# Self-Induced Diffusion in Disordered Nonlinear Photonic Media

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We find that waves propagating in a 1D medium that is homogeneous in its linear properties but spatially disordered in its nonlinear coefficients undergo diffusive transport, instead of being Anderson localized as always occurs for linear disordered media. Specifically, electromagnetic waves in a multilayer structure with random nonlinear coefficients exhibit diffusion with features fundamentally different from the traditional diffusion in linear noninteracting systems. This unique transport, which stems from the nonlinear interaction between the waves and the disordered medium, displays anomalous statistical behavior where the fields in multiple different realizations converge to the same intensity value as they penetrate deeper into the medium.

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Anderson localization is a universal phenomenon, occurring in all linear disordered wave systems. It was proposed to describe the absence of diffusion of electrons in the presence of disorder. Rather than exhibiting diffusion, as expected from particles in a scattering medium, the electron wave becomes localized with an exponentially decaying wave function [1]. Anderson localization is now known to occur in a variety of systems, including electromagnetic (EM) waves [2–7], acoustic waves [8], water waves [9], and ultracold atoms [10]. Among these, localization of light became a popular experimental platform because photons have long coherence times, and unlike electrons, they do not interact with one another [11]. Importantly, disorder in one- and two-dimensional settings always leads to localization, whereas in three dimensions the transport can also be diffusive [12–15].

One of the assumptions Anderson made was that the system is linear; i.e., the waves in the random potential do not interact. For transverse localization of optical waves in dielectrics [5,6,16], as well as for matter waves in the mean-field regime, interactions are manifested as a nonlinear potential term in the Schrödinger-type equation, which is mathematically equivalent to self-focusing of paraxial optical beams [11]. Indeed, localization in the presence of nonlinearities has been observed for optical beams [5,6,11], although its asymptotic behavior is still not fully resolved [17–20]. Despite this extensive research, all studies on waves in disordered systems investigated settings in which the linear potential is disordered, and the nonlinearity is an additional effect. To our knowledge, there have been no studies about a system that is disordered only in its nonlinear (NL) properties, e.g., a system that has a spatially random Kerr coefficient, but is homogeneous in its linear properties.

Here, we study a system that is linearly homogeneous but contains disorder in its NL coefficients and find that the nonlinearly induced disorder gives rise to a unique type of diffusion with unusual statistics and characteristic wave functions. Namely, waves decay in a fundamentally different fashion than the exponential decay characteristic of localization and have completely different statistics. We study a 1D multilayer dielectric system, where each layer has a nonlinear coefficient drawn randomly, with zero mean. Here, the propagating field induces the disorder in the refractive index, and consequently, the disorder level becomes dependent on the (local) intensity. We show that the EM field decays as it penetrates into the disordered medium, but exhibits diffusivelike behavior instead of becoming localized as expected for 1D linear disordered systems. We analyze the statistics of an ensemble of disorder realizations and find that the transmission in all realizations tends to converge to a single value, unlike linear disordered systems which display an increasing variance in transmission [21]. Finally, we examine the case of a saturable NL material and find that it exhibits a distinct transition from exponential decay to diffusivelike transport.

Our system is a 1D multilayer structure [Fig. 1(a)], where the medium has the same linear refractive index $n_0$ everywhere, but a spatially random $n_2$ Kerr coefficient drawn from a uniform distribution, $n_2 \in [-\Delta, \Delta]$. The refractive index in the $m$th layer is $n(z) = n_0 + n_2(m)I(z)$, with $I(z)$ as the EM field intensity. While all layers are of equal width, the system is not periodic because the refractive index varies in a random fashion (with a zero mean) in the presence of light, due to the nonlinearity. The zero mean of the NL coefficient ensures that the effects arise from the nonlinearly induced disorder and not from a change of the average refractive index.

The electric field of the light propagating in a linear 1D multilayer can be found through the transfer matrix method

 

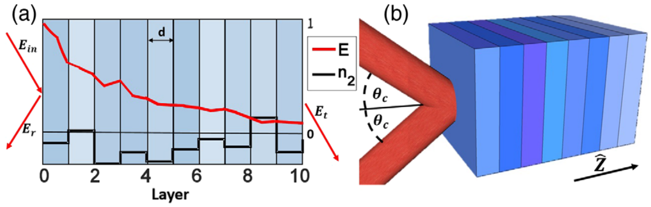


FIG. 1. (a) The 1D dielectric multilayer system and the electric field therein. The nonlinear $n_2$ coefficient of each layer (black) varies randomly, inducing a random change in the refractive index proportional to the local intensity. This nonlinear disorder induces decay in the field amplitude (red). (b) 3D illustration of the system with the light (red) incident at an angle slightly below the critical angle for total internal reflection.

