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Light propagation through multilayer atmospheric turbulence

E. Gershnik, E.N. Ribak *

Department of Physics, Technion-Israel Institute of Technology, Technion City, Haifa 32000, Israel

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Abstract

A new treatment is presented for light propagation through multilayer turbulence. Equations for the intensity and phase of an observed wavefront are derived together with their validity conditions for both single and multiple layer systems. A method for finding the statistics of observed scintillations is presented together with a detailed calculation for a single layer system. © 1997 Elsevier Science B.V.

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1. Introduction

Since the early seventies much attention has been paid to the compensation of wavefront degradation caused by atmospheric turbulence, usually concentrated in a few thin layers [1–6]. To succeed in solving this problem, there is a need for a good theory of light propagation through a system of multiple turbulent layers. Whereas the theory of light propagation through turbulence is well established and confirmed experimentally [7–9], most treatments consider a continuous distribution of turbulence. To investigate the discrete case, two approaches could be used. One is to use the continuous theory, representing the turbulence by some kind of step functions, and the other is to develop the discrete theory from first principles. Despite its simplicity, the first way has some important disadvantages. The main one is that this approach is purely mathematical and does not reveal the underlying physics of the processes involved. In particular, the range of validity of the formulae obtained may not be clear. To avoid these problems we chose to proceed in the second way. This approach has been taken previously by Lee and Harp [10] and Roddier [11]. Lee and Harp consider an action of a thin turbulence layer on a Fourier component of the incoming wave in order to obtain the wavefront statistics for a continuous turbulence. Since they are interested mainly in the continuous case, they integrate the discrete expressions as soon as possible. Thus they do not obtain any theoretical predictions concerning discrete turbulence as such. Roddier in his review paper goes further and applies the Fourier optics formalism to multilayered structures. This enables him to get general convolution formulae for the phase and intensity of the wavefront that has passed a multilayer turbulence. In this paper we intend to go further and use the Fourier optics formalism of light propagation to obtain specific theoretical prediction for the phase and scintillation pattern of a wavefront degraded by a multilayer turbulence. Conditions for the validity of the theory will be clearly indicated. In addition, we shall derive the statistical properties of scintillation patterns resulting from multilayer turbulence.

2. Single turbulence layer

In what follows we shall use the coordinate system with the z-axis lying along the light propagation direction and pointing downward. A turbulence layer is a portion of the atmosphere possessing refractive index variations which lies between $z=0$ and $z=l$, where l is the layer width. The observer on the ground is at $z=h$. Notice that by

* Corresponding author. E-mail: eribak@vmsa.technion.ac.il.

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using such a definition we are not concerned with problems of layer orientation since the layer, so defined, is always perpendicular to the light propagation direction.

Let us consider what happens when a plane monochromatic wave passes through a thin turbulence layer located somewhere in the atmosphere and reaches the observation plane at the ground. By saying that the layer is thin, we assume that the wave acquires only a phase shift as it passes through the layer. Taking the refractive index of the still air to be equal to unity, we can write

$$U(\boldsymbol{r}) = U_0 \exp\left(\mathrm{i} k \int_0^l [1 + \mu(\boldsymbol{r}, z)] \,\mathrm{d} z\right), \tag{1}$$

where $\boldsymbol{r} = (x, y)$ is the two-dimensional lateral position vector, $U(\boldsymbol{r})$ is the wavefront after the turbulence layer, U_0 is the plane wave (independent of $\boldsymbol{r}$) before the layer, k is the wave vector and $\mu(\boldsymbol{r}, z)$ is the refractive index change, produced by turbulence at the point (x, y, z).

We idealize this process, representing the layer by an infinitesimally thin surface lying at $z = 0$. The refractive index of the surface is taken to be the average of the real refractive index over the propagation direction. Thus

$$U(\boldsymbol{r}) = U_0 \exp\{\mathrm{i} k [1 + \bar{\mu}(\boldsymbol{r})] l\}, \tag{2}$$

where

$$\bar{\mu}(\boldsymbol{r}) = \frac{1}{l} \int_0^l \mu(\boldsymbol{r}, z) \,\mathrm{d} z. \tag{3}$$

The constant phase factor in Eq. (2) is entirely unimportant for the present discussion, so we can incorporate it into U_0 and write

$$U(\boldsymbol{r}) = U_0 \mathrm{e}^{\mathrm{i}\varphi(\boldsymbol{r})}, \tag{4}$$

where

$$\varphi(\boldsymbol{r}) = k \bar{\mu}(\boldsymbol{r}) l. \tag{5}$$

Following Tatarskii [7], we shall assume that the refractive index additive due to turbulence is very small, i.e.

$$\mu(\boldsymbol{r}) \ll 1. \tag{6}$$

We also assume that the refractive index changes are randomly distributed in the layer with coherence length ~ 20 cm [9]. So, their average value over the distances of few hundreds meters, which is the probable layer width [3,6], will become very small. This will make φ small despite the big wavevector. Thus, we can assume that

$$\varphi(\boldsymbol{r}) \ll 1. \tag{7}$$

Eq. (4), together with the condition (7), summarize mathematically our concept of weak turbulence layer.

