Topological Properties of Some Classes of Tilings

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Abstract

The topological properties of finite quasiperiodic tilings are examined. We show that the topological winding numbers found in previous works are related to the phase of the diffraction spectrum of a characteristic function. The winding numbers imply the location of the gaps in the spectrum. This ongoing research will be published in [1].

1D Tilings

Substitutions and Atomic Distributions

Define a *binary* substitution rule by

$$\begin{matrix}\sigma(a)=a^{\alpha}b^{\beta}\\ \sigma(b)=a^{\gamma}b^{\delta}\end{matrix} \iff \begin{matrix}a\mapsto a^{\alpha}b^{\beta}\\ b\mapsto a^{\gamma}b^{\delta}\end{matrix}$$

Associate occurrence matrix: $M=\left(\begin{smallmatrix}\alpha&\beta\\ \gamma&\delta\end{smallmatrix}\right)$
Consider only primitive matrices:

- Largest eigenvalue $\lambda_1>1$ (Perron-Frobenius)
- Left and right first eigenvectors are strictly positive

Distribution of letters underlies distribution of atoms:

$$x_0\ \ a\ \ x_1\ \ b\ \ x_2\ \ b\ \ x_3\ \ a\ \ x_4\ \ b\ \ x_5\ \ a\ \ x_6\ \ b\ \ x_7\ \ldots$$

Define atomic density

$$\rho(x)=\sum_k \delta(x-x_k)$$

with distances for a and b given by $\delta_k=x_{k+1}-x_k=d_{a,b}$.
Let $\bar{d}$ be the mean distance and u_k the deviations from the mean. Define

$$x_k=\bar{d}\,k+\delta\,u_k,\qquad \delta\equiv d_a-d_b$$

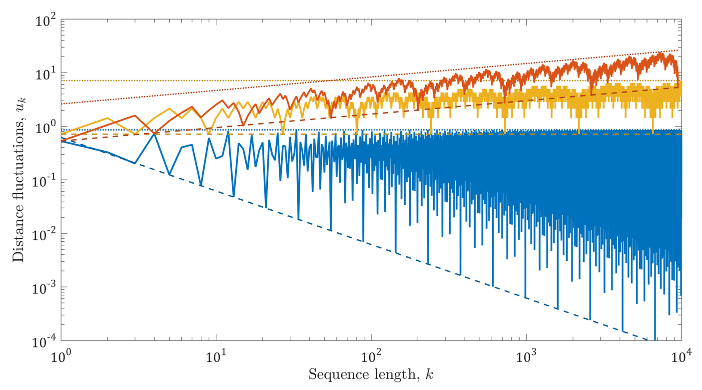
Let $g(\xi)=\sum_k e^{-i\xi x_k}$ be the diffraction pattern, and $S(\xi)=|g(\xi)|^2$ the structure factor. Bragg peak are located at [2, 3]

$$\xi_{m,N}=\frac{\bar{d}}{2\pi}\frac{m}{\lambda_1^N}.$$

Families:

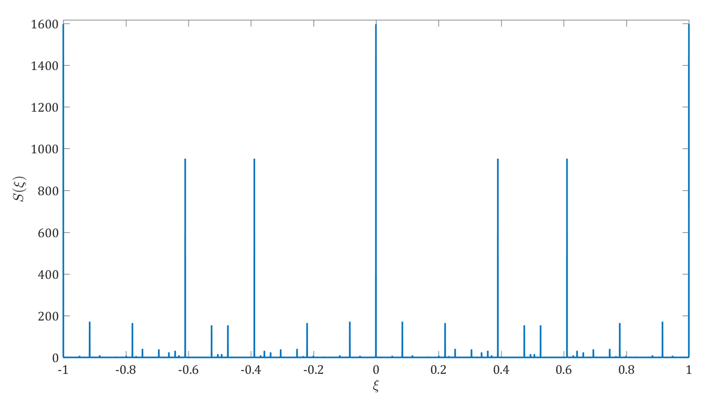
Pisot: The second eigenvalue $|\lambda_2|<1$.

Non-Pisot: The second eigenvalue $|\lambda_2|\geq 1$. Fluctuations u_k are unbounded [4]; there are no Bragg peaks [5].



