Scale anomaly of a Lifshitz scalar: a universal quantum phase transition to discrete scale invariance

Daniel K. Brattan[1], Omrie Ovdat[2] and Eric Akkermans[3]

[1]ICTS, USTC, 96 Jinzhai Road, Hefei, Anhui, 230026 PRC.

[2,3]Technion, Haifa 3200003, Israel.

[1]danny.brattan@gmail.com, [2]somrie@campus.technion.ac.il, [3]eric@physics.technion.ac.il.

Abstract

We demonstrate the existence of a universal transition from a continuous scale invariant phase to a discrete scale invariant phase for a class of one-dimensional quantum systems with anisotropic scaling symmetry between space and time. These systems describe a Lifshitz scalar interacting with a background potential. The transition occurs at a critical coupling λ_c corresponding to a strongly attractive potential.

1. Introduction

Classically scale invariant [1], the Hamiltonians

$$\hat{H}_S = p^2/2m - \lambda/r^2 \ , \ \ \hat{H}_D = \gamma^0\gamma^j p_j - \lambda/r \ , \tag{1}$$

exhibit an abrupt transition in the spectrum at a critical $\lambda = \lambda_c$. For $\lambda < \lambda_c$, the spectrum contains no bound states close to $E = 0$, however, as λ goes above λ_c, an infinite series of bound states appears. Moreover these states arrange themselves in an unanticipated geometric series accumulating at $E = 0$. The existence and geometric structure of the energy levels do not rely on the details of the potential close to its source and is a signature of residual discrete scale invariance. Thus, these Hamiltonians exhibit a quantum phase transition at λ_c between a continuous scale invariant (CSI) phase and a discrete scale invariant phase (DSI). This transition has been associated with Berezinskii-Kosterlitz-Thouless (BKT) transitions [2].

These different Hamiltonians (1) share a similar property - the power law form of the corresponding potential matches the order of the kinetic term. We demonstrated [3] that this property is a sufficient ingredient for the existence of the CSI to DSI transition by considering a generalised class of one dimensional Hamiltonians

$$\hat{H}_N = (p^2)^N - \frac{\lambda_N}{x^{2N}} \ , \tag{2}$$

where N is an integer and λ_N a real coupling.

Corresponding to (2) is the action of a complex scalar field in $(1+1)$-dimensions:

$$\int dt \int_{x=x_0}^{\infty} dx \, \frac{i}{2}\left(\Psi^*\partial_t\Psi - \text{c.c.}\right) + \left|\partial_x^N \Psi\right|^2 - \frac{\lambda_N}{x^{2N}}|\Psi|^2 \ ,$$

where c.c. indicates the complex conjugate. This field theory has manifest Lifshitz scaling symmetry, $(t, x) \mapsto (\Lambda^{2N} t, \Lambda x)$ when $x_0 \to 0$. The scaling exponent of Λ^2 is called the "dynamical exponent" and has value N in this case.

The classical scaling symmetry of (2) implies that if there is one negative energy bound state then there is an unbounded continuum. Thus, the Hamiltonian is non-self-adjoint. To remedy this problem, the operator can be made self-adjoint by applying boundary conditions on the elements of the Hilbert space through the procedure of self-adjoint extension. Alternatively, a suitable cutoff regularisation at $x_0 > 0$ can be chosen to ensure self-adjointness as well as bound the spectrum from below by an intrinsic scale leaving some approximate DSI at low energies. While both these approaches are explored in our paper [3], here we shall discuss only the cut-off approach.

2. Example: $N = 1$

The most general boundary condition consistent with the Hamiltonian:

$$\hat{H}_1 = -d_x^2 + \lambda_1/x^2 \tag{3}$$

being self-adjoint on the space $[x_0, \infty), x_0 > 0$ is

$$\Psi(x_0) + i x_0 \Psi'(x_0) = e^{i\theta}\left(\Psi(x_0) - i x_0 \Psi'(x_0)\right) , \tag{4}$$

where θ is a free parameter. It is not hard to convince oneself using the time dependent Schrödinger equation that choosing this boundary condition sets the probability current at $x = x_0$ equal to zero. Additionally these boundary conditions ensure that $\hat{H}_1$ is symmetric by setting the matrix element $\hat{H}_1 - \hat{H}_1^\dagger$ to zero. This is generally non-zero due to boundary terms at $x = x_0$.

