# Supplementary Information

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## 1. Generality of recurrent structures in complex networks

As we discuss in the paper, our results are relevant to a large class of complex networks, operating by virtue of a variety of interaction rules. That is, we find that many families of interaction rules lead to stable recurrent structures. We provide below several examples, and discuss which kind of interaction rules will not yield recurrent structures.

The interaction rule of the Manakov solitons is a special case of interaction expressed in an interesting formula: it is a bilinear transformation in one variable and a nontrivial nonlinear transformation in the other variable. To compare the Manakov case with some other formulae, we start by writing it here again (as in Fig.1a):

$$y = T_a^g(x) = \frac{\left(1-g+|a|^2\right)x + ga}{g\bar{a}x + 1 + (1-g)|a|^2}$$

$$b = T_x^h(a) = \frac{\left(1-\bar{h}+|x|^2\right)a + \bar{h}x}{\bar{h}\bar{x}a + 1 + \left(1-\bar{h}\right)|x|^2}$$

That is, the output state $y$ is found from an input state $x$ through a bilinear transformation $\mathrm{T}_{\mathrm{a}}^{\mathrm{g}}(x)$, given the other input state is $a$, under node parameter $g$. Equivalently, $\mathrm{T}_{\mathrm{x}}^{\mathrm{h}}(a)$ denotes the bilinear transformation of an input state a into an output state $b$, when the other input state is $x$, and the node parameter is $h$. In our networks, all nodes have the same parameters $g$ and $h$.

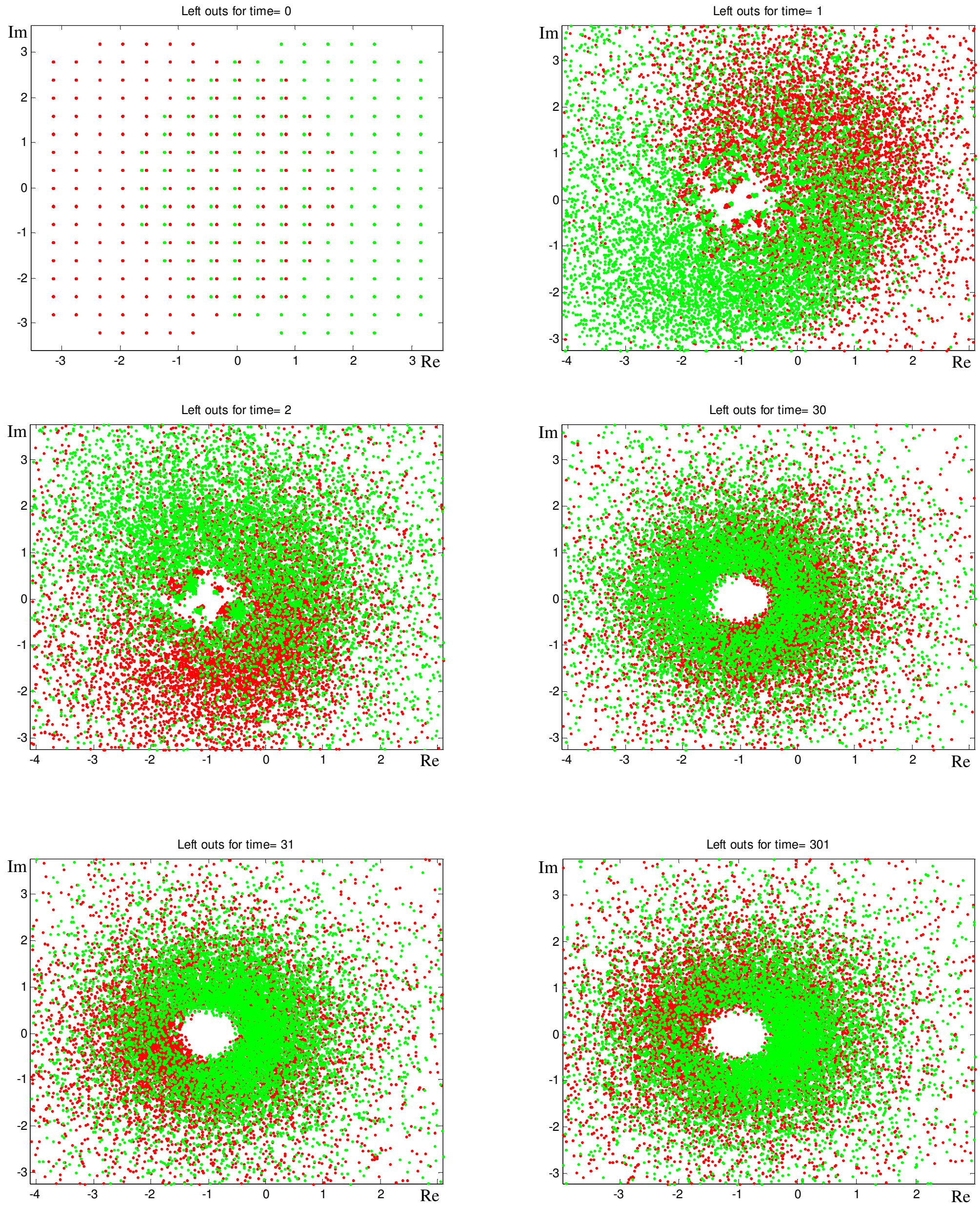
Below are several examples of interactions and their dynamics in a network of 10,000 nodes. In cases A-D below, the Markovian method can be used to obtain a self-consistency integral equation which is converging to the same dynamical recurrent structure as the full-scale simulation. The method is presented in the paper (see Eq. 2) and in section 2 of the supplementary material.