In order to clarify the meaning of condition (7) let us consider what is its meaning for a real atmospheric turbulence layer. For layer width $l \sim 100$ m [3,6] and optical frequencies, condition (7) taken together with the defining Eq. (5) demands that the refractive index additive (averaged over 100 m) should be $\bar{\mu} \ll 10^{-9}$. For the turbulence correlation length $\xi \sim 20$ cm the average index change $\bar{\mu}$ will obviously satisfy this condition for any usual turbulence strength $\mu(x, y, z) = (0.1\text{–}10) \times 10^{-9}$ [8]. The interesting point is that for weak turbulence (large ξ) $\mu(x, y, z)$ will be small by itself, thus making $\bar{\mu}$ even smaller. On the other hand, for strong turbulence ξ will become small (compared to l) and that will compensate for a large refractive index additive. Thus we expect that in any case $\bar{\mu}$ will remain very small, satisfying the above condition.

After passing the layer the wave propagates in a homogeneous and isotropic atmosphere till it reaches the observation plane (at the ground level). For such a propagation the next formula may be used (see Ref. [12], Eq. (3-47))

$$U(\boldsymbol{r}, z = h) = \mathscr{F}^{-1}\left\{\tilde{U}(\boldsymbol{\omega}, z = 0) \exp\left(\mathrm{i} h k \sqrt{1 - \lambda^2 \omega^2}\right)\right\}, \tag{8}$$

where $U(\boldsymbol{r}, z = h)$ denotes the field at vertical distance h down from the turbulence, $\tilde{U}(\boldsymbol{\omega}, z = 0)$ is the two-dimensional Fourier transform of the field at turbulence level, $\mathscr{F}^{-1}$ is the two-dimensional inverse Fourier transform operator, λ is the wavelength and $k = 2\pi/\lambda$ is the wavevector. In what follows we shall generally use the tilde or symbol $\mathscr{F}\{\ \}$ to indicate the Fourier transform operation.

For Eq. (8) to be valid, the following condition must hold [12]

$$h \gg \lambda. \tag{9}$$

For the optical frequencies and propagation distances we are concerned with, this condition is obviously satisfied. Using (7) we can rewrite Eq. (4) as

$$U(\boldsymbol{r}) \equiv U(\boldsymbol{r}, z = 0) \cong U_0(1 + \mathrm{i}\varphi(\boldsymbol{r})). \tag{10}$$

Substituting this relation into Eq. (8) we get

$$\begin{aligned} U(\boldsymbol{r}, h) &= U_0 \mathscr{F}^{-1}\left\{\mathscr{F}\{1 + \mathrm{i}\varphi(\boldsymbol{r})\} \exp\left(\mathrm{i} h k \sqrt{1 - \lambda^2 \omega^2}\right)\right\} \\ &= U_0 \mathscr{F}^{-1}\left\{[\delta(\omega) + \mathrm{i}\tilde{\varphi}(\omega)] \exp\left(\mathrm{i} h k \sqrt{1 - \lambda^2 \omega^2}\right)\right\} \\ &= U_0\Big[\exp(\mathrm{i} h k) \\ &\quad + \mathrm{i}\mathscr{F}^{-1}\left\{\mathrm{i}\tilde{\varphi}(\omega) \exp\left(\mathrm{i} h k \sqrt{1 - \lambda^2 \omega^2}\right)\right\}\Big]. \end{aligned} \tag{11}$$

In the usual Fresnel approximation [12] we assume

$$\lambda^2 \omega^2 \ll 1. \tag{12}$$

So we can expand the exponential kernel as follows,

$$\begin{aligned} &\exp\left(\mathrm{i} h k \sqrt{1 - \lambda^2 \omega^2}\right) \\ &\quad \cong \exp\left(\mathrm{i} h k \left(1 - \tfrac{1}{2}\lambda^2 \omega^2\right)\right) = \exp(\mathrm{i} h k) \exp(-\mathrm{i}\pi h \lambda \omega^2) \\ &\quad = \exp(\mathrm{i} h k)\left(1 - \mathrm{i}\pi h \lambda \omega^2 - \tfrac{1}{2}(\pi h \lambda \omega^2)^2 - \ldots\right). \end{aligned} \tag{13}$$

Now, if we assume that

$$\pi h \lambda \omega^2 \gg \tfrac{1}{2}(\pi h \lambda \omega^2)^2, \tag{14}$$

we can truncate the series in Eq. (13) to get

$$\exp\left(\mathrm{i} h k \sqrt{1 - \lambda^2 \omega^2}\right) \cong \exp(\mathrm{i} h k)(1 - \mathrm{i}\pi h \lambda \omega^2). \tag{15}$$

