INTERPLAY BETWEEN TOPOLOGY AND ENTANGLEMENT IN CONDENSED MATTER

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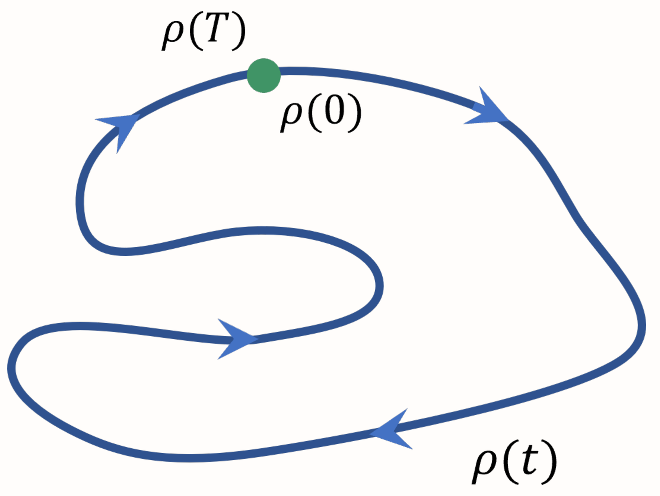
I. ABSTRACT

We study the interplay between topology and quantum entanglement in condensed matter systems. We present results and examples which establish this connection utilizing a toy-model consisting of two qubits, where entanglement is measured by the concurrence and topology shows up as a Berry (geometric) phase. In IX we consider several cases in condensed matter physics where our result might be applicable.

II. TOPOLOGY (VIA GEOMETRIC PHASE)

When a pure state density matrix $\rho(t)$ undergoes cyclic time evolution, the state may gather a gauge invariant phase, the geometric (Aharonov-Anandan [5]) phase.



A cyclic evolution with period T is defined by:

$$\rho(t):\ \rho(0)=\rho(T)$$

$$|\psi(T)\rangle = e^{i(\gamma+\delta(T))}\,|\psi(0)\rangle$$

$\delta(T)$: dynamical phase. γ: gauge invariant geometric phase.

$$\gamma = i\int_0^T dt\left\langle \psi(t)\left|\frac{d}{dt}\right|\psi(t)\right\rangle + \arg\langle\psi(0)|\psi(T)\rangle$$

If we parameterize the Hilbert space $|\psi(P(t))\rangle$ so $P(0)=P(T)$, the second term vanishes and we can integrate over the parameter space instead of time:

$$\gamma = i\oint_P \langle\psi|d|\psi\rangle$$

EXAMPLE: SPIN-1/2 (QUBIT) IN A MAGNETIC FIELD

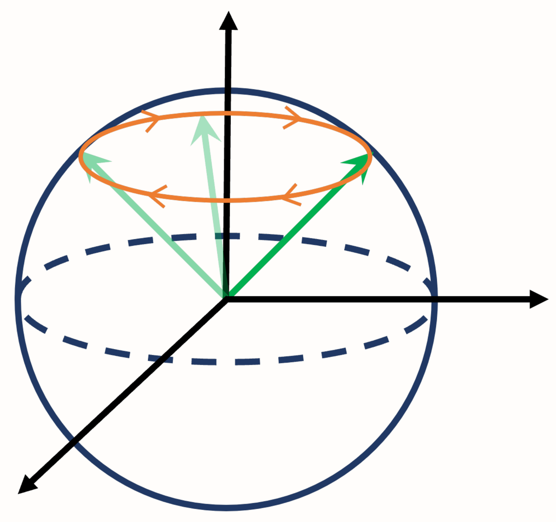
Consider a spin-1/2 particle in a constant magnetic field:

$$|\psi_1(0)\rangle = \cos\frac{\theta}{2}e^{-i\phi/2}|\uparrow\rangle + \sin\frac{\theta}{2}e^{i\phi/2}|\downarrow\rangle$$

$$H_1 = B\hat{z}\cdot\vec{\sigma}$$

$|\psi_1(t)\rangle$ is cyclic with $T_1 = 2\pi/B$:

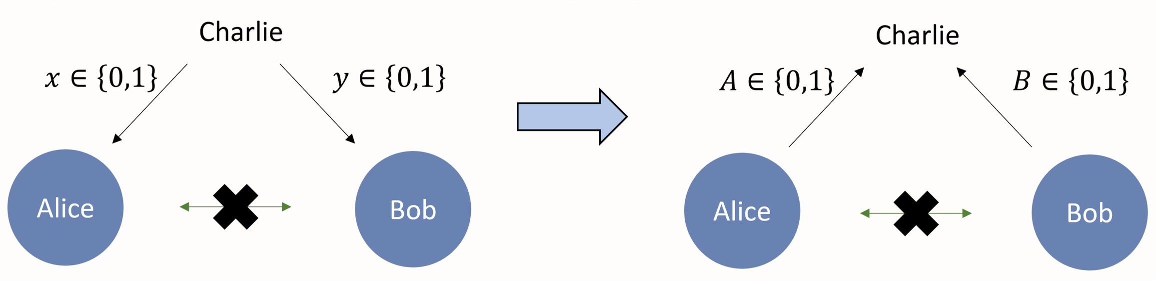
$$\gamma_1 = \pi\cos\theta$$



III. ENTANGLEMENT

XOR GAME

3 players: Alice, Bob and Charlie. Charlie sends a binary messages to Alice and Bob and they reply with binary messages:



