commentary

Fractal optics and beyond

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Fractals, shapes comprised of self-similar parts, are not merely prescribed linear structures. A wide class of fractals can also arise from the rich dynamics inherent to nonlinear optics.

In 1967, Benoit Mandelbrot published a paper that gave birth to the study of fractals, entitled "How long is the coast of Britain? Statistical self similarity and fractional dimension"[1]. According to Mandelbrot, "a fractal is a shape made of parts similar to the whole in some way"[2]. One particularly spectacular example of a fractal in nature is the Romanescu broccoli (Fig. 1).

Although there are a number of different fractal classification systems, one stands out rather distinctly: exact (regular) fractals versus statistical (random) fractals. An exact fractal is "an object which appears self-similar under varying degrees of magnification, in effect, possessing symmetry across scale, with each small part replicating the structure of the whole". Taken literally, when the same object replicates itself on successively smaller scales, even though the number of scales in the physical world is never infinite, we call this object an 'exact fractal'. When, on the other hand, the object replicates itself in only its statistical properties, it is defined as a 'statistical fractal'.

Perhaps the best known example of an exact fractal is the Cantor set fractal, a shape that can be explained by describing its generation. Starting with a single line segment, the middle third is removed to leave behind two segments, each with a length of one-third of the original. From each of these segments, the middle third is again removed, and so on, *ad infinitum*. At every stage of the process, the result is self-similar to the previous stage, which is identical upon rescaling. Of course, this 'triplet set' is not the only possible Cantor set. Any arbitrary cascaded removal of portions of the line segment may form the repetitive structure of an exact fractal. Other famous examples of exact fractals are the Sierpinski triangle and the Koch snowflake.

Statistical fractals have been observed in many physical systems, ranging from material structures such as polymers, aggregation and interfaces, through to biology, medicine, electric circuits, computer interconnects, galactic clusters and stock market price fluctuations[3]. Exact fractals, on the other hand, seem merely to be mathematical constructs, and it is not at all apparent that they exist in nature.


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Figure 1 | Many examples of fractal structures, such as the Romanescu broccoli displayed here, can be found in nature. Experiments are now showing how self-similarity and fractals can be observed in both linear and nonlinear optical systems.

In 1993 Mandelbrot won the prestigious Wolf Prize for "the widespread occurrence of fractals and developing mathematical tools for describing them", which has "changed our view of nature"[4]. This prize was presented in the field of physics, not mathematics. It is therefore fair to ask the question: has our knowledge of fractals really changed the way we view nature? The beauty of self-similarity is clearly present in exact fractals. But can exact fractals be found in physical systems? Mandelbrot presented many fascinating fractal constructs that look exactly like pictures taken by a wonderful photographer, or like scenes from a fantasy movie. The very fact that mathematical constructs can be structured in such artistic ways is undoubtedly very pretty. But do exact fractals have any true significance in physics?

The field of optics was one of the first to demonstrate exact fractals in a physical reality. Various optical settings are known to generate fractals in space. For example, in 1979 Michael Berry showed that diffraction from fractal structures can give rise to a fractal diffraction pattern[5]. In a similar fashion, fractal patterns emerge in many other optical settings, ranging from the lasing modes of unstable resonators[6] through to light diffracting from a binary grating[7].

Omel Mendoza-Yero and colleagues[8] recently demonstrated an interesting application of fractal optics by using an intriguingly simple method for forming temporal fractals from self-similar spatial structures. Their results highlight the interplay between time and space that is so inherent to coherent electromagnetic fields, and also suggest possible applications of 'fractal control' in the important fields of arbitrary waveform generation and pulse shaping. However, one should note that this is a linear system, in which the entire evolution is determined by the initial conditions — hence the results are not really surprising from a fundamental viewpoint. In this sense, what would be surprising and indeed much more unexpected is a nonlinear dynamical system in which the entire dynamics behaves as an exact fractal.

This kind of 'nonlinear thinking' takes us back to early work on self-similarity and fractals in nonlinear optics in the 1980s. The first suggestion that fractals would naturally emerge in nonlinear optical systems was made by Sergei Manakov and Ildar Gabitov, who studied light propagation in an inverted two-level system[9,10]. A decade later, Sunghyuck An and John Sipe suggested that the evolution of Hill gratings is self-similar[11], and Curtis Menyuk, Decio Levi and Pavel Winternitz proposed that the transient regime of stimulated Raman scattering gives rise to self-similar pulses[12]. In all of these cases, there were hints from the experiments that self-similarity dominates the long-term behaviour.

