Collisions of Two Solitons in an Arbitrary Number of Coupled Nonlinear Schrödinger Equations

Marin Soljačić,[1] Ken Steiglitz,[2] Suzanne M. Sears,[3] Mordechai Segev,[4] Mariusz H. Jakubowski,[5] and Richard Squier[6]

[1]*Department of Physics, MIT, Cambridge, Massachusetts 02139, USA*
[2]*Department of Computer Science, Princeton University, Princeton, New Jersey 08544, USA*
[3]*Department of Physics, Princeton University, Princeton, New Jersey 08544, USA*
[4]*Technion-Israel Institute of Technology, Haifa 32000, Israel*
[5]*Microsoft Corporation, Redmond, Washington 98052, USA*
[6]*Department of Computer Science, Georgetown University, Washington, D.C. 20057, USA*

(Received 29 November 2002; published 26 June 2003)

We show that pairwise soliton collisions in $N > 2$ intensity-coupled nonlinear Schrödinger equations can be reduced to pairwise soliton collisions in two coupled equations. The reduction applies to a wide class of systems, including the N-component Manakov system. This greatly simplifies the analysis of such systems and has important implications for the application of soliton collisions to all-optical computing.

DOI: 10.1103/PhysRevLett.90.254102 PACS numbers: 05.45.Yv, 42.65.Tg, 42.81.Dp

Ever since their discovery [1], solitons have fascinated scientists in many widely different fields. Probably the best studied solitons are those of the $(1 + 1)$D cubic nonlinear Schrödinger equation (NLSE), which models propagation in Kerr media, where the nonlinearity is proportional to the intensity of the field. There are two reasons for this. First, the Kerr nonlinearity appears in many different systems. It represents a weak symmetric anharmonicity, which is equivalent to weak saturation in a simple harmonic oscillator. In many cases this is a valid model for the envelope of waves in plasmas, shallow water, deep water, gravity, etc. [2]. The second reason is that Kerr solitons are mathematically elegant—the $(1 + 1)$D cubic NLSE is integrable. Unfortunately, its simplest generalization (to more than a single transverse dimension) has no stable solitons [3]. However, if we also consider the simplest generalization of the nonlinearity (when nonlinearity is an arbitrary function of the field intensity), we can easily find many nonlinearities (e.g., saturable, cubic-quintic, etc.) that support stable solitons. Most centrosymmetric media can be modeled with some of those equations. Consequently, there is an abundance of systems in which exciting multidimensional soliton phenomena (such as collisions, angular momentum, etc.) occur [4].

Recently, motivated by the richness of the new phenomena emerging from generalizations of the $(1 + 1)$D cubic NLSE into higher dimensions and with different forms of nonlinearity, significant interest has been drawn to the next natural generalization of these equations, namely, to multiple equations describing multiple, jointly coupled fields (so-called *coupled* NLSEs). Solitons of coupled NLSEs are called *vector* solitons. In their simplest incarnation, such equations have a coupling term that is a function only of the sum of the intensities of all the fields (*intensity-coupled* NLSEs). Vector solitons were first suggested by Manakov [5], with two nonlinear Schrödinger equations coupled through cubic nonlinearities. This pair of equations is integrable and solvable analytically. Temporal intensity-coupled NLSE solitons were proposed in optical fibers more than a decade ago [6–8], and evidence for their existence has been recently reported [9,10]. In the spatial domain, evidence for the existence of Manakov solitons has also been reported in Kerr media [11] and in photorefractives [12]. In contrast to the Kerr nonlinearity, the photorefractive nonlinearity is saturable, but coincides with the Kerr nonlinearity in the limit of very low intensities [13,14]. Another method of generating spatial Manakov solitons arises from cascading optical rectification and the electro-optic effect [15,16]. Finally, solitons of coupled NLSEs should exist in Bose-Einstein condensates (BEC) of cooled atomic gases, when multicomponent condensates are employed [17–23].

One of the most exciting phenomena associated with solitons is their collisions. In linear media, a localized wave packet propagates through another wave packet completely unaffected by its presence. In contrast, solitons can exchange energy, bounce off each other, spiral around each other, and display many other exciting interaction-associated phenomena [4]. Unfortunately, in the NLSE with Kerr nonlinearity, scalar solitons affect each other only by a phase shift that depends only on the soliton power and velocity, which are both conserved quantities. Thus, when two (scalar) Kerr soliton collisions occur sequentially, the outcome of the first collision does not affect the second collision (except for the uniform phase shift). It then came as a surprise that collision interactions between vector solitons can be very strong [24]. Such strong interactions, besides being fundamentally interesting, have also opened the exciting possibility of soliton applications to the implementation of all-optical logic in a way that does not require fabrication of individual gates [25–28]. Computers based on these "virtual" gates could in principle be built in any medium that supports appropriate solitons, and computation could be embedded in homogeneous materials. Moreover, recent results [29] have demonstrated the feasibility of performing quantum information processing in Bose-Einstein condensates, thus opening up the possibility of using solitons in BEC media to perform quantum information processing.