Alice and Bob win if $A+B \mod 2 = x\cdot y$.
No $A \leftrightarrow B$ communication.
Classically: no strategy with a win rate of over 75%. However, if A and B share two maximally entangled qubits: win rate 85%.

CONCURRENCE BETWEEN TWO QUBITS

$$C(|\psi\rangle) = |\langle\psi|\Theta|\psi\rangle| \in [0,1]$$

Measure of entanglement in two-qubit systems proposed by Wootters [2]. $\Theta = (\sigma_y\otimes\sigma_y)K$ is the antiunitary time reversal symmetry operator on two spin-1/2 particles, K is the antiunitary complex conjugation operator.

Basic Properties

Independent of measuring protocol.
One-to-one correspondence with entanglement entropy.

$$C(|\uparrow\uparrow\rangle)=0,\quad C\left(\frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle+|\downarrow\downarrow\rangle)\right)=1$$

GENERALIZATIONS

Uhlmann [3] shows that the entanglement in general systems are characterized by one or several "concurrences", each based on a different antiunitary symmetry. Specifically, such a construction has been proposed for systems consisting of two spin-1/2 fermions[4], based on the antiunitary particle-hole symmetry P, instead of Θ.

IV. GEOMETRIC PHASE FOR THE 2-QUBIT MODEL

Using the Schmidt decomposition and fixing the gauge of the two qubits, one can write a general 2-qubit pure state as:

$$|\psi\rangle = \cos\frac{\alpha}{2}e^{-i\beta/2}|\hat{n}_1,\hat{n}_2\rangle + \sin\frac{\alpha}{2}e^{i\beta/2}|-\hat{n}_1,-\hat{n}_2\rangle \quad ; \quad (\hat{n}_i(\theta_i,\varphi_i)\cdot\vec{\sigma})\,|\pm\hat{n}_i(\theta_i,\varphi_i)\rangle = \pm|\pm\hat{n}_i(\theta_i,\varphi_i)\rangle$$

1. Effective Spin Algebra

Renaming $|\Uparrow\rangle \equiv |\hat{n}_1\rangle\otimes|\hat{n}_2\rangle$ and $|\Downarrow\rangle \equiv |-\hat{n}_1\rangle\otimes|-\hat{n}_2\rangle$:

$$\tilde{\Sigma}_x = |\Uparrow\rangle\langle\Downarrow| + |\Downarrow\rangle\langle\Uparrow|,\quad \tilde{\Sigma}_y = -i|\Uparrow\rangle\langle\Downarrow| + i|\Downarrow\rangle\langle\Uparrow|$$

$$\tilde{\Sigma}_z = |\Uparrow\rangle\langle\Uparrow| - |\Downarrow\rangle\langle\Downarrow|$$

$$\left[\tilde{\Sigma}_j,\tilde{\Sigma}_k\right] = 2i\varepsilon_{jk\ell}\tilde{\Sigma}_\ell.$$

Observable quantities:

$$\left\langle\vec{\sigma}^{(1),(2)}\right\rangle = \hat{n}_{1,2}\left\langle\tilde{\Sigma}_z\right\rangle^2$$

$$\left\langle\tilde{\Sigma}_x\right\rangle^2 + \left\langle\tilde{\Sigma}_y\right\rangle^2 = \sqrt{1-\left\langle\tilde{\Sigma}_z\right\rangle^2} = \sin\alpha = C$$

2. Rotation Representation

Represent $|\psi\rangle$ with rotation matrices (adapted from [1]):

$$|\psi\rangle = R_3(\alpha,\beta)\,R_1(\theta_1,\varphi_1)\,R_2(\theta_2,\varphi_2)\,|\uparrow\uparrow\rangle$$

$$R_{1,2} = \exp\left(-i\frac{\varphi_{1,2}}{2}\left(\sigma_z^{(1),(2)}\right)\right)\exp\left(-i\frac{\theta_{1,2}}{2}\left(\sigma_y^{(1),(2)}\right)\right)$$

$$R_3 = \exp\left(-i\frac{\beta}{2}\tilde{\Sigma}_z\right)\exp\left(-i\frac{\alpha}{2}\tilde{\Sigma}_y\right)$$

where $\alpha,\ \theta_{1,2}\in[0,\pi]$ and $\beta,\ \phi_{1,2}\in[0,2\pi]$.
Alternatively, defining $|\Uparrow\rangle\equiv|00\rangle$ and $|\Downarrow\rangle\equiv|11\rangle$:

$$|\psi\rangle = R_1(\theta_1,\varphi_1)\,R_2(\theta_2,\varphi_2)\,R_3(\alpha,\beta)\,|\uparrow\uparrow\rangle$$

