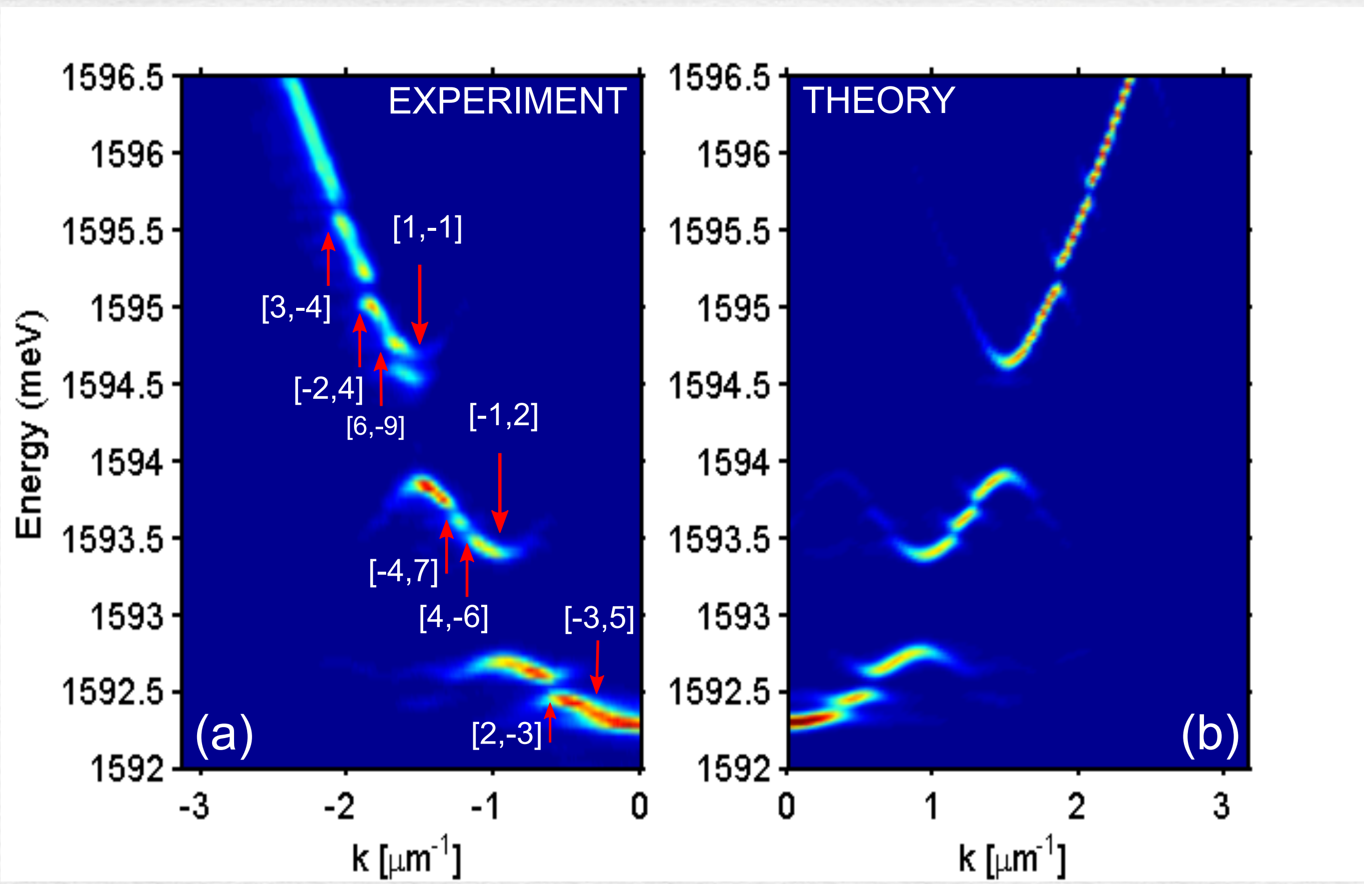
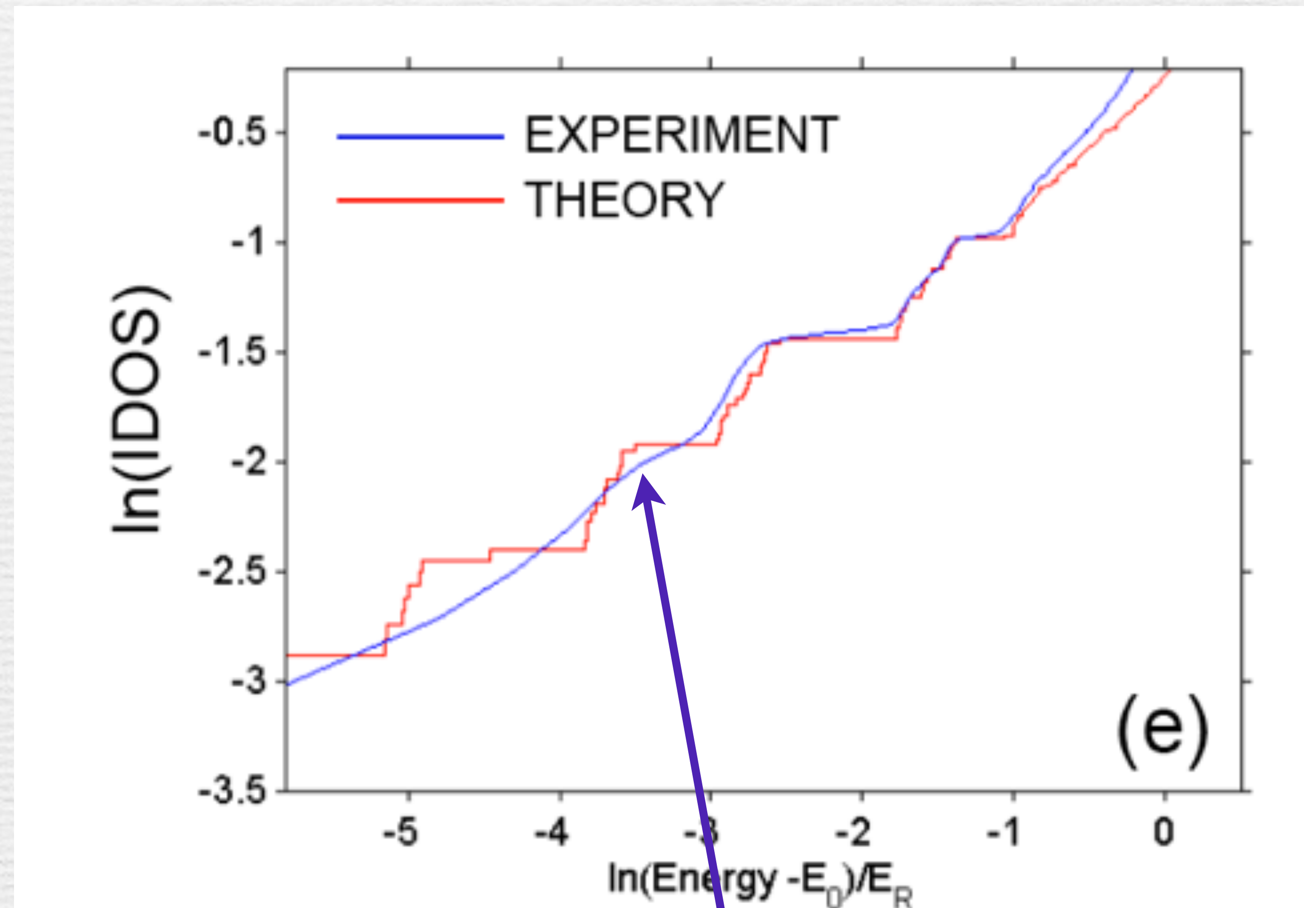
Integrated Density of States-Log-periodic oscillations

