Modulation Instability of Incoherent Beams in Noninstantaneous Nonlinear Media

Marin Soljacic,[1] Mordechai Segev,[2,3] Tamer Coskun,[4,*] Demetrios N. Christodoulides,[4] and Ashvin Vishwanath[1]
[1]*Physics Department, Princeton University, Princeton, New Jersey 08544*
[2]*Physics Department, Technion-Israel Institute of Technology, Haifa 32000, Israel*
[3]*Electrical Engineering Department, Princeton University, Princeton, New Jersey 08544*
[4]*Department of Electrical Engineering and Computer Science, Lehigh University, Bethlehem, Pennsylvania 18015*
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We show that modulation instability can exist with partially spatially incoherent light beams in a noninstantaneous nonlinear environment. For such incoherent modulation instability to occur, the value of the nonlinearity has to exceed a threshold imposed by the degree of spatial coherence.

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Localized wave packets in linear media have a natural tendency to change their shape and broaden as they propagate, since the modes they are composed of propagate at different phase velocities. In nonlinear media, this broadening can be counteracted, resulting in a pulse/beam that does not change its shape during propagation: a soliton [1]. In optics, solitons can be understood as a balance between diffraction (in the spatial domain) or dispersion (in the time domain) and nonlinear self-focusing. In the spatial domain, the light beam elevates through nonlinearity the local index of refraction, thereby creating a waveguide, which in turn guides the beam, thus forming a spatial soliton. Thus far, stable soliton solutions of this sort have been observed in numerous nonlinear systems [2]. Another process, closely related to soliton formation, is modulation instability (MI). During MI, small amplitude and phase perturbations tend to grow exponentially as a result of the combined effects of nonlinearity and diffraction/dispersion. Because of this, a broad optical beam or a quasi-cw pulse tends to disintegrate during propagation [3,4]. Since MI typically occurs in the same parameter region where bright solitons are observed, it can be loosely considered as a precursor to soliton formation.

Until recently, optical spatial solitons were solely coherent entities. However, a recent series of experimental [5] and theoretical [6–10] works has demonstrated solitons made of partially incoherent light: incoherent solitons. Incoherent solitons are multicomponent (multimode) solitons that are made up from modal constituents that are incoherent with one another. They exist in noninstantaneous nonlinear media, when the average phase fluctuation time across the beam (or between modal constituents) is much shorter than the response time of the medium. In this case, the nonlinear change in the refractive index depends only on the time average of the light intensity [5,7]. The existence of incoherent solitons proves that self-focusing is possible not only for coherent wave packets but also for partially spatially/temporally incoherent light [5]. Since MI appears in all systems that support bright solitons, it is natural to wonder whether it also exists for incoherent light beams. In this Letter, we demonstrate, analytically and numerically, the existence of incoherent MI, that is, modulation instability of incoherent wave packets. Incoherent MI occurs when the value of the nonlinearity exceeds a threshold imposed by the degree of spatial coherence. We use analytical and numerical methods to study the properties of incoherent MI in a general self-focusing noninstantaneous medium. We solve for incoherent MI in closed form for input beams with Lorentzian angular power spectra, and illustrate it with Kerr and saturable nonlinearities. We confirm our results with numerical simulations, and further study general cases of input beams along with propagation-evolution effects.

The incoherent light we analyze propagates in the z direction, with its spatial coherence length being much smaller than its temporal coherence length, i.e., the beam is partially spatially incoherent and quasimonochromatic (the wavelength of light λ is much smaller than each of these coherence lengths). The nonlinear material is noninstantaneous, that is, the nonlinear index change is a function of the optical intensity, time averaged over the response time of the medium, τ, which is much longer than the coherence time of the light, t_c. Assuming that the light is linearly polarized and $E(r,z,t)$ is its slowly varying amplitude, we define $B(r_1,r_2,z) = \langle E^*(r_2,z,t)E(r_1,z,t)\rangle$ where the brackets denote the time average (taken over τ). The equation for B, as derived from paraxial wave equation, is [9]

$$\frac{\partial B}{\partial z} - \frac{i}{k}\frac{\partial^2 B}{\partial r\,\partial \rho} = \frac{in_0}{k}\left(\frac{\omega}{c}\right)^2 \{\delta n(r_1,z) - \delta n(r_2,z)\}B\,, \tag{1}$$

where ω is the carrier frequency, k is the carrier wave vector, n_0 is the index of refraction without light present, δn is the tiny nonlinear modification to the refractive index, $r = (r_1 + r_2)/2$ is the middle point coordinate, and $\rho = r_1 - r_2$ is the difference coordinate. $B(r,\rho,z)$ is the spatial correlation function, and $I(r,z) = B(r,\rho = 0,z)$ is the time-averaged light intensity. The definition of B yields $B(r,\rho,z) = B^*(r,-\rho,z)$.

