Composite Multihump Vector Solitons Carrying Topological Charge

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We propose composite solitons carrying topological charge: multicomponent two dimensional [(2 + 1)D] vector (Manakov-like) solitons for which at least one component carries topological charge. These multimode solitons can have a single hump or exhibit a multihump structure. The "spin" carried by these multimode composite solitons suggests 3D soliton interactions in which the particlelike behavior includes spin, in addition to effective mass, linear, and angular momenta.

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Composite (vector) solitons are solitons that consist of two (or more) components that mutually self-trap in a nonlinear medium. Degenerate vector solitons were first suggested by Manakov [1] in the context of the Kerr nonlinearity and recently demonstrated [2] with optical spatial solitons in a AlGaAs waveguide, and later on with optical temporal solitons in fibers [3] and in a fiber laser resonator [4]. Intuitively, vector solitons form when the field components jointly induce a waveguide structure (self-induced potential well) via the nonlinearity, and trap themselves in it by populating the guided (eigen) modes [5]. A key prerequisite for forming a vector soliton is that the interference between the modes does not contribute to the nonlinear index change, Δn; otherwise the induced waveguide is not stationary. In Manakov's original suggestion, this was accomplished by having the field components polarized orthogonally to one another, which eliminates interference altogether. This method of orthogonal polarizations limits the number of vector components to two. Over the years, two additional techniques for eliminating the contribution of interference to the index change (and thus for generating vector solitons) were proposed, both of which require that the nonlinearity has a noninstantaneous temporal response τ. In the first method, each field component is at a different frequency, and the frequency difference between components is much larger than $1/\tau$ [6], whereas in the second method the field components are incoherent with one another [7]. In both cases the phase of the interference terms varies (periodically in the former and randomly in the latter) much faster than the nonlinearity can respond, and therefore interference effects do not contribute to Δn. Stationary propagation of a vector soliton is achieved if, in addition, the field components correspond to guided modes of the waveguide induced by the sum of their intensities. The methods that rely on different frequencies and on mutual incoherence allow the formation of vector (Manakov-like) solitons with many field constituents. In particular, the coherence-based method [7] allows all components to experience the same refractive index and nonlinearity: for example, one can use it to generate true (ideal) Manakov solitons. Subsequently, it has been shown that vector solitons can also form if the field components belong to different modes of their jointly induced waveguide [8,9], provided that, once again, the interference between modes does not contribute to Δn. These solitons are called "multimode solitons" and, in many cases, their total intensity profile possesses multiple humps [10]. Multimode solitons were suggested for temporal vector solitons [8] and for spatial solitons [9]. Last year, they were demonstrated experimentally in photorefractive media by making use of mutual incoherence among the modes to eliminate the contribution of intermodal interference to Δn [10]. The recent progress on vector solitons was paralleled by rapid progress on soliton interactions (collisions), and has motivated new exciting ideas that are truly unique for vector solitons. For example, shape transformations of colliding solitons [11] and energy exchange between colliding vector solitons [12]. However, thus far all the theoretical and experimental studies of vector solitons were about one dimensional $[(1 + 1)\mathrm{D}]$ solitons.

Here, we propose composite solitons carrying topological charge: multicomponent two dimensional $[(2 + 1)\mathrm{D}]$ vector solitons for which at least one component carries topological charge. This analogy between solitons (self-trapped wave packets) and particles acquires now a new feature: the analog of spin, which is manifested in the topological charge carried by each vector soliton. These multimode solitons can have a single hump or exhibit a multihump structure. With soliton collisions in mind, it is obvious that the "spin" (topological charge), the multimode nature, and the multihump structure offer many new exciting possibilities for interactions between two-dimensional vector solitons.

The purpose of this Letter is to propose such solitons and study their parameter range of existence and the diversity of their shapes. To make our theoretical study as analytic as possible, we draw on the so-called thresholding nonlinearity, and employ the *self-consistency* principle presented

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in [9]. We seek composite solitons for which at least one mode carries topological charge. Even though a thresholding nonlinearity has not been encountered yet in physical systems, it serves well as a useful model to understand the *core ideas* involved by enabling an analytic treatment which draws on the modes of a step-index optical fiber. The composite multimode solitons presented here provide the necessary insight to realize these complex structures in a saturable nonlinearity. Following this analytic study, we follow up with a numerical study of solitons in saturable nonlinearity.