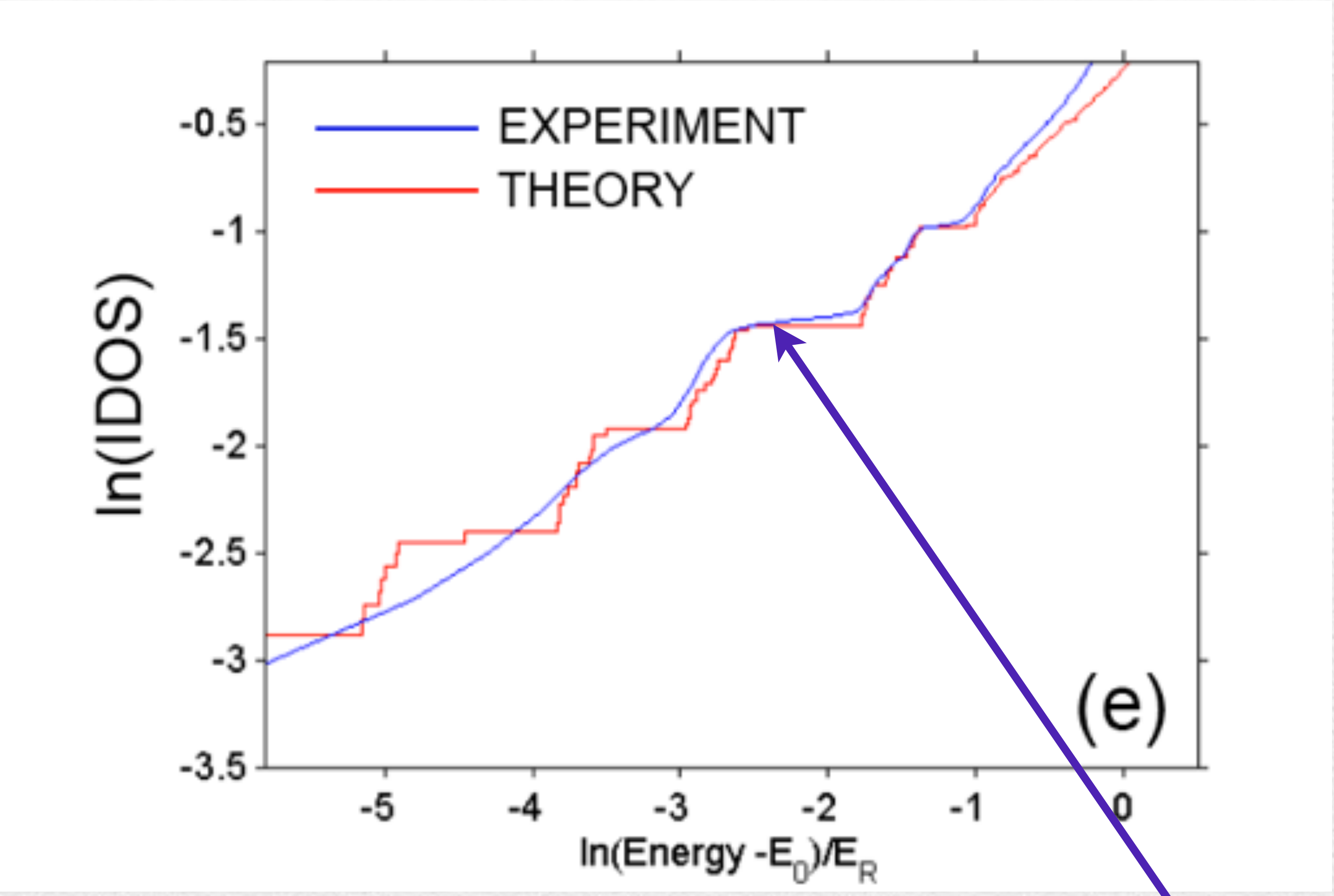
outside $\{p,q\}$ gaps





$$N_\omega(\Delta\omega) = (\Delta\omega)^\alpha \times F\left(\frac{\ln|\Delta\omega|}{\ln b}\right), \qquad \alpha = \frac{\ln a}{\ln b}, \quad F(x+1) = F(x)$$

Log-periodic oscillating structure is the indisputable fingerprint of the underlying fractal structure of the spectrum.



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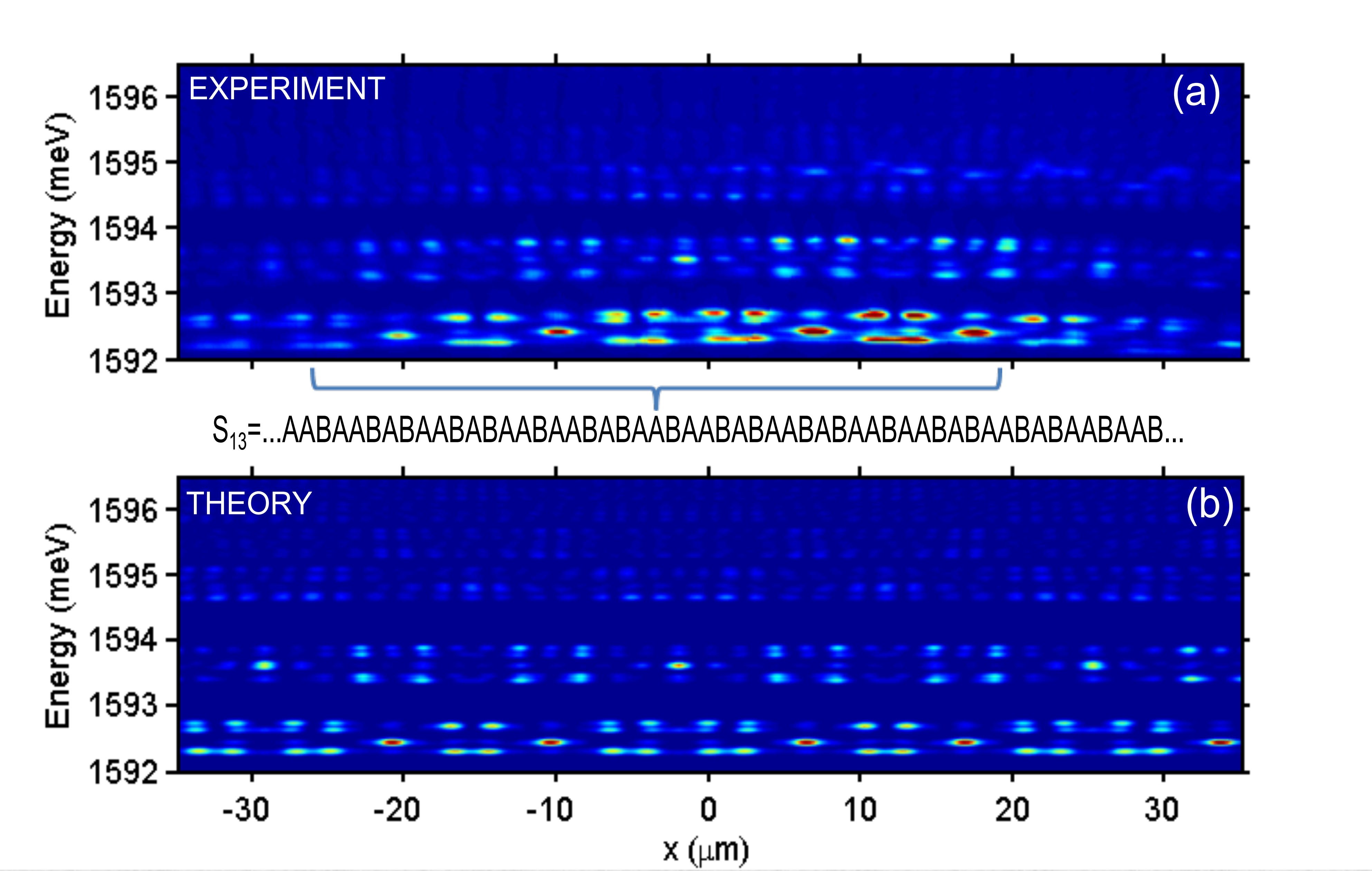
Imaging the modes in real space : spatially and spectrally resolved emission