To study MI, we assume the incident light to have a uniform intensity, except for small intensity perturbations that depend on r and z. Thus, B can be written as $B(r,\rho,z) = B_0(\rho) + B_1(r,\rho,z)$, where $B_1 \ll B_0$,

$B_0(\rho)$ representing the background of a uniform intensity $I_0 = B_0(\rho = 0)$. The dependence of δn on r comes from B_1, so to the lowest order in B_1, $\{\delta n(r_1, z) - \delta n(r_2, z)\} = \kappa\{B_1(r_1, \rho = 0, z) - B_1(r_2, \rho = 0, z)\}$, where $\kappa = d[\delta n(I)]/dI$ evaluated at I_0 is the marginal nonlinear index. In the Kerr case ($\delta n = \gamma I$), so $\kappa = \gamma$. Therefore, linearizing Eq. (1) in B_1 produces

$$\frac{\partial B_1}{\partial z} - \frac{i}{k}\frac{\partial^2 B_1}{\partial r \partial \rho} = \frac{i n_0}{k}\left(\frac{\omega}{c}\right)^2 \kappa \left\{ B_1\left(r + \frac{\rho}{2}, \rho = 0, z\right) - B_1\left(r - \frac{\rho}{2}, \rho = 0, z\right)\right\} B_0(\rho). \quad (2)$$

Up to this point, the discussion applies for any correlation function $B_0(\rho)$. In what follows, we assume that $B_0(\rho)$ has a Lorentzian-shaped k spectrum and obtain closed-form results. Thus, $\hat{B}_0(k_x) = A/[k_x^2 + k_{x0}^2]$, where $\hat{F}(k_x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\rho F(\rho) e^{ik_x\rho}$ is the Fourier transform of $F(\rho)$ for any $F(\rho)$. In this case the normalized angular power spectrum [6] is also Lorentzian $[G_N(\theta) = (\theta_0/\pi)(\theta^2 + \theta_0^2)^{-1}]$ where the angle $\theta = k_x/k$ is in radians. The background intensity is then $I_0 = \pi A/k_{x0}$. The physically acceptable eigenmodes of Eq. (2) can be written as $B_1(r, \rho, z) = \exp(gz)\exp[i(\alpha r + \phi)]L(\rho) + \exp(g^* z)\exp[-i(\alpha r + \phi)]L^*(-\rho)$, where ϕ is an arbitrary real phase that carries no physical significance, α is real, and g is associated with the MI gain. These modes automatically satisfy $B_1(r, \rho, z) = B_1{}^*(r, -\rho, z)$. For each α one can obtain a set of modes $L(\rho)$ needed to describe any perturbation B_1. Defining $M(\rho) = L(\rho)/L(\rho = 0)$, with the boundary condition $M(\rho = 0) = 1$, we get

$$gM(\rho) + \frac{\alpha}{k}\frac{dM(\rho)}{d\rho} + \frac{2\omega\kappa}{c}\sin\left(\frac{\rho\alpha}{2}\right)B_0(\rho) = 0. \quad (3)$$

We are interested in the modes that grow: those with g that are not purely imaginary. We look for the particular and for the homogeneous solutions to Eq. (3). Since we have a physical constraint that $M(\rho)$ has to be bounded for large $|\rho|$, the homogeneous solution has to be zero for the modes that have a real part in g. Therefore, by seeking a particular solution of Eq. (3), we find

$$\hat{M}(k_x)\left(g - i\frac{\alpha}{k}k_x\right) = \frac{i\omega\kappa}{c}\left[\hat{B}_0\left(k_x + \frac{\alpha}{2}\right) - \hat{B}_0\left(k_x - \frac{\alpha}{2}\right)\right]. \quad (4)$$

$M(\rho)$ obtained from Eq. (4) is clearly bounded for large $|\rho|$. Impose $M(\rho = 0) = 1$, or $\int_{-\infty}^{\infty}\hat{M}(k_x)dk_x = 1$:

$$1 = -\frac{\omega\kappa}{c}\int_{-\infty}^{\infty} dk_x\left[\hat{B}_0\left(k_x + \frac{\alpha}{2}\right) - \hat{B}_0\left(k_x - \frac{\alpha}{2}\right)\right] \times \left(\frac{1}{ig + \alpha k_x/k}\right), \quad (5)$$

which gives us a constraint that determines the dispersion relation $g(\alpha)$ for the modes whose g has a real part, for arbitrary $B_0(\rho)$. Note that $\hat{B}_0(k_x)$ is purely real since $B_0(\rho) = B_0{}^*(-\rho)$.