We start from the normalized equations

$$\begin{aligned} i\frac{\partial U}{\partial z} + \nabla_\perp^2 U + F(I)U = 0\,, \\ i\frac{\partial V}{\partial z} + \nabla_\perp^2 V + F(I)V = 0\,, \end{aligned} \tag{1}$$

where U, V are the envelopes of two interacting beams and $\nabla_\perp^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. Equations (1) describe two coupled beams in an optical medium with a *normalized* refractive index change $F(I)$ being zero for $I < I_{\mathrm{th}}$ and constant F_0 otherwise [13]; with the total intensity given by $I = |U|^2 + |V|^2$. We look for a composite vector soliton solution to Eqs. (1) with U and V carrying topological charges zero and m, respectively, in the form

$$U(r,\theta,z) = u(r)e^{i\mu z}\,, \qquad V(r,\theta,z) = v(r)e^{im\theta}e^{i\nu z}\,, \tag{2}$$

where μ,ν are the propagation constants of the vector constituents and vanishing boundary conditions at infinity are imposed. We do not consider here composite solitons in which *both* U and V carry nonzero charge, because those are expected to be unstable even in a saturable nonlinearity [14]. Substituting (2) into (1) yields

$$\begin{aligned} \frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + \kappa_1^2 u = 0\,, \\ \frac{d^2v}{dr^2} + \frac{1}{r}\frac{dv}{dr} + (\kappa_2^2 - m^2/r^2)v = 0\,, \end{aligned} \tag{3}$$

where $\kappa_1^2 \equiv F_0 - \mu > 0$ and $\kappa_2^2 \equiv F_0 - \nu > 0$ with $\mu, \nu > 0$. The solutions to system (3) are

$$u(r) = \begin{cases} u_0 J_0(\kappa_1 r), & 0 \le r \le a\,, \\ u_0 \frac{J_0(\kappa_1 a)}{K_0(\sqrt{\mu}\,a)} K_0(\sqrt{\mu}\,r), & r \ge a\,, \end{cases} \tag{4}$$

$$v(r) = \begin{cases} \eta J_m(\kappa_2 r), & 0 \le r \le a\,, \\ \eta \frac{J_m(\kappa_2 a)}{K_m(\sqrt{\nu}\,a)} K_m(\sqrt{\nu}\,r), & r \ge a\,, \end{cases} \tag{5}$$

where J_m (K_m) are the regular (modified) Bessel functions of the first (second) kind of order m. Here, u_0^2 is the peak intensity of the first component, and $\eta^2 J_m^2$ is the maximum intensity of the second; a is the normalized radius (the so-called V number) of the waveguide jointly induced by the soliton components. The propagation constants μ and ν satisfy the eigenvalue equations

$$\begin{aligned} \frac{\sqrt{\mu}\,K_1(\sqrt{\mu}\,a)}{K_0(\sqrt{\mu}\,a)} - \frac{\kappa_1 J_1(\kappa_1 a)}{J_0(\kappa_1 a)} = 0\,, \\ \frac{\sqrt{\nu}\,K_{m-1}(\sqrt{\nu}\,a)}{K_m(\sqrt{\nu}\,a)} + \frac{\kappa_2 J_{m-1}(\kappa_2 a)}{J_m(\kappa_2 a)} = 0\,. \end{aligned} \tag{6}$$

Equations (6) correspond to the eigenvalue equations of a weakly guiding step-index fiber. The polarization states of these modes were discussed extensively [15]. We impose the *self-consistency* condition on the vector components; i.e., the total intensity at the margins of the induced waveguide is equal to the threshold intensity

$$I_{\mathrm{th}} = u_0^2 J_0^2(\kappa_1 a) + \eta^2 J_m^2(\kappa_2 a)\,. \tag{7}$$

