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Fast Fourier demodulation

Yuval Carmon[a] and Erez N. Ribak[b]
Department of Physics, Technion, Haifa 32000, Israel

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We present a fast Fourier demodulation method for calculating the distortion in a repetitive pattern. The technique is based on applying digital demodulation, then using only the anti-Hermitian part of the pattern in Fourier space. After demodulation, we are left with the Fourier transform of the sought phase information only. Using also the Hermitian part, we would have gotten the object itself. We investigate the boundaries of the technique, as related to aberration content, amplitude variations, and sensitivity to noise. © *2004 American Institute of Physics.* [DOI: 10.1063/1.1759770]

The Fourier demodulation technique is convenient when small irregularities in a quasiregular grid are investigated. Takeda[1] used a one-dimensional Fourier method to calculate the fringe phase. Roddier and Roddier[2] described a method for acquiring the two-dimensional phase and amplitude of a fringe visibility function. Lately a technique was developed for two-dimensional Fourier demodulation in order to acquire the phase gradient components from a Hartmann–Shack wave front sensor[3] (from here on CR). Basically the multidimensional demodulation technique works as follows:

(1) Fourier transform (FT) the data.
(2) Apply a rigid translation so that the designated sidelobe will be centered on the origin in Fourier space;
(3) apply a low pass filter;
(4) inverse FT; and
(5) extract the phase argument of the inverse transform, to yield the desired phase derivative result:
(6) FT the phase derivative results.

The above technique suffers from inefficiency, since many FTs must be applied before acquisition of the FT of the phase result. In order to reduce the number of transforms, we developed a technique in which under some constraints, the FT of the phase can be calculated using steps (1)–(3) alone. Only steps (2)–(3) have to be repeated per dimension in the multidimensional case. One more semistep has to be added, but its complexity is negligible relative to the other steps.

After applying an FT to the data, Fourier demodulating, and applying a low pass filter (steps 1–3) we are left with $\Im(Ve^{i\xi})$, where $\Im$ is the Fourier transform operator, V is the pattern amplitude, and ξ is the information we are interested in obtaining (CR). ξ can correspond to different physical entities: in a Hartmann–Shack sensor it corresponds to one of the phase gradient components, while in a structured light measurement it corresponds directly to the height of the measured model.

If we assume that ξ is small, then using the Taylor theorem to first order on the imaginary and real parts separately we obtain

$$\mathrm{Im}\{e^{i\xi}\}=\xi-\frac{\tilde{\xi}^3}{6},\quad \mathrm{Re}\{e^{i\xi}\}=1-\frac{\hat{\xi}^2}{2},$$
$$\text{where}\ |\tilde{\xi}|<|\xi|_{\max},|\hat{\xi}|<|\xi|_{\max}, \tag{1}$$

where $|F|_{\max}$ is the maximum of the absolute of F.

We denote $V=A_0+\delta A$, where A_0 is the mean intensity. Let us calculate the ratio of error to signal on the phase, and limit this ratio by η: using Eq. (1) we get the following condition:

$$\frac{|A_0\xi-\mathrm{Im}\{Ve^{-i\xi}\}|}{A_0\xi}=\frac{1}{\xi}\left|\xi-\left(1+\frac{\delta A}{A_0}\right)\left(\xi-\frac{\tilde{\xi}^3}{6}\right)\right|$$
$$<\left|\frac{\delta A}{A_0}\right|_{\max}+\frac{|\xi|^2_{\max}}{6}+O(|\xi|^4_{\max})<\eta. \tag{2}$$

Next we calculate the ratio of error to signal on the amplitude, and limit that ratio by σ

$$\frac{|A_0-\mathrm{Re}\{Ve^{-i\xi}\}|}{A_0}=\left|1-\left(1+\frac{\delta A}{A_0}\right)\left(1-\frac{\hat{\xi}^2}{2}\right)\right|$$
$$<\left|\frac{\delta A}{A_0}\right|_{\max}+\frac{|\xi|^2_{\max}}{2}+O(|\xi|^4_{\max})<\sigma. \tag{3}$$

If these conditions are met, we can write the following equations for the anti-Hermitian and Hermitian parts of the Fourier transform:

$$\frac{\Im(Ve^{i\xi})-\Im^*(Ve^{i\xi})}{2A_0}\approx\Im(\xi),$$
$$\frac{\Im(Ve^{i\xi})+\Im^*(Ve^{i\xi})}{2}\approx\Im(A_0).$$

We see that if ξ is small and adiabatic (is not under sampled), and the pattern amplitude variability is small enough, the FT of ξ can be obtained by applying steps (1), (2), and (3), and applying an additional step [3(b)] that discards the Hermitian part of the transform. The complexity of this additional step is negligible relative to the other steps in the method, so overall efficiency is improved.