[22]. This method was developed for linear media, where it accounts for all the scattering between the different layers. However, this formalism is ill-suited to handle nonlinearity. Here, the presence of the field changes the refractive index according to the local intensity, and in turn, the field itself is affected (backscatters and decays) due to the self-induced disorder. The problem needs to be solved self-consistently: finding the field structure that induces the exact nonlinear variations in the refractive index that, in turn, give rise to the same field structure. We use a numeric iterative algorithm, where in each iteration the field of the previous iteration is used to calculate the refractive index at each location. Then, the refractive index is used to recalculate the fields inside the system, and we again calculate the induced refractive index until convergence to a self-consistent solution. This procedure is similar to the method of finding solitons [23].

The strength of the effect depends on the size of the index variation $\Delta n$. Physically, the optical Kerr effect in the transparent wavelengths range of bulk dielectrics is weak; hence, the index change can realistically only reach values of $\Delta n \approx 5 \times 10^{-3}$, typically lower. We overcome this by using an angle of incidence close to the critical angle for total internal reflection [Fig. 1(b)], where localization becomes substantially stronger [24,25]. In this regime, the effects presented here can be studied in the laboratory [25].

Examples of transport in a nonlinearly disordered multilayer system are shown in Fig. 2, along with comparison to localization via linear random variation of the refractive index in a multilayer of the same layer width $d$. For concreteness, we take $d = 0.25$ $\mu$m and $\lambda = 1$ $\mu$m.

Let us begin with recalling the established results of Anderson localization. Figure 2(a) shows the linear case of a random index variation drawn from $\Delta n = [-5, 5] \times 10^{-3}$ on the background of an ambient refractive index $n_0 = \sqrt{3}$. The figure shows the intensity as a function of propagation distance for two specific realizations of the disorder (two specific cases of randomly chosen $\Delta n$ in the specified range). As expected from waves in linearly disordered media, the fields decay exponentially as they penetrate into

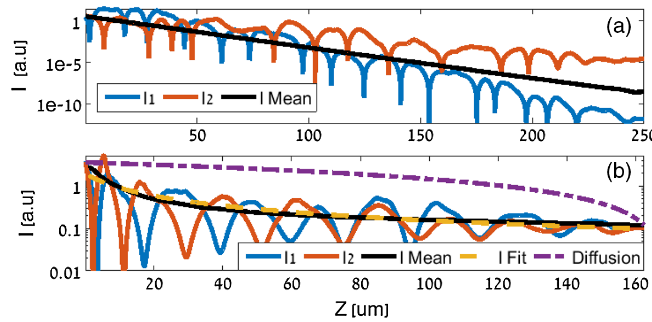


FIG. 2. Propagation of waves in multilayer structures with disorder in their nonlinear properties results in evolution strikingly different than in linear disordered systems. This figure shows the evolution of waves in (a) linear and (b) nonlinear disordered systems, for waves entering from the left. Red and blue lines: light intensity as a function of propagation distance $z$ for two specific realizations of disorder. Black line: ensemble-averaged intensity. Yellow line: power-law fit to the ensemble-average results. The decay of the ensemble-averaged intensity in the linear system (a) is exponential, conforming to Anderson localization, whereas the decay in the nonlinear system (b) is of a power law. The nonlinear system is also different than linear diffusive systems [purple dashed line in (b)].

the disordered structure. Figure 2(a) also shows (black curve) the ensemble-average intensity (averaged over 10 000 realizations of the disorder), from which the exact decay rate can be extracted.