Examples

Fibonacci: $a\mapsto ab$, $b\mapsto a$. It is Pisot, $M=\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$, $\lambda_1=\frac{1}{2}(\sqrt{5}+1)\equiv\tau$ the golden ratio and $\lambda_2=-\tau^{-1}$. Bragg peaks: $\xi_{p,q}=\frac{\bar{d}}{2\pi}(p+q\tau)$.



Thue-Morse: $a\mapsto ab$, $b\mapsto ba$. Here it is Pisot, $M=\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$, $\lambda_1=2$ and $\lambda_2=0$. Bragg peaks: $\xi_{m,N}=\frac{\bar{d}}{2\pi}\frac{m}{2^N}$.

Bratteli Diagrams

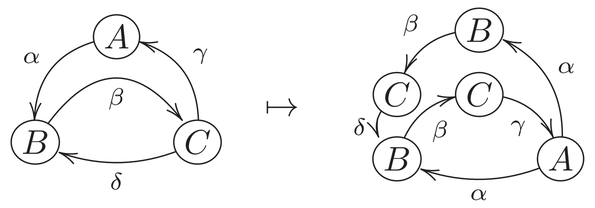
Consider the infinite word

$$w_1^{\infty}=\sigma^{\infty}(a)=abaababaabaababaababaabaababaabab\ldots.$$

Next, consider the *collared* graph of the *two-letter* words $A=aa$, $B=ab$, $C=ba$ such that

$$w_2^{\infty}=BCABCBCABCABCBCABCBCABCABCBCABC\ldots$$

(see left graph below), with $\alpha=AB$, $\beta=BC$, $\gamma=CA$, $\delta=CB$.
Calculate the boundary operators, where $\partial_0(A)=0$ and $\partial_1(\overrightarrow{e}=\overrightarrow{AB})=B-A$.
Represent them as matrices and calculate the cohomology groups $H^i(G)=\frac{\ker\partial_i'}{\operatorname{im}\partial_{i+1}'}$.
Consider the inflation rule:



Calculate its inflation matrices A_0' and A_1'.
Calculate ζ-function $\zeta(z)=\exp\sum_{m=1}^{\infty}\frac{|\operatorname{fix}\sigma^m|}{m}z^m$. The 1D ζ-function is given by [6]