Transforming Eq. (14) we obtain

$$\lambda^2 \omega^2 \ll 2\lambda / \pi h. \tag{16}$$

Comparing Eq. (16) with Eq. (12) we obtain that the condition of Eq. (16) is at least as severe as that of Fresnel diffraction (for h equal to just a few wavelengths) and for distances of kilometers it is much stronger. This means that if we wish to use the condition of Eq. (16) we can safely omit the Fresnel diffraction condition. Eq. (16) may also be written as

$$\sin(\theta) \cong \theta \ll \sqrt{2\lambda/\pi h}, \tag{17}$$

where θ is the usual off-axis angle, related to the Fourier frequency by $\omega = \sin\theta/\lambda$. The validity of the above condition was already discussed by Ribak et al. [13]. There it is shown that for the layer height of ~ 8 km and optical frequencies, condition (17) requires that the angular size of the star should be smaller than 1.4″. Bright stars and small sodium beacons approach this limit. We shall assume in what follows that (17) holds.

Substituting the result of Eq. (15) into Eq. (11) we obtain

$$\begin{aligned} U(\boldsymbol{r},h) &= U_0\big[\exp(\mathrm{i}hk) \\ &\quad + \mathrm{i}\mathscr{F}^{-1}\{\tilde{\varphi}(\boldsymbol{\omega})\exp(\mathrm{i}hk)(1-\mathrm{i}\pi h\lambda\omega^2)\}\big] \\ &= U_0\exp(\mathrm{i}hk)\big[1+\mathrm{i}\varphi(\boldsymbol{r}) \\ &\quad + \pi h\lambda\mathscr{F}^{-1}\{\tilde{\varphi}(\boldsymbol{\omega})\omega^2\}\big]. \end{aligned} \tag{18}$$

The last term in the square brackets may be transformed according to the following well-known formula

$$\nabla^2 f(\boldsymbol{r}) = -4\pi^2\mathscr{F}^{-1}\{\tilde{f}(\boldsymbol{\omega})\omega^2\}, \tag{19}$$

where ∇^2 denotes the two-dimensional Laplacian operator. Thus we obtain

$$U(\boldsymbol{r},h) = U_0\exp(\mathrm{i}hk)\left[1+\mathrm{i}\varphi(\boldsymbol{r}) - \frac{h\lambda}{4\pi}\nabla^2\varphi(\boldsymbol{r})\right]. \tag{20}$$

This equation is the main result of this section. It gives the wavefront at any distance from the turbulent layer in terms of the wavefront incident on it. Thus, we can consider the layer as a sort of a blackbox device whose response to the monochromatic plane wave is given by Eq. (20). From here it follows that

$$I(\boldsymbol{r},h) = I_0\left\{\left[1-\frac{h\lambda}{4\pi}\nabla^2\varphi(\boldsymbol{r})\right]^2 + \varphi^2\right\}, \tag{21}$$

where $I(\boldsymbol{r}, h)$ and I_0 denote intensity at point $(\boldsymbol{r}, h)$ and initial (pre-turbulence) wavefront intensity respectively.

Now let us make the final assumption that

$$\frac{h\lambda}{4\pi}\nabla^2\varphi(\boldsymbol{r}) \ll 1. \tag{22}$$

For optical wavelengths and distances of less than tens of kilometers which are usually involved, this condition obviously holds even for great phase curvatures (see also the discussion following Eq. (37)).

To obtain a numerical estimate of the above condition let us take, again, the layer width and altitude to be $l \sim 100$ m and $h \sim 8$ km. Then from condition (22) taken together with Eq. (5), we obtain that $\nabla^2\overline{\mu} \ll 3\times10^{-6}$ m^{-2}. For the same layer parameters condition (7) demands that $\overline{\mu} \ll 10^{-9}$ (see the discussion following (7)). To this end, we may assume $\nabla^2\overline{\mu} \sim \overline{\mu}/\xi^2$, where ξ is the correlation length of the refractive index fluctuations. For $\xi \sim 20$ cm [9] we shall have, then, $\nabla^2\overline{\mu} \ll 3\times10^{-8}$ m^{-2}. So we see that (22) is likely to hold for most parts of the layer.

Returning to Eq. (21) and using conditions (7) and (22) we obtain

$$\frac{I(\boldsymbol{r},h)}{I_0} = 1 - \frac{h\lambda}{2\pi}\nabla^2\varphi(\boldsymbol{r}), \tag{23}$$

or using (22) again

$$\ln\frac{I(\boldsymbol{r},h)}{I_0} = -\frac{h\lambda}{2\pi}\nabla^2\varphi(\boldsymbol{r}). \tag{24}$$

The only conditions for validity of this equation are (7), (17) and (22). Eq. (24) connects the scintillation pattern as observed on the ground with the curvature of the phase in the turbulent layer. This equation (as well as its generalization of Eq. (41)) can also be obtained from the conventional continuous theory [7,8] by substituting a δ-function to represent a turbulence layer. However, such an approach would not provide us with any indication of the validity range of Eq. (24). Using the present derivation we are able to state clearly the conditions of its validity (formulae (7), (17) and (22)). Moreover, in the case when condition (22) does not hold we can resort to the more general Eq. (20) to obtain the corresponding results. Thus we do not only propose a new derivation of the old results but a whole method of dealing with the multilayer turbulence structures. The value of the present derivation, as we shall see, lies also in the possibility of expanding it to non-monochromatic illumination and to systems of multiple turbulence layers.