(A). A modified Manakov transformation which is bilinear in both inputs:

$$y = \frac{(1-g+a)x + ga}{gx + 1 + (1-g)a} \qquad g = \bar{h} = 0.8 - 0.4i$$

$$b = \frac{\left(1-\bar{h}+x\right)a + \bar{h}x}{\bar{h}a + 1 + \left(1-\bar{h}\right)x}$$

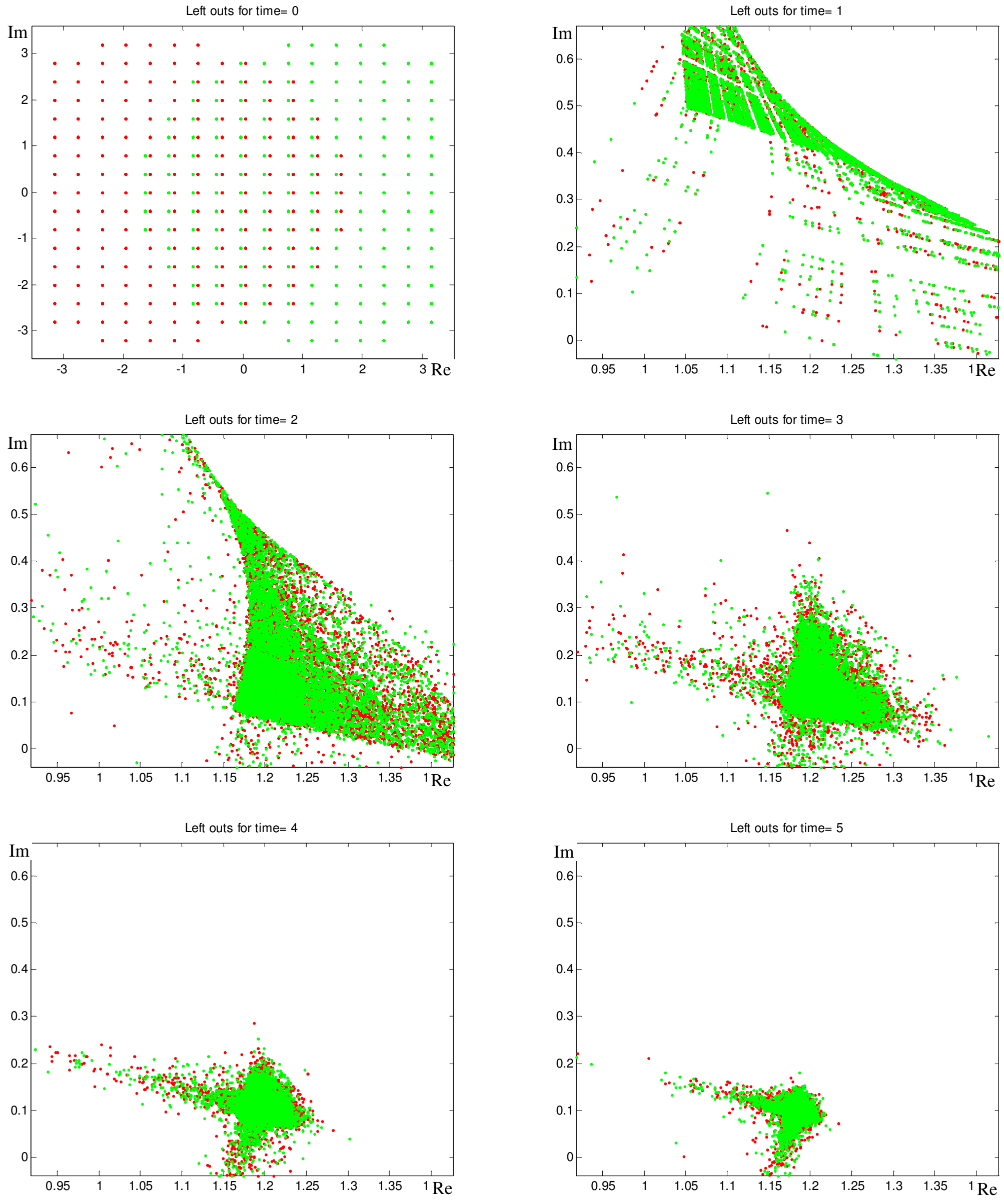
The initial conditions are chosen from two probability distributions that are shifted from one another by a real number. The dynamics evolves into a ring-shaped structure that is not isotropic but is whirling around the point $(-1,0)$. See figures below. Several other simulations reveal that different initial conditions evolve into other recurrent structures.



(B). A modified Manakov transformation which is nonlinear in both inputs, and depends on the modulus of the states:

$$\begin{aligned} y &= \frac{\left(1-g+|a|^2\right)|x|^2+g|a|^2}{g|x|^2|a|^2+1+(1-g)|a|^2} \\ b &= \frac{\left(1-\bar{h}+|x|^2\right)|a|^2+\bar{h}|x|^2}{\bar{h}|a|^2|x|^2+1+\left(1-\bar{h}\right)|x|^2} \end{aligned} \qquad g=\bar{h}=0.8-0.4i$$