Around a decade later, two of us (Mordechai Segev and Marin Soljačić, then at Princeton), together with Menyuk, made the connection between solitons and fractals[13]. Solitons are self-trapped wavepackets that interact with one another in a manner similar to the way that particles do. In some cases, solitons are self-similar; that is, their shape is universal and their amplitude scales with their width. This happens, for example, for temporal solitons in optical fibres and for one-dimensional spatial solitons, both in

Kerr-type nonlinear media. Our idea was that nonlinear soliton-supporting systems could evolve under non-adiabatic conditions to give rise to self-similarity and fractals. Such fractals were believed to be observable in many systems, in which their existence would depend on two requirements[13]. First, the system should possess neither a natural length scale (that is, the physics should be the same on all scales) nor a natural scale in the parameter range of interest. Second, the system should undergo abrupt, non-adiabatic changes in at least one of its properties. Of course, the first requirement could physically occur only over a finite range, but that range should yield at least several generations of the self-similar structure.

Immediately after the connection between solitons and fractals was made[13], we, together with our co-workers at Princeton, proposed experimental systems in which Cantor set soliton fractals might be observable[14]. The goal was to design a system for generating the first exact fractals in nonlinear optics, perhaps even the first man-made exact fractals in nature. The idea involved launching a high-power pulse into a sequence of pre-designed dispersion-managed fibres[14]. The pulse would break up into 'daughter solitons' that are self-similar to one another in the sense that they can be mapped (by a change of scale only) onto one another because they all have the same shape. If, however, an adequately abrupt change is made to a property of the medium such as the dispersion coefficient, each of the daughter solitons would undergo the same breakup experienced by the initial mother pulse and generate even smaller 'granddaughter solitons'. Successive changes to the medium properties thus create successive generations of solitons on successively smaller scales (Fig. 2). The resulting structure after every breakup is self-similar with the products of the first breakup. Successive generations of breakups lead to a structure that is self-similar on widely varying scales, and each part breaks up again in a structure that replicates the whole. The entire structure is therefore a fractal comprised of solitons[14]. What is intriguing in this scheme is that the entire dynamics is self-similar; the entire dynamic process of pulse breakup repeats successively on a smaller and smaller scale, forming a 'dynamic nonlinear fractal'. This concept of linking fractal behaviour with soliton dynamics attracted significant attention from the wider physics community. Important early results obtained by Jianke Yang and Yu Tan from the University of Vermont demonstrated that collisions between specific types of solitons give rise to a fractal structure[15].

The idea of generating soliton fractals was so appealing that we (Segev and Soljačić) spent quite some time experimenting with

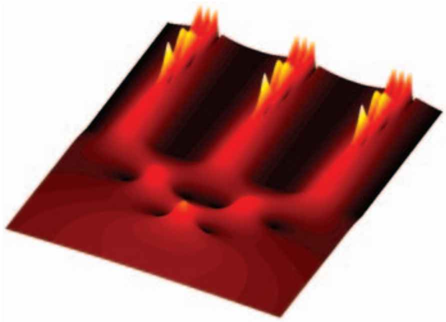

Figure 2 | Nonlinear Cantor set generation was first proposed from self-similar patterns of soliton-splitting.

wavepackets in space and in time, although generating exact fractals in nonlinear optics remained a challenge. Success was found 8,707 miles away in Auckland, New Zealand, where the third among us (John M. Dudley), together with his co-workers, made the first successful demonstration of self-similar scaling in a fibre amplifier. This development facilitated the design and interpretation of experiments that are now considered to be the first example of fractal scaling in nonlinear optics[16]. Subsequent experiments reported self-similar evolution in a wider range of fibre systems, and important work led by Frank Wise at Cornell University in the USA showed how self-similarity can be combined with optical feedback to create a self-similar laser[17]. More recent research by Ömer Ilday and co-workers at Bilkent University in Turkey led to the development of a laser cavity that combines two very different classes of nonlinear dynamics: self-similar propagation in one branch and soliton propagation in another[18]. The resilience of this laser to noise (due to the attractive nature of the nonlinear propagation in both branches) provides greatly improved stability and represents a beautiful example of how exploiting nonlinear dynamics can yield new technologies that are not possible in strictly linear systems.