By assuming a Lorentzian k spectrum $\hat{B}_0(k_x) = A/[k_x^2 + k_{x0}^2]$ in Eq. (5), a contour integration yields the following result for the mode that grows, if g is bigger than zero:

$$\frac{g}{k} = -(k_{x0}/k)(|\alpha|/k) + (|\alpha|/k)\sqrt{\frac{\kappa I_0}{n_0} - \left(\frac{\alpha}{2k}\right)^2}, \quad (6)$$

where κ represents the marginal nonlinear index change because of the constant background intensity, and $k_{x0}/k = \theta_0$. The result of Eq. (6) clearly demonstrates that the MI growth rate is substantially affected by the coherence of the source. Moreover, in the limit $k_{x0} \to 0$, it correctly reduces to the well-known result of coherent MI [3,4]. Even more importantly, Eq. (6) indicates that for a given degree of coherence, MI occurs only when the quantity κI_0 exceeds a specific threshold; incoherent MI exists *only if* $\kappa I_0/n_0 > \theta_0^2$, whereas when $\kappa I_0/n_0 < \theta_0^2$ MI is *entirely eliminated.* Thus, the more incoherent a source is, the larger κI_0 (marginal index change) required to induce MI. Computer simulations suggest that this trend is *universal* and is independent of the angular power spectrum. Having found $g(\alpha)$, one can then easily determine the intensity of the perturbation $I_1(r, z) = B_1(r, \rho = 0, z)$.

To apply the result of Eq. (6) for Kerr nonlinearity $\delta n(r) = \gamma I$, we set $\kappa = \gamma$ and present it graphically in Fig. 1. In this case, $\kappa I_0 = \delta n$, so the larger the nonlinear index change, the stronger the MI growth. However, the MI growth rate can be analytically determined for any type of nonlinearity. Of particular significance is the saturable nonlinearity which occurs, for example, in photorefractives [11], and in homogeneously broadened 2-level systems (atomic rubidium vapor) [12], in which the nonlinear index change is of the form $\delta n(r) = \gamma I(r)/[1 + I(r)/I_{\text{sat}}]$. For the saturable case, we find that the growth rate is given by Eq. (6) with $\kappa = \gamma/[1 + I_0/I_{\text{sat}}]^2$. As with the Kerr nonlinearity, incoherent MI exists, once again, only above a specific threshold for κI_0. From the result, it is apparent that saturation suppresses MI by a factor of $1/[1 + I_0/I_{\text{sat}}]$. Figure 2 displays graphically typical results with the saturable nonlinearity.

To verify our analytical findings and to further explore incoherent MI, we use computer simulations. In particular, the intensity/correlation MI dynamics of Eq. (1) are investigated by means of the coherent density approach [6]. The power Fourier spectrum of the intensity fluctuations growing on top of the constant incoherent background is used to identify the spatial frequencies that exhibit maximum gain. Figure 3(a) shows the evolution of the power spectra of the intensity fluctuation when the angular power spectrum of the source is Lorentzian, and the nonlinearity is Kerr type. In this example, $\theta_0 = 9.6$ mrad, $\gamma I_0 = 5 \times 10^{-4}$, $n_0 = 2.3$, $\lambda = 0.5\ \mu$m in vacuum. These results indicate that maximum MI gain is attained