A natural question is how the complexity of collisions of vector solitons is affected in going from $N = 2$ to $N > 2$ components. Recent analysis by Kanna and Lakshmanan in [30] has yielded explicit solutions for collisions in N-component Manakov vector systems with $N > 2$ components. Having such solutions is indeed an important and fascinating result, but it does not make the predictions of interactions between Manakov solitons easy in any sense. In fact, a major effort was carried out by Kanna and Lakshmanan to find expressions for the evolution of two and three solitons. The result, albeit analytic, remains complicated and does not reveal the physical collision properties in a clear way. On the other hand, for the two-component Manakov system, the authors of [25] have developed a linear fractional transformation that characterizes a collision between two Manakov solitons in a very simple and intuitive way that highlights all the physical properties of a complex collision. The main purpose of this Letter is to point out that for many cases of interest, including many N-component Manakov systems, two-soliton collisions in such higher-component systems can be reduced to two-soliton collisions in two-component systems [31]. The reduction does not rely on integrability, and applies to any intensity-coupled NLSE. This reduction of complexity significantly eases the analysis, provides useful intuition, and allows us to apply the considerable accumulated knowledge of collisions in two-component Manakov to N-component Manakov systems (including providing us with a simple analytic expression for the outcome of soliton collisions). Most important, this result shows that in contrast to what might be naïvely expected, and to what was previously conjectured [30], no new complexity appears in the collision of two solitons by expanding the number of coupled equations beyond two.

The coupled nonlinear Schrödinger equations (CNLS) that describe our system are

$$i\frac{\partial \vec{q}}{\partial t} + \nabla_T^2 \vec{q} + 2f(|\vec{q}|^2)\vec{q} = 0, \tag{1}$$

where $\vec{q}(t, \vec{r}) = [q_1(t, \vec{r}), \ldots, q_N(t, \vec{r})]$ are the N complex fields of the system, f is a real-valued function of a single real variable, the subscript T denotes the directions orthogonal (transverse) to the direction of propagation, $\vec{r}$ are the coordinates transverse to the propagation direction, and t denotes the propagation direction. The form of the nonlinearity present in Eq. (1) appears in any centrosymmetric, slightly anharmonic, system when the various field components interact only through the combined intensity of all the fields. For example, in optics, a soliton in Eq. (1) is composed of N mutually incoherent yet jointly self-trapped fields, with the total intensity of the soliton equal to the sum of the intensities of the component fields. This is consistent with the interpretation of the soliton as a vector in N-dimensional space, with orthogonal components represented by the amplitudes of the component fields. As another example, multicomponent Bose-Einstein condensates [17,18] can often be modeled by this system.

We first state two important properties of Eq. (1) above. These are well known, but we include them for completeness.

Property 1.—The total energy in each of the N fields (components) is conserved.

Proof: Multiply the kth component of Eq. (1) above by q_k^*, $k = 1, \ldots, N$, and subtract from the complex conjugate of the same equation, yielding

$$i\left(q_k^*\frac{\partial q_k}{\partial t} + q_k\frac{\partial q_k^*}{\partial t}\right) + (q_k^*\nabla_T^2 q_k - q_k\nabla_T^2 q_k^*) = 0. \tag{2}$$

Next, integrate this expression over the entire transverse volume. The two terms inside the second set of parentheses can be integrated by parts, which gives us four terms instead. Two of these terms cancel each other identically, while the other two can be transformed into surface integrals. This surface is infinitely far away, so the fields and their derivatives are all zero there, and thus those terms vanish also. We conclude that the total energy in each component is an invariant of motion:

$$\frac{\partial}{\partial t}\int_{\text{trans vol}} |q_k(t, \vec{r})|^2\, d^\nu \vec{r} = 0, \tag{3}$$

where ν is the dimensionality of the transverse space. ■

The second proposition states the invariance of solutions under unitary transformation.