The resulting phase for a general time evolution is:

$$\gamma = \frac{1}{2}\oint_P\sqrt{1-C^2}\,(d\beta + \cos\theta_1 d\varphi_1 + \cos\theta_2 d\varphi_2)$$

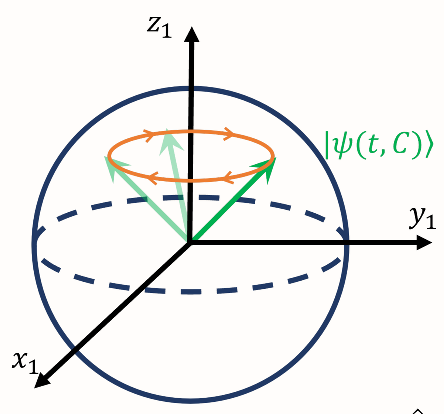
V. TWO QUBITS IN A MAGNETIC FIELD

Consider a general 2-qubit state evolving under $H_2 = B\sigma_z\otimes I$:

$$|\psi_2(t)\rangle = R_1(\theta_1,\varphi_1+2Bt)\,R_2(\theta_2,\varphi_2)\,R_3(\alpha,\beta)\,|\uparrow\uparrow\rangle$$

$|\psi_2(t)\rangle$ is periodic: $T=2\pi/B$:

$$\gamma_2 = \pi\sqrt{1-C^2}\cos\theta_1$$



This phase is gathered by the motion of the first qubit in its' Bloch sphere.

As the initial state is isotropic our result is true for any $\hat{B}$ direction:

$$\gamma_3 = \pi\sqrt{1-C^2}\hat{B}\cdot\hat{n}_1$$

VI. INTERACTIONS BETWEEN QUBITS

The exchange Hamiltonian is $H_4 = \lambda\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)}$. For simplicity, we choose

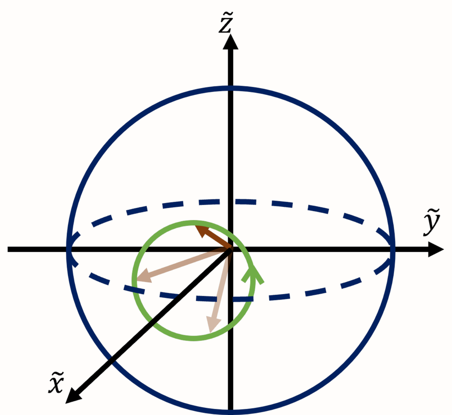
$$|\psi_4(t=0)\rangle = \cos\frac{\alpha}{2}e^{-i\beta/2}|\uparrow\downarrow\rangle + \sin\frac{\alpha}{2}e^{i\beta/2}|\downarrow\uparrow\rangle$$

Using the effective SU(2) with $|\Uparrow\rangle\equiv|\uparrow\downarrow\rangle$ and $|\Downarrow\rangle\equiv|\downarrow\uparrow\rangle$:

$$H_4 = 2\lambda\tilde{\Sigma}_x + \lambda\left(|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|\right)$$

An effective magnetic field in the $\hat{x}$ direction. The phase for a single period $T_4=\pi/\lambda$ is:

$$\gamma_4 = \pi\sin\alpha\cos\beta = \pi C(t=0)\cos\beta$$



VII. INCOMMENSURATE PERIODS

Consider a system with both a magnetic field and an exchange interaction: $H_5 = H_2 + H_4$.
Choosing $|\psi_5(t=0)\rangle = \cos\frac{\alpha}{2}e^{-i\beta/2}|\uparrow\downarrow\rangle + \sin\frac{\alpha}{2}e^{i\beta/2}|\downarrow\uparrow\rangle$, time evolution is periodic with $T_5 = 2\pi/\sqrt{B^2+4\lambda^2}$. The resulting phase for a single period is:

$$\gamma_5 = 2\pi\sin\alpha\left(\frac{2\lambda}{\sqrt{B^2+4\lambda^2}}\cos\beta + \frac{B}{\sqrt{B^2+4\lambda^2}}\right)$$

Never trivial when $B/\lambda\notin\mathbb{Q}$ for $\alpha\neq 0,\pi$.