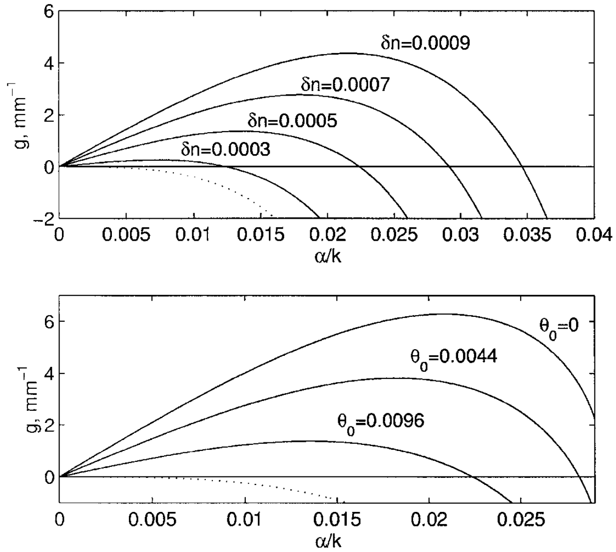

FIG. 1. Incoherent MI for Kerr nonlinearity: growth rate of perturbations vs the perturbation wavelengths. The light wavelength in vacuum is 500 nm, and the refractive index $n_0 = 2.3$. The background uniform intensity has a Lorentzian shaped angular power spectrum of width θ_0. The nonlinear index change due to the background is given by δn. In the upper plot, we show gain curves for a few δn's, with a fixed $\theta_0 = 0.0096$ rad; the dashed line in the plot has δn marginally small enough so that MI just disappears. In the lower plot, we show gain curves for a few θ_0's, with a fixed $\delta n = 0.0005$; the dashed line in the plot has θ_0 marginally large enough so that MI just disappears.

at a spatial frequency $\alpha/k \approx 0.0135$, with a peak value of $g = 1.37\ \mathrm{mm}^{-1}$, both in an excellent agreement with that predicted from Eq. (6) as also depicted in Fig. 1 for the same set of parameters. Note that in this case the spatial frequency where maximum MI gain occurs remains invariant during propagation. After a certain distance (in this example, after 9 mm) additional subbands emerge as in the case of coherent MI [13]. This is "secondary" MI: modulation instability for which the first amplified instability peak acts as a "pump" and plays the role of B_0. Numerical simulations confirm another prediction of the analytic result: *the existence of a threshold for incoherent MI.* The numerical study also provides information about the evolution of incoherent MI from input beam of angular power spectra different than the Lorentzian shape. For example, Fig. 3(b) depicts information similar to Fig. 3(a) when the source angular power spectrum is Gaussian $[G_N(\theta) = (1/\pi^{1/2}\theta_0)\exp(-\theta^2/\theta_0^2)]$. In this case $\theta_0 = 9.6$ mrad, $\gamma I_0 = 2.5 \times 10^{-4}$, $n_0 = 2.3$, and $\lambda = 0.5\ \mu$m. The MI spectra initially evolve in a fashion similar to the Lorentzian case, but, after some propagation distance the perturbation grew substantially (and our analytic approximation $B_1 \ll B_0$ no longer holds), and the MI spectrum does not exhibit a clear peak but broadens with propagation. As observed from Eq. (5), the MI growth highly depends on the shape of the angular power spectrum of the source.

We emphasize that one can use our logic up to Eq. (6) in order to obtain analytical understanding of MI with an arbitrarily shaped angular power spectrum, where one needs to solve the integral in Eq. (5) numerically (except for the Lorentzian case, where one can obtain closed-form solutions as we did). When we apply this idea for an input beam of a Gaussian power spectrum [Fig. 3(b)], we obtain an excellent agreement between our analytic and numeric results. The analytic expansion also captures the fact that there always exists a threshold κI_0 for incoherent MI to occur.

Before closing, we wish to emphasize that this MI result cannot be obtained using a ray optics (transport) approach [8], which assumes that the source is fully incoherent (or the period of the perturbation is much larger than the transverse coherence length). In this picture, each ray behaves like a particle that follows a trajectory determined by the local index of refraction, which has a role of potential. The local index of refraction is in turn determined by the local density of the rays. Therefore, this picture is very similar to a gravitational system of many particles. Unfortunately, for partially spatially incoherent optical beams, the spatial frequency that grows fastest is beyond this regime. Nevertheless, studying this approach is instructive because it connects the incoherent MI system with systems as different as galaxy formation, and one can use it to predict driven MI (induced MI).