To study the range of existence and shape diversity of the composite multihumped solitons we concentrate on two cases for which the first component always carries zero charge, whereas the second one can have either charge 1 or 2. We denote such composite structures by $(0,1)$ and $(0,2)$, respectively. By employing the self-consistency condition (7) on the $(0,1)$ configuration and solving Eq. (6), we find multiple branches of existence curves of composite vector solitons as shown in Fig. 1. These composite structures have *fundamentally* different shapes on different groups of branches [16]. In branches I and II, the total intensity profile changes by increasing η, ($\eta = 0$ being the scalar case), from single hump (SH) to double hump (DH) (on branch I), and from triple hump (TH) to four humps (FH) (on branch II). In Fig. 2 we provide several examples of $(0,1)$ composite solitons of the branches I and

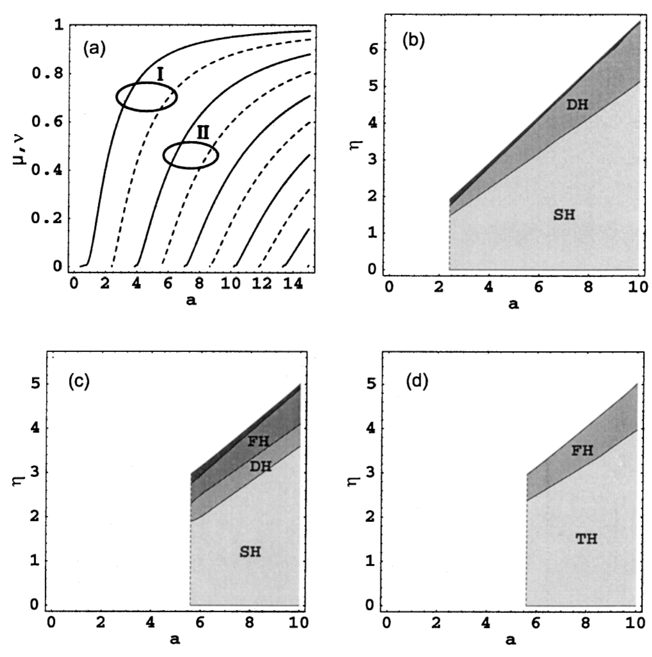

FIG. 1. (a) Propagation constants μ (solid) and ν (dashed) as a function of the normalized radius a of the induced waveguide (the V number) ; (b)–(d) existence regions for composite $(0,1)$ solitons at different branches: (b) branch I, (c) mixed branch, and (d) branch II.

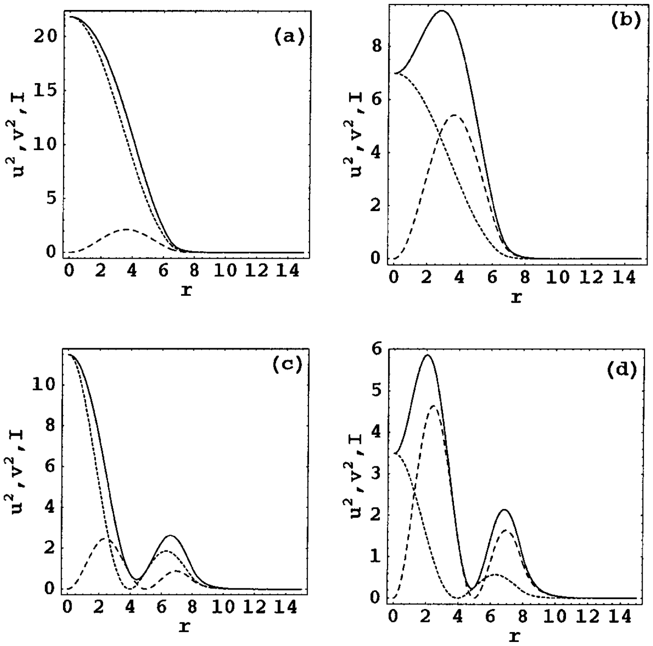

FIG. 2. u^2 (dotted), v^2 (dashed), and total intensity I (solid) for the $(0, 1)$ case. The parameters are (a) $a = 6.5$, $\eta = 2.5$ (branch I); (b) $a = 6.5$, $\eta = 4$ (branch I); (c) $a = 8$, $\eta = 2.7$ (branch II); and (d) $a = 8$, $\eta = 3.7$ (branch II).

