Multimode Incoherent Spatial Solitons in Logarithmically Saturable Nonlinear Media

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We show that multimode incoherent spatial solitons are possible in logarithmically saturable nonlinear media. The mode-occupancy function associated with these soliton states is found to obey a Poisson distribution. Our analysis indicates that two approaches, i.e., the dynamic coherent density description as well as static self-consistent multimode method lead to exactly the same results. Closed form solutions are obtained for $(1 + 1)$D as well as for $(2 + 1)$D circular and elliptical incoherent solitons. [S0031-9007(98)05547-1]

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To date, the nonlinear process of optical beam self-focusing has been investigated by means of coherent or laser sources [1]. Only very recently has the first experimental observation of incoherent self-trapping been reported in literature [2]. In this experiment, a monochromatic partially spatially incoherent light beam was found to self-trap in a biased photorefractive crystal. As in the case of coherent photorefractive spatial solitons, this was achieved via the saturable photorefractive drift nonlinearity [3,4]. More specifically, at a certain bias voltage the incoherent beam was found to self-trap whereas above or below this limit the beam underwent compression or expansion (diffraction), respectively. Even more importantly, the experiment suggested that incoherent spatial solitons may indeed be feasible in biased photorefractives. These results are by themselves of great significance since they shed new light on the complex dynamics of incoherent optical beams in a nonlinear environment. Shortly after, self-trapping of incoherent white light was also observed in the same material system [5].

In order to explain the results of Ref. [2], we have recently approached this problem by introducing a theory based on the so-called coherent density [6]. In this description, the underlying evolution model takes the form of a nonlinear Schrödinger-like integro-differential equation provided that at the origin the coherent density is appropriately scaled with respect to the angular power spectrum of the incoherent source. The results of this study were found to be in good agreement with the experimental data of Ref. [2]. Moreover, using this same approach it has also been shown that $(1 + 1)$D incoherent spatial solitons are in fact possible in saturable nonlinear media of the logarithmic type [7]. It is important to note, however, that, in general, the coherent density model is by nature better suited to describe dynamic evolution and thus it does not readily lend itself toward identifying stationary solutions such as incoherent spatial solitons. With the exception of the saturable logarithmic model (perhaps the only exactly soluble model in the coherent density description), the question of whether incoherent spatial solitons also exist or not in other saturable nonlinear systems (such as in biased photorefractives) cannot be easily addressed. In order to circumvent this problem, we have very recently developed an alternative theory which is essentially a self-consistent incoherent multimode approach [8]. In brief, in this procedure, an incoherent spatial soliton is sought, which intensitywise is a superposition of all the modes self-consistently guided in its nonlinearly induced waveguide [9]. By doing so, we have obtained $(1 + 1)$D photorefractive incoherent spatial solitons, the conditions necessary for their existence, as well as their coherence characteristics [8]. At first sight, the two theories [6,8] may seem to have very little in common. Thus it is important to ask whether the two approaches are mutually consistent or lead to the same results. Moreover, it is interesting to know whether $(2 + 1)$D incoherent spatial solitons are also possible in saturable nonlinear media.

In this Letter we show that multimode incoherent spatial solitons are possible in logarithmically saturable nonlinear media. The mode-occupancy function of these soliton states is found to obey a Poisson distribution. Our analysis demonstrates that, in this case, the two theories, i.e., the dynamic coherent density description and the static self-consistent multimode method, lead to identical results. Two-dimensional circular and elliptical incoherent spatial solitons along with their characteristics are also obtained in closed form. Even though the saturable logarithmic nonlinearity differs from the photorefractive, it provides nevertheless a platform (perhaps the only platform) upon which the equivalence of the two previously mentioned approaches can be established in closed form. Finally, as recently argued by Snyder and Mitchell [10] and by Shen [11], simplifications of this sort in terms of "accessible" nonlinearities can provide valuable insight and still maintain the characteristic features of the underlying physical process.

We begin our analysis by considering a saturable nonlinear medium of the logarithmic type, similar to that

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previously considered by Snyder and Mitchell in their study of "mighty morphing" spatial solitons and bullets [12], i.e.,

$$n^2(I) = n_0^2 + n_2 \ln(I/I_t), \tag{1}$$

where n_0 is the linear refractive index of the material, n_2 is a positive dimensionless coefficient associated with the strength of the nonlinearity, and I_t is a threshold intensity. To avoid any logarithmic singularities we assume that the $\ln(I/I_t)$ nonlinearity results from the more realistic $\ln(1 + I/I_t)$ model in the limit $I \gg I_t$. Let the optical beam propagate in the z direction. We also make the important assumption that the nonlinearity responds much slower than the characteristic phase fluctuation time across the incoherent beam so as to avoid speckle-induced filamentation instabilities [13]. Thus, in this regime the material will experience only the time-averaged beam intensity as in the case of photorefractives. Furthermore, let the electric component of the optical field be written in terms of a slowly varying envelope U, i.e., $E = U\exp(ikz)$, where $k = k_0 n_0$ and $k_0 = 2\pi/\lambda_0$. In that case, it can readily be shown that the envelope U in this nonlinearly induced waveguide evolves according to