Summary-Further directions

- Coupling of a quantum emitter to a fractal quasi-continuum leads to an unusual decay dynamics.
- The decay exhibits scaling properties related to the discrete scaling symmetry of the quasi-continuum.
- The experimental study of a macroscopic coherent polariton gas in a Fibonacci cavity allows for a quantitative study of a fractal singular continuous energy spectrum : spectral function, wave functions and gap labeling.

Further directions

- Long time dynamics of wave packets with a quasi-continuum fractal spectrum. Log-periodic oscillations.

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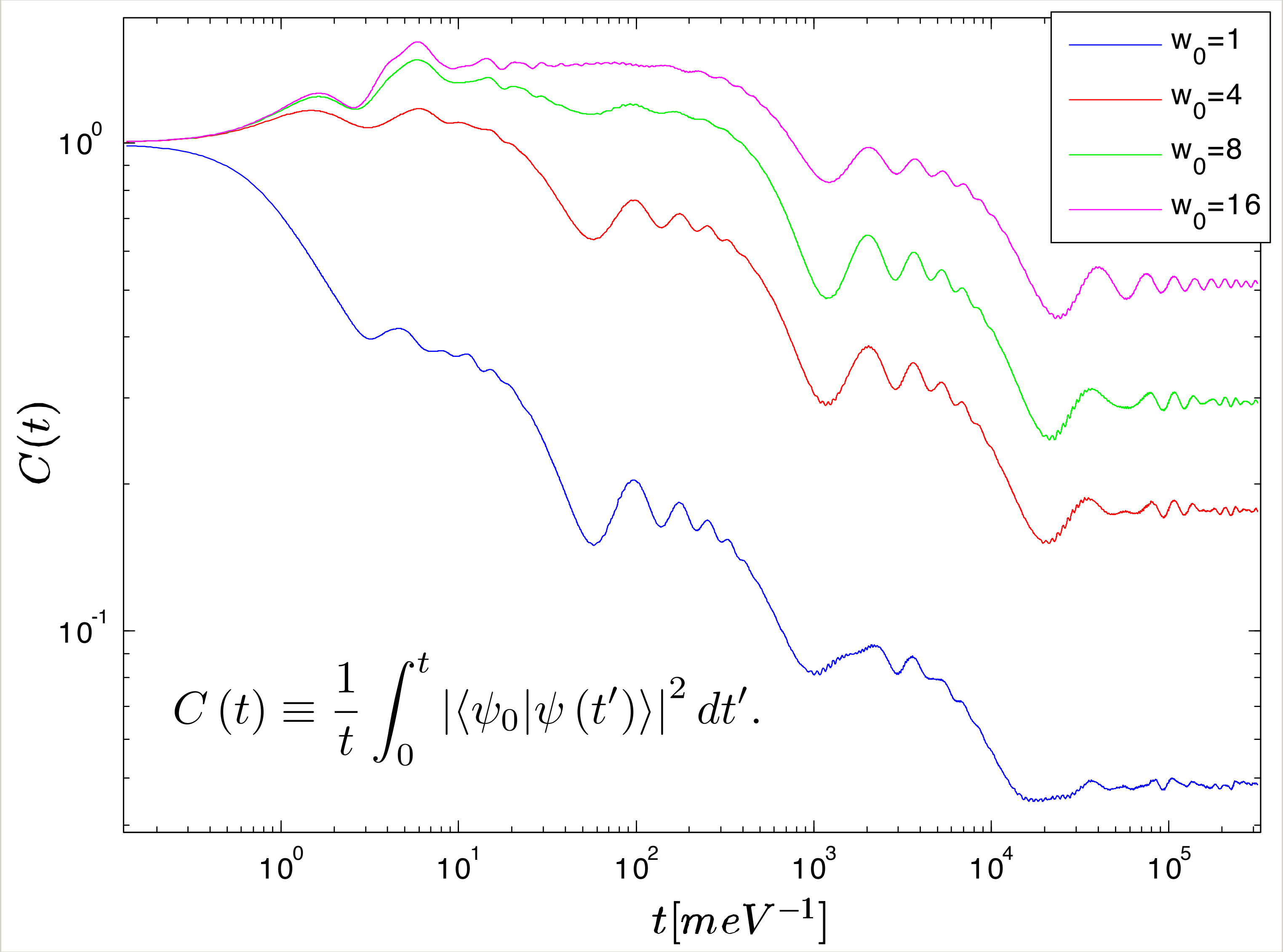
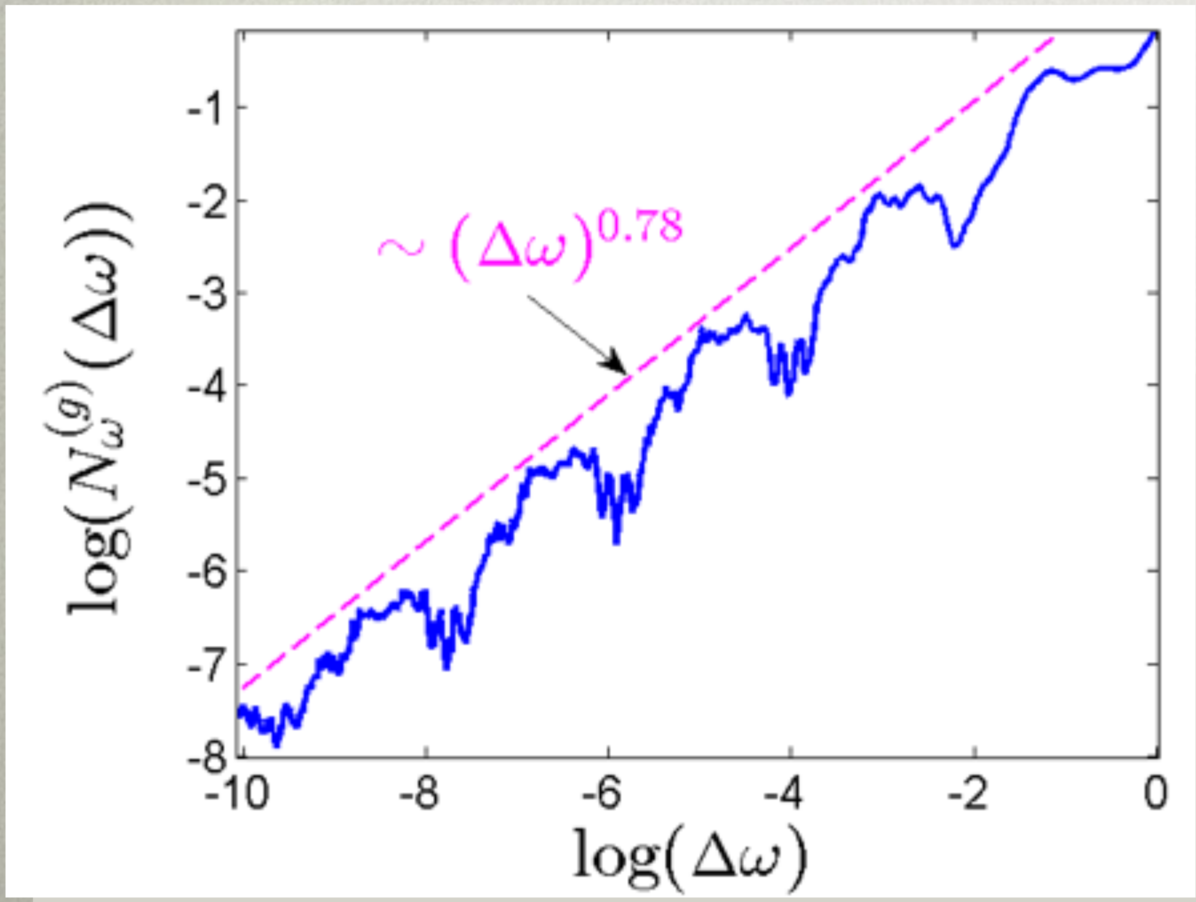
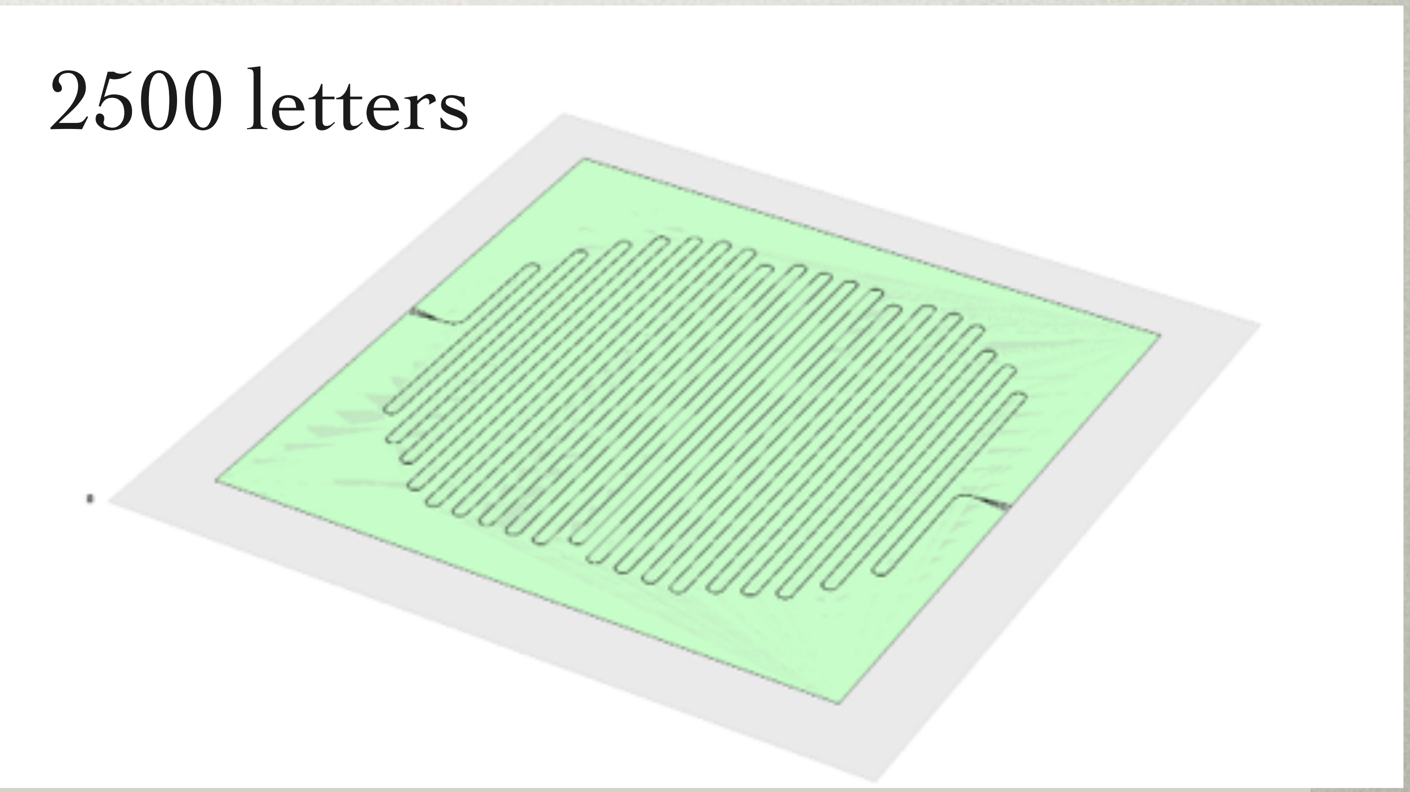