$$\zeta_{1D}(z)=\frac{\det(I-zA_0')}{\det(I-zA_1')}$$

Name	Substitution: Rule σ_1	Substitution: Occurrence M_1	Substitution on Doublets: Rule σ_2	Substitution on Doublets: Occurrence M_2	Self Properties: Eigenvalue λ_1	Self Properties: Char. Polynomial	Cohomology: $H^0(G)$	Cohomology: $H^1(G)$	Zeta Function $\zeta(z)$	Gap Labeling Theorem		Properties: Pisot char.	Properties: Periodicity
Fibonacci	$0\mapsto 01$, $1\mapsto 0$	$\begin{pmatrix}1&1\\1&0\end{pmatrix}$	$a\mapsto bc$, $b\mapsto bc$, $c\mapsto a$	$\begin{pmatrix}0&1&1\\0&1&1\\1&0&0\end{pmatrix}$	τ	$\lambda^2-\lambda-1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\frac{1-z}{1-z-z^2}$	$p+q\cdot\tau$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Cantor Set	$0\mapsto 010$, $1\mapsto 111$	$\begin{pmatrix}2&1\\0&3\end{pmatrix}$	$a\mapsto aba$, $b\mapsto ccb$, $c\mapsto ccc$	$\begin{pmatrix}2&1&0\\0&1&2\\0&0&3\end{pmatrix}$	3	$\lambda^2-5\lambda+6=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{(1-2z)(1-3z)}$	$\frac{k}{3^N}$	$k,N\in\mathbb{Z}$	not primitive	limit-quasiperiodic
Non-Pisot	$0\mapsto 0001$, $1\mapsto 011$	$\begin{pmatrix}3&1\\1&2\end{pmatrix}$	$a\mapsto aabc$, $b\mapsto aabc$, $c\mapsto bdc$, $d\mapsto bdc$	$\begin{pmatrix}2&1&1&0\\2&1&1&0\\0&1&1&1\\0&1&1&1\end{pmatrix}$	$\tau+2$	$\lambda^2-5\lambda+5=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{1-5z+5z^2}$	$\frac{p+q\cdot\tau}{5^N}$	$p,q,N\in\mathbb{Z}$	non-Pisot	limit-quasiperiodic
Periodic	$0\mapsto 01$, $1\mapsto 01$	$\begin{pmatrix}1&1\\1&1\end{pmatrix}$	$a\mapsto ab$, $b\mapsto ab$	$\begin{pmatrix}1&1\\1&1\end{pmatrix}$	2	$\lambda^2-2\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^1$	$\frac{1-z}{1-2z}$	$\frac{k}{2}$	$k\in\mathbb{Z}$	Pisot	periodic
Thue-Morse	$0\mapsto 01$, $1\mapsto 10$	$\begin{pmatrix}1&1\\1&1\end{pmatrix}$	$a\mapsto bc$, $b\mapsto bd$, $c\mapsto ca$, $d\mapsto cb$	$\begin{pmatrix}0&1&1&0\\0&1&0&1\\1&0&1&0\\0&1&1&0\end{pmatrix}$	2	$\lambda^2-2\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{(1-2z)(1+z)}$	$\frac{k}{3\cdot 2^N}$	$k,N\in\mathbb{Z}$	Pisot	aperiodic
Period Doubling	$0\mapsto 01$, $1\mapsto 00$	$\begin{pmatrix}1&1\\2&0\end{pmatrix}$	$a\mapsto bc$, $b\mapsto bc$, $c\mapsto aa$	$\begin{pmatrix}0&1&1\\0&1&1\\2&0&0\end{pmatrix}$	2	$\lambda^2-\lambda-2=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{(1-2z)(1+z)}$	$\frac{k}{3\cdot 2^N}$	$k,N\in\mathbb{Z}$	non-Pisot	limit-quasiperiodic
Circle Sequence	$0\mapsto 202$, $1\mapsto 02202$, $2\mapsto 01202$	$\begin{pmatrix}1&0&2\\2&0&3\\2&1&2\end{pmatrix}$	$a\mapsto dbd$, $b\mapsto dbd$, $c\mapsto bedbd$, $d\mapsto acdbe$, $e\mapsto acdbd$	$\begin{pmatrix}0&1&0&2&0\\0&1&0&2&0\\0&2&0&2&1\\1&1&1&1&1\\1&1&1&2&0\end{pmatrix}$	τ^3	$\lambda^3-3\lambda^2-5\lambda-1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{(1+z)(1-4z-z^2)}$	$\frac{1}{2}(p+q\cdot\tau)$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Rudin-Shapiro	$0\mapsto 02$, $1\mapsto 32$, $2\mapsto 01$, $3\mapsto 31$	$\begin{pmatrix}1&0&1&0\\0&0&1&1\\1&1&0&0\\0&1&0&1\end{pmatrix}$	$a\mapsto bf$, $b\mapsto be$, $c\mapsto he$, $d\mapsto hf$, $e\mapsto ac$, $f\mapsto ad$, $g\mapsto gd$, $h\mapsto gc$	$\begin{pmatrix}0&1&0&0&0&1&0&0\\0&1&0&0&1&0&0&0\\0&0&0&0&1&0&0&1\\0&0&0&0&0&1&0&1\\1&0&1&0&0&0&0&0\\1&0&0&1&0&0&0&0\\0&0&0&1&0&0&1&0\\0&0&1&0&0&0&1&0\end{pmatrix}$	2	$\lambda^4-2\lambda^3-2\lambda^2+4\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^9$	$\frac{1-z}{(1-2z)(1-2z^2)(1+z)}$	$\frac{k}{2^N}$	$k,N\in\mathbb{Z}$	non-Pisot	aperiodic
Luck Ternary #1	$0\mapsto 01$, $1\mapsto 02$, $2\mapsto 012$	$\begin{pmatrix}1&1&0\\1&0&1\\1&1&1\end{pmatrix}$	$a\mapsto ac$, $b\mapsto ac$, $c\mapsto be$, $d\mapsto be$, $e\mapsto ade$	$\begin{pmatrix}1&0&1&0&0\\1&0&1&0&0\\0&1&0&0&1\\0&1&0&0&1\\1&0&0&1&1\end{pmatrix}$	2.247	$\lambda^3-2\lambda^2-\lambda+1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{1-2z-z^2+z^3}$	$p+q\cdot\lambda_1+r\cdot\lambda_1^2$	$p,q,r\in\mathbb{Z}$	Pisot	quasiperiodic
Luck Ternary #2	$0\mapsto 2$, $1\mapsto 0$, $2\mapsto 12$	$\begin{pmatrix}0&0&1\\1&0&0\\0&1&1\end{pmatrix}$	$a\mapsto c$, $b\mapsto a$, $c\mapsto be$, $d\mapsto bc$, $e\mapsto bd$	$\begin{pmatrix}0&0&1&0&0\\1&0&0&0&0\\0&1&0&0&1\\0&1&1&0&0\\0&1&0&1&0\end{pmatrix}$	1.4656	$\lambda^3-\lambda^2-1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{1-z-z^3}$	$p+q\cdot\lambda_1+r\cdot\lambda_1^2$	$p,q,r\in\mathbb{Z}$	Pisot	quasiperiodic