Returning back to Eq. (20) we can calculate also the wavefront phase $\psi(\boldsymbol{r}, h)$ which is equal to

$$\begin{aligned} \psi(\boldsymbol{r},h) &= \arg\{U(\boldsymbol{r},h)\} = hk + \arctan\left(\frac{\varphi}{1-(h\lambda/4\pi)\nabla^2\varphi}\right) \\ &\cong hk + \varphi(\boldsymbol{r}), \end{aligned} \tag{25}$$

where we have used (7) and (22) again. Eq. (25) means that, apart from a constant additive from a straight propagation, the phase does not change in the first order as a result of propagation after the turbulence layer (provided conditions (7), (17) and (22) hold).

Eqs. (23) and (25) give us the amplitude and the phase of the wave $U(\boldsymbol{r}, h)$. So using (22) we can write

$$\begin{aligned} U(\boldsymbol{r},h) &= U_0\left[1-\frac{h\lambda}{2\pi}\nabla^2\varphi(\boldsymbol{r})\right]^{1/2} \mathrm{e}^{\mathrm{i}[hk+\varphi(\boldsymbol{r})]} \\ &\cong U_0\left[1-\frac{h\lambda}{4\pi}\nabla^2\varphi(\boldsymbol{r})\right]\mathrm{e}^{\mathrm{i}[hk+\varphi(\boldsymbol{r})]}. \end{aligned} \tag{26}$$

This is another, more convenient form of Eq. (20).

3. Non-coherent illumination

Let us now consider the case when the wave incident on the turbulence layer is not monochromatic. Inasmuch as our method has many things in common with the geometrical optics approach [8] we may suppose intuitively that the results obtained will not change in the non-monochromatic case apart from the integration over the whole spectrum. However, in order to check whether this will not introduce any additional constraints on the system or any undesired effects we will explicitly derive all formulae. We can still use Eq. (26) for each monochromatic component of the wave reaching the ground level. The full time-dependent wavefront will be given by

$$U(\boldsymbol{r},h,t) = \int U_0(\lambda)\left[1 - \frac{h\lambda}{4\pi}\nabla^2\varphi(\boldsymbol{r},\lambda)\right] \times \mathrm{e}^{\mathrm{i}[2\pi h/\lambda + \varphi(\boldsymbol{r},\lambda)]}\,\mathrm{e}^{-\mathrm{i}2\pi ct/\lambda}\,\mathrm{d}\lambda \equiv \int A(\boldsymbol{r},\lambda)\,\mathrm{e}^{-\mathrm{i}2\pi ct/\lambda}\,\mathrm{d}\lambda, \quad (27)$$

where c is the velocity of light, t is the time and we have allowed for the wavelength dependence of the phase φ. The instantaneous intensity is then

$$I(\boldsymbol{r},h,t) = \int\int A(\boldsymbol{r},\lambda)\,A^*(\boldsymbol{r},\lambda_1) \times \exp\left[2\pi\mathrm{i}ct\left(\frac{1}{\lambda_1} - \frac{1}{\lambda}\right)\right]\mathrm{d}\lambda\,\mathrm{d}\lambda_1. \quad (28)$$

The observed intensity is equal to the time average of Eq. (28) over the relaxation time of the detector. The exponent in the integral vanishes for any available detection time unless the two frequencies λ and λ_1 are equal. Thus, we obtain

$$\tau\bar{I}(\boldsymbol{r},h) = \int |A(\boldsymbol{r},\lambda)|^2\,\mathrm{d}\lambda = \int I_0(\lambda)\left[1 - \frac{h\lambda}{4\pi}\nabla^2\varphi(\boldsymbol{r},\lambda)\right]^2\mathrm{d}\lambda, \quad (29)$$

where τ is the detection time, $\bar{I}$ is the average intensity and $I_0(\lambda) = |U_0(\lambda)|^2$ is the initial intensity spectrum. Using (22) we obtain

$$\tau\bar{I}(\boldsymbol{r},h) = \int I_0(\lambda)\,\mathrm{d}\lambda - \frac{h}{2\pi}\nabla^2\int I_0(\lambda)\varphi(\boldsymbol{r},\lambda)\,\lambda\,\mathrm{d}\lambda, \quad (30)$$

where we interchanged the Laplacian and integration since they are acting on different variables. It follows that

$$\frac{\tau\bar{I}(\boldsymbol{r},h)}{\int I_0(\lambda)\,\mathrm{d}\lambda} = 1 - \frac{h}{2\pi}\nabla^2\langle\lambda\varphi(\boldsymbol{r},\lambda)\rangle, \quad (31)$$

where

$$\langle\lambda\varphi(\boldsymbol{r},\lambda)\rangle = \frac{\int I_0(\lambda)\varphi(\boldsymbol{r},\lambda)\,\lambda\,\mathrm{d}\lambda}{\int I_0(\lambda)\,\mathrm{d}\lambda}. \quad (32)$$