Further directions

- Long time dynamics of wave packets with a quasi-continuum fractal spectrum. Log-periodic oscillations.
- **Spontaneous emission** : tunnel junction and/or squbit in a microwave fractal resonator (J. Gabelli, Orsay) : Notion of photons- counting statistics-zero point motion with fractal spectra.





Let us conclude with something a bit weird…

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A simulator for quantum Einstein gravity

Quantum gravity

Einstein general relativity based on Einstein-Hilbert action is a highly successful effective field theory on length scales larger than

$$l_{Pl} = \left(\frac{\hbar G_N}{c^3} \right)^{1/2} \approx 10^{-33}\, cm$$

- Newton's constant:

$$G_N = 6.67 \times 10^{-11} \frac{\mathrm{m}^3}{\mathrm{kg\,s}^2}$$

Is it possible to promote it to a fundamental microscopic quantum theory of the gravitational interaction and space time structure ?

What are the relevant degrees of freedom at the Planck scale?

Which aspects of spacetime are dynamical at the Planck scale: geometry? topology? dimensionality?

Basic tool : sum over histories

$$\int \mathcal{D}g \, e^{-S[g]}$$

Each path is a 4-dimensional, curved space time geometry "g" which can be thought of as a 3-dim., spatial geometry developing in time.

associated with each "g" is given by the corresponding Einstein-Hilbert action $S[g]$

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$$S[g] = \frac{1}{16\pi G_N} \int d^4x \sqrt{g} \, (-R + 2\Lambda)$$

- Newton's constant: $G_N = 6.67 \times 10^{-11} \frac{\mathrm{m}^3}{\mathrm{kg\, s}^2}$
- cosmological constant: $\Lambda \approx 10^{-35}\ \mathrm{s}^{-2}$

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<u>The fundamental problem</u>

…) a functional integral over all metrics "g" on a space time.

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The fundamental problem
…) a functional integral over all metrics "g" on a space time.

Non renormalisable in perturbation theory. Very unfortunate !

A hard problem ! Several approaches on the market.

The options

- Leave the framework of quantum field theory :
  String theory, spin foams,…

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  M. Reuter, F. Saueressig

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  M. Reuter, F. Saueressig
- Statistical field theory (dynamical triangulations)

  Ambjorn, Jurkewicz, R. Loll.