Property 2.—If $q(t, \vec{r})$ *satisfies Eq.* (1), *then so does* $Uq(t, \vec{r})$, *where U is any (constant) unitary matrix.*

Proof: This follows directly from that fact that the form of our nonlinear term respects the symmetry of the unitary group, $U(N)$. ■

Now consider a collision of two N-component solitons. Before the collision, the two solitons are well separated, are moving at some angle towards each other, and can therefore be written in the form

$$\vec{q}(t \to -\infty, \vec{r}) = \vec{\alpha}_1\psi_1(t \to -\infty, \vec{r}) + \vec{\alpha}_2\psi_2(t \to -\infty, \vec{r}), \tag{4}$$

where we write a lowest-order soliton of Eq. (1) as $\vec{\alpha}\psi(t \to -\infty, \vec{r})$. By "lowest-order soliton," we mean the soliton that has no nodes and has the same modal profile for all components. The intensity profile of this soliton does not depend on the propagation distance. For example, in the case of the (1 + 1)D cubic coupled nonlinear Schrödinger equation, ψ has the usual sech shape.

The simple (but key) observation is that the vectors $\vec{\alpha}_1$ and $\vec{\alpha}_2$ lie in a 2D complex plane within a complex N-dimensional space. Consequently, we are free to pick a unitary matrix (the choice of this matrix is typically not unique) that maps $\vec{\alpha}_1$ to $\vec{\alpha}'_1 = U\vec{\alpha}_1 = (a, 0, \ldots, 0)$ and $\vec{\alpha}_2$ to $\vec{\alpha}'_2 = U\vec{\alpha}_2 = (b, c, 0, \ldots, 0)$, where a, b, and c are complex numbers. That is, we can "rotate" the coordinate system so that all the energy of the two colliding solitons resides in the first two coordinates. We will denote the rotation of any vector $\vec{v}$ by $\vec{v}' = U\vec{v}$.

A concrete construction of the matrix U is as follows. Pick the orthonormal basis $(\vec{\beta}_1, \vec{\beta}_2)$ in the $\vec{\alpha}_1$-$\vec{\alpha}_2$ plane, which has $\vec{\beta}_1$ parallel to $\vec{\alpha}_1$ and $\vec{\beta}_2$ orthogonal to $\vec{\beta}_1$ (a Gram-Schmidt orthonormalization). The conditions that $U\vec{\beta}_1 = (1, 0, \ldots, 0)$ and $U\vec{\beta}_2 = (0, 1, \ldots, 0)$ determine the first and second columns of U^{-1}, respectively. These conditions ensure that $\vec{\alpha}_1$ and $\vec{\alpha}_2$ are mapped as required. It is a necessary and sufficient condition for U^{-1} to be unitary that its columns be orthonormal, so the rest of the columns of U^{-1} can be filled in easily, which then also determines the unitary U.

Now, since Eq. (1) conserves the total energy in each field individually, we are guaranteed that after the collisions, $\vec{q}'(t \to +\infty, \vec{r})$ will still be only a two-component field (components for $k = 3, \ldots, N$ will be zero). [This can also be seen by direct inspection of Eq. (1).] Therefore, the complexity of this particular collision has been reduced from an N-component problem to a two-component problem. Once we obtain the solution $\vec{q}'(t \to +\infty, \vec{r})$, the simple transformation $U^{-1}\vec{q}'(t \to +\infty, \vec{r})$ gives us $\vec{q}(t \to +\infty, \vec{r})$.

Collisions of the kind described above are particularly interesting for their potential use in soliton computing [25,27,28,31]. Consider the case when after the collision, the two solitons are well separated, are moving at an angle away from each other, and can be written in the form $\vec{q}(t \to +\infty, \vec{r}) = \vec{\alpha}_A\psi_A(t \to +\infty, \vec{r}) + \vec{\alpha}_B\psi_B(t \to +\infty, \vec{r})$. This holds exactly for some integrable cases of Eq. (1), and is an excellent approximation for many other physically important cases of Eq. (1) that are nearly integrable, such as arise in media with saturating nonlinearities when the collision angle (transverse velocity) is greater than the critical angle for total internal reflection within each "induced potential" (see, for example, the review of such spatial soliton systems in [4]). When this is the case, solitons can be used to carry information and perform logic operations solely through pairwise collisions [32]. The reduction above plus the simple linear fractional transformation of states for the two-component Manakov case [25] then provides a powerful analysis tool, while such a transformation is not available for $N > 2$ components.