Specifically, for the Hartmann–Shack pattern the constraints are

[a]Also at: Shamir Optical Industries, Shamir, Upper Galilee 12135, Israel.
[b]Author to whom correspondence should be addressed; electronic mail: eribak@physics.technion.ac.il

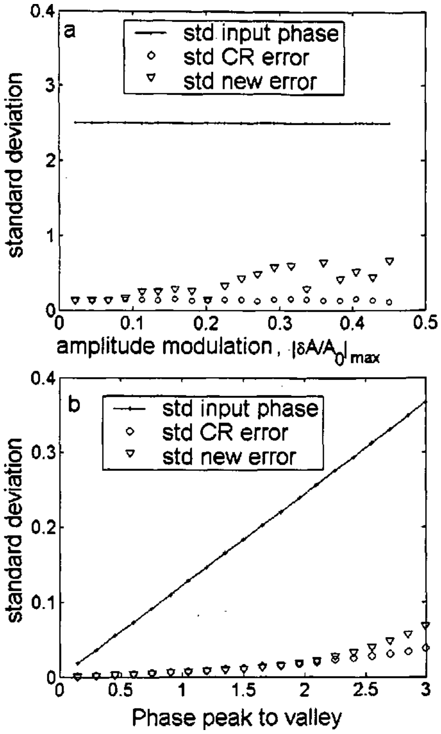

FIG. 1. Performance of both algorithms under varying circumstances. We used 200×200 pixels with a Hartmann–Shack lenslet array of 20×20 lenslets. In the top frame, the amplitude modulation is increased while the aberration content remains constant. It is seen that our algorithm's fidelity deteriorates linearly. In the bottom frame, the aberration becomes increasingly larger and the amplitude remains constant. The fidelity of the reconstruction is not significantly reduced in our algorithm until a peak to valley of two radians in the input phase is reached.

$$\left|\frac{\delta A}{A_0}\right|_{\max} + \frac{|F\phi_x|^2_{\max}}{6} < \eta, \quad \left|\frac{\delta A}{A_0}\right|_{\max} + \frac{|F\phi_y|^2_{\max}}{6} < \eta. \quad (4)$$

$$\left|\frac{\delta A}{A_0}\right|_{\max} + \frac{|F\phi_x|^2_{\max}}{2} < \sigma, \quad \left|\frac{\delta A}{A_0}\right|_{\max} + \frac{|F\phi_y|^2_{\max}}{2} < \sigma. \quad (5)$$

F is the focal length of the lenslet array, and ϕ_x, ϕ_y are the gradient components of the phase.

We simulated the method: A random wave front phase and amplitude were generated, and a Hartmann–Shack pattern simulated. We used the CR algorithm and our algorithm to reconstruct the wave front. It was confirmed that as long as the aberration content is relatively small, and the amplitude is close to constant [Fig. 1 and Eq. (4)], the fidelity of both algorithms is similar. Sensitivity to noise content was also tested, and it was found that it is the same as in the CR algorithm.

One advantage of CR over conventional Hartmann–Shack centroiding is that it can handle a larger class of wave aberrations, since there is no pre-designated subaperture inside which the Hartmann spots need fit.[5] If the wave aberration measured is very large, special unwrapping algorithms must be used, making the algorithm less robust.[6,7] Our method never needs to be unwrapped so the calculation never fails totally. However, relatively large errors may occur [Fig. 1(a)]. In very complex cases, this is an improvement over CR.

In many applications, the aperture of the system is not rectangular. In this case one must write $V=P(A_0+\delta A)$ where P is an aperture function (1 inside the aperture). After applying steps [1–3(b)] we are left with the convolution $\Im(PA_0)^*\Im(\xi)$. For nonrectangular apertures this can introduce some error near the edge. If S is the size of the aperture, and s is the sampling segment size, the error increases with s/S. A method was proposed[4] in which the two gradient components of the phase are calculated from a Hartmann–Shack pattern using the traditional centroiding technique. These gradient components are then extended beyond the aperture limits in such a way that the mixed derivatives theorem is not violated. This technique is not applicable in our case since we calculate the gradient components in Fourier space, not real space. We reduced this error to some extent by extrapolating the actual Hartmann pattern by one spot period on each side before applying a Fourier transform to it.[8]

A Fourier reconstructor can be used to compute the wave front from the FTs of the gradient components. It is shown[4] that for large adaptive optic systems this method is more efficient than the reconstructor method that is usually used, because of the high efficiency of the fast Fourier transform. Our method can further speed the calculation, since we offer a way of calculating the FT of both gradient components using one transform only.

Because of the technique's simplicity and speed, we started to implement it in an adaptive optics system. Usually the gradients of the wave front are multiplied by a reconstructor matrix (which is the inverse of the mirror elements' response to voltage). This product is the voltage vector that is sent to the mirror to correct the aberration. The realization of this method now involves the following steps:

Calculate the reconstructor matrix so that it accepts as inputs the FTs of the gradient components, instead of the gradient components in real space. Then, during the adaptive optics loop, repeat the following steps:

(a) Acquire the Hartmann image;
(b) go through stages 1–3b to calculate the FTs of the two gradient components, as described above; and
(c) multiply the result with the previously calculated reconstructor matrix, to yield the desired voltage correction vector.

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