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Quantum state tomography with a single measurement setup

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The field of quantum information relies on the crucial issue of characterizing quantum states from measurements. This is performed through a process called quantum state tomography (QST). However, QST requires a large number of measurements, each derived from a different physical observable corresponding to a different experimental setup. Changing the setup results in unwanted changes to the data, prolongs the measurement, and impairs assumptions made about noise. *Here, we propose to overcome these drawbacks by performing QST with a single observable.* A single observable can often be realized by a single setup, thereby considerably reducing the experimental effort. However, the information contained in a single observable is insufficient for full QST. To overcome the lack of sufficient measurements in a single observable, we increase the system dimension by adding an ancilla that couples to the information in the system and exploit the fact that the sought state is often close to a pure state. We demonstrate our approach on multiphoton states by recovering structured quantum states from a single observable in a single experimental setup.

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1. INTRODUCTION

The fields of quantum information and quantum computation have attracted considerable interest in recent years. From fundamental quantum mechanics (the measurement process [1], description [2], the role [3] of entanglement, and the information content of quantum measurements [4]) to applied physics (molecular energy calculations [5] and experimental quantum key distribution [6]), quantum information and computations have helped raise questions and aid in the investigations of many topics in physics and related fields. Rooted in Feynman's ideas [7], the concept of a quantum computer has scaled and evolved, yielding programmable computation units [8,9]. In particular, optical quantum computations, based on the concepts of Knill, Laflamme, and Milburn [10] and cluster states [11–14], have recently witnessed experimental realizations of larger and more complex systems in terms of photon numbers [14,15]. Alternatives to the common quantum circuit model have also been suggested [11,16], among them a recent linear nonuniversal scheme for quantum computing [17–21], which is thought to yield a real quantum advantage over classical information processing already with near-future technology.

Whether exploring fundamental quantum mechanics or advancing quantum computations, the ability to characterize a quantum state from measurements is a principal component in the fields of quantum information and computation. As the fundamental description of a quantum system is given by the density matrix, the characterization amounts to identifying all elements of this matrix. The density matrix, a positive semidefinite, trace-normalized matrix, allows for the prediction of every experimental result, thus providing the full description of a quantum system. To recover the density matrix, quantum state tomography (QST) is usually performed. In this process, the density matrix is measured by a large number of observables, each corresponding to a different experimental setup. Each observable is associated with a Hermitian operator, whose eigenvalues are the outcome of the measurements, whereas their probabilities are derived from the eigenstates. By choosing a suitable set of observables, known as a "tomographically complete set of observables," the elements of the density matrix can be recovered from the measurements. However, the process of QST requires a large number of measurements, realized in multiple experimental setups. Generally, for a system of dimension d, the density matrix is described by $d^2 - 1$ real parameters. Since each observable yields at most d measurement outcomes, a tomographically complete set of observables consists of at least d observables, each corresponding to a different experimental setup. Naturally, the need to change the physical setup (even by rotating a wave plate or a polarizer) increases the duration of the experiment and hampers the integrity of the state. Perhaps even more importantly, the need to carry out many variations of the experimental setup often impairs the assumptions made on the noise. That is, in all quantum optics experiments, the flux of entangled states is

low. Hence, the integration times are long (minutes and longer). Consequently, the detection process always assumes that the noise does not change during the experiment. Clearly, varying the experimental setup due to the need to measure multiple observables, as well as having to carry out a large number of measurements, hampers these assumptions. Naturally, it would be highly desirable to be able to recover quantum states using a single observable in a single experimental setup. However, as we explain below, measurements obtained from a single observable are known to be insufficient for full QST. Therefore, the recovery of quantum states from a single observable has never been realized experimentally.