Next, we examine the nonlinear system, where the nonlinear coefficient varies randomly in the range $n_2 = [-5, 5] \times 10^{-3}$ $\mathrm{cm}^2/\mathrm{W}$ and the initial intensity of the wave is $I(z = 0) = 1$ $\mathrm{W/cm}^2$. The intensity is chosen to yield at the input face of the structure a mean index change of $\langle|\Delta n_{\mathrm{NL}}|\rangle = \langle|n_2(z = 0)|I(z = 0)\rangle \approx 5 \times 10^{-3}$, to facilitate a comparison with the linear case of Fig. 2(a). Figure 2(b) shows two examples of nonlinear transport and the ensemble average. The field in all realizations decays as it penetrates into the multilayer and so does the ensemble average. We find that the decay rate of the intensity in this nonlinear self-induced disorder scheme is much weaker than in the linear disorder case and does not have an exponential dependence. Rather, the decay here has a power-law dependence, which highlights a major difference between the transport via nonlinear disorder and localization. Clearly, the transport in our system does not exhibit the main feature associated with Anderson localization: exponential decay. In our system, as the light intensity decays while penetrating into the structure, so does the magnitude of the random light-induced variations in the refractive index. This interplay between the NL and the decaying intensity introduces a negative feedback mechanism. The disorder becomes weaker as the wave penetrates deeper: a higher disorder therefore causes a more rapid decay in the intensity as the wave propagates, leading to lower levels of disorder in the subsequent layers.

Consequently, the backscattering mechanism becomes weaker deeper into the structure, resulting in power-law decay, as shown in Fig. 2(b).

Our finding of a power-law decay rate in our nonlinear system calls for a comparison with disordered systems where the transport of waves is diffusive. Generally, in linear systems giving rise to diffusive transport of light, the intensity transmission has a power-law dependence on the propagation distance, with transport inversely proportional to the length of the diffusive medium [26]. Our nonlinear system is fundamentally different—not only in its physical origins (being strictly nonlinear as opposed to fully linear, as diffusive systems generally are), but also in its most important features: decay rate and statistics.

To better understand these findings, we investigate an analytic toy model, using the WKB approximation. In a linear disordered system with a fixed level of disorder $A$ (real positive number), localization predicts exponential decay with a rate that depends on $A$ [12]. Hence, the mean intensity $\bar{I}(z) = \langle |E(z)|^2 \rangle$ ($\langle \cdots \rangle$ denoting ensemble average) obeys $[d\bar{I}(z)/dz] = -A\bar{I}(z) \Rightarrow \bar{I}(z) = I_0 \exp(-Az)$. In our case of strictly NL disorder, the level of disorder is no longer fixed, but depends on the intensity, and through it on $z$: $A(z) = A_0 n_2 \bar{I}(z)$, where the intensity decays with $z$ and affects the level of disorder. Under WKB, we assume that at every position $z$ the wave function experiences (for some short distance) a constant level of disorder and therefore exhibits decay according to the "local disorder level." This approximation is valid while the length scale on which the disorder $A(z)$ varies is much larger than the wavelength of the light. The mean intensity in this NL system is therefore governed by $[d\bar{I}(z)/dz] = -A(z)\bar{I}(z) \Rightarrow [d\bar{I}(z)/dz] = -A_0 \bar{I}\,\bar{I} \Rightarrow \bar{I}(z) = [1/(A_0 z + C)]$.

This solution is a power-law decaying wave function, with a power law of (−1). Indeed, in our simulations, we find a mean (ensemble-averaged) power-law decay [yellow curve in Fig. 2(b)]. The exact power-law decay found in the analytic toy model agrees with the mean intensity found in simulation [black curve in Fig. 2(b)]. The only significant discrepancy between the curves is near the entrance plane, where the nonlinear index change varies rapidly, violating WKB approximation. Consequently, the power transmission through such a NL disordered medium also decays as a function of the structure length with the same power law as the intensity. The longer the structure is, the lower the transmission through it.

The observation of a power-law decay can also be explained through a nonlinear diffusion argument. Diffusion is a measure of the strength of the disorder and the diffusion coefficient $D$ is inversely proportional to the variance of the refractive index $D = D_0 \mathrm{var}(\Delta n)^{-1}$. In ordinary diffusive systems, $D$ is constant, and the mean intensity decays linearly with distance. In our case, the diffusion coefficient is dependent on the variance of the refractive index and through it on the light intensity: $D = D_0/\mathrm{var}(\Delta n) \approx D_0 \bar{I}^{-2}$. If we insert this intensity dependent diffusion coefficient into the 1D steady-state diffusion equation, we get $D(\bar{I})\partial_z \bar{I} = \mathrm{const} \Rightarrow D_0 \bar{I}^{-2} \partial_z \bar{I} = \mathrm{const} \Rightarrow \bar{I}(z) = (1/c_1 + c_2 z)$, where the constant is proportional to the power flux through the medium (see details in the Supplemental Material [27]). Thus, when considering the dependence of the disorder on intensity, the diffusion equation yields a power-law solution, identical to the result of our toy model, and describing our simulation results [Fig. 2(b)].

