Efimov physics–fractal tilings–quantum Einstein gravity and their relations

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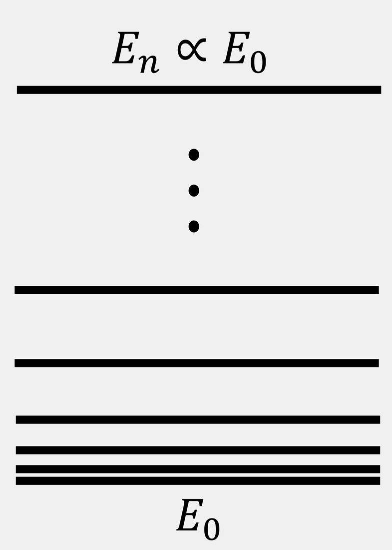
I. Abstract

We consider limit cycle solutions in Efimov physics and show it is possible to recover these, using specific choices of tilings. This new and unexpected discovery provides an opening to study semi-classical solutions of quantum Einstein gravity (QEG).
Can you forsee the relation between these three remote fields?

II. Efimov Physics

Efimov effect [4]: Three quantum particles interacting via attractive short-range interactions enjoy an infinite geometric series of three-body bound states accumulating at the ground state energy E_0, when the interactions are unable to support pair-wise bound states (at the limit of low energy scattering, infinite scattering length).

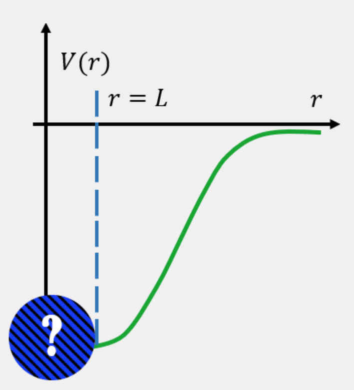


The effect arises when the range of interaction vanishes compared to the scattering length, generically $E \simeq 0$ (S-wave with $\ell = 0$). Such systems lack a characteristic length scale, are independent of direction and precise details of the interactions, and thus belong to the ***universal*** class of *Efimov physics*. With a large scattering length, the effective dynamics can be mapped to a radial Schrödinger equation with a universal potential $-\zeta/r^2$, $\zeta = s_0^2 + 1/4$, $|s_0| \approx 1.006$, whose d-dimensional generalization is

$$\left(\frac{d^2}{dr^2} + \frac{d-1}{r}\frac{d}{dr} + \frac{\xi}{r^2} + \frac{2\mu E}{\hbar^2}\right)\psi(\mathbf{r}) = 0,$$

where $\xi = 2\mu\zeta$, μ is the reduced mass and $\mathbf{r}$ is a function of the separation between the particles.
A quantum phase transition [1]: When does the $-\xi/r^2$ Schrödinger equation have Efimov bound states solutions? Since the equation is scale-free, rescaling a bound solution $\{E, \psi(\mathbf{r}, E)\} \to \{E/\lambda^2, \psi(\lambda\mathbf{r}, E/\lambda^2)\}$ still solves the equation. Therefore, there is no ground state.

We remedy this issue by regularizing with some *arbitrary* short range potential, imposing the continuity condition



$$L\frac{\psi'(L)}{\psi(L)} \equiv g(L),$$

breaking the continuous scale invariance (CSI) of the equation at the new characteristic length scale $r = L$. One can derive from the Schrödinger equation and continuity condition a set of renormalization group (RG) flow equations,

$$L\frac{d\xi}{dL} = 0, \quad L\frac{dg}{dL} = (2-d)g - g^2 - \xi.$$

The flow has two fixed points,

$$g_\pm = -\sqrt{\xi_c} \pm \sqrt{\xi_c - \xi},$$

with $\xi_c = (d-2)^2/4$. For $\xi \le \xi_c$ these are real fixed points. At $L \to \infty$ the stable fixed point g_+ restores the CSI phase of the system with a single ground state $E = 0$. However for $\xi > \xi_c$ the fixed point are complex and the flow has a **limit cycle** [2],

$$g(L) = g\left(e^{\frac{n\pi}{\sqrt{\xi - \xi_{cr}}}}L\right),$$

giving the discrete scale invariant (DSI) Efimov bound states,

$$E_n = E_0 e^{-\frac{n\pi}{\sqrt{\xi - \xi_{cr}}}}.$$

References

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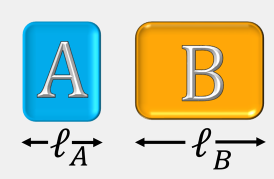
III. Substitutions and Fractals

We know periodic chrystals are assembled from repeated fundamental elements. We can think of these basic pieces as "letters" in an infinite "word". What happens when the distribution of letters is not periodic, but deterministic with a-periodic rules?