Here, ***we propose recovering the quantum state using a single observable, corresponding to a single experimental setup***. Since a single observable of dimension d can only yield d measurement outcomes, whereas $d^2 - 1$ are needed for the complete recovery of the density matrix, relying on a single observable implies that information will be missing. To overcome this lack of information while keeping the measurement number to the necessary minimum, we may rely on prior information, which should be as generic as possible in order to maintain the applicability of our approach to a large class of settings. With this in mind, we exploit generic prior information: that the sought quantum states are close to pure states. This prior is natural in quantum information because many of its applications deal with mapping pure states onto other pure states. Of course, in a physical system, the states are not ideally pure due to noise in the generation, manipulation, and detection of the quantum states. Nonetheless, the states are in most cases still close to pure states. Hence, the density matrix has a small number of nonzero eigenvalues, i.e., the eigenvalues are sparse.

Exploiting sparsity is at the heart of the field of compressed sensing (CS) [22–24], a very active area of research within signal processing, which enables reconstruction of information from incomplete measurements by exploiting sparse priors. More recently, CS has been brought into the quantum domain for the purpose of reducing the number of measurements necessary in QST [25] and in quantum process tomography [26], enabling much more efficient tomography. Other approaches to reduce the number of measurements required for QST come in the form of adaptive QST [27–30] and utilizing artificial neural networks [31,32]. CS was further used in wave function measurements [33,34], measurements of complementary observables [33], weak measurements [35,36], characterization of incoherent light [37], holography [38] and ghost imaging [39]. The general concept of using sparsity to solve underdetermined inverse problems has opened the door for a wide range of applications in various fields, ranging from sub-Nyquist sampling [40], subwavelength imaging [41–44], phase retrieval [42,43,45–47], to ankylography [48], ptychography [49], and quantum state recovery from low-order correlations [50].

The sparsity naturally arising in quantum information comes in the form of the quantum states, which, for most applications, tend to be pure states or close to pure states. Such states are of interest for a variety of reasons. First, pure states have zero entropy. Thus, as a random variable, they contain the most information and many theoretical results in quantum information are derived for these states. Second, the purity $\mathcal{P}(\rho)$ of a state ρ is invariant under unitary transformations. The purity is defined as $\mathcal{P}(\rho) = \mathrm{Tr}(\rho^2)$ (ρ being the density matrix), with $\mathcal{P} = 1$ for a pure state and $\mathcal{P} < 1$ for mixed states. Thus, under time evolution of a closed system, pure states always remain pure states. These are the reasons that most applications of quantum information ideally deal with mapping pure states onto other pure states. Of course, a realistic experimental scenario also contains uncertainties and imperfections, and the system is not always closed. Thus, the resulting states in experiments are not perfectly pure, but they nevertheless can often be described by states that are close to pure states. Finally, some quantum channels, describing noise processes and open system evolution, map pure states to states that can be approximated by relatively pure states. In the language of the density matrix, a pure state is described by a rank-1 density matrix, whereas a relatively pure state is described by a low-rank density matrix, having a small number of nonzero (sparse) eigenvalues relative to the system dimension. Accordingly, a state that can be approximated by a relatively pure state has a small number of significant eigenvalues. All these states fall under the category of sparse (or compressible) states and are addressed by our method.

An essential requirement for CS recovery to work well is that each measurement carries information. This is achieved by performing measurements in a basis that is least correlated (so-called the "least coherent" in the language of CS) with the basis providing the sparse representation. In the field of optics, for example, two such bases naturally occur in the form of real space and Fourier space. For a sparse signal in real space, measurements performed in Fourier space are good for CS and vice versa. Thus, in the spirit of CS, we introduce mixing between the physical modes in the system. In the context of indistinguishable photons in discrete spatial modes, the mixing is realized by a random, linear coupler, which can be experimentally realized by beam splitters [51] or by coupled modes in integrated photonics [8,20,52,53]. The coupler corresponds to the single observable and describes the single experimental setup in this scenario. By using a single coupler, the disadvantages of changing the experimental setup are avoided.

However, measurements taken with a single observable do not contain enough information for recovering the state of a quantum system (the density matrix). This is because a single observable of dimension d yields at most d different measurement outcomes, whereas even a pure state has more degrees of freedom ($2d - 1$), let alone relatively pure states, which require even more measurements. To overcome the lack of measurements, we add an ancilla in a known state to the state we wish to recover. This requires a short explanation about ancillas in quantum information.