II (corresponding to Figs. 1b and 1d). There u (v) carries zero (unity) charge, and $u^2(r), v^2(r)$ and the total intensity, $I = u^2(r) + v^2(r)$ are plotted for different values of η. The situation seems to be different on the mixed branch. By increasing η we observe a transition from SH to DH, then to FH (see Fig. 1c and specific examples in Figs. 3a, 3b, and 3c).

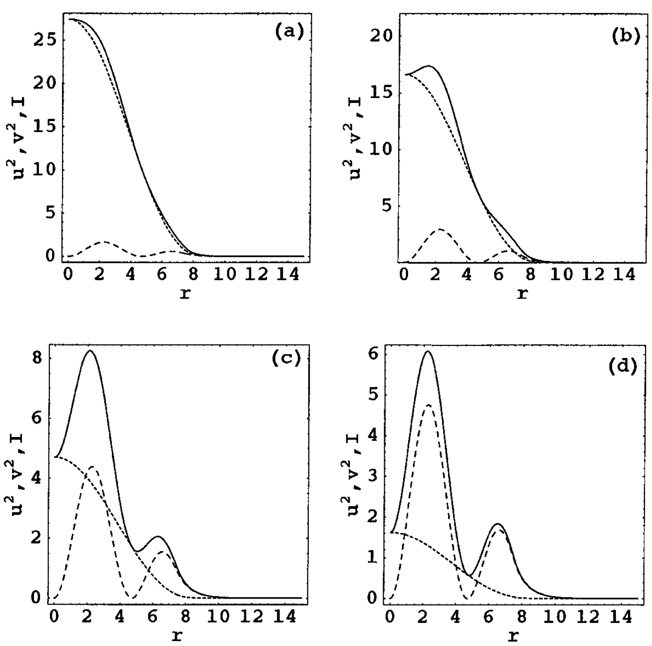

FIG. 3. As in Fig. 2 with parameters $a = 7.5$ (mixed branch) and $\eta = 2.2$ (a), 2.95 (b), 3.6 (c), and 3.75 (d).

Further increasing η, we find a narrow range of parameters, where the induced waveguide corresponds to a double core fiber (see Fig. 3d) for which $I < I_{th}$ in some regions of $r < a$. This region corresponds to the dark area in Fig. 1c and exists also in branches I and II, although the region is narrower (Figs. 1b and 1d). Next, for composite solitons with $(0, 2)$ topological charges, we solve the eigenvalue problem (6) with $m = 2$. Contrary to the $(0, 1)$ case where composite solitons exist in the first branch, here they exist starting from the lowest mixed and second branches (see Fig. 4a). In the mixed branch, the composite soliton total intensity profile changes from SH to TH (Figs. 4b, 5a, and 5b). On the other hand, in the second branch, starting with small values of η we find total intensity with TH shape along which $I < I_{th}$ in some regions of $r < a$. By crossing a line of critical values of η, the composite soliton intensity profile still exhibits TH shape but with $I(r < a) > I_{th}$ for all r (Figs. 4c and 5d).

The method we employ to find composite solitons of the structures $(0, 1)$ and $(0, 2)$ can be further used to find composite two-component solitons of structure $(0, m)$ where m is any integer, and it represents the topological charge of the V component. In the same manner one can find even higher composite structures: those that have three (or higher) field components. In fact, we have found them but they will be described elsewhere.