The boundary condition (4) can be rewritten as

$$\frac{x_0 \Psi'(x_0)}{\Psi(x_0)} = \tan\left(\frac{\theta}{2}\right) \qquad \theta \neq \pm\pi \ . \tag{5}$$

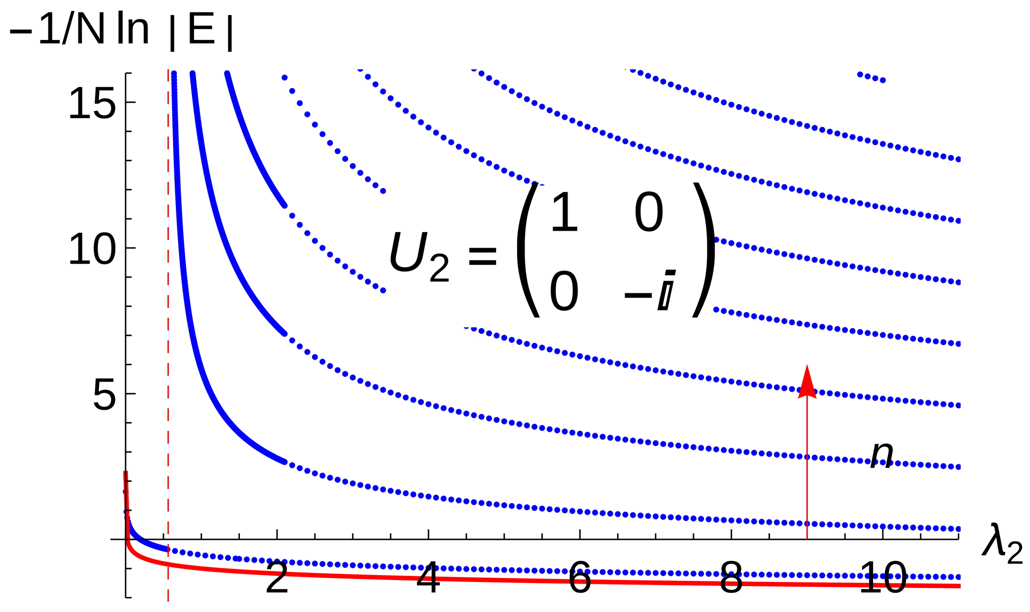

Figure 1: *Flow of the bound state energy E against λ_2 for a cut-off position $x_0 = e^{-1}$ and boundary condition displayed. For $\lambda_2 < \lambda_{2,c} = 9/16$ there is an isolated bound state. The solid red line indicates an analytic lower bound on the negative energies. The dotted red line at $\lambda_{2,c}$ indicates where the first pair of complex roots appears, above which we can see the geometric tower abruptly appearing.*

The cases of $\pm\pi$ correspond to Dirichlet and Neumann conditions for the wavefunction at the cut-off and can be thought of as limits. Taking small energies, $\epsilon = |E|^{1/2} \ll x_0^{-1}$, is equivalent to taking the cut-off to zero where for $N = 1$ the wavefunction for $\lambda_1 > \lambda_{1,c} = 1/4$ has the form:

$$\Psi(x_0) = \tilde{A}\left(\frac{\epsilon x_0}{2}\right)^{\frac{1}{2}} |\Gamma(-i\nu_1)|^{\frac{1}{2}} \cos\left(\nu_1 \ln\left(\frac{\epsilon x_0}{2}\right) + \frac{\phi_1}{2}\right) + \mathcal{O}^{\frac{3}{2}}(x_0) \ , \tag{6}$$

$$e^{i\phi_1} = \frac{\Gamma(-i\nu_1)}{\Gamma(i\nu_1)} \ , \qquad \nu_1 = \sqrt{\lambda_1 - \frac{1}{4}} \ , \tag{7}$$

with $\tilde{A}$ some normalisation constant.