This result is surprising because 1D systems with linear disorder cannot be described through diffusion or the diffusion equation. Rather, localization "triumphs," and the result always is absence of diffusion. Our finding highlights that the self-induced disorder in our nonlinear system results in diffusion rather than localization of light. The nonlinear self-induced diffusion is also fundamentally different than diffusion in a linear disordered system, because the NL process relies on the negative feedback between the wave and the disorder. To demonstrate that, we studied what happens in our system when we disconnect that feedback: we examined light propagation in a linear system for which the disorder magnitude decays in the exact same fashion as our NL system [Fig. 2(b)]. As shown in [27], without the feedback mechanism, the field intensity does not decay as the nonlinearly induced diffusion of Fig. 2(b). Rather, it decays at a much lower rate. Fundamentally, it is the interplay between the field intensity and the induced refractive index that gives rise to this special kind of nonlinear diffusion, which was never studied before.

Next, we study the statistical features of the self-induced diffusion. While the ensemble average yields the typical behavior [Fig. 2(b)], the wave function in each disorder realization decays in a different fashion, and as the light propagates, the difference between the decay of the wave functions in each realization can grow considerably. Figure 3 shows the histogram of the transmission logarithm and the variance for an ensemble of realizations, at different positions along the $z$ axis. In a linear disordered system [Fig. 3(a)], the histograms are shaped as a Gaussian, from which the variance of the transmission can be calculated. After propagating a short distance, the histogram is relatively narrow (blue line), indicating low variance between the transmission of different realizations in the ensemble. But as the light penetrates deeper, the histograms become wider and their center shifts to lower levels of transmission. The widening of the histograms indicates a larger variance in the ensemble realizations. In the linear systems, the variance of the transmission increases linearly with $z$ [Fig. 3(c)], as known from 1D Anderson localization [21], indicating an increasing difference between realizations.

In sharp contrast to linear disordered systems, the nonlinearly disordered medium exhibits completely different statistics. Surprisingly, from our simulations we find that not only does the variance not increase linearly, but it

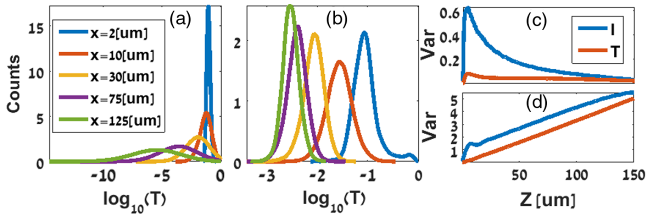


FIG. 3. (a),(b) Histogram of $\log_{10}$(Transmission), for an ensemble of linear and nonlinear (respectively) disordered multilayer structures of different lengths $L$. For both the linear and the NL cases, the histograms are Gaussian, and their centers shift to lower values as the structure becomes longer. However, in the linear case the distribution becomes wider as $L$ increases (a), whereas in the NL case it becomes narrower (b). (c),(d) Variance of $\log_{10}$(Transmission) (red) and intensity (blue) as a function of $L$, for linear (c) and nonlinear (d) disorder. For linear disorder the variance increases linearly, whereas for NL disorder the variance first jumps from 0 to 0.08 very close to the entrance face, but then starts to decrease, reaching zero asymptotically. Consequently, the intensities of the optical field in all realizations of NL disorder coalesce to a single value asymptotically.

actually decreases as a function of propagation distance $z$. Figure 3(b) shows the histograms for the ensemble in the NL disordered multilayer. In contrast to the linear case, the histogram describing the ensemble after a short propagation distance (blue) is the widest, indicating the largest variance. As the light penetrates deeper into the medium, the histograms become narrower, indicating a decrease in variance. The dependence of the variance on $z$ in the NL multilayer is shown in Fig. 3(d). There is an initial "jump" of the variance at the very beginning of propagation, but from there on the variance in transmission decreases. This is another consequence of the negative feedback mechanism: the variance in transmission decreases because realizations with strong initial decay experience less disorder and localization henceforth, and vice versa. In stark contrast to the divergence of the variance in the linear system, the NL disordered multilayer displays a convergence of all realization to a single value.