To illustrate the reduction in the N-component Manakov case [$f(I) = I$ in Eq. (1) for (1 + 1)D], we consider the collision of two three-component solitons, with corresponding component vectors $p = (1/\sqrt{17})(2\sqrt{2}, 2\sqrt{2}, 1)$ and $q = (1/\sqrt{17})(3\sqrt{2}, 3\sqrt{2}, -7)$, and velocities ± 2. A unitary matrix U that maps these to solitons with energy in only the first two coordinates is

$$U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}, \tag{5}$$

which yields $Up = (1/\sqrt{17})(4, 1, 0)$ and $Uq = (1/\sqrt{17})(6, -7, 0)$. A beam-propagation program using the split-step Fourier method was used to integrate the collision of the three-component solitons p and q in Eq. (1), and the results are shown in Fig. 1. This example was chosen to illustrate the transfer of energy between components; in fact, the peak energy of p in the third component is increased by a factor of 16.6 by the collision with q.

To check our reduction procedure, a (two-component) collision between Up and Uq was simulated, the results transformed by U^{-1}, and the energy at the peaks of the solitons compared with the direct integration above using p and q. The peak energies of the three components all agree to within at most 0.4% error, which is the order of accuracy expected from the numerical integration method used.

The linear fractional state transformation in [25] predicts the results of the 2D collision as follows. The polarization state ρ is defined as the ratio of the first and

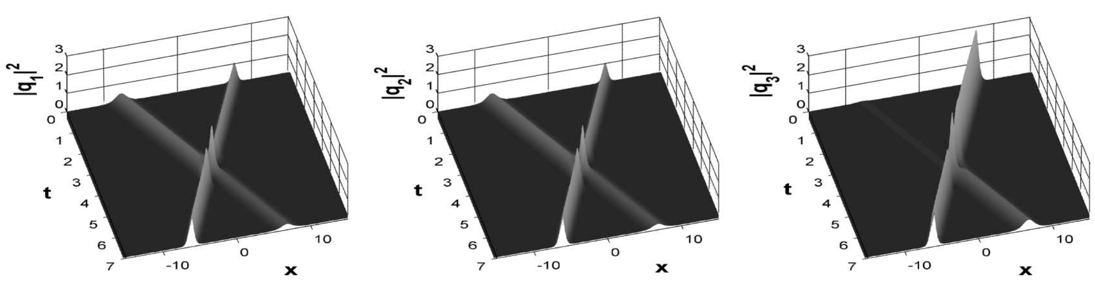

FIG. 1. Collision of two three-component solitons as described in the example. Shown are $|q_1|^2$, $|q_2|^2$, and $|q_3|^2$, respectively. The scale is normalized. Each component of each soliton interacts with the same component of the other soliton (shown in the figure), and also with the intensity of the other two components (this interaction is not explicitly visible in the figure). Note the transfers of energy between solitons, especially pronounced in the third component.

second components, and the state of a right-moving soliton after the collision of a right-moving soliton in state ρ_1 and a left-moving soliton in state ρ_2 is

$$\rho_1' = \frac{a\rho_1 + b}{c\rho_1 + d}, \quad (6)$$

where $a = (1 - h^*)/\rho_2^* + \rho_2$, $b = h^* \rho_2/\rho_2^*$, $c = h^*$, and $d = (1 - h^*)\rho_2 + 1/\rho_2^*$. The coefficient $h^* = (k_2 + k_2^*)/(k_1 + k_2^*)$, where k_1 and k_2 are the usual soliton parameters. (An analogous formula gives the state of the left-moving collision product.) In our example, $k_1 = 1 + i$, $k_2 = \sqrt{5} - i$, and the soliton states are $\rho_1 = 4$ and $\rho_2 = -6/7$. The prediction of Eq. (6) was checked against the numerical simulation with error consistent with the accuracy of the simulation. In summary, we have found the results of a collision of two solitons in the three-component Manakov system using the simple analytic formulas available for the two-component Manakov system. Following our procedure, one can analytically predict the outcome of the collision of two N-component Manakov solitons, for arbitrary N.

The reduction described here has a very simple geometric intuition behind it: A unitary coordinate transformation can always be found that transforms all the energy in a collision of two solitons to a two-dimensional complex subspace, and no energy will leave that subspace during the collision. The reduction does not rely on integrability, and applies to a much wider class of coupled systems than the N-component Manakov system. Consequently, any characterization of pairwise collisions using polarization state in 2-CNLS, including the succinct transformations for the two-component Manakov system, carry over to N-CNLS. One important, perhaps counterintuitive, implication of our reduction is that the collision of two solitons in N-component systems is analytically no more complex than such collisions in two-component systems.