Using Parseval's theorem we can rewrite Eq. (31) as

$$\frac{\bar{I}(\boldsymbol{r},h)}{\bar{I}_0} = 1 - \frac{h}{2\pi}\nabla^2\langle\lambda\varphi(\boldsymbol{r},\lambda)\rangle, \quad (33)$$

where $\bar{I}_0$ is the initial intensity averaged over time τ. Because of the rapid variation of the intensity with time we can assume that the average over τ is equal to the average over infinite time. The last term in Eq. (33) is very small compared to unity because φ and λ are small. So we can take the logarithm of both sides and write

$$\ln\frac{\bar{I}(\boldsymbol{r},h)}{\bar{I}_0} = -\frac{h}{2\pi}\nabla^2\langle\varphi(\boldsymbol{r},\lambda)\lambda\rangle. \quad (34)$$

We see that for an incoherent illumination the observed intensity will obey the same spatial equation as for the coherent one (cf. Eqs. (24) and (34)). The only difference is that for the incoherent case we should average all the wavelength dependent parameters over the initial spectrum. This means, in particular, that no special color effects will be observable in the scintillation pattern (to the order of accuracy of the present treatment) apart from the possible overall average color. We have obtained also that no new constraints are introduced if we wish to extend the monochromatic expressions for the non-monochromatic case.

4. Multiple layers

In this section we shall expand the above treatment to the realistic case when more than one layer is present. We shall show how using the present method one can easily obtain the expressions for the multilayer system knowing exactly which is their validity range. Again, here we shall apply the most widely used turbulence conditions in order to be able to compare our results with the conventional ones in the limit of continuous turbulence. However, the derivation may easily be repeated for other conditions that do not necessarily fall into the framework of the conventional theory.

In order to extend the previous treatment to a system of multiple layers we need to allow for the wave incident on the layer to vary spatially. That would be the case when a wavefront perturbed by one layer reaches the second. In all other respects the treatment will be quite similar to that of Section 1. Each layer is assumed to add to the passing wavefront a phase $\varphi_n(\boldsymbol{r})$, where n is the layer number. The only additional condition we impose on a multilayer system would be that the distance between any two adjacent layers will be greater than few wavelengths (see Eq. (9)). This will enable us to use the diffraction formula of Eq. (8) again.

Consider a wave that has passed a layer, traveled in still air to the next one and passed it too. Accordingly, this wave will now be given by an analogue of Eq. (4)

$$U_n(\boldsymbol{r},s_n)=U_{n-1}(\boldsymbol{r},s_n)\,e^{i\varphi_n(r)} \cong U_{n-1}(\boldsymbol{r},s_n)(1+i\varphi_n(\boldsymbol{r})), \quad (35)$$

where s_n is the altitude of the nth layer, $U_n(\boldsymbol{r}, s_n)$ is the wave immediately after the nth layer, $U_{n-1}(\boldsymbol{r}, s_n)$ is the incident wave as a result of the action of the previous layer and φ_n is the phase added by the last layer. Now we can proceed exactly as in the previous section, the only difference being the fact that now the pre-layer wavefront is position dependent. We shall use the conditions of (7), (17) and (22) again, whenever possible, assuming that they hold for each layer. Then Eq. (11) transforms to

$$\begin{aligned}U_n(\boldsymbol{r},s_n+h) &= \mathscr{F}^{-1}\left\{\left[\tilde{U}_{n-1}+i\mathscr{F}\{U_{n-1}\varphi_n\}\right]\exp\left(ihk\sqrt{1-\lambda^2\omega^2}\right)\right\}\\ &\cong e^{ihk}\mathscr{F}^{-1}\left\{\left[\tilde{U}_{n-1}+i\mathscr{F}\{U_{n-1}\varphi_n\}\right](1-i\pi h\lambda\omega^2)\right\}\\ &= e^{ihk}\left[U_{n-1}(1+i\varphi_n)+\frac{ih\lambda}{4\pi}\nabla^2U_{n-1} - \frac{h\lambda}{4\pi}\nabla^2(\varphi_nU_{n-1})\right], \quad (36)\end{aligned}$$

where h is the distance between the nth layer and the observation plane (or the next layer). The Laplacian is, as before, a two-dimensional one (i.e. with respect to $\boldsymbol{r}$ only). After some algebra the last equation can be written as

$$\begin{aligned}U_n(\boldsymbol{r},s_n+h) &= e^{ihk}\left[U_{n-1}\left(1-\frac{h\lambda}{4\pi}\nabla^2\varphi_n+i\varphi_n\right) + \frac{ih\lambda}{4\pi}\nabla^2U_{n-1}(1+i\varphi_n)-\frac{h\lambda}{4\pi}\nabla U_{n-1}\cdot\nabla\varphi_n\right]\\ &\cong e^{ihk}\left[U_{n-1}\left(1-\frac{h\lambda}{4\pi}\nabla^2\varphi_n\right)e^{i\varphi_n}+\frac{ih\lambda}{4\pi}e^{i\varphi_n}\nabla^2U_{n-1} - \frac{h\lambda}{4\pi}\nabla U_{n-1}\cdot\nabla\varphi_n\right], \quad (37)\end{aligned}$$

where the last step followed from condition (7).