Substitutions

A substitution tiling is defined by [3]

an alphabet (for simplicity, 2 letters):



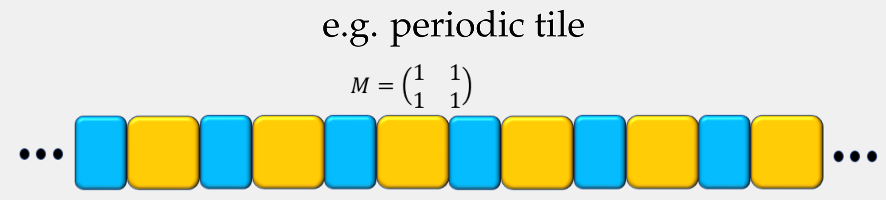
and a substitution rule:

$$A \mapsto A^\alpha B^\beta$$
$$B \mapsto A^\gamma B^\delta$$
$$\Rightarrow M \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \alpha, \beta, \gamma, \delta \in \mathbb{N}$$

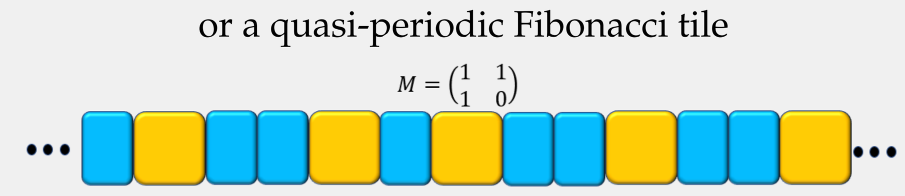
The sequence inflates by replacing the letters repeatedly. Here A is replaced by α times A and β times B.

M determine the evolution of the densities of A, B in the tiling. After many iterations, the substitution process generates an infinite periodic/quasi-periodic/fractal sequence,

e.g. periodic tile



or a quasi-periodic Fibonacci tile



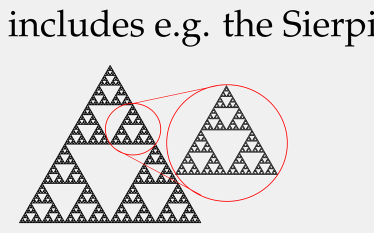
A *generalized* substitution [5], with $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, replaces the letters by tiling "potentials", $A, B \to V_A, V_B$ and fills the space by a pattern of potentials, on which a "test particle" accumulates work.

Fractals

Self similar fractal functions are defined by

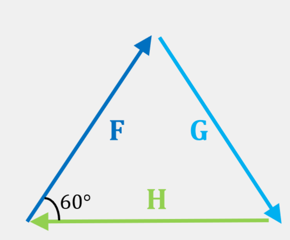
$$\boxed{f(a^n x) = b^n f(x) \quad \text{for } \textbf{fixed } a, b}$$

This definition includes e.g. the Sierpisnki gasket

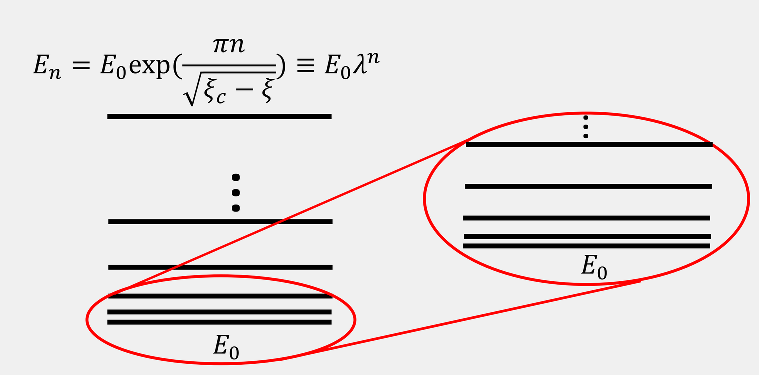


generated by the substitution [8]:

$F \mapsto FGFHF$
$G \mapsto GG$
$H \mapsto HH$



Efimov bound states are a self similar fractal as well:



A RG flow determines the evolution of the couplings similar to how the substitution matrix M determines the evolutions of the letter densities in the tiling.