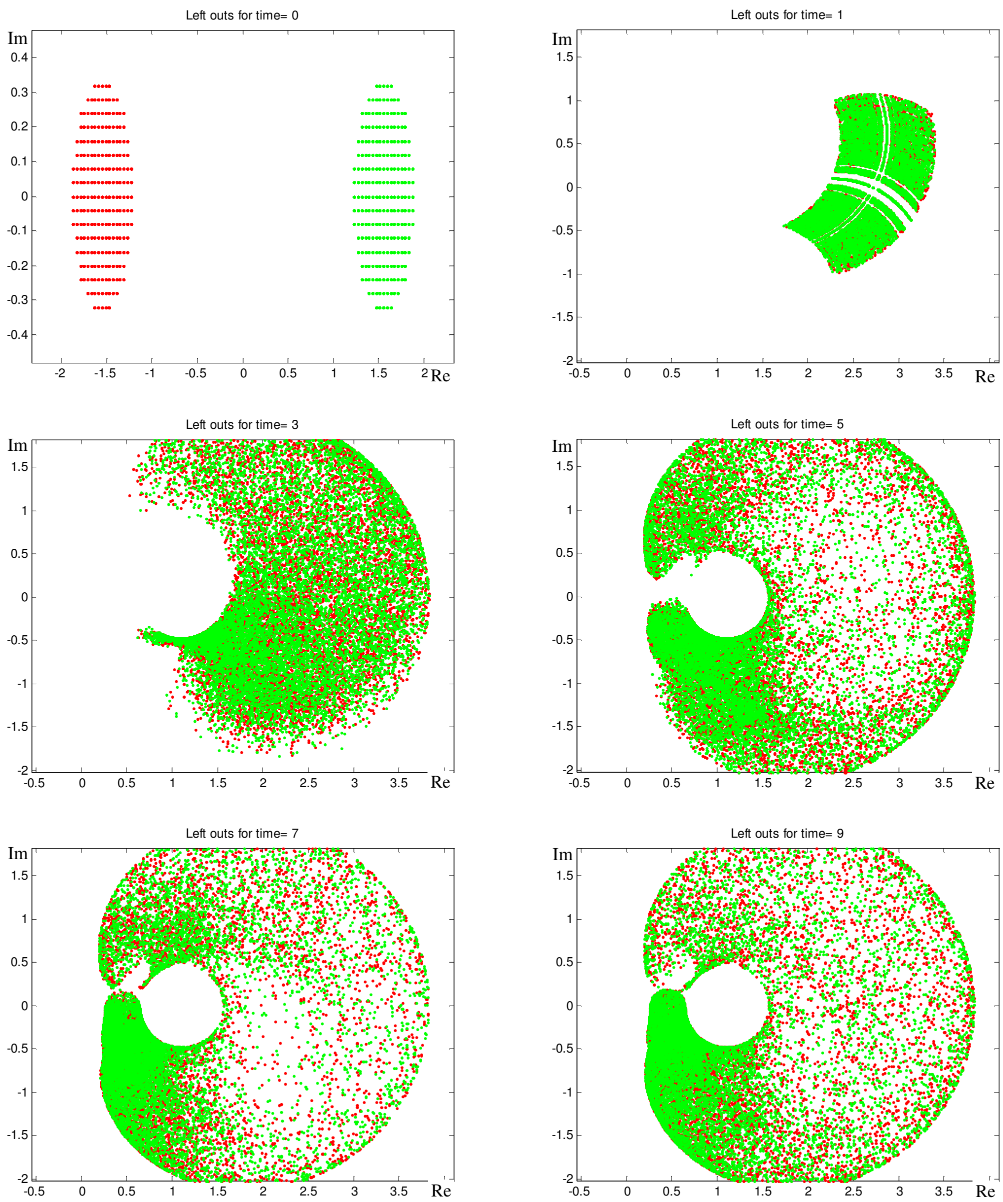
The initial conditions are chosen from the same distributions as in (A). The dynamics evolves into a triangular-shaped recurrent structure that is shrinking to the point $(1.1513, 0.0756)$. See figures below. Several other simulations reveal that different initial conditions evolve into other recurrent structures that are always shrinking into other basins of attractions. Note that this particular result – of a structure shrinking into a point - does not contradict the general concept that the networks evolve into recurrent dynamical structures, because a fix-point dynamic is just a special case of a recurrent shape.




(C). A formula that depends on the phases of the states, confining all states to an intersection of semi-circles in the complex plane:

$$y = e^{(1-g)x/\bar{x} + g\,a/\bar{a}} \qquad g = \bar{h} = 0.8 - 0.4i$$
$$b = e^{\bar{h}\,x/\bar{x} + (1-\bar{h})a/\bar{a}}$$

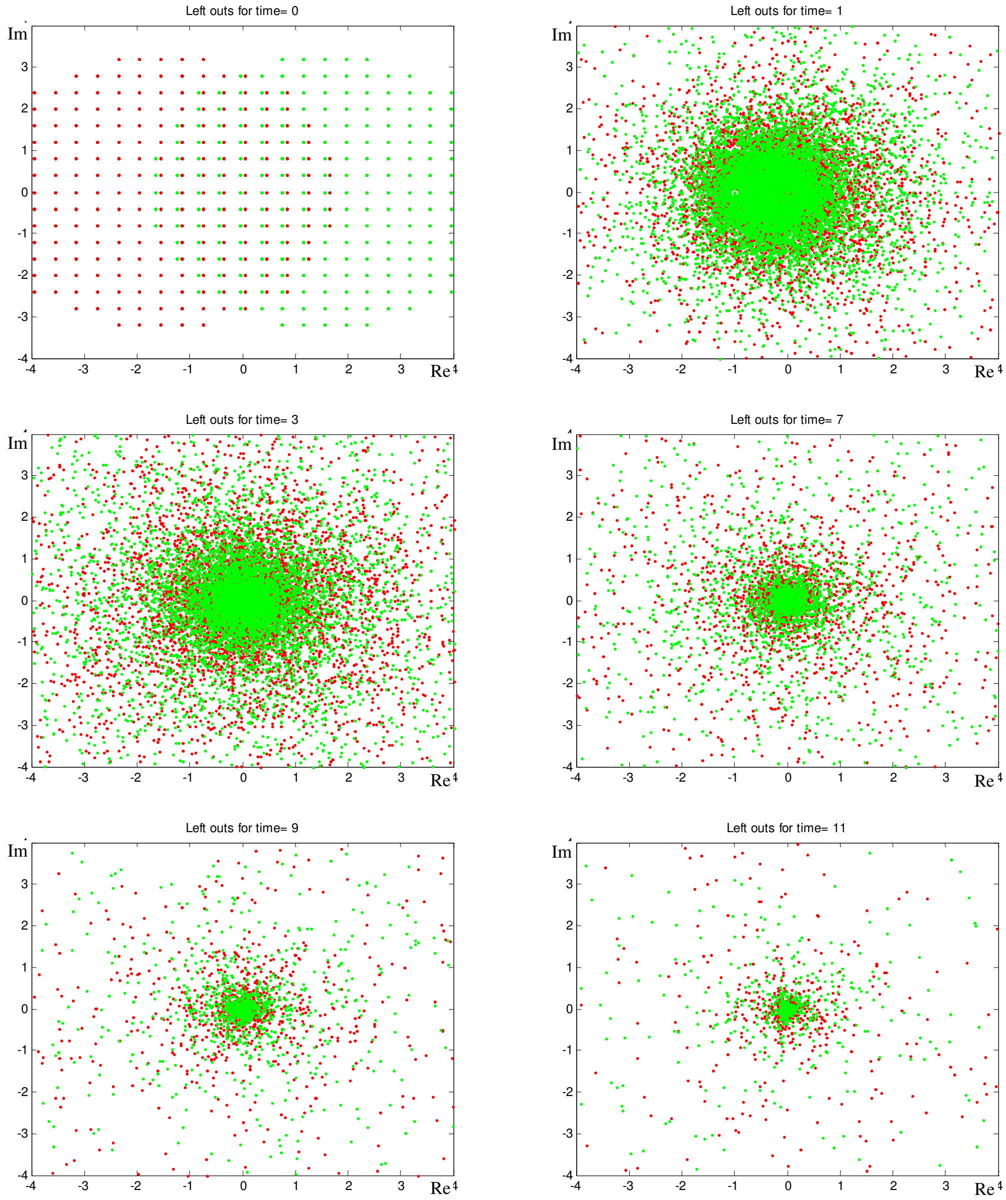
The initial conditions are chosen from similar distributions to (A). The dynamics evolves in a restricted ring around the point $(1.1513, 0.0756)$, where the recurrent structure is generated within the ring. See figures below. Several other simulations reveal that different initial conditions evolve into the same self-similar structure, where different choice of $g$ give different restricting semi-circles.