$$i\frac{\partial U}{\partial \xi} + \frac{1}{2}\left(\frac{\partial^2 U}{\partial s^2} + \frac{\partial^2 U}{\partial \eta^2}\right) + \frac{\alpha^2}{2}\ln(I_N)U = 0, \tag{2}$$

where $I_N = I/I_t$ is a normalized intensity and in Eq. (2) we have used normalized coordinates and quantities; that is, $s = x/w_0$, $\eta = y/w_0$, $\xi = z/kw_0^2$, $\alpha^2 = n_2(k_0 w_0)^2$, where w_0 is an arbitrary spatial scale or spot size.

We first employ the self-consistent incoherent multimode description in the case when the incoherent beam is one dimensional or planar. For the time being, let us assume that an incoherent spatial soliton of a Gaussian intensity profile exists, i.e., $I_N = r\exp(-s^2)$, where r is an intensity ratio with respect to I_t. From these last assumptions, Eq. (2) takes the form

$$i\frac{\partial U}{\partial \xi} + \frac{1}{2}\frac{\partial^2 U}{\partial s^2} + \frac{\alpha^2}{2}[\ln(r) - s^2]U = 0. \tag{3}$$

The allowed modes in this parabolic waveguide can then be easily obtained in terms of Gauss-Hermite functions [14] and thus the optical field can be expressed through superposition, i.e., $U = \sum_{m=0}^{\infty} c_m u_m$, where

$$u_m = H_m(\alpha^{1/2}s)\exp(-\alpha s^2/2)\exp(i\beta_m\xi) \tag{4}$$

and c_m are the mode-occupancy coefficients that vary randomly in time. In Eq. (4), $H_m(x)$ are Hermite polynomials, $\beta_m = (1/2)[\alpha^2\ln(r) - (2m+1)\alpha]$ and $m = 0, 1, 2, \ldots.$ The time-averaged intensity of this beam can then be obtained from

$$I_N = \langle |U|^2\rangle = \sum_{m=0}^{\infty} \langle |c_m|^2\rangle |u_m|^2, \tag{5}$$

where we made use of the fact that the time average of the cross-interference terms is zero ($\langle c_i c_j^*\rangle \propto \delta_{ij}$) under incoherent excitation [8,15]. Yet, at this point, we still do not know the mode-occupancy function $\langle |c_m|^2\rangle$ that self-consistently leads to the incoherent Gaussian soliton $I_N = r\exp(-s^2)$ assumed in the very beginning of this analysis. Let the mode occupancy, as in the case of quantum mechanical coherent or Glauber states [14,16], be described by a Poisson distribution [17]; that is,

$$\langle |c_m|^2\rangle = \hat{r}\exp(-p/2)\frac{(p/2)^m}{m!}, \tag{6}$$

where p is the Poisson parameter and $\hat{r}$ is a constant to be determined. From Eqs. (4)–(6) and with the aid of Mehler's formula [18],

$$\sum_{m=0}^{\infty}\frac{(p/2)^m}{m!}H_m(x)H_m(y) = \frac{1}{\sqrt{1-p^2}}\exp\left[\frac{2pxy - p^2(x^2+y^2)}{1-p^2}\right], \tag{7}$$

one can then obtain the following result:

$$I_N = \frac{\hat{r}\exp(-p/2)}{\sqrt{1-p^2}}\exp\left[-\alpha s^2\left(\frac{1-p}{1+p}\right)\right]. \tag{8}$$

Thus the assumed Poisson distribution self-consistently leads to the incoherent spatial soliton $I_N = r\exp(-s^2)$ provided that $\hat{r} = r\sqrt{1-p^2}\exp(p/2)$ and

$$p = \frac{\alpha - 1}{\alpha + 1}. \tag{9}$$

From Eq. (7) it is evident that, in this case, the positive Poisson parameter p must be below unity ($p < 1$) for this incoherent soliton to exist. Or alternatively, from Eq. (9), $\alpha > 1$ which implies that at a given wavelength λ_0, the quantity $n_2^{1/2}w_0$ has a cutoff, below which no incoherent spatial solitons are allowed.