Periodic 1-2	$0\mapsto 011$, $1\mapsto 011$	$\begin{pmatrix}1&2\\1&2\end{pmatrix}$	$a\mapsto acb$, $b\mapsto acb$, $c\mapsto acb$	$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$	3	$\lambda^2-3\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^1$	$\frac{1-z}{1-3z}$	$\frac{k}{3}$	$k\in\mathbb{Z}$	Pisot	periodic
Periodic 1-3	$0\mapsto 0111$, $1\mapsto 0111$	$\begin{pmatrix}1&3\\1&3\end{pmatrix}$	$a\mapsto accb$, $b\mapsto accb$, $c\mapsto accb$	$\begin{pmatrix}1&1&2\\1&1&2\\1&1&2\end{pmatrix}$	4	$\lambda^2-4\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^1$	$\frac{1-z}{1-4z}$	$\frac{k}{4}$	$k\in\mathbb{Z}$	Pisot	periodic
Periodic 1-4	$0\mapsto 01111$, $1\mapsto 01111$	$\begin{pmatrix}1&4\\1&4\end{pmatrix}$	$a\mapsto acccb$, $b\mapsto acccb$, $c\mapsto acccb$	$\begin{pmatrix}1&1&3\\1&1&3\\1&1&3\end{pmatrix}$	5	$\lambda^2-5\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^1$	$\frac{1-z}{1-5z}$	$\frac{k}{5}$	$k\in\mathbb{Z}$	Pisot	periodic
Periodic 2-3	$0\mapsto 00111$, $1\mapsto 00111$	$\begin{pmatrix}2&3\\2&3\end{pmatrix}$	$a\mapsto abddc$, $b\mapsto abddc$, $c\mapsto abddc$, $d\mapsto abddc$	$\begin{pmatrix}1&1&1&2\\1&1&1&2\\1&1&1&2\\1&1&1&2\end{pmatrix}$	5	$\lambda^2-5\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^1$	$\frac{1-z}{1-5z}$	$\frac{k}{5}$	$k\in\mathbb{Z}$	Pisot	periodic
Periodic 2-5	$0\mapsto 0011111$, $1\mapsto 0011111$	$\begin{pmatrix}2&5\\2&5\end{pmatrix}$	$a\mapsto abddddc$, $b\mapsto abddddc$, $c\mapsto abddddc$, $d\mapsto abddddc$	$\begin{pmatrix}1&1&1&4\\1&1&1&4\\1&1&1&4\\1&1&1&4\end{pmatrix}$	7	$\lambda^2-7\lambda=0$	$\mathbb{Z}^1$	$\mathbb{Z}^1$	$\frac{1-z}{1-7z}$	$\frac{k}{7}$	$k\in\mathbb{Z}$	Pisot	periodic
Golden Mean	$0\mapsto 10$, $1\mapsto 0$	$\begin{pmatrix}1&1\\1&0\end{pmatrix}$	$a\mapsto cb$, $b\mapsto ca$, $c\mapsto b$	$\begin{pmatrix}0&1&1\\1&0&1\\0&1&0\end{pmatrix}$	τ	$\lambda^2-\lambda-1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\frac{1-z}{1-z-z^2}$	$p+q\cdot\lambda_1$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Silver Mean	$0\mapsto 100$, $1\mapsto 0$	$\begin{pmatrix}2&1\\1&0\end{pmatrix}$	$a\mapsto cab$, $b\mapsto caa$, $c\mapsto b$	$\begin{pmatrix}1&1&1\\2&0&1\\0&1&0\end{pmatrix}$	$\sqrt{2}+1$	$\lambda^2-2\lambda-1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\frac{1-z}{1-2z-z^2}$	$p+q\cdot\lambda_1$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Copper Mean	$0\mapsto 1000$, $1\mapsto 0$	$\begin{pmatrix}3&1\\1&0\end{pmatrix}$	$a\mapsto caab$, $b\mapsto caaa$, $c\mapsto b$	$\begin{pmatrix}2&1&1\\3&0&1\\0&1&0\end{pmatrix}$	$\frac{\sqrt{13}}{2}+\frac{3}{2}$	$\lambda^2-3\lambda-1=0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\frac{1-z}{1-3z-z^2}$	$p+q\cdot\lambda_1$	$p,q\in\mathbb{Z}$	Pisot	quasiperiodic
Marginal	$0\mapsto 001$, $1\mapsto 011$	$\begin{pmatrix}2&1\\1&2\end{pmatrix}$	$a\mapsto abc$, $b\mapsto abc$, $c\mapsto bdc$, $d\mapsto bdc$	$\begin{pmatrix}1&1&1&0\\1&1&1&0\\0&1&1&1\\0&1&1&1\end{pmatrix}$	3	$\lambda^2-4\lambda+3=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1}{1-3z}$	$\frac{k}{2\cdot 3^N}$	$k,N\in\mathbb{Z}$	non-Pisot	limit-quasiperiodic
Binary non-Pisot	$0\mapsto 01$, $1\mapsto 000$	$\begin{pmatrix}1&1\\3&0\end{pmatrix}$	$a\mapsto bc$, $b\mapsto bc$, $c\mapsto aaa$	$\begin{pmatrix}0&1&1\\0&1&1\\3&0&0\end{pmatrix}$	$\frac{\sqrt{13}}{2}+\frac{1}{2}$	$\lambda^2-\lambda-3=0$	$\mathbb{Z}^1$	$\mathbb{Z}^3$	$\frac{1-z}{1-z-3z^2}$	$\frac{p+q\cdot\lambda_1}{3^N}$	$p,q,N\in\mathbb{Z}$	non-Pisot	limit-quasiperiodic
Ternary non-Pisot	$0\mapsto 2$, $1\mapsto 0$, $2\mapsto 101$	$\begin{pmatrix}0&0&1\\1&0&0\\1&2&0\end{pmatrix}$	$a\mapsto e$, $b\mapsto f$, $c\mapsto b$, $d\mapsto a$, $e\mapsto cad$, $f\mapsto cac$	$\begin{pmatrix}0&0&0&0&1&0\\0&0&0&0&0&1\\0&1&0&0&0&0\\1&0&0&0&0&0\\1&0&1&1&0&0\\1&0&2&0&0&0\end{pmatrix}$	1.5214	$\lambda^3-\lambda-2=0$	$\mathbb{Z}^1$	$\mathbb{Z}^6$	$\frac{1}{1-z^2-2z^3}$	$\frac{p+q\cdot\lambda_1+r\cdot\lambda_1^2}{2^N}$	$p,q,r,N\in\mathbb{Z}$	non-Pisot	limit-quasiperiodic

