Spatial supercontinuum generation in nonlinear photonic lattices

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We show that two Bloch modes launched into a nonlinear photonic lattice evolve into a comb or a supercontinuum of spatial frequencies, exhibiting a sensitive dependence on the difference between the quasi-momenta of the two initially excited modes. This phenomenon results from four-wave mixing combined with exchanges of momentum between the optical field and the lattice.

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Wave propagation in nonlinear periodic structures exhibits many interesting phenomena.[1] The periodicity of the refractive index alters diffraction, while the nonlinearity may lead to various effects such as the formation of lattice solitons[2–4] and modulation instability.[2,5] Such systems have been a central theme in science for more than half a century, starting from the famous work of Fermi, Pasta, and Ulam[6] (FPU) and continuing nowadays as an extensive field of research.[7] In their work, FPU excited one mode of a *discrete* periodic system and studied the way energy spreads among the modes of the linear lattice under the influence of nonlinearity. Instead of the anticipated equipartition of energy among the different lattice modes, the system evolved in an almost periodic fashion, exhibiting near recurrence to the initial state.

Here we study how energy spreads among the linear [Floquet–Bloch (FB)] modes of a *continuous* periodic system, under the action of nonlinearity. We address the following question: starting with the excitation of two FB modes (of quasi-momenta k_1 and k_2), to which other FB modes will the energy flow and how? We show that, through four-wave mixing (FWM), energy spreads to more and more FB modes that are evenly spaced in reciprocal space. After sufficiently large propagation distances, the energy distribution among the FB modes attains a comb or a continuum structure, displaying a sensitive dependence on the momentum difference between the two initial modes. The dynamics described here also applies to other nonlinear periodic systems, such as nonlinear photonic crystals and Bose–Einstein condensates in periodic lattices.

The propagation of a coherent wave in a general linear system is best understood by considering the evolution of the linear modes of the system. Each one of these modes evolves in a trivial way, merely accumulating phase at a constant rate as it propagates, with no energy exchange between modes. The linear modes of a periodic system are the FB modes.[8] According to the FB theorem these modes can be written as $A_{nk}(x,z)=\exp(i\beta_{nk}z)\varphi_{nk}(x)=\exp(i\beta_{nk}z)\times\exp(ikx)u_{nk}(x)$, where $u_{nk}(x+D)=u_{nk}(x)$ are periodic functions with the same period D as the periodicity of the system; k is the quasi-momentum (QM) of the mode, which may be chosen to be in the first Brillouin zone (BZ): $-G/2\le k\le G/2$ ($G\equiv 2\pi/D$ is the width of the BZ); n is the band number; and β_{nk} is the propagation constant of this mode. If nonlinearity is introduced into the system, the linear modes become coupled. Let us consider a 1D waveguide array that is periodic in the transverse dimension x and invariant along the z axis, displaying a Kerr nonlinearity. The evolution of a coherent wavepacket in this system is described by the dimensionless nonlinear Schrödinger equation

$$i\frac{\partial A}{\partial z}+\frac{\partial^2 A}{\partial x^2}+\Delta n(x)A\pm|A^2|A=0, \tag{1}$$

where A is the slowly varying amplitude of the field and $\Delta n(x)$ is a linear refractive index with periodicity D. Expanding A as a superposition of the FB modes $A(x,z)=\Sigma_{n'k'}a_{n'k'}(z)\varphi_{n'k'}(x)\exp(i\beta_{n'k'}z)$, substituting A into Eq. (1), multiplying by the complex conjugate of φ_{nk}, and integrating over x yields coupled-mode equations for the amplitudes $a_{nk}(z)$:

$$\begin{aligned} i\frac{\mathrm{d}a_{nk}(z)}{\mathrm{d}z} &\pm \sum_{n1k1}\sum_{n2k2}\sum_{n3k3} a_{n1k1}(z)a_{n2k2}(z)a^*_{n3k3}(z) \\ &\times C_{n1k1n2k2n3k3nk} \\ &\times \exp[i(\beta_{n1k1}+\beta_{n2k2}-\beta_{n3k3}-\beta_{nk})z]=0, \end{aligned} \tag{2}$$

using the orthogonality of the FB modes. The tensor $C_{n1k1n2k2n3k3nk}$ is defined by the overlap integral, $\int\varphi_{n1k1}(x)\varphi_{n2k2}(x)\varphi^*_{n3k3}(x)\varphi^*_{nk}(x)\mathrm{d}x$, integrated over the whole lattice. Thus, quartets of modes are coupled, and for each quartet the extent of energy transfer depends on the overlap integral between the four modes of the quartet. In the case of FB modes, owing to their symmetry properties, these overlap integrals vanish for most combinations of modes; so energy is transferred only between groups of modes that have special relations between their QM. Utilizing the properties of the FB modes, the tensor C can be written as

$$C_{n1k1n2k2n3k3nk} = C_0 \int_{\text{one cell}} u_{n1k1}(x) u_{n2k2}(x) u^*_{n3k3}(x) \times u^*_{nk}(x) \exp[i(k1 + k2 - k3 - k)x] \mathrm{d}x, \tag{3}$$