Using known inputs to a system to improve its probing is a widely used concept in optical detection, from spectral interferometry [54] to optical homodyne detection [55,56] and integrated photonic schemes [57–59]. In the quantum context, consider a quantum system of m modes carrying information with a coupler that couples (mixes) the information in the modes. An ancilla is the addition of m' new modes with known inputs (say, zero input), but the evolution in the system couples these modes with the modes of the original system. The total number of modes where the measurements take place is therefore $M = m + m'$. The input state of the extended system is a tensor product of the original state ρ_0 and the ancilla state ρ_{ancilla}, such that $\rho_{\mathrm{in}} = \rho_0 \otimes \rho_{\mathrm{ancilla}}$. Because the input is a simple product state, its mutual information [$\mathcal{I}(A{:}B) = \mathcal{S}(A) + \mathcal{S}(B) - \mathcal{S}(A, B)$, where $\mathcal{S}(\rho) = -\mathrm{Tr}(\rho \log \rho)$ is the von Neumann entropy] is zero:

$\mathcal{S}(\rho_0 \otimes \rho_{\text{ancilla}}) = \mathcal{S}(\rho_0) + \mathcal{S}(\rho_{\text{ancilla}}) \Rightarrow \mathcal{I}_{\text{in}}(\text{original:ancilla}) = 0$. However, after the evolution, the state is $\rho_{\text{out}} = \mathcal{U}\rho_0 \otimes \rho_{\text{ancilla}}\mathcal{U}^\dagger$, where $\mathcal{U}$ is the evolution operator in the system. Whenever the evolution couples between the ancilla and the original system, such that the output state is not a pure tensor product anymore, $\rho_{\text{out}} \neq \rho_0^{\text{out}} \otimes \rho_{\text{ancilla}}^{\text{out}}$, the mutual information at the output is larger than zero. Physically, this means that the ancilla effectively increases the number of measurements (see Supplement 1). In our system specifically, the dimension of the ancilla should be large enough so as to account for the missing information to facilitate the recovery of the input state from a single observable. In photonics, the ancilla is conveniently realized by adding vacuum ports at the input. Surely, increasing the system dimension to the full $d^2 - 1$ degrees of freedom (for a generic density matrix of dimension d) is possible, such that a single observable accounts for all the measurements required for QST. However, for large values of d, the number of required measurements and the number of required ancilla modes for full QST can be very large. Here, using the prior knowledge that the input state is sparse allows us to considerably reduce the required dimensional increase of the ancilla.

2. METHODS

With the notions of sparsity, the addition of the ancilla, and the mixing between the degrees of freedom in mind, we can formulate the problem. Our goal is to recover a density matrix ρ_0 of dimension d and rank r (unknown but small relative to d) from measurements of a single observable A. Consider a system of N photons in m ports [Fig. 1(a), with $N = 3$ and $m = 4$], having dimension $d = \binom{m-1+N}{N}$ [in Fig. 1(a), $d = 20$]. We add vacuum ports [Fig. 1(b)], realizing the ancilla in the state $|\psi_{\text{ancilla}}\rangle = |0\rangle$, such that the total number of ports is $M > m$ ($M - m$ vacuum ports are added, in Figs. 1(b) and 1(c), where $M = 8$ and $M - m = 4$). The dimension of the system is now $D = \binom{M-1+N}{N} > d$, and the state of the joint system is described by the D-dimensional density matrix $\rho = \rho_0 \otimes \rho_{\text{ancilla}} = \rho_0 \otimes |0\rangle\langle 0|$. The mixing is realized by a linear, random coupler U of M ports, introduced between the input and the number-resolving detectors at the output [Fig. 1(c)], causing the state to evolve according to

$$\rho_0 \otimes |0\rangle\langle 0| \mapsto \mathcal{U}(\rho_0 \otimes |0\rangle\langle 0|)\mathcal{U}^\dagger, \tag{1}$$