Spectral Properties of Tilings

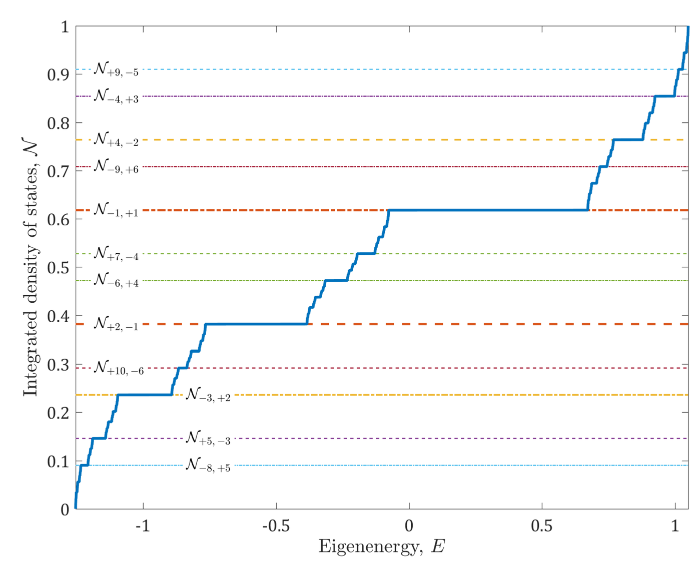
Consider a $1D$ discrete tight-binding equation