It is now interesting to examine the dependence of the diffusive transport on the optical wavelength. In linear disordered Anderson-localized systems, the decay length $L$ (distance after which the intensity decays to $1/e$ of its initial value) in the long wavelength limit is usually proportional to $\lambda^2$, while near the critical angle this dependence is linear or close to it [24]. Our simulations show that near the critical angle the power of the wavelength dependence is 1.2, as shown by the red curve in Fig. 4(a), together with a power-law fit of $L(\lambda) = A\lambda^{1.2}$ (blue). This result is similar to the results from 1D linear disordered systems near the critical angle [24,25,35]. It is yet unknown exactly why these results are similar, as they arise from different origins (linear vs nonlinear disorder), while the only common aspects are that both systems are one-dimensional and near critical angle. In a similar vein,

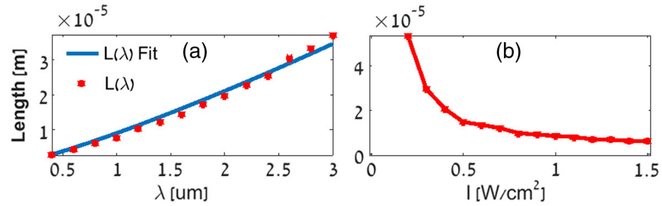


FIG. 4. (a) Decay length as a function of wavelength (for layer width of 250 nm) found from simulations (red points) showing best fit to a power-law dependence (blue), $L(\lambda) \approx A\lambda^{1.2}$, near the critical angle. (b) Decay length as a function of initial light intensity. As the initial light intensity increases, the magnitude of disorder increases as well, and the scattering process becomes more prominent.

we examine the effect of the initial light intensity on the decay length in our nonlinear multilayer [also defined as $I(L) = I(0)e^{-1}$]. The magnitude of the disorder is proportional to the intensity; hence, it is expected that as the initial intensity decreases, so will the disorder magnitude. Figure 4(b) shows the decay length as a function of the initial light intensity: as the intensity increases, the decay length decreases, because the magnitude of the disorder increases, and the scattering process becomes more prominent. However, even though a high intensity beam decays faster than a low intensity beam, the output power of the high intensity beam is still higher than that of the low intensity beam.

Finally, we study the transport in a multilayer system where the nonlinearity is saturable, $\Delta n(z) = \Delta n_0 I/(I + I_{\rm sat})$, with $\Delta n_0$ being the maximum index change (which varies randomly from layer to layer) and $I_{\rm sat}$ is the saturation intensity (kept constant). This is an important class of nonlinearity, which appears in many physical systems [36]. In this kind of disordered nonlinearity, we find a distinct transition between localization and diffusive transport, as the light intensity crosses $I_{\rm sat}$. As we show in [27], if $I \gg I_{\rm sat}$, the nonlinearity is saturated and therefore the refractive index change is independent of the value of the intensity, and thus the disorder level remains constant. As a result, in this regime the system undergoes localization with exponential decay of the wave function. Because of this decay, $I$ decreases, until eventually the system undergoes a transition to the $I < I_{\rm sat}$ regime, where the refractive index change and the disorder levels depend again on the value of the intensity. There, we once again find the anomalous negative feedback mechanism that results in diffusive transport with power-law decay. The details, statistics, and transition from the high to low intensity regimes are described in [27].

In conclusion, we analyzed the transport of light in a new kind of system, where only the nonlinear coefficients are disordered. The unique interplay between the light intensity and the induced disorder creates negative feedback, causing the linear localization process to break down and the decay of the wave functions to change from exponential to power law, manifesting self-induced diffusion. The statistical attributes of an ensemble of multilayer structures of different realizations of the disorder are dramatically different from those of linear systems: the variance and the light intensity in all realizations of disorder coalesces to a single value. These results raise interesting questions. Is this type of self-induced diffusive transport possible in higher-dimensional systems, e.g., for transverse localization in the paraxial regime [5,6] or in full 3D? This kind of self-induced diffusion will also occur for Bose-Einstein condensation in the mean-field regime (e.g., as in [10]), but would it also occur for disordered quantum systems displaying many-body interactions [37,38]?

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