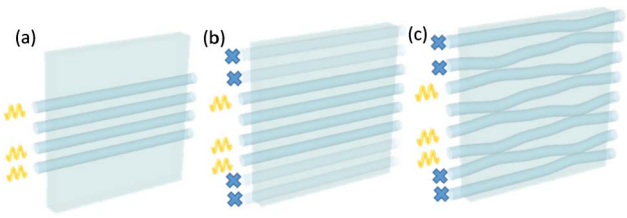

Fig. 1. (a) A system of N photons in m ports, with $N = 3$ and $m = 4$. The input state is assumed to be sparse, that is close to a pure state. (b) In our scheme, the dimension is increased by the addition of an ancilla, taking the form of vacuum ports in a photonic system. (c) The mixing between the degrees of freedom is realized by a random, linear coupler in the larger system.

where $\mathcal{U} \in U(D)$ is the evolution of the entire system, original state, and ancilla, dictated by the coupler $U \in U(M)$, such that $\mathcal{U} \neq \mathcal{U}_{\text{original}} \otimes \mathcal{U}_{\text{ancilla}}$. See Supplement 1 for further details.

The measurements performed are correlation measurements of N photons. This set of measurements, along with the linear coupler, define the observable that we use:

$$A = \sum_i i\mathcal{U}^\dagger|\{n\}^i\rangle\langle\{n\}^i|\mathcal{U} \in \mathbb{C}^{D\times D}, \tag{2}$$

where $A^\dagger = A$ is the single observable and $|\{n\}^i\rangle = |n_1^i n_2^i \ldots n_M^i\rangle$ is the ith Fock state with n_q^i photons in port q (see the details in Supplement 1). Experimentally, these measurements describe the N-fold correlation measurements after the linear coupler. The problem now translates into finding $\rho_0 \in \mathbb{C}^{d\times d}$, a positive semidefinite matrix $\rho_0^\dagger = \rho_0$, $\rho_0 \geq 0$ (an Hermitian matrix with nonnegative eigenvalues) with unit trace $\text{Tr}(\rho_0) = 1$, having the lowest rank and conforming to the measurements

$$\begin{aligned} y_i &= \text{Tr}(\rho\mathcal{U}^\dagger|\{n\}^i\rangle\langle\{n\}^i|\mathcal{U}) \\ &= \langle\{n\}^i|\mathcal{U}\rho\mathcal{U}^\dagger|\{n\}^i\rangle, \qquad i \in \{1, \ldots, D\}. \end{aligned} \tag{3}$$

We emphasize that the number of measurements here is $D = \binom{M-1+N}{N}$, and each measurement is repeated N_{repeat} times to obtain an approximation of the value of $\text{Tr}(\rho|\{n\}^i\rangle\langle\{n\}^i|)$. However, since they are all derived from a single realization of the coupler, all of these measurements can be realized in a single experimental setup.

The density matrix we wish to find is the solution to the problem

$$\begin{aligned} &\min_{\rho_0} \text{rank}(\rho_0) \\ &\text{subject to } \rho_0^\dagger = \rho_0, \qquad \rho_0 \geq 0, \qquad \text{Tr}(\rho_0) = 1 \\ &|\text{Tr}(\rho_0 \otimes |0\rangle\langle 0|A_i) - y_i| \leq \varepsilon, \qquad i = 1, \ldots, D. \end{aligned} \tag{4}$$

Here, $A_i = \mathcal{U}^\dagger|\{n\}^i\rangle\langle\{n\}^i|\mathcal{U}$ is the spectral decomposition of the single observable [Eq. (2)]. This optimization problem is related to the matrix completion problem [60–62] with the additional constraints stemming from the physical nature of the object that we wish to recover and the measurements derived from the single observable. This problem is not convex due to the rank objective. Thus, to find the density matrix ρ_0, we utilize the logdet [63] approach, in which the nonconvex rank is replaced by the logarithm of the matrix determinant, which in turn is linearized to yield the following iterative algorithm:

$$\begin{aligned} &\min_{X_k} \text{Tr}(X_{k-1} + \delta I)^{-1} X_k, \\ &\text{subject to } X_k \geq 0, \qquad \text{Tr}(X_k) = 1, \qquad X_k^\dagger = X_k, \\ &|\text{Tr}(X_k \otimes |0\rangle\langle 0|A_i) - y_i| \leq \varepsilon, \qquad i = 1, \ldots, D. \end{aligned} \tag{5}$$

Here, in iteration k, we look for the matrix X_k, where X_{k-1} is the solution of the previous iteration and δ is a small regularization parameter. The parameter ε is related to the measurement noise. The algorithm step described in Eq. (5) is a semidefinite program. The result of the minimization of the linearized objective is the minimization of the logdet function at each iteration, since logdet is concave, meaning that the function curve lies beneath its tangent. Therefore, $\text{logdet}(X_k + \delta I)$ converges to a local minimum of $\text{logdet}(X + \delta I)$ [63]. Other methods can be utilized as well; see the details in Supplement 1.