Substituting (6) into (5) we find

$$\tan\left(\frac{\theta}{2}\right) = \sqrt{\lambda_1 - \frac{1}{2}} \frac{\cos\left(\nu_1 \ln\left(\frac{\epsilon x_0}{2}\right) + \frac{\phi_1 + \alpha}{2}\right)}{\cos\left(\nu_1 \ln\left(\frac{\epsilon x_0}{2}\right) + \frac{\phi_1}{2}\right)} \ , \tag{8}$$

$$e^{i\alpha} = \frac{\frac{1}{2} + i\nu_1}{\frac{1}{2} - i\nu_1} \ . \tag{9}$$

Fixing θ we can numerically solve this equation for some E. Taking this as a reference energy, the symmetry of the right hand side ensures that $E \exp(-2\pi n/\nu_1)$ is also a solution where $n \in \mathbb{N}$ is required for the approximation (6) to apply. Hence, for $N = 1$ the cut-off regularisation gives approximate discrete scale invariance at low energies.

3. General N

For $N > 1$ the matrix element $\hat{H}_N - \hat{H}_N^\dagger$ involves more derivatives of the wavefunction than the $N = 1$ case. However, it can readily be diagonalised into an expression proportional to:

$$\vec{\Phi}^+(x_0)^\dagger \cdot \vec{\Psi}^+(x_0) - \vec{\Phi}^-(x_0)^\dagger \cdot \vec{\Psi}^-(x_0) \tag{10}$$

where

$$x_0^{k-1} d_x^{k-1} \Psi(x_0) = \Psi_k^+(x_0) + \Psi_k^-(x_0) \ ,$$
$$x_0^{2N-k} d_x^{2N-k} \Psi(x_0) = e^{i\pi(k-\frac{1}{2})}\left[\Psi_k^+(x_0) - \Psi_k^-(x_0)\right] \ .$$

The general boundary conditions at $x = x_0$ that make $\hat{H}_N$ self-adjoint are

$$\vec{\Psi}^+(x_0) = U_N \vec{\Psi}^-(x_0) \tag{11}$$

for some arbitrary unitary matrix: U_N.

The general solution to the energy eigenvalue equation for a decaying wavefunction at infinity with boundary conditions given by (11) can be given analytically in terms of generalised hypergeometric functions.

As an illustration of the appearance of the geometric tower at $N > 1$, consider figs. 1 and 2. The former plots ϵ for $N = 2$ against λ_2. It is plain that as soon as $\lambda_2 > 9/16$ (the dotted red line) there is a sudden transition from an isolated bound state to a tower of states. Similarly fig. 2 plots the logarithm of E_n/E_{n+1} for $N = 3$ as a function of λ_3 at low ϵx_0. The result, shown by the blue points in fig. 2, is a good match with π/ν_3 with ν_3 defined by (**??**).