(D). Interaction rule constituting of simple division:

$$y = {}^{x}\!/_{a}$$
$$b = {}^{a}\!/_{x}$$

The initial conditions are chosen from the same distributions as (A). The dynamics is converging to the origin and diverging to infinity in the same time (the same states can move from one basin to the other in each interaction). Since infinity is a part of the extended complex plane, one can say that the dynamic converges into a double fix-point: the origin and the complex infinity. See figures below. It is easy to see that the structure shrinks, but there are other states that are diverging outside of the plot scope. Several other simulations reveal that different initial conditions evolve into the same double fix-point basin of attraction.




(E). A trivial remark, which is nevertheless important, refers to the convergence in the cases of "non-mixing" interaction rules. By "non-mixing" we mean formulae in which each of the output states depends on only one of the input states. For example, for any arbitrary $f$ the interaction rule:

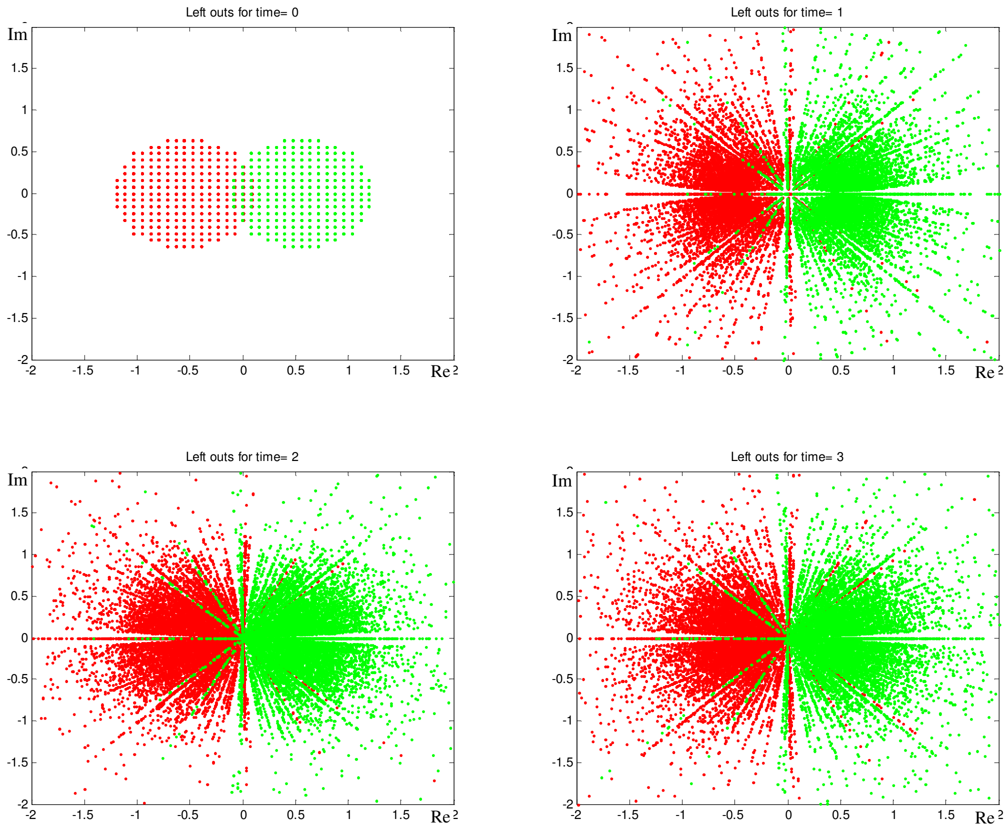
$$y = f(x)$$
$$b = f(a)$$

is "non-mixing". Such a formula may lead to non-converging dynamics, since the topology does not affect the dynamics. Instead, each state evolves according to the series: $x, f(x), f(f(x)), f(f(f(x))), \ldots$. In such a case, we are back within the boundaries of the standard dynamical processes theory, where it is well known that different $f$ and different initializations can exhibit a variety of dynamics: from fix-point through limit cycles to strange attractors and chaos. How will such interaction act in a network? Any initial state will evolve according to its own trajectory following its own autonomous dynamic. No recurrent structure will emerge.