The phase in a layer cannot vary appreciably over distances shorter than ξ, the correlation length of the turbulence layer [9]. Thus, we can write $\nabla\varphi\sim 1/\xi$ and $\nabla^2\varphi\sim 1/\xi^2$, where φ stands for the phase in any layer. Let us estimate the order of magnitude of the expressions containing ∇U_{n-1} and ∇^2U_{n-1}. To do so we can use the results obtained for single layer turbulence. From Eq. (23) we obtain that $U\sim U_0(1-h\lambda\nabla^2\varphi)$, where all symbols have the meaning as in Section 1. Taking the gradient and Laplacian we have $\nabla U\sim -U_0h\lambda/\xi^3$ and $\nabla^2U\sim -U_0h\lambda/\xi^4$. These relations give us an estimate of the relative magnitude of both derivatives. To this end we may assume that $U_{n-1}\sim U_0$ and substitute the above estimates into Eq. (37). For the usual values [1,5] of the correlation length ($\sim$ 10 cm), wavelength ($\sim 10^{-7}$ m) and interlayer distance ($<$ 10 km) [3,5,6] we obtain that the last two terms in the bracket in Eq. (37) can be safely neglected. However, for stronger turbulence and, thus, smaller correlation lengths this may be not possible for very large distances. In what follows we shall consider only the case when turbulence is not very strong and we can neglect these terms.

We remain with

$$U_n(\boldsymbol{r},s_n+h)=e^{i(hk+\varphi_n)}U_{n-1}\left(1-\frac{h\lambda}{4\pi}\nabla^2\varphi_n\right). \quad (38)$$

This formula is the analogue of Eq. (26) for the multilayer case. Accordingly, the intensity is given by

$$I_n(\boldsymbol{r},s_n+h)=I_{n-1}(\boldsymbol{r},s_n)\left(1-\frac{h\lambda}{2\pi}\nabla^2\varphi_n(\boldsymbol{r})\right). \quad (39)$$

We see that with regard to intensity, each layer in a multilayer system behaves exactly as one single layer of Section 1. For the full system of N layers shown in Fig. 1, we immediately obtain

$$I(\boldsymbol{r})=I_0\prod_{n=1}^{N}\left(1-\frac{h_n\lambda}{2\pi}\nabla^2\varphi_n(\boldsymbol{r})\right), \quad (40)$$

where $I(\boldsymbol{r})$ is the final intensity at the ground level, I_0 is the intensity of the initial monochromatic wave coming from the star to the first layer, h_n is the distance between layers n and $n+1$, whereas h_N denotes the distance from the last layer to the observation plane. In logarithmic form this equation (using condition (22)) reduces to

$$\log\frac{I(\boldsymbol{r})}{I_0}=\sum_{n=1}^{N}\log\left(1-\frac{h_n\lambda}{2\pi}\nabla^2\varphi_n(\boldsymbol{r})\right) \cong -\sum_{n=1}^{N}\frac{h_n\lambda}{2\pi}\nabla^2\varphi_n(\boldsymbol{r}). \quad (41)$$

Returning to Eq. (38) we can find the phase of the observed wavefront

$$\psi_n(\boldsymbol{r},s_n+h)=\arg\{U_n(\boldsymbol{r},s_n+h)\} = \psi_{n-1}(\boldsymbol{r},s_n)+kh+\varphi_n(\boldsymbol{r}). \quad (42)$$

We see that the phase is simply added at each layer, so that the final phase is, actually, a sum of the phases of all layers (ignoring an unimportant constant),

$$\psi(h)=\sum_{n=1}^{N}\varphi_n(\boldsymbol{r}). \quad (43)$$

Eqs. (40) and (43) are the generalizations of Eqs. (23) and (25) for a multilayer system. We have seen that they are valid for any attainable distances between layers from few wavelengths to hundreds of kilometers. The distance of a few wavelengths is of course a theoretical idealization. In practice layers do not have a rigid boundary, so we can interpret the above condition as simply saying that Eqs. (40) and (44) are valid for continuous turbulence too. Actually, the same equations for a continuous case are long known (see Ref. [7], Eqs. (6-18) and (6-20) and Ref.

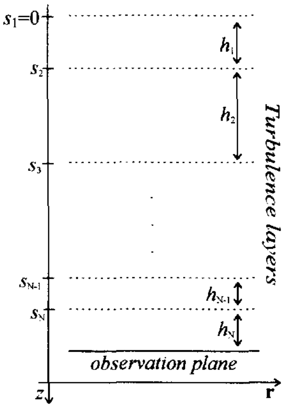

Fig. 1. Turbulence layers encountered by down-propagating wave front.