Having found composite solitons, a reasonable question to ask is the following: are they stable, and if not, are they at least observable? The work presented here deals with the thresholding nonlinearity, which does not lend itself to stability analysis. We have therefore studied the existence of composite solitons with a real saturable nonlinearity, i.e., $F(I) = -1/(1 + I)$, which represents, e.g., nonlinearities in an isotropic homogeneously broadened two-level system. Using a relaxation code, we find the wave functions of composite two-component solitons, and they have the same features as the solitons of the thresholding nonlinearity, including the multihump structure, except that now one can use the results to study stability and collisions. We find (numerically) that, at least in the SH case, these structures are stable against deviation of at least 2% in initial amplitudes and at least 5% relative displacement of the vector components. We have checked all of these within a propagation distance in excess of 100 physical

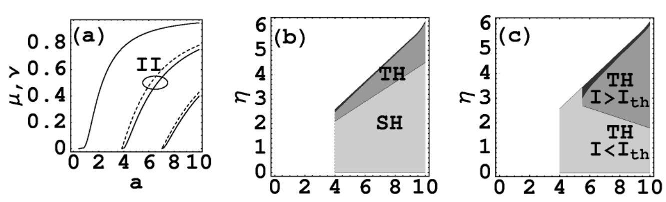

FIG. 4. (a) Propagation constants μ (solid) and ν (dashed) as a function of the normalized waveguide radius (the V number) a. (b)–(c) Existence regions for composite $(0, 2)$ solitons at the mixed branch (b) and at branch II (c).

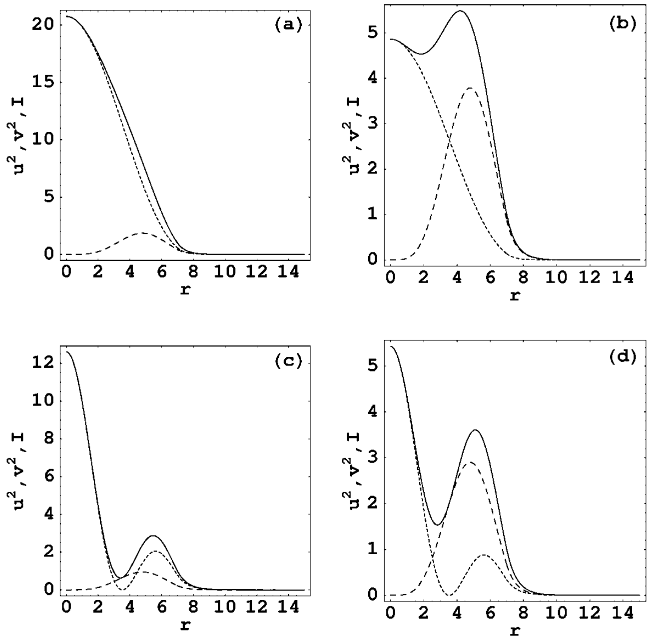

FIG. 5. As in Fig. 2 but for the $(0, 2)$ case. The parameters are (a) $a = 7$, $\eta = 2.8$ (mixed branch); (b) $a = 7$, $\eta = 4$ (mixed branch); (c) $a = 7$, $\eta = 2$ (branch II); and (d) $a = 7$, $\eta = 3.5$ (branch II).

diffraction lengths. Thus, we safely conclude that at least the SH composite solitons in a true saturable nonlinearity should be observable. This fact bears much importance because the higher-order modes, standing alone, are unstable [17]. This resembles $(1 + 1)$D multimode multihump solitons that were found numerically [10] and semianalytically [18] to be stable, at least in the SH and the DH cases, even though the higher-order modes, standing alone, are unstable. One can find, of course, regions of instability, especially for the multihump cases. This is not surprising, because regions of instability can be found also in the $(1 + 1)$D multihump cases [18,19]. Based on our numerical simulations for a true saturable nonlinearity, it seems very promising that single-hump $(2 + 1)$D composite solitons should be stable (or at least observable) for large propagation distances, and there seems to be a good chance that DH $(2 + 1)$D composite solitons will be observable as well. And indeed we are now conducting experiments to observe them.

In conclusion, we have predicted the existence of composite solitons that carry topological charges, with the hope to study collisions that involve not only energy, linear, and angular momenta, but also the equivalent of spin.

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