(F). Another special family of interaction rules includes all formulae where mixing does occur but does not affect the whole state, i.e., where some conservation law does exist, but it involves each single state separately, rather than the dynamics in the whole system. A typical example is:

$$y = \frac{x}{|x|^2 + |a|^2}$$
$$b = \frac{a}{|a|^2 + |x|^2}$$

in which, one can easily see that $\arg(y) = \arg(x)$, and therefore the complex phase of each state will never change. The outcome of dynamics in a network with this interaction rule is important to understand since it stands right in the middle between the "non-mixing" family and the "converging to recurrent structures" family. For the above example, the dynamic is confined to lines intersecting the origin, since the phases cannot change. However, on each of those lines, a one dimensional recurrent structure will emerge. See figures below.




Following these examples, one can make a general statement: an arbitrary interaction rule will always have two kinds of effects on the input states in a complex network: a mixing effect which leads to a recurrent structure, and a non-mixing effect that evolves independently, irrespective of the other states in the network. Since the interacting states are multidimensional (2 dimensions in the case of complex numbers), we expect a mixture of those two dynamic evolutions. In general, the interaction rule will separate the state into several sub-dimensions where each will evolve according to one of the two possibilities for the evolution dynamics – mixing or non-mixing.

Finally, following the examples given above, and many other cases we have tested, we believe the results to be valid for networks of wider families of topologies, i.e. topologies where not all nodes are restricted to have 2 inputs / 2 outputs. For example, a network in which more than two particles can collide in each node will most likely also exhibit recurrent dynamics, since the topology will be complex as well. Topology relying on nodes with higher degree of connectivity will strengthen the argument of independence between consecutive interactions, needed for the validity of the Markovian method. More specifically, in such networks the values in sequential time steps are even more likely to be independent.

## 2. The Markovian method for complex networks – proof and example

In this section, we prove the self-consistency equation presented in the paper (Eq. 2). This equation is derived specifically for the Manakov interaction rule. However, similar derivation works for other interaction rules as well. We call it the Markovian method since it is based on the same assumption as the "Markov chain" characterizing random processes. We also give a comparison between the networks dynamics calculated via direct simulation and the evaluated probability density of states.

We calculate the probability density for the output state $y$ as a function of the densities of the input states $x,a$. We denote $P_X, P_A, P_Y, P_B$ for the probability densities of $x,a,y,b$. The letter $D$ indicates some arbitrary domain in the complex plane. Since $y = T_a^g(x)$ is a bilinear transform in $x$, it is invertible (in $x$) and we can write $x = T_a^{g,inv}(y)$. Thus,

$$\Pr(y \in D) = \Pr\left(T_a^g(x) \in D\right) = \Pr\left(x \in T_a^{g,inv}(D)\right) = \int_{a \in C} \int_{x \in T_a^{g,inv}(D)} P_{X,A}(x,a)\,dx\,da \ .$$

Now we make the only assumption needed for this calculation – statistical independence of the input states $x,a$. Under this assumption, $P_{X,A}(x,a) = P_X(x)P_A(a)$ hence

$$\Pr(y \in D) \underbrace{=}_{\substack{independency\\ assumption}} \int_{a \in C} \int_{x \in T_a^{g,inv}(D)} P_X(x)P_A(a)\,dx\,da$$

$$\underbrace{=}_{x \to T_a^{g,inv}(\lambda)} \int_{a \in C} \int_{\lambda \in D} P_X\left(T_a^{g,inv}(\lambda)\right)P_A(a)\left|Jacobian_{x \to T_a^{g,inv}(\lambda)}\right| d\lambda\,da$$

where the last substitution of the variable $x \to T_a^{g,inv}(\lambda)$ is needed to get the integral borders back onto the domain $D$.

For complex functions, one can prove that $\left|Jacobian_{x \to T_a^{g,inv}(\lambda)}\right| = \left|\frac{\partial T_a^{g,inv}(\lambda)}{\partial \lambda}\right|^2$ ,

which yields

$$\Pr(y \in D) = \int_{\lambda \in D} \int_{a \in C} P_X\left(T_a^{g,inv}(\lambda)\right) P_A(a) \left|\frac{\partial T_a^{g,inv}(\lambda)}{\partial \lambda}\right|^2 da\, d\lambda \ .$$

Since this is correct for any domain $D$, the density of $y$ is:

$$P_Y(y) = \int_{a \in C} P_X\left(T_a^{g,inv}(y)\right) P_A(a) \left|\frac{\partial T_a^{g,inv}(y)}{\partial y}\right|^2 da \ .$$

For the Manakov transformation $T_a^{g,inv}(y) = \frac{\left(1+(1-g)|a|^2\right)y - ga}{-g\bar{a}y + (1-g) + |a|^2}$ , hence:

$$P_Y(y) = \int_A P_X\left(\frac{\left(1+(1-g)|a|^2\right)y - ga}{-g\bar{a}y + (1-g) + |a|^2}\right) P_A(a) \left|\frac{(1-g)\left(1+|a|^2\right)^2}{\left[g\bar{a}y - |a|^2 - (1-g)\right]^2}\right|^2 da \, .$$

A similar expression can be derived for $P_B(b)$.

Note that this process is a nonlinear convolution of the probabilities of $x, a$. It can be done in a similar fashion for any other interaction rule (although obviously, a transformation which is invertible in at least one of its variable, is preferred).

In order to get the self-consistency formula (Eq. 2 from the paper) we still have one step to go. We treat all the inputs states of the network as random variables of the same probability. This argument is reasonable since the network topology is completely random, hence any particular position of any specific node is irrelevant. We now add the notation of the discrete time $n$, expecting the probability density of the states to change after each time step due to interactions. This yields

$$P_{n+1}^{Left}(y) = P_{X,n+1}(y) = P_{Y,n}(y) = \int_A P_{A,n}(a) P_{X,n}\left(\frac{\left(1+(1-g)|a|^2\right)y - ga}{-g\bar{a}y + (1-g) + |a|^2}\right)\left|\frac{(1-g)\left(1+|a|^2\right)^2}{\left[g\bar{a}y - |a|^2 - (1-g)\right]^2}\right|^2 da$$