$$-(\psi_{k+1}+\psi_{k-1})+V_k\psi_k=2E\psi_k$$

The gaps in the integrated density of states are given by [7]

$$\mathcal{N}_{m,N}=\frac{1}{c}\frac{m}{\lambda_1^N}\pmod 1,\qquad m,N\in\mathbb{Z}.$$

- λ_1 is the first eigenvalue of M.
- c is the gcd of the first eigenvectors of $M_1=M\,[v_1=(e_1\ e_2\ \cdots)]$, the collared $M_2\,[v_2=(f_1\ f_2\ \cdots)]$, and λ_1.



Finite Structures

Scattering Matrix

Inspect a system modulated in quasiperiodic manner. Solve the wave equation

$$-\frac{d^2\psi}{dx^2}-k_0^2v(x)\psi(x)=k_0^2\psi(x)$$

with scattering boundary conditions.

so that $\left(\begin{smallmatrix}\overleftarrow{o}\\ \overrightarrow{o}\end{smallmatrix}\right)=\left(\begin{smallmatrix}\overrightarrow{r}(k)&t(k)\\ t(k)&\overleftarrow{r}(k)\end{smallmatrix}\right)\left(\begin{smallmatrix}\overrightarrow{\imath}\\ \overleftarrow{\imath}\end{smallmatrix}\right)\equiv\mathcal{S}\left(\begin{smallmatrix}\overrightarrow{\imath}\\ \overleftarrow{\imath}\end{smallmatrix}\right)$.
The $\mathcal{S}$-matrix is unitary and can be diagonalized to

$$\mathcal{S}\mapsto\begin{pmatrix}e^{i\phi_1}&0\\0&e^{i\phi_2}\end{pmatrix}$$

so that $\det\mathcal{S}=e^{2i\delta(k)}$ with $\delta(k)=\frac{1}{2}(\phi_1(k)+\phi_2(k))$.
Using the Krein-Schwinger formula, find the density of states,

$$\varrho(k)-\varrho_0(k)=\frac{1}{2\pi}\operatorname{Im}\frac{d}{dk}\ln\det\mathcal{S}(k)$$

the integrated density of states reads

$$\mathcal{N}(k)-\mathcal{N}_0(k)=\frac{1}{\pi}\delta(k).$$

A the infinite sequence w^{∞} is characterized by F_N words $w_i\subset w^{\infty}$ of size F_N. Scattering matrices are related by a unitary transformation [8, 9]

$$S(w_j)=U^{\dagger}(i,j)\,S(w_i)\,U(i,j)$$

with $U(i,j)=U(\phi_i-\phi_j)$ and a gauge field ϕ.