[8]). So, as we expected at the limit of continuous turbulence our derivation produces the same results as the current theory [7–9].

5. Determination of the intensity statistics

To demonstrate the use of the above theory we shall determine the correlation function of intensity fluctuations in the observation plane when only one layer is present. Such a calculation has been already made for continuous turbulence [7–9] and here, again, we may achieve similar results by representing the turbulence as a δ-function. But, as stated before, this approach will not provide us with any information concerning the validity of the formulae obtained. The method proposed here has the advantage of relative simplicity compared with the integral derivations of Refs. [7,8] and may be extended directly to turbulence layers which obey equations others than Eq. (41).

We assume that the phase fluctuations of the wave after passing the layer are approximately isotropic. Then their structure function may be defined as follows

$$D_\varphi(\boldsymbol{r}) = \langle [\varphi(\boldsymbol{r}_0) - \varphi(\boldsymbol{r}_0+\boldsymbol{r})]^2 \rangle, \tag{44}$$

where $\langle\ \rangle$ denotes an ensemble average over different realizations. In what follows we shall assume that this ensemble average can be interchanged with spatial average, i.e. with integration over $\boldsymbol{r}_0$ divided by the area of integration. We shall also assume that a similar structure function exists for the intensity fluctuations at the observation plane. In view of Eq. (24) we shall be interested, however, in the fluctuations of logarithmic intensity χ, which is defined as

$$\chi(\boldsymbol{r}) = \ln(I(\boldsymbol{r})/I_0). \tag{45}$$

That is, we wish to find

$$D_\chi(\boldsymbol{r}) = \langle [\chi(\boldsymbol{r}_0) - \chi(\boldsymbol{r}_0+\boldsymbol{r})]^2 \rangle. \tag{46}$$

First we shall find the Fourier transform of Eq. (46),

$$\tilde{D}_\chi(\boldsymbol{\omega}) = \int \mathrm{d}\boldsymbol{r}\, \mathrm{e}^{-2\pi i \boldsymbol{\omega}\cdot\boldsymbol{r}} \frac{1}{S} \int \mathrm{d}\boldsymbol{r}_0 [\chi(\boldsymbol{r}_0) - \chi(\boldsymbol{r}_0+\boldsymbol{r})]^2, \tag{47}$$

where S is the area of averaging (we assume it to be finite but very large to avoid the unnecessary complications of dealing with convergence problems). After interchanging the order of integrations, we obtain

$$\begin{aligned}\tilde{D}_\chi(\boldsymbol{\omega}) &= \frac{1}{S}\int \mathrm{d}\boldsymbol{r}_0 \int \mathrm{d}\boldsymbol{r}\, \mathrm{e}^{-2\pi i \boldsymbol{\omega}\cdot\boldsymbol{r}} [\chi^2(\boldsymbol{r}_0) + \chi^2(\boldsymbol{r}_0+\boldsymbol{r}) \\ &\quad - 2\chi(\boldsymbol{r}_0)\chi(\boldsymbol{r}_0+\boldsymbol{r})] \\ &= \frac{1}{S}\int \mathrm{d}\boldsymbol{r}_0 [\delta(\boldsymbol{\omega})\chi^2(\boldsymbol{r}_0) \\ &\quad + \mathrm{e}^{2\pi i \boldsymbol{\omega}\cdot\boldsymbol{r}_0}\tilde{\chi}^2(\boldsymbol{\omega}) - 2\mathrm{e}^{2\pi i \boldsymbol{\omega}\cdot\boldsymbol{r}_0}\chi(\boldsymbol{r}_0)\tilde{\chi}(\boldsymbol{\omega})].\end{aligned} \tag{48}$$

Performing the integration on $\boldsymbol{r}_0$ (it is assumed to be so large that we can take infinite integrals where possible) we get

$$\tilde{D}_\chi(\boldsymbol{\omega}) = \delta(\boldsymbol{\omega})\left(\langle \chi^2 \rangle + \frac{1}{S}\tilde{\chi}^2(\boldsymbol{\omega})\right) - \frac{2}{S}|\tilde{\chi}(\boldsymbol{\omega})|^2. \tag{49}$$

Noting that the expression multiplying the delta function has meaning only when $\boldsymbol{\omega} = 0$, and remembering that the Fourier transform at zero frequency is just the integral of the transformed function we obtain

$$\tilde{D}_\chi(\boldsymbol{\omega}) = 2\delta(\boldsymbol{\omega})\langle \chi^2 \rangle - \frac{2}{S}|\tilde{\chi}(\boldsymbol{\omega})|^2. \tag{50}$$

There is nothing in this derivation which uses the function χ specifically. Thus we can write a similar expression for any function for which similar assumptions could be made. In particular for φ we have

$$\tilde{D}_\varphi(\boldsymbol{\omega}) = 2\delta(\boldsymbol{\omega})\langle \varphi^2 \rangle - \frac{2}{S}|\tilde{\varphi}(\boldsymbol{\omega})|^2. \tag{51}$$