Could a substitution process generate the Efimov spectrum?
Yes!

V. Quantum Einstein Gravity

Back to our initial question.
QEG as candidate for a quantum field theory of gravity is unique in two aspects:

1. The quantum action is supressed at low energy scales by a "mass term"

$$\Delta S_k \sim h\mathcal{R}_k h,$$

where h is the metric, $\mathcal{R}_{k^2 \gg p^2} \sim k^2$ and $\mathcal{R}_{k^2 \ll p^2} \sim 0$, and p is the energy scale we probe. This "weight" narrows the band of momenta contributing to the path integral.

2. The use of background gauge, which splits

$$g = \overbrace{\bar{g}}^{\text{background}} + \overbrace{h}^{\text{dynamical}},$$

ensures not only gauge covariance (in this case, diffeomorphism invariance) but also avoids pre-supposing a metric – predictions (e.g. causality) are physically justified *results*. All observables must be independent of the choice of $\bar{g}$.

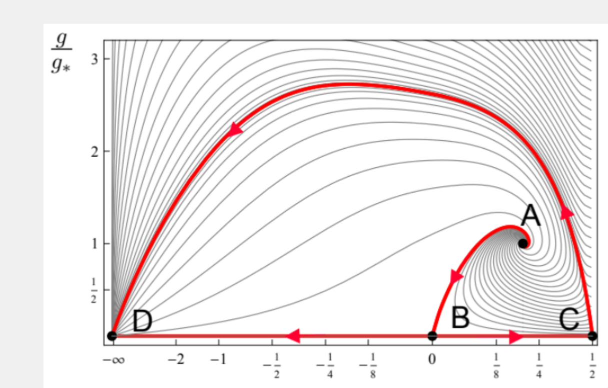
The resulting metric solutions $\langle h_k \rangle$ are **scale dependent**. Within QEG one must choose a truncation for the action, e.g. the Einstein-Hilbert (EH) truncation

$$S = \frac{1}{16\pi G_k}\int d^d x \sqrt{\bar{g}}\{-R + 2\Lambda_k\},$$

where R is the Ricci curvature, and G_k, Λ_k are the scale dependent gravitational and cosmological constants, respectively. If one chooses the background metric to be the d-sphere, $R = d(d-1)/r^2$ and the equations of motion are [5]

$$\left(D^2 - \frac{d(d-1)^2(d-4)}{r^2} - 2\Lambda_k\right)\mathrm{tr}(h) = 0,$$

where D^2 is the covariant Laplacian. Just like in Efimov physics, the operator acting on tr$h$ scales like $1/r^2$! Litim and Satz got this behavior numerically in [6]:



$\Rightarrow$ QEG in the EH truncation with a spherical background metric (and perhaps a wider variety) belongs in the *universality class* of Efimov physics!

IV. The Mapping

The Mapping

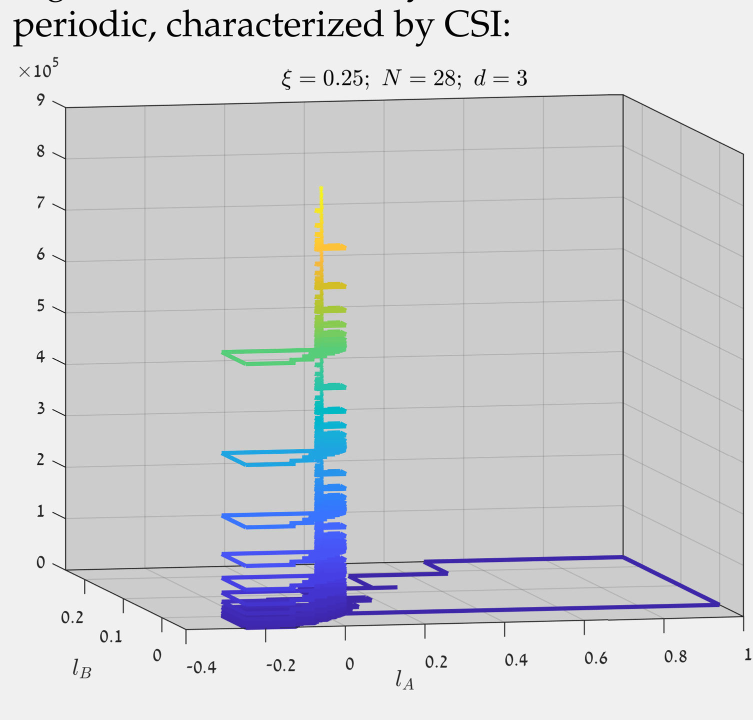
To extract substitution steps from the Efimov RG flow, we translate the coupling to tilings A, B such that $g(\ln L) \equiv g_k = \ell_k^A/\ell_k^B$. Near a fixed point, the steps are small, $g_k \approx g_{k+1}$. We define the mapping and find the rule [5]