Characteristic Function

Take a characteristic function

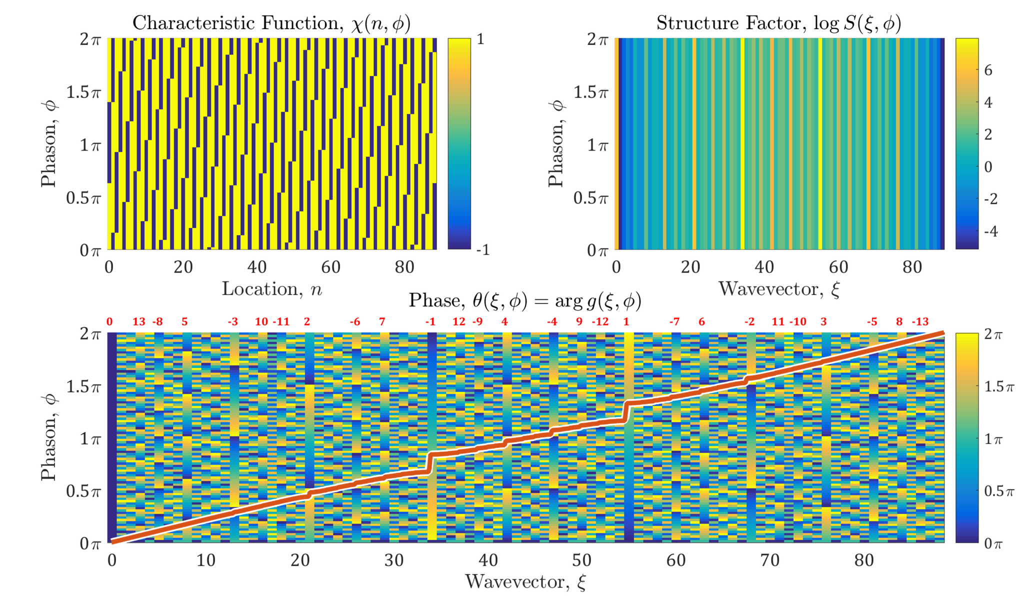
$$\chi(n,\phi)=\operatorname{sign}\left[\cos\left(2\pi n\lambda_1^{-1}+\phi\right)-\cos\left(\pi\lambda_1^{-1}\right)\right]$$

with $n=0\ldots F_N-1$ and $[0,2\pi]\ni\phi\to\phi_\ell=\frac{2\pi}{F_N}\ell$.
Take its discrete Fourier transform w.r.t. n [1],

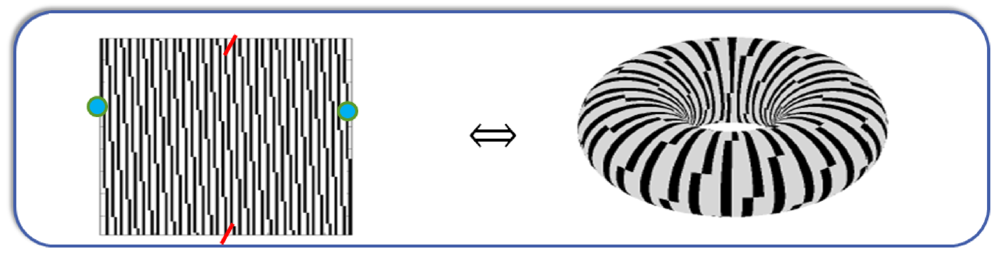
$$g(\xi,\phi)=\sum_{n=0}^{F_N-1}\omega^{-\xi n}\chi(n,\phi),\qquad \omega=e^{\frac{2\pi i}{F_N}}.$$

Calculate the structure factor $S(\xi,\phi)=|g(\xi,\phi)|^2$ and phase $\theta(\xi,\phi)=\arg g(\xi,\pi)$.
The winding number at ξ_0 reads

$$W_{\xi_0}=\frac{1}{2\pi}\int_0^{2\pi}\frac{\partial\theta(\xi=\xi_0,\phi)}{\partial\phi}\,d\phi$$



The characteristic function $\chi(n,\phi)$ admits periodic (toroidal) boundary conditions



Algebraic Structures

Let [1]

$$s_0(n)=\chi(n,0)=\operatorname{sign}\left[\cos(2\pi n/\lambda_1)-\cos(\pi/\lambda_1)\right].$$

Let $\mathcal{T}[s_0(n)]=s_0(n+1)$ be a translation operator. Define

$$\Sigma_0=\begin{pmatrix}s_0\\ \mathcal{T}[s_0]\\ \cdots\\ \mathcal{T}^{F_N-1}[s_0]\end{pmatrix}\implies\Sigma_0(n,\ell)=\mathcal{T}^{\ell}[s_0(n)]$$

Consider row permuted Σ_r

$$\Sigma_r=U_r\Sigma_0,\quad U_r(m,\ell)=\delta_{\ell,\,m\,r\,F_{N-1}}\implies\Sigma_r(n,\ell)=\mathcal{T}^{m(\ell,r)}[s_0(n)].$$

here, U_r is a unitary operator satisfying an algebraic group structure

$$U_rU_s=U_sU_r=U_{r\,s\,F_{N-2}}\pmod{F_N}.$$

The group structure is isomorphic to $\mathbb{Z}/F_N\mathbb{Z}$.