The information evolves in our setting as follows. Consider an ensemble of quantum states of varying ranks $r \in \{1, \ldots, d\}$. Each state undergoes a depolarization channel $\rho_0 \mapsto (1-\mu)\rho_0 + \frac{\mu}{d} I_d$, realizing noise in the state. The resulting state has r significant eigenvalues. Then, the density matrix evolves in the linear coupler, which has a large dimension and provides mixing between the ports. The information propagates to the output ports, which are where the measurements of the single observable are taken. The measured single observable includes noise added to it. This is the "hardware" defining our system.

Next, we demonstrate the power of our scheme to recover quantum states of rank r (unknown but small relative to the dimension of the system). The input to our recovery procedure is the noisy single-observable measurements $y_i, i \in \{1, \ldots, D\}$. The algorithm (described in Supplement 1) results in the recovered density matrix ρ_{rec}. To evaluate the recovery performance, we compare ρ_{rec} to the original density matrix ρ_0 by means of the fidelity between the two states $\mathcal{F}(\rho_0, \rho_{\text{rec}}) = \text{Tr}\sqrt{\rho_0^{1/2}\rho_{\text{rec}}\rho_0^{1/2}}$.

An example of the original and recovered density matrices is presented in Fig. 2(a). The density matrix of rank 2 describes three photons in three input ports. The ancilla used consists of four vacuum ports, yielding a total of seven output ports. The original and recovered density matrices match very well, with a fidelity of 0.96. To test the performance of our methodology, we generate an ensemble of mixed density matrices of various ranks.

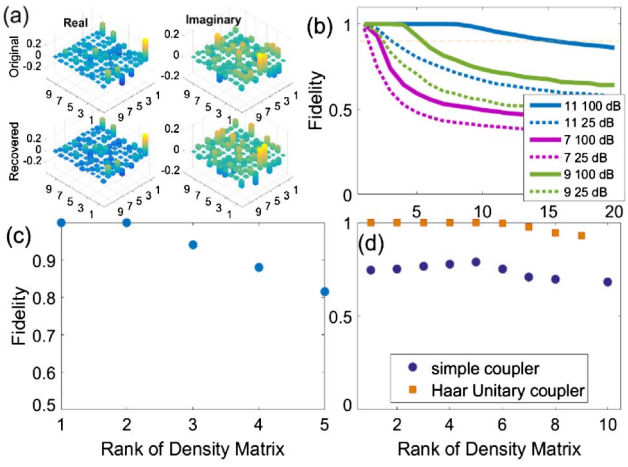

Fig. 2. (a) Recovery example of a rank-2 density matrix describing $N = 3$ photons in $m = 3$ input ports and $M = 7$ output ports. The state is recovered with fidelity 0.96. (b) Mean fidelity of the state recovered from measurements of a single observable versus rank of input state. Solid curves: recovery with practically no noise (SNR of 100 dB) using 7, 9, and 11 output ports. Dotted curves: same as the solid curves but with depolarization noise added to the state and measurement noise of 25 dB added to the measurements. The plots show the average over 200 realizations of the density matrix and 10 realizations of the random coupler for each point. The measurements used here are only a portion (21%–71%) of the measurements required for full QST. (c) Mean fidelity versus rank of input state, describing $N = 3$ photons in $m = 7$ input ports and $M = 16$ output ports, averaged over 15 realizations of the density matrix. Here, the dimension of the system is large, $d = 84$. The number of measurements in a single setup in this scenario is 11% of the measurements required for full QST. (d) Comparison between a fully mixing coupler (randomly sampled from the Haar measure) and a simpler coupler, consisting of identical evanescently coupled waveguides. The simpler coupler does not facilitate recovery from a single observable, whereas the Haar coupler performs well up to rank 6.