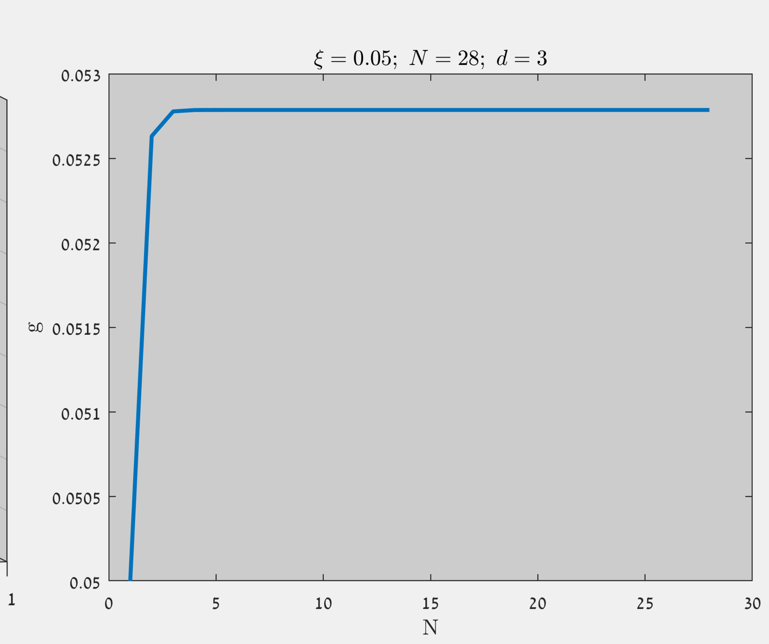
$$\boxed{\begin{pmatrix} \ell_k^A \\ \ell_k^B \end{pmatrix} \mapsto \begin{pmatrix} \ell_k^A \\ \ell_k^B \end{pmatrix}, \quad \begin{pmatrix} \ell_{k'}^A \\ \ell_{k'}^B \end{pmatrix} = \underbrace{\begin{pmatrix} 2-d & -\xi \\ 1 & 0 \end{pmatrix}}_{M}{}^{k'-k} \begin{pmatrix} \ell_k^A \\ \ell_k^B \end{pmatrix}}$$

The plots visualize mapped sequences alongside the flow of the coupling g_k. From beginning (bottom) to end (top), every A with length ℓ_k^A in the sequence is a line drawn in direction sign $(\ell_k^A)\,\hat{x}$ and length $|\ell_k^A|$, and the same for B on $\hat{y}$.

Fixed points of flow $\xi \le \xi_c$

Eigenvectors of M satisfy $g_{k'} = g_k, \quad \forall k, k' \Rightarrow$ a sequence mapped from a fixed point is fully periodic, characterized by CSI:





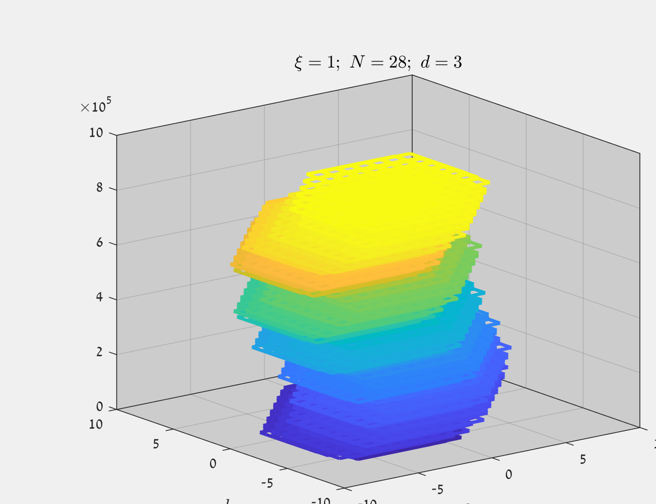
Fixed points $\Longleftrightarrow$ *Real* eigenvalues