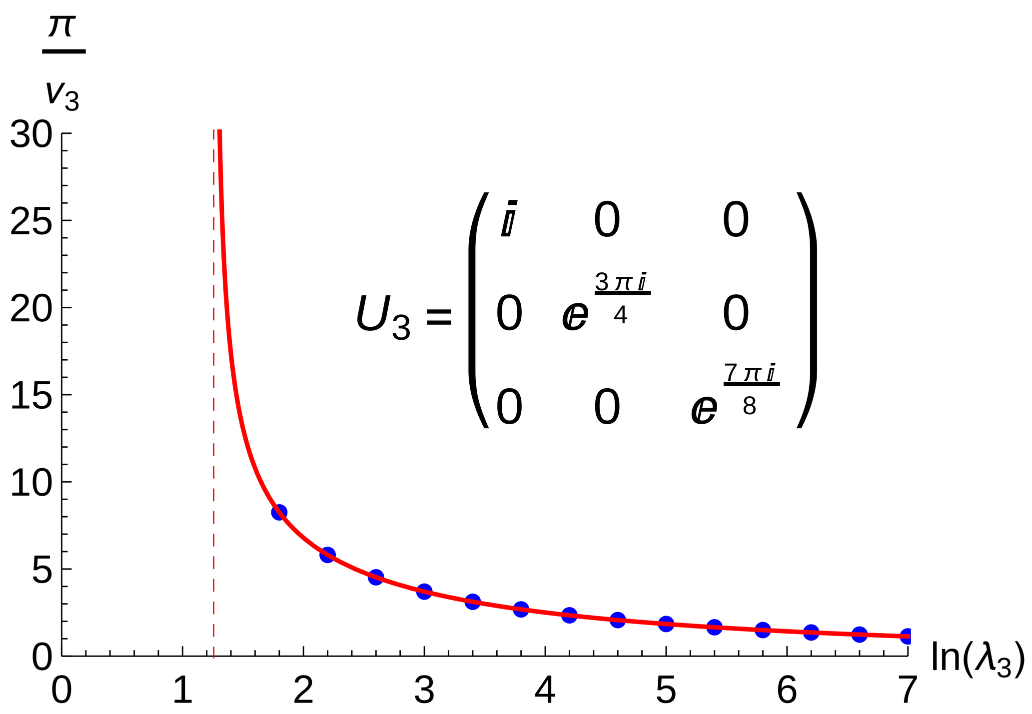

Figure 2: *A plot of $\ln E_n/E_{n+1} \propto \pi/\nu_3$ against $\ln\lambda_3$ for a cut-off position $x_0 = e^{-1}$ and boundary condition displayed. The solid red line indicates our analytic expression for π/ν_3 while the blue dots are calculated by numerically determining the gradient of $\log E_n/E_{n+1}$ against n for several n corresponding to $\epsilon x_0 \ll 1$. The red dotted line indicates the critical λ_3.*

For general N, $\lambda_N > \lambda_{N,c}$ and small enough energies we argued [3] that one always finds DSI with the scaling defined in (**??**) using a small ϵ expansion. Determining the energy eigenstates analytically for arbitrary boundary conditions is made difficult for $N > 1$ due to the presence of multiple distinct complex roots in the small energy expansion. However, one pair makes a contribution to the solution that decays more slowly as we consider small bound state energies than any other and derive an approximation in this limit.

To see this, note that the leading contributions to the general decaying, negative energy, solution at $\epsilon x_0 \ll 1$ have the form

$$\Psi(x) = \sum_{i=1}^{N} \tilde{\phi}_i \left(\frac{x}{x_0}\right)^{\Delta_i} + {G_1}^N \tilde{\phi}_N (\epsilon x_0)^{2i\nu_N} \left(\frac{x}{x_0}\right)^{\Delta_{N+1}}$$

where ϵx_0 only enters the leading term through a phase and all other contributions to O_i from the $\tilde{\phi}_i$ drop out as they come with ϵx_0 to a real positive power. The displayed terms above are the relevant ones at low energies for solving (11). Moreover these leading terms are invariant under the discrete scaling transformation and thus we have DSI. As a result, applying (11) will necessarily give the energy spectrum (**??**) for $\epsilon x_0 \ll 1$.

We can use our expression (**??**) for ν_N in terms of λ_N to find:

$$E_n = -E_0 e^{-\frac{N\pi n \alpha_N}{\sqrt{\lambda_N - \lambda_{N,c}}}} \left(1 + \mathcal{O}(\lambda_N - \lambda_{N,c})\right) \ , \tag{12}$$

characteristic of the BKT scaling, where all subleading terms vanish for $N = 1$.

With the above considerations we can say that a CSI to DSI transition is a generic feature of our models and universal in that it is independent of the completion of the potential near the origin. Thus, the Hamiltonian (2) need only be effective for the consequences of DSI to be relevant.

Acknowledgements

The work of DB was supported in part by the Israel Science Foundation under grant 504/13 and is currently supported by a key grant from the NSF of China with Grant No: 11235010. This work was also supported by the Israel Science Foundation Grant No. 924/09. DB would like to thank the Technion and University of Haifa at Oranim for their support.

References

[1] Roman W Jackiw. *Diverse topics in theoretical and mathematical physics.* World Scientific, 1995.

[2] David B. Kaplan, Jong-Wan Lee, Dam T. Son, and Mikhail A. Stephanov. Conformality lost. *Phys. Rev. D*, 80:125005, Dec 2009.

[3] Daniel K. Brattan, Omrie Ovdat, and Eric Akkermans. Scale anomaly of a Lifshitz scalar: a universal quantum phase transition to discrete scale invariance. 2017.