The matrices are sampled from a product measure of the eigenvalues and eigenvectors, where the uniform measures on the unit simplex and the unitary group are used, respectively (see the details in Supplement 1). The recovery fidelity without measurement noise, averaged over 200 random realizations of the density matrix for each rank and 10 realizations of the random coupler, is shown in Fig. 2(b) (solid lines). For each recovery, measurements from a single coupler are used, as the different realizations are employed for the sake of averaging (see Supplement 1). The different solid curves correspond to a varying number of output ports $M \in \{7, 9, 11\}$. As expected, the fidelity is very high for low ranks, describing states that are relatively close to pure states. The fidelity grows with the increase of the system dimension through the ancilla (which also increases the dimension of the single observable). The fraction of measurements used, out of the total number of measurements required for full QST, is 21%, 42%, and 71% for $M = 7$, 9, and 11, respectively. For example, for rank 2 (almost pure states), we can recover the quantum state with only 21% of the total measurements, while for rank 6, we would need 71% in this small system.

The dotted lines in Fig. 2(b) describe the average recovery fidelity in a noisy scenario. Here, depolarization noise of 2% is added to the state, and measurement noise of 25 dB is added to the measurements. Once again, the recovery from a single setup works well for low-rank density matrices describing states that are close to pure states. Importantly, the recovery does not depend on the exact realization of the coupler, as long as it is sampled from the correct distribution (see Supplement 1 for details). Motivated by boson sampling [17], the theory and experimental realization of such Haar random linear optical couplers have been developed significantly [52,64]. In Fig. 2(c), the mean recovery fidelity is shown for a state of a larger dimension. Here, the density matrices describe $N = 3$ photons in $m = 7$ input ports and $M = 16$ output ports. The dimension of the system is $d = 84$, meaning that the number of measurements required for full QST is $d^2 - 1 = 7055$. However, as Fig. 2(c) shows, we recover the state with $D = 816$, meaning that high-fidelity reconstruction is achieved with only 11% of the measurements required for full QST. In Fig. 2(d), we compare the recovery fidelity of a Haar random coupler to that of a simpler coupler, consisting of uniform, evanescently coupled waveguides with nearest-neighbor coupling only. The simple waveguide array (1D photonic lattice) offers some mixing between the degrees of freedom but at a level considerably lower than the Haar random coupler. As evident in the figure, the simpler coupler fails to allow the recovery from a single observable, in a single setup, whereas the Haar coupler performs perfectly up to rank 6. As seen in the results for a large dimension [Fig. 2(c)] compared to the results in Fig. 2(b), as the dimension of the original state increases, low-rank states become amenable for recovery from a smaller portion of the measurements required for complete QST (~0.11 of the measurements for $d = 84$ comparing to ~0.21–0.71 of the measurements for $d = 20$). This result is expected from the theory of CS, where for ideal measurement matrices, the number of measurements required to recover a rank-r state is $N_{\text{measurements}} \geq O(rd\,\text{poly}[\log d])$ [25,65], where $\text{poly}[\log d]$ is a polynomial in $\log d$. In our case, $N_{\text{measurements}} = D = \binom{M-1+N}{N}$, where M is the number of output ports and N is the number of photons. Therefore,

we expect that as the dimension increases, the sparsity will enable recovery of the state from a single observable measurement with a smaller portion of the measurements required for full QST.

The recovery fidelity in Figs. 2(b)–2(d) is obtained using the input state, known in simulations, and comparing it to the recovered state. However, in experiments, the input state is not known. Thus, some method is required to assess the correctness of the obtained solution as well as make sure that sufficient measurements have been taken. We suggest the following procedure to assess whether the obtained solution can be trusted—given the coupler (which determines the number of measurements), would that coupler suffice to recover the quantum state at high fidelity, or would a larger coupler be required. To that end, after the measurements are recorded (using an upper bound on the expected rank to fully harvest the reduction in the number of degrees of freedom) and a solution is obtained, we artificially discard a small portion (5%) of the measurements. Then, we attempt to recover the density matrix from the smaller sample. If the two solutions are similar (in the Frobenius norm sense, for example), then we consider the solution trustworthy. This method is based on the observation that if enough measurements are used, the solution should not be sensitive to a small reduction in the number of measurements.