where the coefficient C_0 is given by

$$C_0 = \exp\left[\frac{i\Delta k(N-1)D}{2}\right] \frac{\sin(\Delta k N D/2)}{\sin(\Delta k D/2)} = \exp\left[\frac{i\Delta k(N-1)D}{2}\right] N \sum_{n=-\infty}^{\infty} \mathrm{sinc}\left[\frac{ND}{2}(\Delta k - nG)\right] \times \begin{cases} 1 & N \text{ is odd} \\ (-1)^n & N \text{ is even}' \end{cases} \tag{4}$$

where $\Delta k \equiv k_1 + k_2 - k_3 - k_4$ and N is the number of lattice sites. That is, the coefficient C_0 is a sum of sinc functions centered around the origin in k space and around every reciprocal lattice vector. The first zero of each sinc occurs exactly at $1/N$ of the width of the BZ, and in the limit of an infinite lattice, the sinc function becomes a delta function. For a finite lattice with periodic boundary conditions, the sinc function is nullified exactly at the allowed k values (apart from $k=0$). Hence, quartets of modes are coupled if and only if $\Delta k \equiv 0, \pm mG$ (m is an integer), which is a QM conservation rule up to an integer multiple of a reciprocal lattice vector. Surely this result is anticipated: the QM's value is fixed just up to a reciprocal lattice vector, and the nonlinear term in Eq. (1) yields FWM.[9–12] In addition, Eq. (2) shows that energy is transferred efficiently when the phase-matching condition $\beta_{n1k1} + \beta_{n2k2} - \beta_{n3k3} - \beta_{n4k4} \approx 0$ is satisfied. Although the results are presented in one transverse direction, they can be readily generalized to higher dimensions, and also for other nonlinearities (saturable, quadratic, etc.).

Up to this point our discussion is general and does not specify the initial conditions. From this point on we consider an initial condition where two FB modes are excited at $z=0$, and we examine how a nonlinear interaction leads to redistribution of energy into the other (initially nonpopulated) FB modes. Implementing the QM conservation rule for this case, one can see that two new FB modes with QM $k=2k_1-k_2$ and $k=2k_2-k_1$ will be generated. When the propagation distance is large enough, the nonlinear interaction causes a cascaded excitation of modes, where every newly generated FB mode interacts with the FB mode that excited it, and their interaction generates modes with QM $k_1 - n\Delta k_{12}$ and $k_2 + n\Delta k_{12}$ ($n=1,2\ldots$, $\Delta k_{12} \equiv k_2 - k_1$). If the resulting QM k is outside the first BZ (i.e., $k > G/2$ or $k < -G/2$), the FB mode to which energy is transferred is folded back into the BZ to $k-G$ (or to $k+G$). The coupling is mainly to the band of the originally excited FB modes because of a higher value of the integral of Eq. (3) (describing the fourfold overlap among the wave functions of the four modes) and better phase matching. Since the value of the QM is meaningful up to an integer multiple of the reciprocal lattice vector, the QM space has the topology of a torus (which in one dimension is a circle). In one dimension, this implies that if the difference between the QM of the two initial FB modes Δk_{12} is commensurable with the width of the BZ (i.e., $\Delta k = \alpha G$ with a rational $\alpha = m/n$, m and n being coprime integers), then the nonlinear interaction gives rise to a comb of FB modes, consisting of only n modes in each band [as an example, see Fig. 1(a), for which $\alpha = 1/10$]. On the other hand, if Δk_{12} is incommensurable with G, then the nonlinear interaction excites an infinite dense set of modes[13] [Fig. 1(d)]. Clearly, the dynamics described above is sensitive to initial conditions because the commensurable initial conditions are densely embedded between the incommensurable ones. For a finite lattice with N sites, instead of commensurability the relevant criterion is the value of the greatest common divisor (GCD) of $\Delta k_{12}/(2\pi/DN)$ and N. The number of excited states in this case is $N/\mathrm{GCD}(\Delta k_{12}/(2\pi/DN), N)$.

To quantify the difference between the commensurate and the incommensurate cases, we calculate the participation number $\mathrm{PN} = 1/\Sigma_k p_k^2$, where $p_k = |\hat{A}(k)|^2 / \int |\hat{A}(k)|^2 \mathrm{d}k$ and $\hat{A}(k)$ is the Fourier transform of A. This function measures the number of appreciably excited modes and is approximately equal to the exponent of the Shannon entropy.[14] In both cases, PN evolves initially in the same fashion, starting with the value 2 and increasing as more and more modes become populated (Fig. 2). After some energy flows at least one cycle around the BZ (which in our example occurs at $z \approx 16$), the evolution of PN differs dramatically between the two cases: in the commensurate case it stays within the range 5–10 (bounded by the denominator of $\alpha \equiv \Delta k_{12}/G = 1/10$), whereas in the incommensurate case it increases rapidly.