The final result is a set of two discrete-time integral equations:

$$\begin{cases} P_{n+1}^{Left}(y) = \int_A P_n^{Right}(a) P_n^{Left}\left(\frac{\left(1+(1-g)|a|^2\right)y - ga}{-g\bar{a}y + (1-g) + |a|^2}\right)\left|\frac{(1-g)\left(1+|a|^2\right)^2}{\left[g\bar{a}y - |a|^2 - (1-g)\right]^2}\right|^2 da \\ P_{n+1}^{Right}(b) = \int_A P_n^{Left}(x) P_n^{Right}\left(\frac{\left(1+(1-\bar{h})|x|^2\right)b - \bar{h}x}{-\bar{h}\bar{x}b + (1-\bar{h}) + |x|^2}\right)\left|\frac{(1-\bar{h})\left(1+|x|^2\right)^2}{\left[\bar{h}\bar{x}b - |x|^2 - (1-\bar{h})\right]^2}\right|^2 dx \end{cases}$$

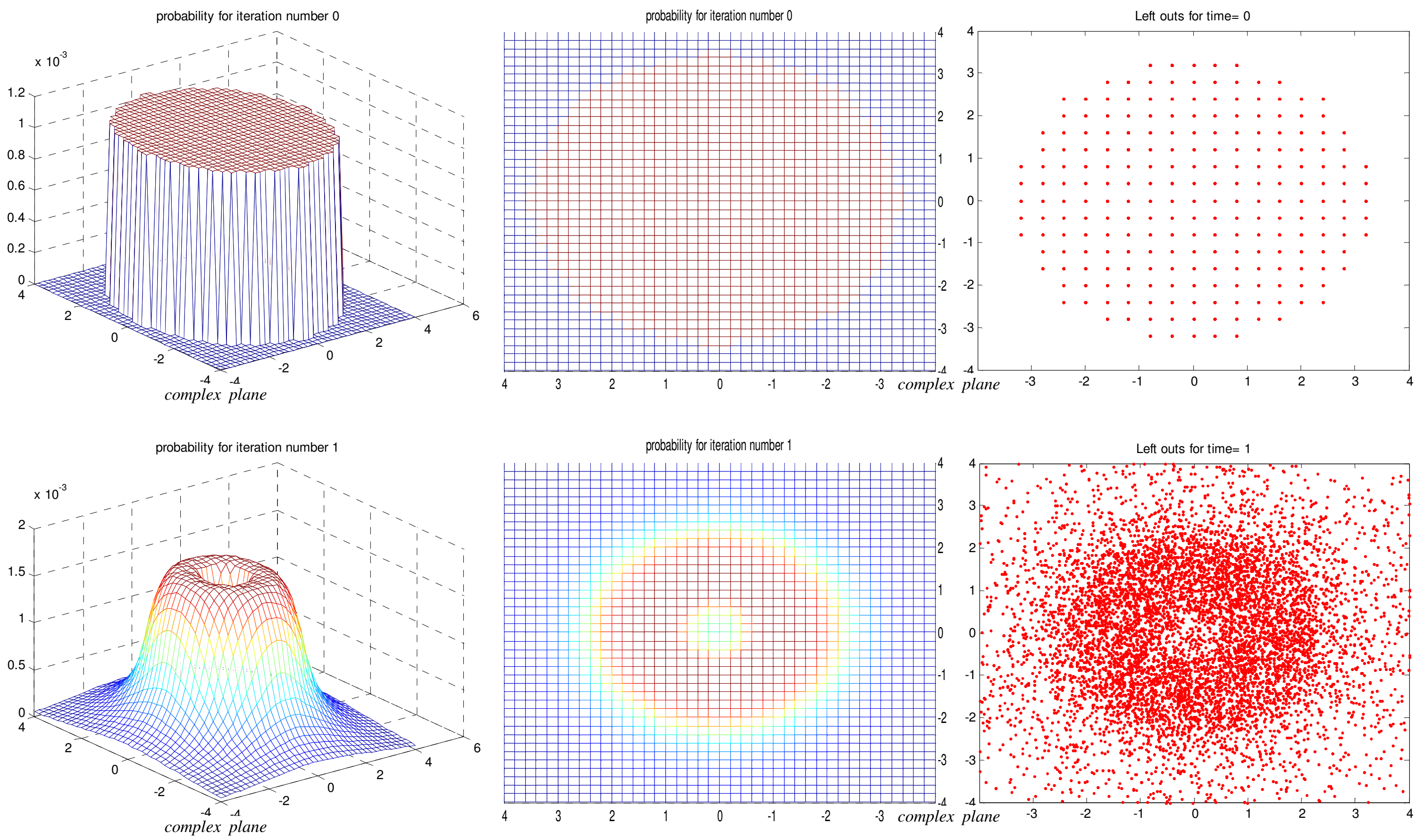
The solutions for these equations are expected to converge to the same recurrent structure exhibited by the same network under a full numerical simulation. The particular case of symmetric initial conditions ($P_0^{Left} = P_0^{Right}$) and symmetric interactions ($g = \bar{h}$), yields the reduced equation:

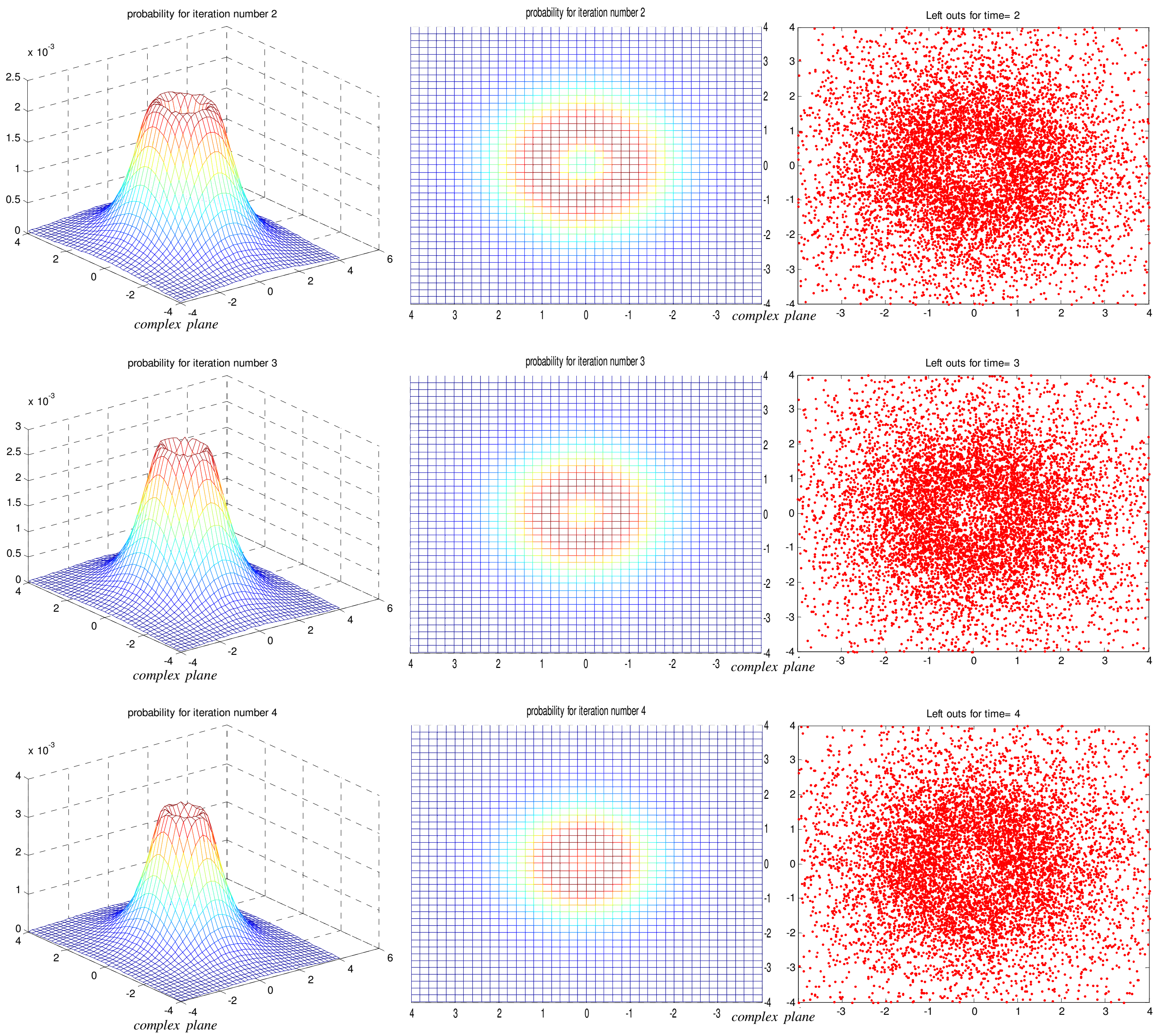
$$P_{n+1}(y) = \int_A P_n(a) P_n\left(\frac{\left(1+(1-g)|a|^2\right)y - ga}{-g\bar{a}y + (1-g) + |a|^2}\right)\left|\frac{(1-g)\left(1+|a|^2\right)^2}{\left[g\bar{a}y - |a|^2 - (1-g)\right]^2}\right|^2 da$$

We call it the self-consistency equation, since we can look for the solution of $P_{n+1} = P_n$ which will be the recurrent structure.

This method has much in common with tools used in Markov chain processes, which are widely used for linear processes. The underlying assumption here is the statistical independence of the present state on the previous state, without any memory. So although our transformation is nonlinear and the probabilities are not discrete, we refer to this method as the "Markovian method" throughout the paper and the supplementary material.

To illustrate the importance of the above equation, we apply it for the symmetric case and compare its results with those of a simulated network of 20,000 nodes. The figure bellow shows the states of the network in the complex plane (on the right column), with the density of the Markovian method (on the left and middle columns). The same dynamics is found in both cases. Hence, the recurrent dynamics of the network can be found by solving the equation self-consistently, instead of simulating the evolution dynamics of the entire network.





probability for iteration number 2
x 10^-3
complex plane
probability for iteration number 2
complex plane
Left outs for time= 2
probability for iteration number 3
x 10^-3
complex plane
probability for iteration number 3
complex plane
Left outs for time= 3
probability for iteration number 4
x 10^-3
complex plane
probability for iteration number 4
complex plane
Left outs for time= 4

## 3. Background information on Manakov solitons

Manakov solitons were studied extensively, and closed-form solutions were found even for interacting Manakov solitons ([8]). By defining an appropriate state, which is a complex number, one can write an explicit transformation for this state before and after interaction. We call it "the interaction rule" and it is written here (as in Fig. a1) with two other representations. Further reading on Manakov solitons can be found in [M. H. Jakubowski, Ph.D. thesis, Princeton University, 1998]

$$y = \frac{\left(1-g+|a|^2\right)x+ga}{g\bar{a}x+1+(1-g)|a|^2} = \frac{\left(\frac{(1-g)}{\bar{a}}+a\right)x+g\frac{a}{\bar{a}}}{gx+\frac{1}{\bar{a}}+(1-g)a} = \frac{(a-(1-g)\tilde{a})x-ga\tilde{a}}{gx-\tilde{a}+(1-g)a}$$

$$b = \frac{\left(1-\bar{h}+|x|^2\right)a+\bar{h}x}{\bar{h}\bar{x}a+1+\left(1-\bar{h}\right)|x|^2} = \frac{\left(\frac{\left(1-\bar{h}\right)}{\bar{x}}+x\right)a+\bar{h}\frac{x}{\bar{x}}}{\bar{h}a+\frac{1}{\bar{x}}+\left(1-\bar{h}\right)x} = \frac{\left(x-\left(1-\bar{h}\right)\tilde{x}\right)a-\bar{h}x\tilde{x}}{\bar{h}a-\tilde{x}+\left(1-\bar{h}\right)x}$$