This method for evaluating the correctness of the recovery does not require using the input state or taking more measurements for the evaluation process. The results are presented in Fig. 3. By following this approach, the yellow areas are those for which the solution can be trusted. They are obtained without any averaging, only using the measurements and the measurement matrix, both of which are at the hands of the experimentalist. The solid lines are taken from Fig. 2(b). Comparing the two, this method captures the ranks for which the recovery is perfect, without using the input state. In particular, for each coupler size [7 in Fig. 3(a) and 9 in Fig. 3(b)], a single density matrix realization has been generated for every rank. The measurements are calculated (with a single coupler realization), and a recovery is obtained with all the measurements and with a reduced set of measurements, where the discarded 5% of the measurements are chosen randomly. Then, the norm difference (normalized) is calculated. If it is smaller than 10^{-2}, the solution is considered trustworthy (the corresponding rank is shaded). See Supplement 1 for further details and extensions.

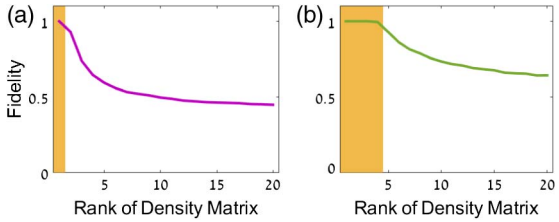

Fig. 3. Fidelity of the recovery of the quantum states from Fig. 2(b) in settings with (a) seven and (b) nine output ports. Solid lines: recovery fidelity obtained by comparing to the original states, known in our simulation but not available in experiments. Shaded areas: the ranks for which our scheme to gain confidence in the solution yields the answer "trustworthy" without knowing the input state. Without knowledge of the original state, the shaded areas match the ranks for which perfect recovery is achieved from only a fraction of the measurements, using the single observable, in a single setup.

3. RECOVERING QUANTUM STATES WITH CLICK DETECTORS

Following the success of our methodology to recover the quantum states from a single observable, we wish to tackle another problem having to do with recovering quantum states from partial measurements: recovering the quantum state using click detectors (i.e., detectors that cannot resolve the number of photons detected). Generally, photon-number-resolving detectors are required for performing QST of multiphoton states. However, such detectors often exhibit poor performance (primarily in terms of sensitivity and fall times, which affects the sampling rate), and currently the majority of quantum optics experiments are carried out with click detectors. One possible avenue for optical QST with click photodetectors is to mimic their functionality by the addition of beam splitters [66]. However, this method introduces further losses to the system and is not scalable to larger numbers of modes and photons. A different approach has recently been shown to obtain the counting statistics of light [57,58].

As we now show, our scheme works even in the extreme case of combining a single-observable with click detectors. That is, we recover the full density matrix of a multiphoton state with a fixed number of photons, from a single observable and without number-resolving detectors. For an N photon state in m ports and $m > N$, we use regular click detectors, detecting the presence of more-than-zero photons in each detection event. We use only the detection events involving N different clicks. In these events, counting and detecting are the same. Now, we use a partial set of measurements: $y_i = \langle\{n\}^i|\mathcal{U}\rho\mathcal{U}^\dagger|\{n\}^i\rangle$, as before, only with $i \in I$, such that $I = \{i \in \{1, \ldots, D\}|n_q^i \in \{0, 1\}\forall\ q \in \{1, \ldots, M\}\}$ and $|I| = \binom{M}{N}$. Naturally, these events do not contain all the information [67] needed for quantum state reconstruction. However, by using our scheme, we can overcome this deficit and recover the full density matrix from such very partial measurements.