The study of self-similarity and fractal dynamics has now become an important area of research in nonlinear fibre optics. New classes of fibre and improved measurement techniques are now enabling experiments to test the early ideas of this field. For instance, the development of gas-filled photonic crystal fibre has finally allowed theoretical work on self-similarity in Raman scattering[12] to be confirmed experimentally[19]. Even more recently, studies into the higher-order regimes of nonlinear modulation instability have revealed cascaded pulse splitting[20] of a type very close to that of the original prediction in ref. 14 for the Cantor set fractal. Soliton fractals have also been observed in systems beyond optics. The first of these was the work of Mingzhong Wu, Boris Kalinikos and Carl Patton at Colorado State University in collaboration with Lincoln Carr at the Colorado School of Mines, USA, who demonstrated exact fractals with spin-wave solitons in magnetic films[21]. Soliton fractals are now also being explored throughout various systems in nature.

We are now 45 years after Mandelbrot published his highly influential paper and almost 20 years after his Wolf Prize. The progress made in nonlinear optics over the past decade adds an important flavour to fractals because it proves that fractals — even exact fractals — can emerge naturally by virtue of interactions; that is, the nonlinear interplay between systems that is manifested in the plethora of light–matter interactions underlying nonlinear optics. There must be many other natural nonlinear systems in which fractals evolve naturally with self-similar nonlinear dynamics. In this respect, the statement made during the Wolf Prize ceremony was not only correct but visionary. Fractals are not merely prescribed linear structures; rather, nonlinear systems can give rise to dynamic fractals in which the entire evolution is self-similar on many scales. We are confident that the beauty of self-similarity has many more features yet to be discovered. □

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References

1. Mandelbrot, B. *Science* **156,** 636–638 (1967).
2. Mandelbrot, B. B. *Enc. Phys. Sci. Tech.* **5,** 579–593 (1987).
3. Addison, P. S. *Fractals and Chaos: An illustrated course* (IOP, 1997).
4. http://www.wolffund.org.il/full.asp?id=116
5. Berry, M. V. *J. Phys. A* **12,** 781–797 (1979).
6. Karman, G. P., Mcdonald, G. S., New, G. H. C. & Woerdman, J. P. *Nature* **402,** 138 (1999).
7. Berry, M. V. & Klein, S. *J. Mod. Opt.* **43,** 2139–2164 (1996).
8. Mendoza-Yero, O. *et al. Opt. Lett.* Doc. ID:156405 (2012).
9. Manakov, S. V. *Sov. Phys. JETP* **56,** 37–44 (1982).
10. Gabitov, I. R. & Manakov, S. V. *Phys. Rev. Lett.* **50,** 495–498 (1983).
11. An, S. & Sipe, J. E. *Opt. Lett.* **16,** 1478–1480 (1991).
12. Menyuk, C. R., Levi, D. & Winternitz, P. *Phys. Rev. Lett.* **69,** 3048–3051 (1992).
13. Soljacic, M., Segev, M. & Menyuk, C. R. *Phys. Rev. E* **61,** R1048 (2000).
14. Sears, S. *et al. Phys. Rev. Lett.* **84,** 1902–1905 (2000).
15. Yang, J. & Tan, Y. *Phys. Rev. Lett.* **85,** 3624–3267 (2000).
16. Fermann, M. E. *et al. Phys. Rev. Lett.* **84,** 6010–6013 (2000).
17. Ilday, F. Ö., Buckley, J. R., Clark, W. G. & Wise, F. W. *Phys. Rev. Lett.* **92,** 213902 (2004).
18. Oktem, B., Ülgüdür, C. & Ilday, F. Ö. *Nature Photon.* **4,** 307–311 (2010).
19. Nazarkin, A., Abdolvand, A., Chugreev, A. V. & Russell, P. St. J. *Phys. Rev. Lett.* **105,** 173902 (2010).
20. Erkintalo, M. *et al. Phys. Rev. Lett.* **107,** 253901 (2011).
21. Wu, M., Kalinikos, B. A., Carr, L. D. & Patton, C. E. *Phys. Rev. Lett.* **96,** 187202 (2006).