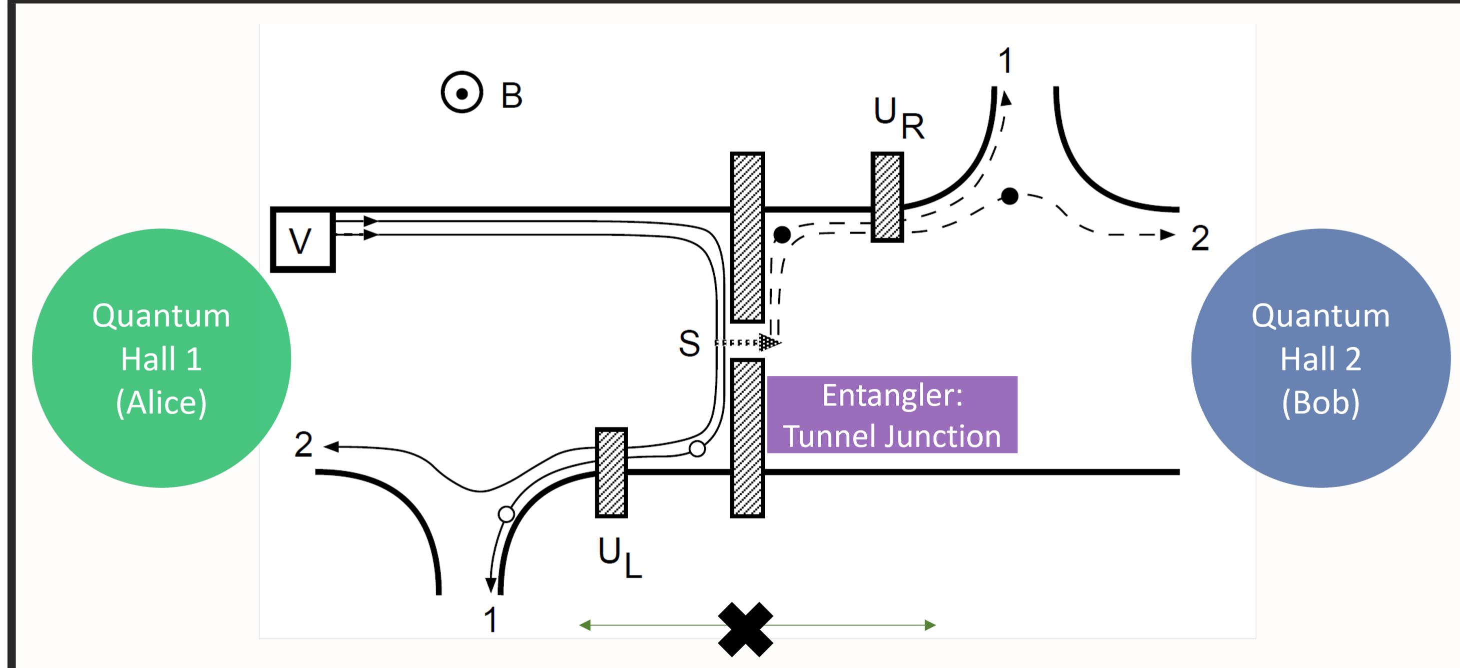
VIII. GENERALIZATIONS

Our result motivates a generalization to systems of a qubit interacting with a spin-N particle ($\mathbb{C}^2\otimes\mathbb{C}^N$ Hilbert spaces). For example, consider $\mathbb{C}^2\otimes\mathbb{C}^3$, a qubit and a three-level system. The Schmidt decomposition allows to write any state $|\phi\rangle$ as:

$$|\phi\rangle = \cos\frac{\alpha}{2}|\hat{n}_1,+\hat{m}_2\rangle + \sin\frac{\alpha}{2}e^{i\beta}|-\hat{n}_1,-\hat{m}_2\rangle$$

where the spin-1 states $|\pm\hat{m}_2\rangle$ are defined following a nested-sphere parameterization [6]. It is now possible to use the effective SU(2) to characterize a general time evolution.

IX. ENTANGLEMENT OF TOPOLOGICAL MODES IN CONDENSED MATTER



Beenakker et al [7]

Play a XOR game in condensed matter:

Qubit $\leftrightarrow$ Topological edge states

Interactions $\leftrightarrow$ Tunnel junction

Shot Noise $\leftrightarrow$ Spin measurements

Alice and Bob "speak" at the junction, and then measured separately.
In the QHE example [7], Wootters concurrence for distinguishable qubits is used, but does it measure entanglement in this system?

Geometric phase is a specific case of topological invariants. Can our results capture topological condensed matter systems?

New topological phenomena arising from interactions?

Can interactions between two topological systems bring forth new invariants and edge states?

Entanglement of indistinguishable electrons in distinguishable states?

What is the correct definition of entanglement for these systems? What protocol should be used to measure it?

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