Figure 4(a) shows an example of a rank-2 state describing $N = 3$ photons in $m = 3$ input ports. Noise of 25 dB is added to the measurements, and the state is recovered from a single observable, without number-resolving detectors. In Fig. 4(b), the recovery fidelity versus the rank of the density matrix describing $N = 3$ photons in $m = 4$ input ports, $M = 11$ output ports, and measurement noise of 25 dB, without number-resolving

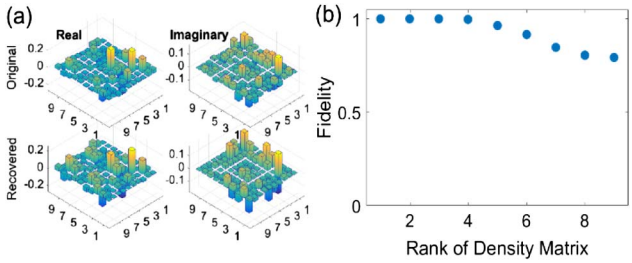

Fig. 4. (a) Recovery example of a rank-2 density matrix describing $N = 3$ photons in $m = 3$ input ports, $M = 8$ output ports, and with 25 dB of measurement noise. The state is recovered with 0.93 fidelity from measurements using click detectors. (b) Recovery fidelity with click detectors of density matrices of $N = 3$ photons in $m = 4$ output ports and $M = 11$ output ports, corresponding to 41% of the measurements required for QST, in a single setup.

detectors. Low-rank states are recovered from the noisy measurements, in a single setup, without number-resolving detectors.

In the state recovery using click detectors, the number of measurements depends on the system characteristics as $N_{\text{measurements}} = \binom{M}{N}$, where M is the number of output ports and N is the number of photons. Here as well, we expect that the sparsity will reduce the portion of measurements required to recover the state with click detectors as the dimension of the original state d increases.

4. DISCUSSION AND CONCLUSIONS

In our scheme, to obtain state reconstruction with a single observable, in a single experimental setup, a larger system is required. This fact raises the question of a possible increase in the total loss (absorption) in the system. However, we reduce the number of measurements, relative to the full number required for tomography, by using sparsity. Hence, the number of added ports, as well as the absorption increase, is kept low. Furthermore, recent experimental advances have demonstrated a random Haar coupler with 9 ports, which is nearly unitary, having total transmissivity of more than 99%—including insertion loss, which is irrelevant for the vacuum ports [53]. In addition, a new decomposition method of Haar unitaries is expected to reduce the losses even further [68].

Finally, the two aspects of our approach, namely, performing tomography with a single observable and in a single setup, are augmented by a third point of view. Often, QST is formulated in the language of positive operator valued measurements (POVMs) [69], which are generalized quantum measurements. In POVM language, measurements are described not by observables but rather by a set of positive semidefinite operators $\{E_i\}, E_i^\dagger = E_i, E_i \geq 0$, which sum up to the identity $\sum_i E_i = I$. The probability for each result is given by $p(i|\rho) = \mathrm{Tr}(\rho E_i)$. This is a generalization of the observables-related measurements since the POVM elements need not be orthogonal. Our method can be thought of as an efficient approach to the Neumark theorem [70,71], using CS to reduce the dimension of the ancilla, which is used to realize general quantum measurements (POVMs) using projective measurements. See Supplement 1 for further details.

In conclusion, we showed that the generic prior knowledge of having a low-rank density matrix (sparsity) can be used to recover the complete quantum state from measurements of a single observable, often corresponding to a single experimental setup, in an efficient manner. This is achieved by adding an ancilla to the original state and introducing a random linear coupler between the input state and the measurements. We further used these ideas to recover the complete density matrix of a state (with a fixed number of photons) with click detectors. We have shown how the main ideas of the scheme can be implemented in a system of N photons in m input modes. However, these ideas can be extended to any system supporting the addition of an ancilla and enough mixing between the degrees of freedom in the form of interactions. A natural development of our scheme would be to try to optimize the linear coupler, such that it is ideal in the sense of CS. A further development is the addition of mixing between fixed photon number subspaces in the Hilbert space, aiding in the recovery of density matrices describing multiphoton states with a varying photon number, with click detectors.

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See Supplement 1 for supporting content.

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