$$\tilde{x} = \frac{-1}{\bar{x}} \qquad \tilde{a} = \frac{-1}{\bar{a}}$$

Where we define $g,h = \left(k_{1,2}+\bar{k}_{1,2}\right)/\left(k_{2,1}+\bar{k}_{1,2}\right)$, with $k_{1,2}$ being the complex parameters that are invariant during the collisions (and are hence unchanged for the entire network), for the left and right solitons, respectively. The real and imaginary parts of $k_{1,2}$ represent the power and the velocity of the soliton (respectively), hence collisions of two Manakov solitons of equal power result in $g=\bar{h}$. Mathematically,

$$\begin{aligned} g &= \frac{k_1+\bar{k}_1}{k_2+\bar{k}_1} \\ h &= \frac{k_2+\bar{k}_2}{k_1+\bar{k}_2} \end{aligned} \quad \Rightarrow \quad \begin{gathered} \textit{if } \operatorname{Re}\{k_1\}=\operatorname{Re}\{k_2\}=W \\ \textit{and } \operatorname{Im}\{k_2\}-\operatorname{Im}\{k_1\}=2\Delta \end{gathered} \quad \Rightarrow \quad \begin{aligned} g &= \frac{2W}{2W+2i\Delta} = \left(1+i\Delta/W\right)^{-1} \\ h &= \frac{2W}{2W-2i\Delta} = \left(1-i\Delta/W\right)^{-1} \end{aligned}$$

## 4. Deriving the closed-form solution of the butterfly network

Here we explain how to derive the conservation law of the butterfly network. The topology of the butterfly network is shown in Fig. 1 in the paper. The conservation law is a circle on which all possible states are restricted to reside. This circle is not changing throughout the entire network dynamic; hence the only degree of freedom is the phase of a state on the circle.

Any three points on a plane uniquely define a circle, unless they are on a line. Therefore, in the complex plane, a generalized circle is usually defined to be a circle or a line. A useful formula to obtain the center of the circle as a function of the three (complex) points:

$$CircleCenter = \frac{\begin{vmatrix} |z_1|^2 & |z_2|^2 & |z_3|^2 \\ z_1 & z_2 & z_3 \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} z_1 & z_2 & z_3 \\ 1 & 1 & 1 \\ \overline{z_1} & \overline{z_2} & \overline{z_3} \end{vmatrix}}$$

Defining the trajectories of the initial states $x_0, a_0$ to be $x_0, x_1, x_2, x_3 ...\quad a_0, a_1, a_2, a_3 ...$ and remembering they all depend only on $x_0, a_0, g, h$, we now find the center of the circle created from $x_0, a_0, x_1$ using the above formula. This requires some nontrivial algebra, and is best done by using Mathematica to do symbolic calculations, which yields

$$CircleCenter = \frac{a_0\left(1 + x_0\overline{a}_0\right) + x_0\left(1 + a_0\overline{x}_0\right)}{\left(1 + x_0\overline{a}_0\right) + \left(1 + a_0\overline{x}_0\right)}$$

Note that the obtained center does not depend on $g$, hence symmetry leads to the fact that $a_1$ is also restricted to the same circle. To go further, we define

$$C(a,x) = \frac{a(1+x\bar{a}) + x(1+a\bar{x})}{(1+x\bar{a}) + (1+a\bar{x})}.$$

Consider solitons of equal power, which gives $g = \bar{h}$. This implies complete symmetry in the interaction rules. Furthermore, it allows us to prove $C(a_1, x_1) = C(a_0, x_0)$ using some more algebra (that can best be done symbolically using Mathematica), and the definition of $g = \left(1 + i\Delta/W\right)^{-1}$. This last result is actually very strong, because it holds for any initial states $a_0, x_0$. Thus, an equivalent statement is $C(a_{i+1}, x_{i+1}) = C(a_i, x_i)$ proving that the circle center is a constant of the dynamic. The proof is completed by using induction with the above statement as the inductive step and $C(a_0, x_0)$ as the base case.

$$C(a_i, x_i) = \frac{a_i(1+x_i\bar{a}_i) + x_i(1+a_i\bar{x}_i)}{(1+x_i\bar{a}_i) + (1+a_i\bar{x}_i)} = const$$

The circle radius can be calculated by any of $R = |a_i - C| = |x_i - C|$.

A natural question to ask next refers to the phases of the states on the circle. We do not have a closed-form expression for the phases as of yet, but we do know that the two phases of $a_i, x_i$ are related by an invertible function. They can be related by extracting one of them from $C(a_i, x_i)$. Consequently, after stating the initial conditions $x_0, a_0$ for the butterfly network, only one (real) degree of freedom remains. Hence, just one complex phase characterizes the entire butterfly network.

For unequal soliton power ($g \neq \bar{h}$), two distinct circles are restricting the left and right states separately. Their close form analytical expressions and conservation laws will be presented elsewhere.

## 5. The role of noise in the dynamics of the Solitonet

Noise may play an important role in the dynamics of complex systems. It is therefore natural to ask about the role of noise in Solitonets. The answer has two parts, one having to do with the role of noise in the actual waves (i.e., launching non-ideal solitons), and the other with adding noise to the complex states defining the Manakov solitons.

(A). **Adding noise to the actual waves comprising the solitons**. The Solitonets discussed in the paper are not only networks constructed from nodes connected through some mathematical interaction rules, but they actually represent physical entities: two-component self-localized waves described by integrable equations. Having noise in the system where such waves are propagating is therefore natural. However, Manakov solitons are known to be stable ([8]). More specifically, the noise will excite other modes of the system, of which some are radiation modes that will propagate away from the interaction region (out of the system). All parts of the noise that do not disappear, in the long run will change the state of the soliton (see part 2).

(B). **Adding noise to the complex states of the networks**. Here, noise will have a minor effect on the stochastic structure. Such noise will be generally averaged out (although not necessarily in a linear way), before affecting the final structure. More importantly, the dependence on the recurrent state on the initial conditions is only through the stochastic density of the initial conditions (as discussed earlier), so adding noise can anyway affect the recurrent state only if the noise broadens the variance of the entire initial density. And even then it will just be equivalent to "clean" initial conditions with an increased variance. Altogether, noise is expected to have only minor effects on the recurrent state of a network.