Dynamically generated four-dimensional quantum universe, obtained from a path integral over causal spacetimes

The Spectral Dimension of the Universe is Scale Dependent

J. Ambjørn,[1,3,*] J. Jurkiewicz,[2,†] and R. Loll[3,‡]

[1]*The Niels Bohr Institute, Copenhagen University, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark*
[2]*Mark Kac Complex Systems Research Centre, Marian Smoluchowski Institute of Physics, Jagellonian University, Reymonta 4, PL 30-059 Krakow, Poland*
[3]*Institute for Theoretical Physics, Utrecht University, Leuvenlaan 4, NL-3584 CE Utrecht, The Netherlands*
(Received 13 May 2005; published 20 October 2005)

We measure the spectral dimension of universes emerging from nonperturbative quantum gravity, defined through state sums of causal triangulated geometries. While four dimensional on large scales, the quantum universe appears two dimensional at short distances. We conclude that quantum gravity may be "self-renormalizing" at the Planck scale, by virtue of a mechanism of dynamical dimensional reduction.

$$Z(t) = Tr\, e^{-\Delta t} = \int dx \langle x | e^{-\Delta t} | x \rangle = \sum_{\lambda} e^{-\lambda t}$$

Heat kernel

$$Z_d(t) = \int_{Vol.} d^d x P_t(x,x) = \frac{Volume}{(4\pi D t)^{d/2}}$$

measure the spectral dimension

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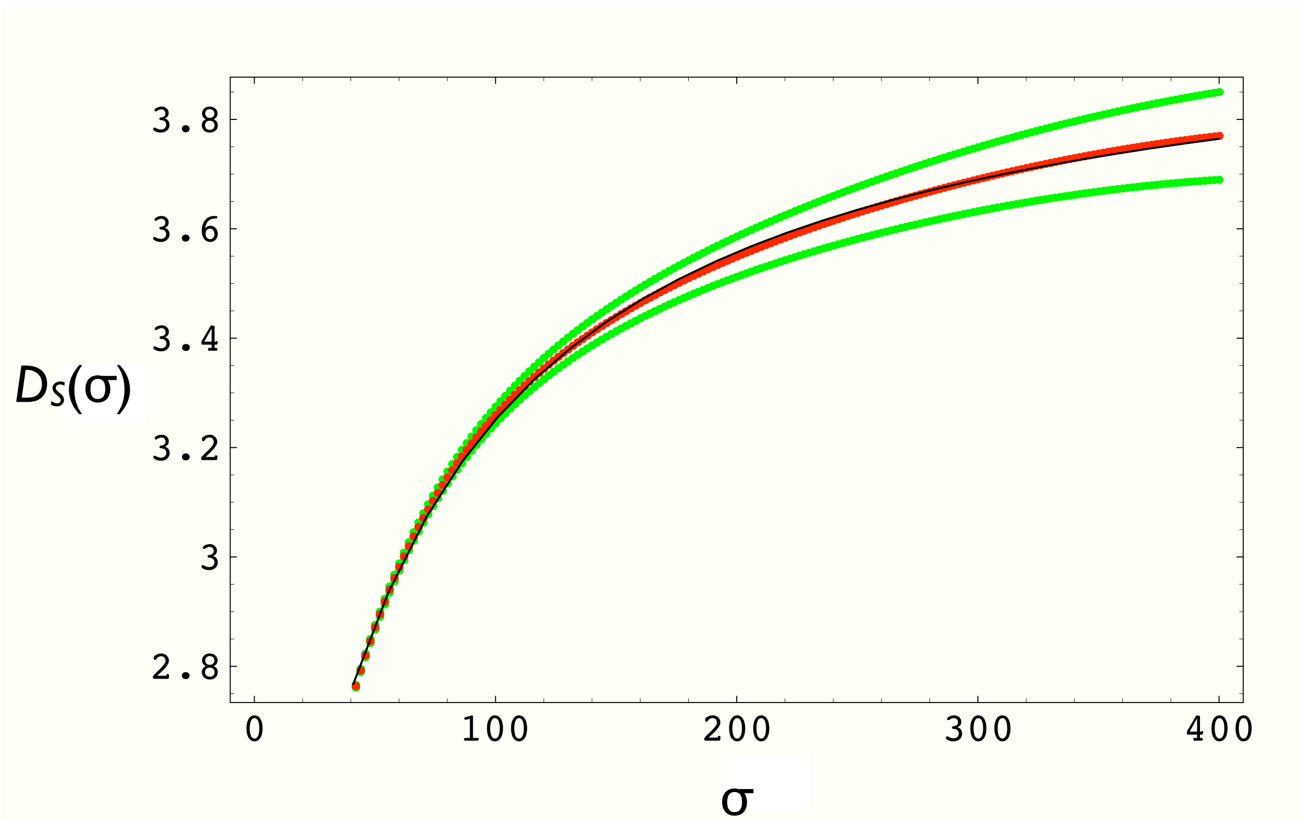

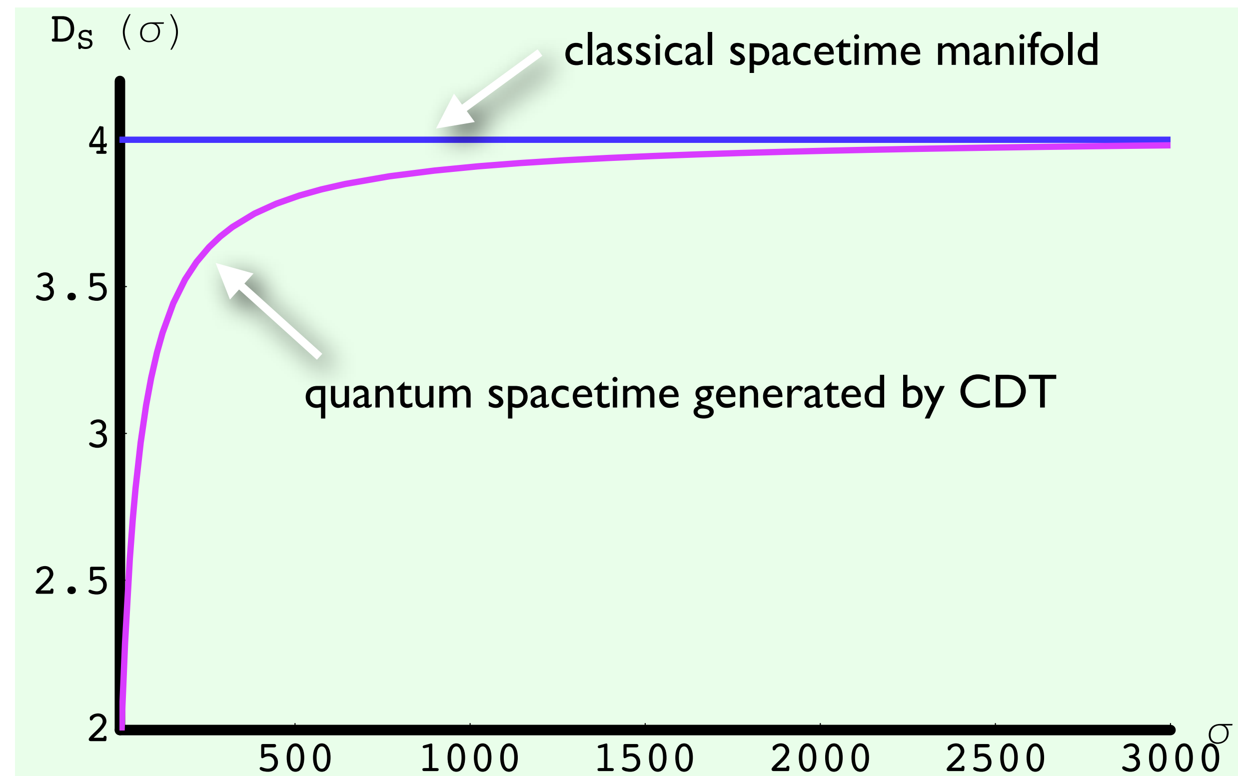

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measure the spectral dimension