To conclude, we have shown that modulation instability exists in partially incoherent systems, and that its existence requires the marginal nonlinear index change times the background intensity, κI_0, to be above a well-defined threshold. *This is in a marked difference with coherent MI, since there does not exist a similar threshold for coherent MI.* The κI_0 is determined by the spatial degree of coherence (angular power spectrum). To emphasize the fundamental importance of this result, recall that partially incoherent light is a system in which the "quasiparticles"

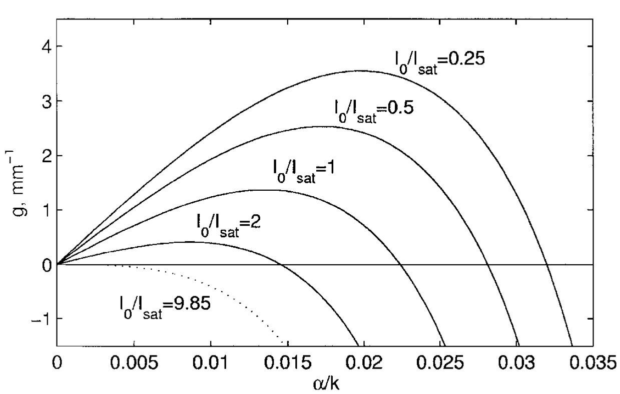

FIG. 2. Incoherent MI for a saturable nonlinearity: perturbations growth rate vs perturbation wavelengths. $\lambda = 500$ nm in vacuum, and the refractive index $n_0 = 2.3$. The background uniform intensity has a Lorentzian angular power spectrum of width $\theta_0 = 0.0096$ rad. The nonlinear index change due to the background is $\delta n = 0.001$. In the saturable case, there is additional suppression due to the saturation. The gain curves are plotted for various degrees of saturation; the dashed curve is the case when the saturation is marginally large enough so that MI just disappears.

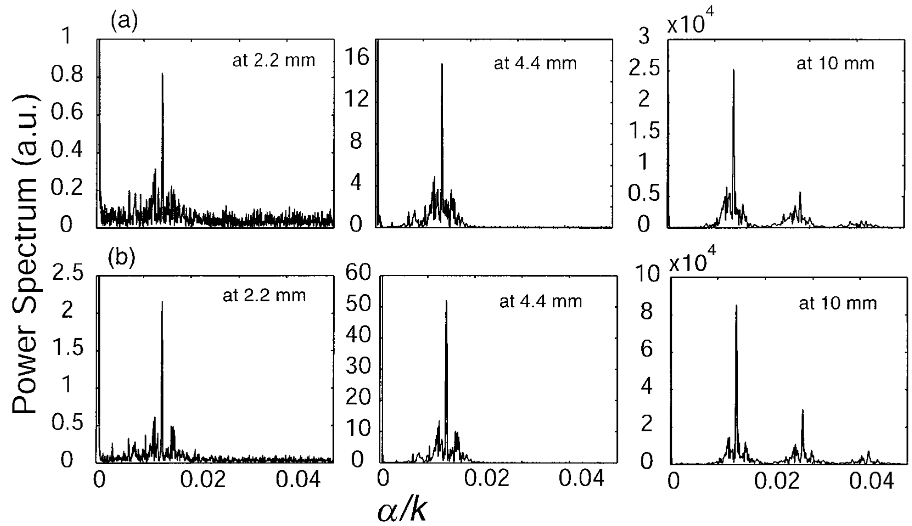

FIG. 3. Power spectrum of an incoherent beam during propagation in Kerr nonlinearity when (a) the source power spectrum is Lorentzian with $\theta_0 = 0.0096$ rad and $\delta n = 0.0005$; (b) the source power spectrum is Gaussian with $\theta_0 = 0.0096$ rad and $\delta n = 0.000\,25$.

are only weakly phase correlated (with the extreme case being a fully incoherent system in which the "quasiparticles" are fully noncorrelated). Yet this weakly correlated system exhibits features characteristic of phase transition: above a well-defined threshold, it collapses and forms "clusters" (filaments). We also emphasize that, since modulation instability is a precursor of solitons, it is to some extent an even more universal phenomenon than solitons. In fact, MI appears in all systems supporting solitons [14] and also in systems in which solitons have not been identified as of yet. We have identified here MI in an incoherent system, which is fundamentally a system in which repulsion forces are much weaker than the attraction forces [2]. Since nature is full of nonlinear systems in which incoherent wave packets exist (e.g., optics [15], plasma physics [16]), we expect that these systems will exhibit MI as well. We believe that this work lays the foundations for instabilities and pattern formation in any nonlinear incoherent system in nature. Furthermore, incoherent MI links to other related but qualitatively different phenomena, such as galaxy formation. From all of these arguments, one thing is obvious: there are many more new exciting features that are intimately related to incoherent modulation instability and are yet unraveled, calling for future research.

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*Present address: Pamukkale University, Denizli, Turkey.

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