From Eq. (24) we obtain

$$\tilde{\chi}(\boldsymbol{\omega}) = 2\pi h\lambda\omega^2\tilde{\varphi}(\boldsymbol{\omega}). \tag{52}$$

Substituting this into Eq. (50) and solving together with Eq. (51) for $\tilde{D}_\chi(\boldsymbol{\omega})$ we get

$$\begin{aligned}\tilde{D}_\chi(\boldsymbol{\omega}) &= 2\delta(\boldsymbol{\omega})\left[\langle \chi^2 \rangle - (2\pi h\lambda\omega^2)^2\langle \varphi^2 \rangle\right] \\ &\quad + (2\pi h\lambda\omega^2)^2\tilde{D}_\chi(\boldsymbol{\omega}).\end{aligned} \tag{53}$$

Noting again that the expression in square brackets has meaning only for $\boldsymbol{\omega} = 0$ we obtain

$$\tilde{D}_\chi(\boldsymbol{\omega}) = 2\delta(\boldsymbol{\omega})\langle \chi^2 \rangle + (2\pi h\lambda\omega^2)^2\tilde{D}_\chi(\boldsymbol{\omega}). \tag{54}$$

Taking the inverse Fourier transform we arrive at the expression for the structure function of the log-intensity fluctuations

$$D_\chi(\boldsymbol{r}) = 2\langle \chi^2 \rangle + (h\lambda/2\pi)^2\nabla^4 D_\varphi(\boldsymbol{r}). \tag{55}$$

From the usual relation between the structure and correlation functions (when such exists) [9] we obtain

$$\Gamma_\chi(r) = -\tfrac{1}{2}(h\lambda/2\pi)^2\nabla^4 D_\varphi(r), \tag{56}$$

where $\Gamma_\chi(r)$ denotes the correlation function of the log-intensity fluctuations. An analogous expression for continuous turbulence was obtained by Reiger [8]. Thus we see that our results correspond to those of the continuous theory. However, as before the method presented here may be easily extended to systems with other governing equations whereas with integral formulation it becomes a much harder task.

Specifically for Kolmogorov turbulence [7,9]

$$D_\varphi(r) = C_n^2 r^{2/3}, \quad r \gg l_0 \tag{57}$$

we get

$$\Gamma_\chi(r) = -\frac{1}{2}\left(\frac{8}{9}\right)^2 C_n^2\left(\frac{h\lambda}{2\pi}\right)^2 r^{-10/3}, \quad r \gg l_0. \tag{58}$$

We see that for distances longer than the inner scale l_0, the log-intensity is anti-correlated, and as the distance increases, the correlation falls to zero as expected. Intensity fluctuations in this model are small (see Section 4). Thus, because we are dealing with log-intensity fluctuations we may assume that $\Gamma_\chi(0) = \langle \chi^2 \rangle \approx 1$. For reasonable values of the constants ($h \sim 10$ km, $\lambda \sim 0.5$ μm, $C_n^2 \sim 10^{-15}$ $m^{-2/3}$, $l_0 \sim 1$ mm) [9] we have $\Gamma_\chi(l_0) \sim 10^{-12}$. Thus the correlation length of the log-intensity is actually much smaller than l_0, and for all practical purposes the logarithmic intensity may be considered to be white noise. We know that for a large number of layers the log-intensity statistics should become Gaussian. This is because according to Eq. (41) the final log-intensity is composed of a large sum of terms. Each of these terms is a white noise as we have shown above. So by virtue of the central limit theorem we should obtain the normal distribution in the overall log-intensity. Thus, the intensity itself will be log-normal. This is in accordance with the result of Tatarskii [7], who proves it for continuous turbulence. We see that the statistics of the observed intensity may serve as an indicator whether the atmospheric turbulence in a specific location is concentrated in a few layers or is continuous with altitude.

6. Conclusions

We have presented a general method for determining the optical properties of systems of multiple turbulence layers. Although our results may be obtained by the conventional continuous theory, our method offers many advantages. Firstly for every result one obtains there are clear conditions of its validity. Secondly the derivation of all results may easily be repeated for systems which do not obey the conditions of weak turbulence and, thus, are out of reach for any approach that uses geometrical optics a priori or relies in its basis on the weakness of the turbulence.

We have derived the formulae for intensity and phase fluctuations of a wavefront that have passed multilayer turbulence. The equations obtained reveal many interesting features. First we should note that the equations for logarithmic intensity are differential ones. This is in contrast to conventional, continuous turbulence theory that results in integro-differential equations which are much harder to handle. For example, the derivation of statistical properties of the observed wavefront reduces, as we have shown, to a very simple procedure. Another important feature is the connection between the curvatures of the phase fluctuations in the turbulence layers and the resulting intensity. This may enable us to apply the well-established solutions of Poisson's equation to find the phase fluctuations of the wavefront. It may turn out to be very important in adaptive optics systems. Such methods should bear close resemblance to curvature sensing as formulated by Roddier [14]. One such method was recently proposed by Ribak et al. [13] for a system of two turbulence layers.

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