The Spectral Dimension of the Universe is Scale Dependent

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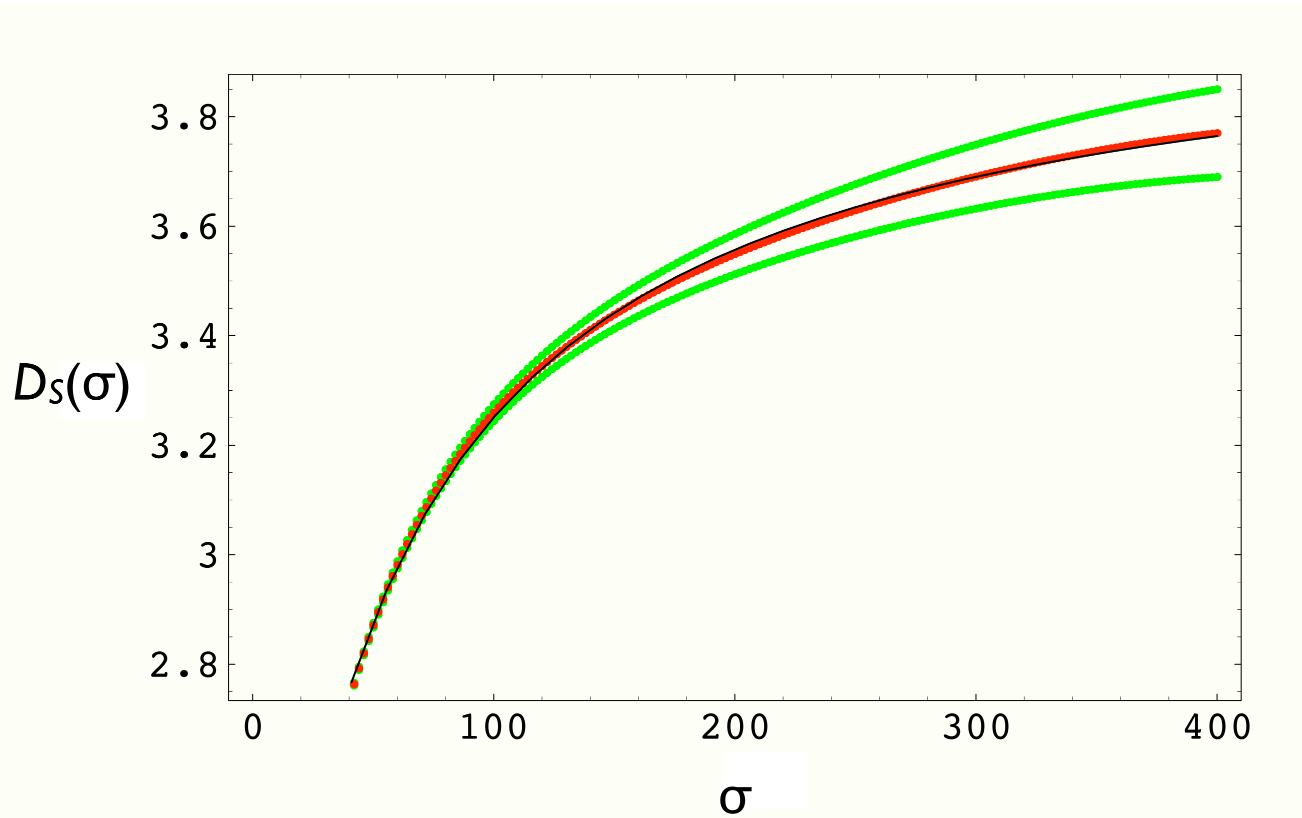

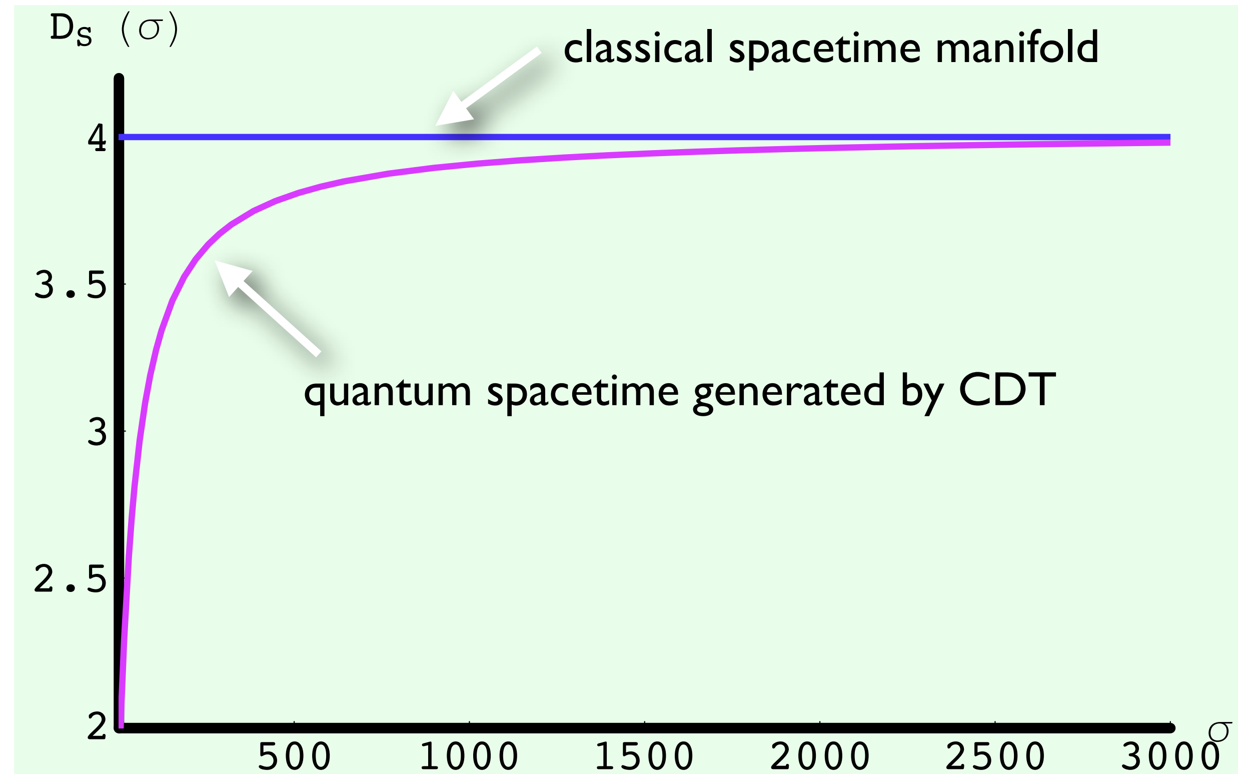

$$Z_d(t) = \int_{Vol.} d^d x P_t(x,x) = \frac{Volume}{(4\pi D t)^{d/2}}$$

measure the spectral dimension

More precisely,

$$D_S(\sigma) \to 4.02 \pm 0.1 \text{ as } \sigma \to \infty, \quad D_S(\sigma) \to 1.82 \pm 0.25 \text{ as } \sigma \to 0$$

The other option : non perturbative renormalisation group flow analysis (M. Reuter, F. Saueressig, 2012)