We do not claim that there cannot be utility or interest in using soliton collisions in the N-CNLS system for $N > 2$. For one thing, the reduction described here does not apply to simultaneous collisions of three or more solitons in N-CNLS (it applies only to pairwise collisions). Nor can it be used for the case when the intensity distribution is different in different components (e.g., the first component has no nodes while the second component has nodes). Furthermore, even if the collisions are well separated, each pairwise collision must be analyzed in its own 2D subspace. In fact, we can see that if we have three N-CNLS solitons, we can always find a 3D subspace to which the system can be transformed; for four N-CNLS solitons, we can always find a 4D subspace, and so on. This consideration does give us an upper bound on how complex these more general collisions can be.

The model we study can readily be experimentally tested with spatial solitons in optics: using photorefractive materials, or liquid crystals. Alternatively, it can be tested on matter-wave solitons in Bose-Einstein condensates.

This work was supported in part by the MURI project on optical spatial solitons.

[1] J. Scott Russell, in *Report on Waves, Proceedings of the Meeting of the British Association for the Advancement of Science, York, 1842–1843* (John Murray, London, 1844), p. 311.
[2] E. Infeld and G. Rowlands, *Nonlinear Waves, Solitons, and Chaos* (Cambridge University Press, Cambridge, U.K., 1990).
[3] P. L. Kelley, Phys. Rev. Lett. **15**, 1005–1008 (1965).
[4] G. I. Stegeman and M. Segev, Science **286**, 1518–1523 (1999).
[5] S. V. Manakov, Sov. Phys. JETP **38**, 248 (1974).
[6] D. N. Christodoulides and R. I. Joseph, Opt. Lett. **13**, 53 (1988).
[7] M. V. Tratnik and J. E. Sipe, Phys. Rev. A **38**, 2011 (1988).
[8] C. R. Menyuk, J. Opt. Soc. Am. B **5**, 392 (1988).
[9] Y. Barad and Y. Silberberg, Phys. Rev. Lett. **78**, 3290 (1997).
[10] S. T. Cundiff *et al.*, Phys. Rev. Lett. **82**, 3988–3991 (1999).
[11] J. U. Kang *et al.*, Phys. Rev. Lett. **76**, 3699 (1996).
[12] D. N. Christodoulides *et al.*, Appl. Phys. Lett. **68**, 1763 (1996); Z. Chen *et al.*, Opt. Lett. **21**, 1436 (1996).
[13] M. Segev *et al.*, Phys. Rev. Lett. **73**, 3211 (1994).
[14] D. N. Christodoulides and M. I. Carvalho, J. Opt. Soc. Am. B **12**, 1628 (1995).
[15] V. Steblina *et al.*, J. Opt. Soc. Am. B **17**, 2026 (2002).
[16] J. P. Torres *et al.*, Opt. Commun. **213**, 351 (2002).
[17] M. R. Matthews *et al.*, Phys. Rev. Lett. **83**, 2498 (1999).
[18] B. D. Esry *et al.*, Phys. Rev. Lett. **78**, 3594 (1997).
[19] D. S. Hall *et al.*, Phys. Rev. Lett. **81**, 1539–1542 (1998).
[20] J. Stenger *et al.*, Nature (London) **392**, 687 (1998).
[21] V. M. Perez-Garcia and J. Garcia-Ripoll, Phys. Rev. Lett. **84**, 4264–4267 (2000).
[22] J. William and M. Holland, Nature (London) **401**, 568 (1999).
[23] P. Ohberg and L. Santos, Phys. Rev. Lett. **86**, 2918–2921 (2001).
[24] R. Radhakrishnan, M. Lakshmanan, and J. Hietarinta, Phys. Rev. E **56**, 2213–2216 (1997).
[25] M. H. Jakubowski, K. Steiglitz, and R. Squier, Phys. Rev. E **58**, 6752–6758 (1998).
[26] C. Anastassiou *et al.*, Phys. Rev. Lett. **83**, 2332 (1999).
[27] C. Anastassiou *et al.*, Opt. Lett. **26**, 1498–1500 (2001).
[28] K. Steiglitz, Phys. Rev. E **63**, 016608 (2001).
[29] A. Sorensen, L.-M. Duan, J. I. Cirac, and P. Zoller, Nature (London) **409**, 63 (2001).
[30] T. Kanna and M. Lakshmanan, Phys. Rev. Lett. **86**, 5043–5046 (2001).
[31] S. Sears *et al.*, in *Nonlinear Guided Waves and Their Applications, Topical Meeting Optical Society of America, Dijon, France, 1999* (Optical Society of America, Washington, DC, 1999).
[32] M. H. Jakubowski, K. Steiglitz, and R. Squier, Multiple Valued Logic **6**, 439–462 (2001).