Lemma. For $\phi_\ell=\frac{2\pi}{F_N}\ell$ with $n,\ell=0\ldots F_N-1$ one has $\chi(n,\phi_\ell)=\Sigma_1(n,\ell)$.

The discrete Fourier transform of Σ_1 reads

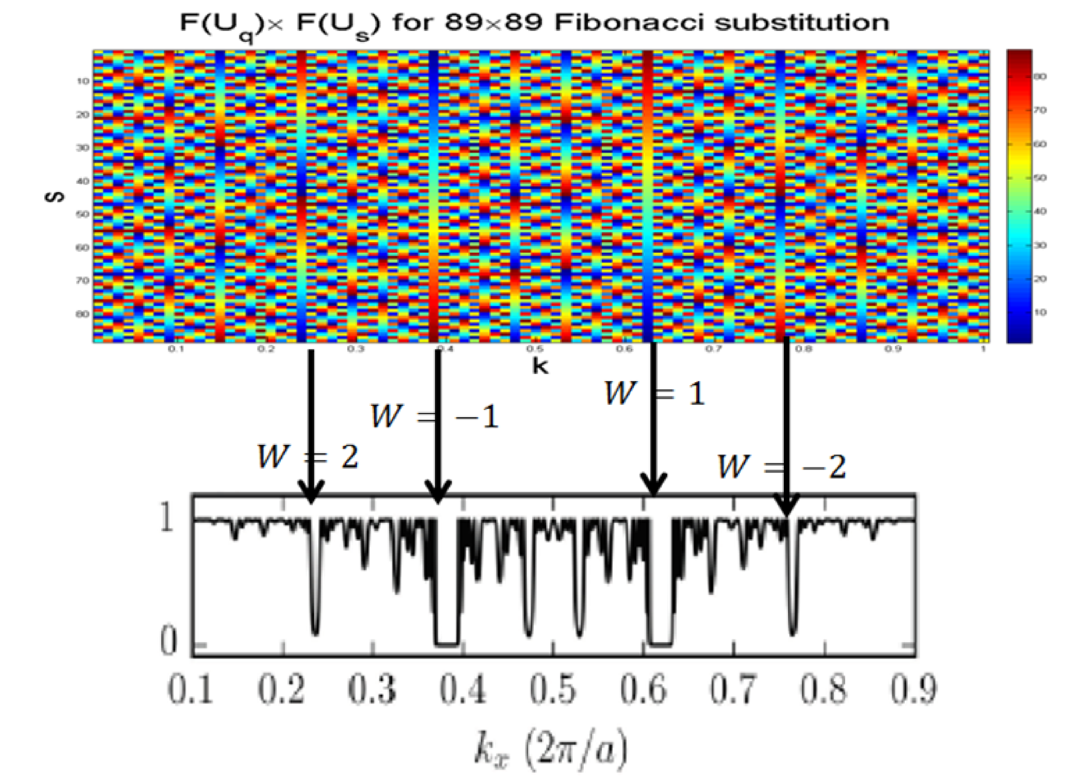
$$G(\xi,\ell)\equiv\sum_{n=0}^{F_N-1}\omega^{-\xi n}\,\Sigma_1(n,\ell)=\omega^{m(\ell)\xi}\,\varsigma_0(\xi).$$

The structure factor $S(\xi,\phi)=|\varsigma_0(\xi)|^2$ is ϕ-independent. The phase reads

$$\Theta(\xi,\ell)\equiv\arg\omega^{m(\ell)\xi}=\frac{2\pi}{F_N}\ell\,\frac{\xi}{F_{N-1}}.$$

Corollary. For any $\xi_q=qF_{N-1}$ one has the (discrete) winding

$$\Theta(\xi_q)=\frac{2\pi}{F_N}\ell\,q\implies W_{\xi_q}=q.$$



Conclusions

- One dimensional quasiperiodic tilings show correspondence between the diffraction pattern and the spectrum.
- Constructing Bratteli diagrams reveals the topological nature of the tiling via the cohomology group $H^1(G)$ and the $\zeta(z)$ function.
- In finite systems, the phason relates different structures of the same size via the scattering matrix.
- The phason determines a winding number in finite systems.
- An algebraic structure is induced by the action of the characteristic function, and defines the winding numbers.

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This work was supported by the Israel Science Foundation Grant No. 924/09.

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