Asymptotic Safety, Fractals, and Cosmology*

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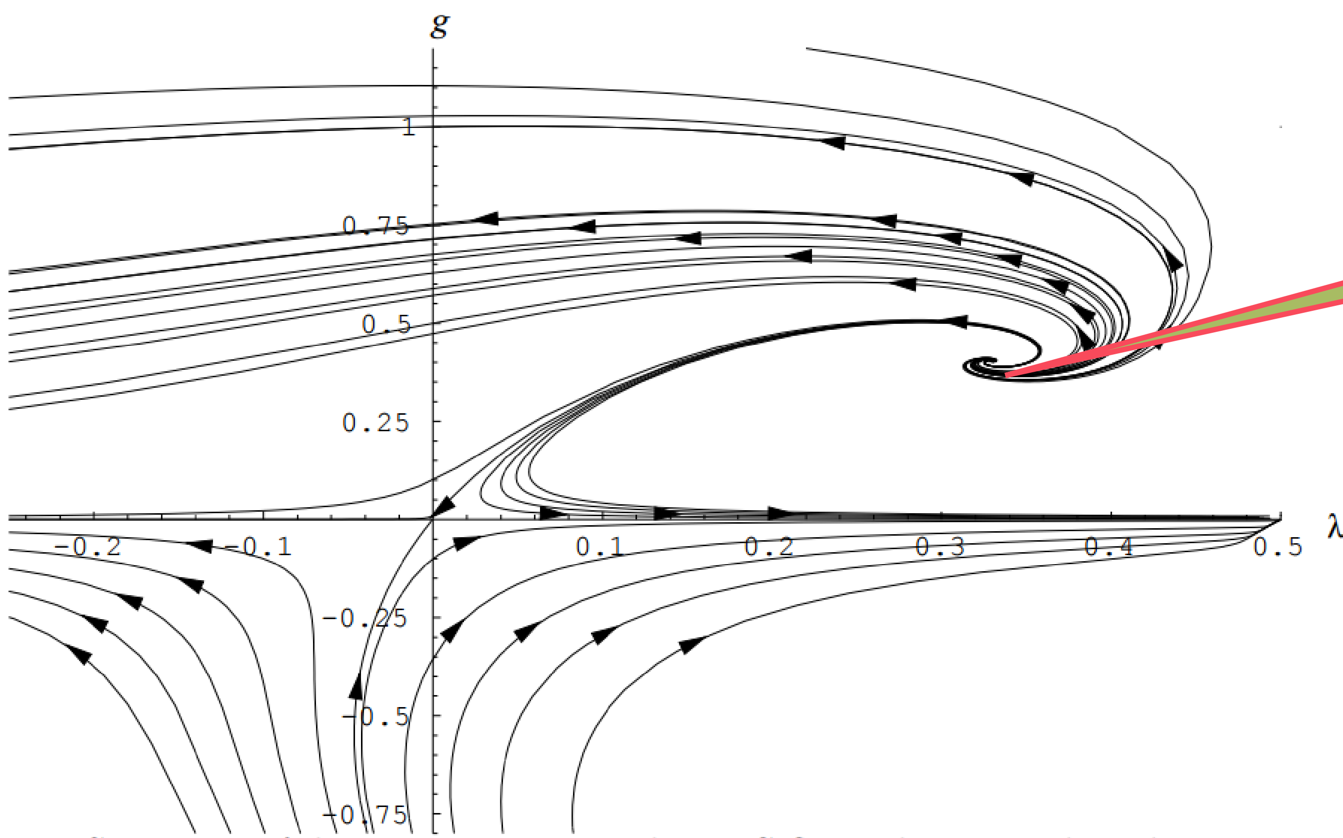
Abstract

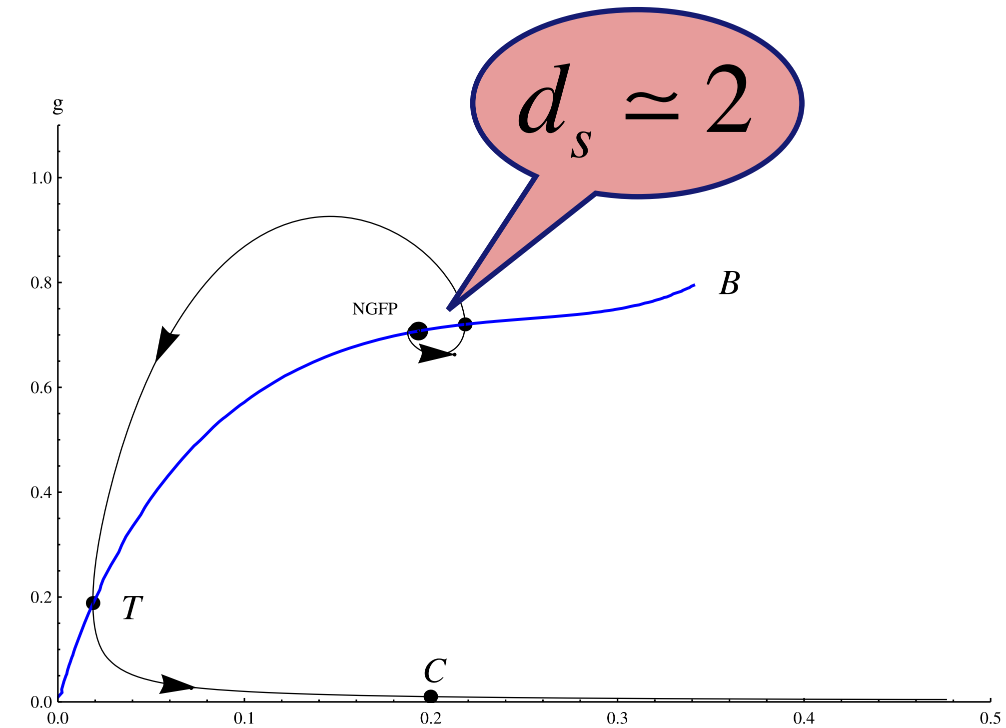
These lecture notes introduce the basic ideas of the Asymptotic Safety approach to Quantum Einstein Gravity (QEG). In particular they provide the background for recent work on the possibly multifractal structure of the QEG space-times. Implications of Asymptotic Safety for the cosmology of the early Universe are also discussed.

Running coupling constants:

Newton constant G_k, dimensionless: $g(k) = k^{d-2} G_k$

cosmological constant Λ_k, dimensionless: $\lambda(k) = k^{-2} \Lambda_k$





$$Z_d(t) = \int_{Vol.} d^d x P_t(x,x) = \frac{Volume}{(4\pi D t)^{d/2}}$$

Summarise

A quasi-periodic dielectric stack

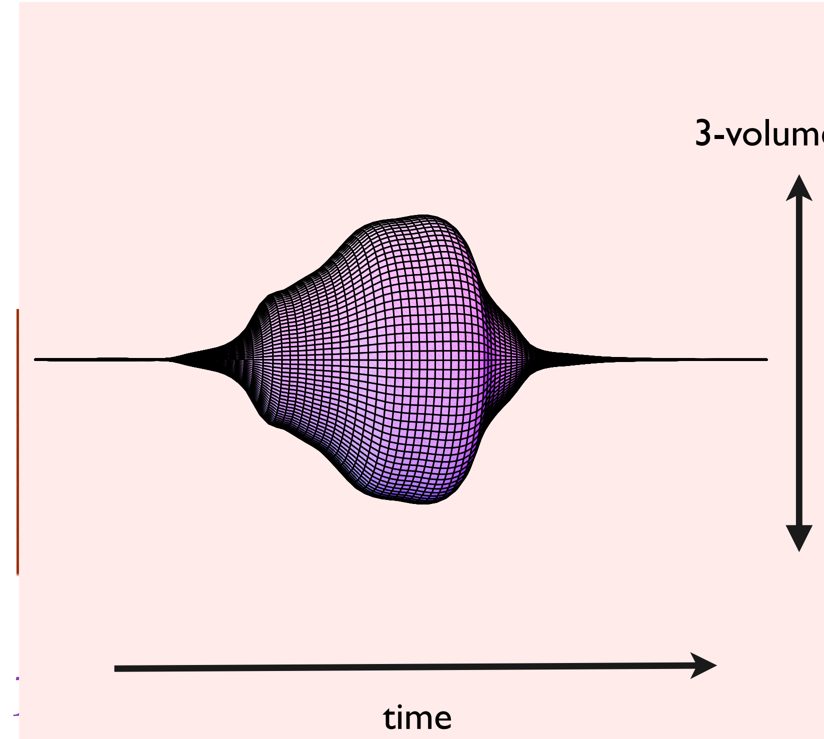
does not have a geometric fractal structure, but...