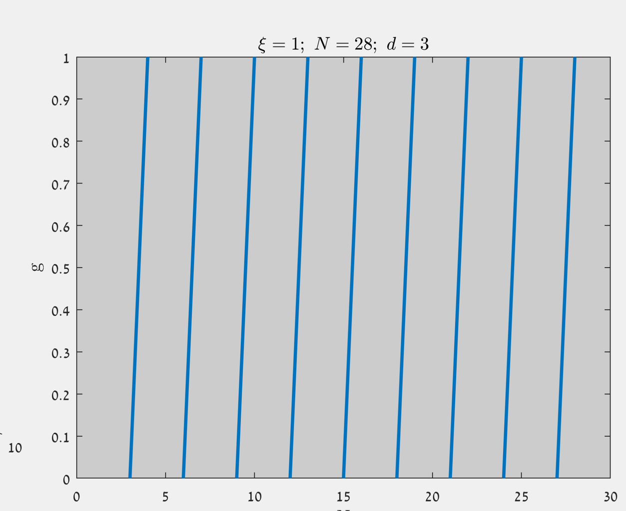
Limit cycle $\xi > \xi_c$

Complex conjugate eigenvalues of M imply DSI of g_k:

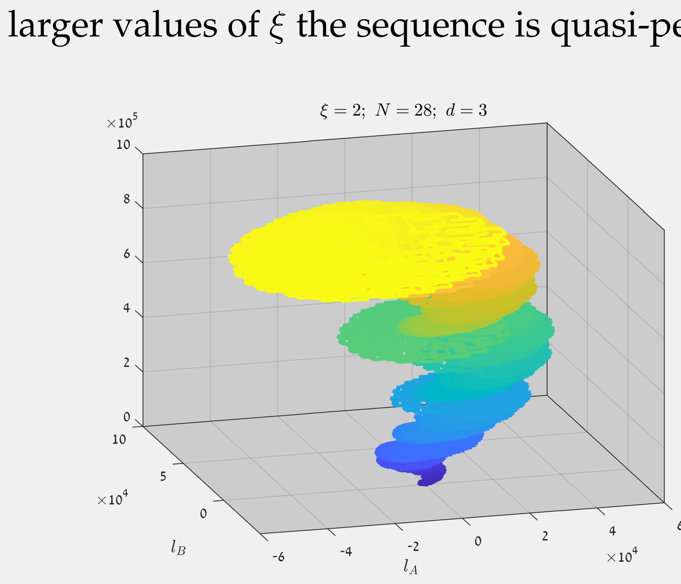
$$g_k = g_{k'} \iff k' - k = \frac{\sqrt{\xi_c}\pi n}{\sqrt{\xi - \xi_c}}, \quad n = 1, 2 \ldots$$

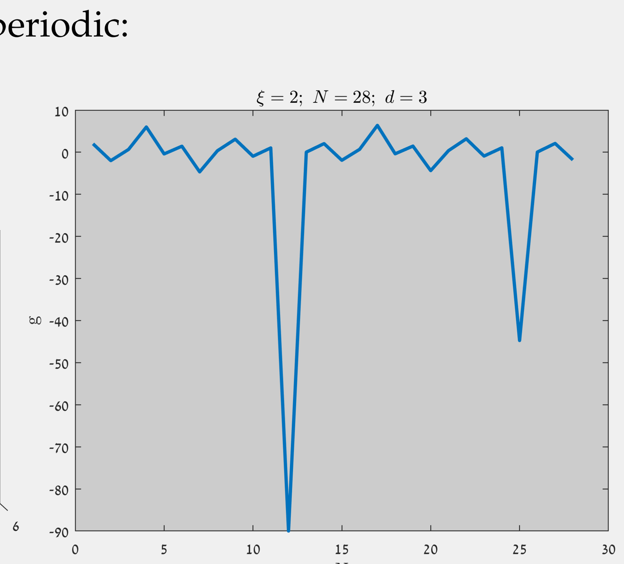
$\Rightarrow$ A sequence mapped from a limit cycle is a **fractal**:





For larger values of ξ the sequence is quasi-periodic:





Limit cycle $\Longleftrightarrow$ Complex *conjugate* eigenvalues