To contrast this situation with propagation in a homogeneous medium, we launch two plane waves of

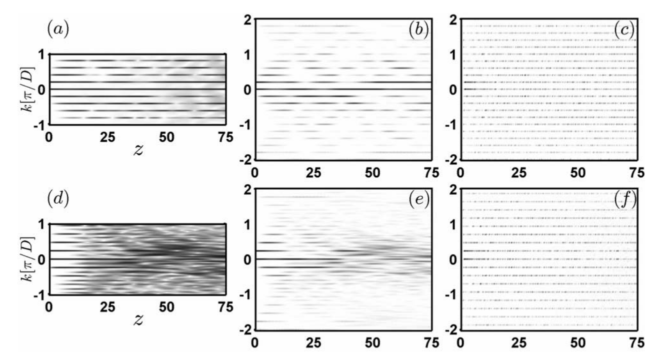

Fig. 1. Spatial power spectrum of the light, evolving with propagation. Excitation of two FB modes with QM difference (a), (b) commensurate or (d), (e) incommensurate with the width of the BZ. (a), (d) One-band FB power spectrum; (b), (e) Fourier power spectrum; (c), (f) Evolution of the Fourier power spectrum of the field when two plane waves are launched into a nonlinear homogeneous medium. In all panels the horizontal axis is the propagation (z) axis and the vertical axis is the (quasi-) momentum (k_x) axis.

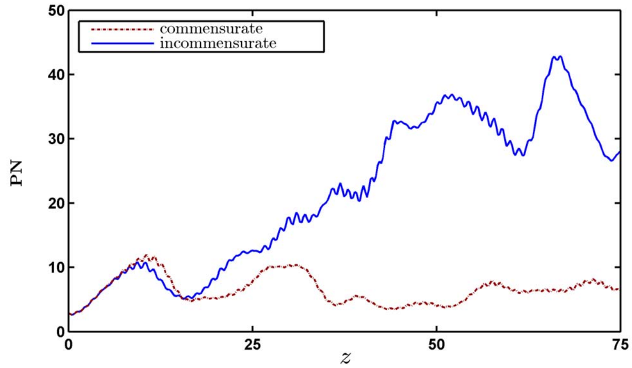

Fig. 2. (Color online) Participation number as a function of propagation for incommensurate (solid curve) and commensurate (dashed curve) QM difference.

the same amplitudes and k's of Figs. 1(a) and 1(d) into a homogeneous medium with the same nonlinearity and compare the evolution in Fourier space between the nonlinear propagation in the lattice [Figs. 1(b) and 1(e)] and in the homogeneous medium [Figs. 1(c) and 1(f)]. Since every FB mode is a comb in Fourier space, the comb (continuum) evolving in FB space is a comb (continuum) in Fourier space [Figs. 1(a) and 1(b)] [or Figs. 1(d) and 1(e)]. In contrast to the lattice case, for propagation in homogeneous media, the nonlinear interaction in both cases always populates a discrete set of plane waves, i.e., a comb in Fourier space. That is, in all our simulations, the two plane waves interacting via FWM in homogeneous media never evolve into a continuum and never display any sensitive dependence on momentum difference. We note that for both self-focusing and self-defocusing nonlinearities, the nonlinear dynamics and supercontinuum generation (SCG) are generically the same. In addition, we note that the introduction of a "prismatic" term (i.e., linear in x) into the linear refractive index does not change the physical behavior described above, since it causes only a collective transport of the energy in the reciprocal space with a k-independent rate. Hence, fulfillment of the commensurability condition is unaffected by this term. This means that SCG also appears in settings where (linear) Bloch oscillations occur,[15,16] with the results being similar to those without the prismatic term.

The maximum propagation distance for studies on SCG generated by the mechanism described above is limited by nonlinear interaction with noise, giving rise to the appearance of modulation instability. This phenomenon is inevitable numerically as well as experimentally, because the noise is never zero: the nonlinearity amplifies the noise and gives rise to the formation of a periodic pattern (determined by the interplay between lattice dispersion and nonlinearity[2]), thereby affecting the energy redistribution among the FB modes. Here these noise-driven effects manifest themselves by excitation of a continuum of modes for both the commensurate and incommensurate cases, diminishing the difference between the two generic cases. We also reemphasize that our studies here are performed in a continuous periodic medium, rather than discrete as in the original FPU problem. Naturally, one may ask how the energy flows among lattice modes for different types of excitation (single-mode excitation, etc.). In this sense, the problem that we addressed is the continuous version of the FPU problem. Altogether, understanding SCG, modulation instability, and other nonlinear phenomena in continuous nonlinear periodic systems should have important implications for the long-standing FPU problem of energy equipartition in nonlinear periodic systems.

In summary, we have shown that two optical Bloch modes interacting nonlinearly in a photonic lattice redistribute their energy through FWM and folding in reciprocal space, populating a comb of an increasing number of evenly spaced FB modes (if the momentum difference between the initial modes is commensurate with the lattice momentum) or a supercontinuum of FB modes (if the initial momentum difference is incommensurate with the lattice momentum). This behavior is independent of the sign of the nonlinearity and does not occur in homogeneous nonlinear media. Finally, we suggest implementing these ideas with Bose–Einstein condensates in optical lattices.

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