its spectrum has a fractal structure :

$$N_{\omega}(\Delta\omega) = (\Delta\omega)^{\alpha} \times F\left(\frac{\ln|\Delta\omega|}{\ln b}\right), \qquad \alpha = \frac{\ln a}{\ln b}, \quad F(x+1) = F(x)$$

Spectral fractal dimension

Summarise



d=4 space-time manifold

does not have a geometric
structure, but…

its spectrum has a fractal structure :

$$N_\omega(\Delta\omega) = (\Delta\omega)^\alpha \times F\left(\frac{\ln|\Delta\omega|}{\ln b}\right), \qquad \alpha = \frac{\ln a}{\ln b}, \quad F(x+1) = F(x)$$

Spectral fractal dimension $\alpha \to d_s \simeq 2$

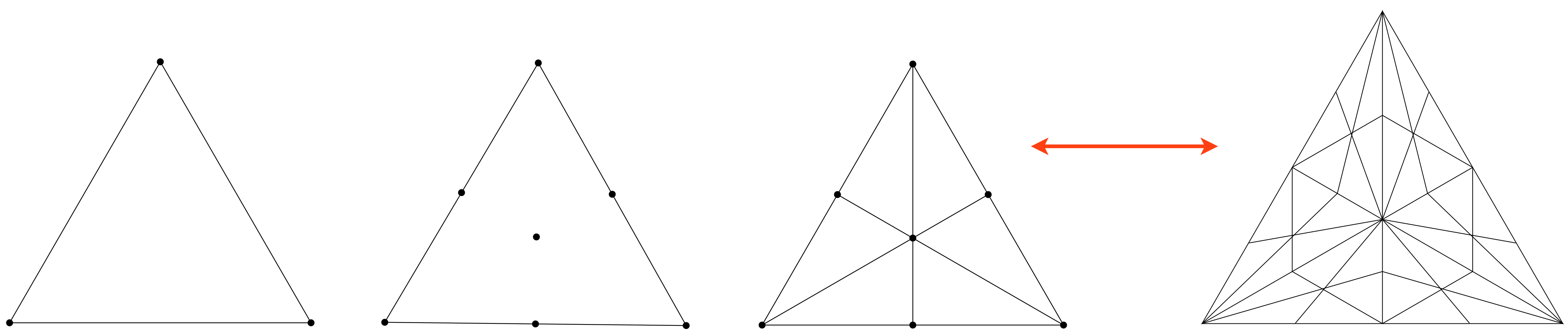
Is it possible to “mimic” time dimension

Not so simple to find one with $d_s \simeq 2$

Is it possible to "mimic" time [dimension]

Not so simple to find one with $d_s \simeq 2$

One serious contender : barycentric fractal



$d_s \simeq 1.74$

Simulator for quantum Einstein gravity at Planck length -
allows to measure/calculate other physical quantities not accessible otherwise

Apparently not that weird...

Apparently not that weird…

F. Englert proposed a very similar idea back in 1986.

F. Englert et al. / Metric space-time

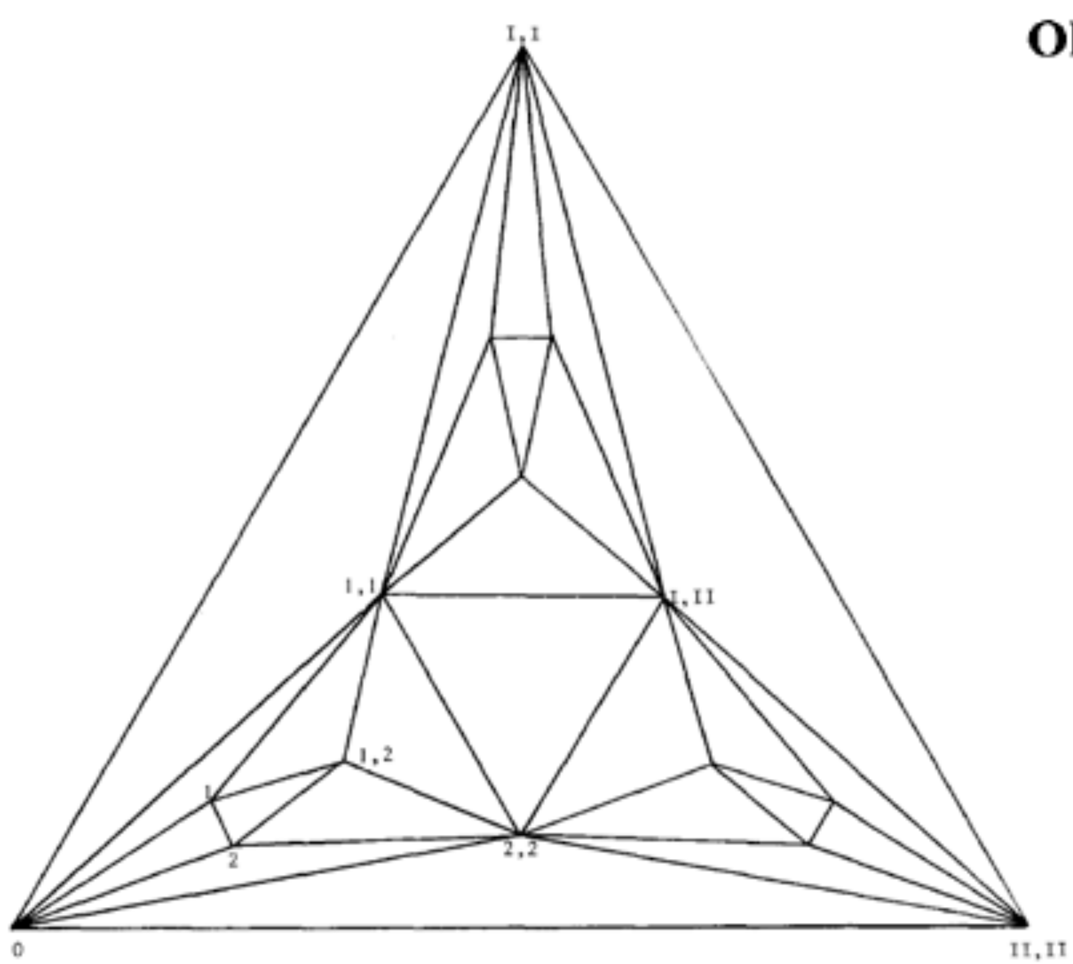


Fig. 10. A metrical representation of the two first iterations of a 2-dimensional 2-fractal corresponding to the euclidean fixed point. Vertices are labelled according to fig. 4.

METRIC SPACE-TIME AS FIXED POINT OF THE RENORMALIZATION GROUP EQUATIONS ON FRACTAL STRUCTURES

Thank you for your attention.