To establish the equivalence between this result [Eqs. (8) and (9)] with that previously obtained using the coherent density approach [7] one must resort to diffraction data. More specifically, during linear diffraction ($n_2 = 0$), the intensity of each Gauss-Hermite mode evolves according to [14]

$$|u_m(s,\xi)|^2 = \frac{1}{\sqrt{1+\alpha^2\xi^2}}\exp\left(-\frac{\alpha s^2}{1+\alpha^2\xi^2}\right) \times H_m^2\left[\frac{\alpha^{1/2}s}{\sqrt{1+\alpha^2\xi^2}}\right]. \tag{10}$$

By keeping in mind that the mode-occupancy function $\langle |c_m|^2\rangle$ satisfies the Poisson distribution of Eq. (6) and by employing again Eqs. (7) and (9), we find that upon diffraction the normalized intensity profile I_D is given by

$$I_D = \frac{r}{\sqrt{1+\alpha^2\xi^2}}\exp\left(-\frac{s^2}{1+\alpha^2\xi^2}\right). \tag{11}$$

Equation (11) clearly shows that during diffraction this incoherent beam remains Gaussian. Given the fact that the initial modulation function [6] is Gaussian, i.e., $\phi_0(s) = \exp(-s^2/2)$, and that the diffraction behavior of this statistically stationary beam obeys a convolution integral [Eq. (3) of Ref. [6]], it then becomes immediately apparent that the angular power spectrum $G_N(\theta)$ of the incoherent source must also be Gaussian. In this regard, let $G_N(\theta) = (\pi^{1/2}\theta_0)^{-1}\exp(-\theta^2/\theta_0^2)$, where θ_0 is the width of the incoherent angular power spectrum. Thus, had we treated the diffraction regime within the coherent density approach [6], we would have arrived at the following result:

$$I_D = \frac{r}{\sqrt{1 + (1 + V^2)\xi^2}}\exp\left(-\frac{s^2}{1 + (1 + V^2)\xi^2}\right), \tag{12}$$

where $V \equiv kw_0\theta_0$. Since diffractionwise, the results of Eqs. (11) and (12) are the same, we conclude that $\alpha^2 = 1 + V^2$ or that

$$w_0 = \frac{1}{k_0\sqrt{n_2 - n_0^2\theta_0^2}}. \tag{13}$$

The result of Eq. (13) is exactly what has been previously obtained [Eq. (9) of Ref. [7]] using the coherent density approach. At a more fundamental level, our analysis demonstrates that the two theories, i.e., the dynamic coherent density description as well as the self-consistent multimode method, lead in this case to exactly the same results. In the coherent limit $\theta_0 \to 0$ or $\alpha \to 1$, the soliton spot size reduces to $w_0 = (k_0 n_2^{1/2})^{-1}$ as previously found by Snyder and Mitchell [12]. In this same limit, Eq. (9) suggests that the Poisson parameter $p \to 0$ and thus the beam itself is single moded or fully coherent. Moreover, the width θ_0 of the angular power spectrum of the incoherent source and the Poisson parameter p are now related through the following expression: $p = [(1 + V^2)^{1/2} - 1]/[(1 + V^2)^{1/2} + 1]$. Of course these incoherent spatial soliton states exist as long as $p < 1$ or $\alpha > 1$ or equivalently $\theta_0 < n_2^{1/2}/n_0$.

In the same vein, one may show that two-dimensional incoherent spatial solitons also exist in this nonlinear system. To do so, let us assume in general an elliptic Gaussian incoherent beam

$$I_N = r\exp[-(s^2 + \sigma^2\eta^2)], \tag{14}$$

where the parameter σ is associated with its degree of ellipticity. Substituting Eq. (14) in Eq. (2) we get

$$i\frac{\partial U}{\partial\xi} + \frac{1}{2}\left(\frac{\partial^2 U}{\partial s^2} + \frac{\partial^2 U}{\partial\eta^2}\right) + \frac{\alpha^2}{2}[\ln(r) - s^2 - \sigma^2\eta^2]U = 0. \tag{15}$$

The optical field U in this elliptic paraboloid potential can be obtained as a superposition of the allowed Gauss-Hermite modes, i.e., $U = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_{m,n}u_{m,n}$, where

$$u_{m,n} = H_m(\alpha^{1/2}s)H_n(\alpha^{1/2}\sigma^{1/2}\eta) \times \exp[-(\alpha/2)(s^2 + \sigma\eta^2)]\exp(i\beta_{m,n}\xi), \tag{16}$$

where $\beta_{m,n} = (\alpha/2)[\alpha\ln(r) - (2m + 1) - \sigma(2n + 1)]$. The overall intensity of this beam can then be found from

$$I_N = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\langle|c_{m,n}|^2\rangle|u_{m,n}|^2, \tag{17}$$

where we have assumed again that under incoherent excitation $\langle c_{ij}c_{kl}^*\rangle \propto \delta_{ik}\delta_{jl}$ [15]. If we now allow the mode occupancy $\langle|c_{m,n}|^2\rangle$ to obey a double Poisson distribution

$$\langle|c_{m,n}|^2\rangle = \hat{r}\exp\left[-\frac{1}{2}(p + q)\right]\frac{(p/2)^m(q/2)^n}{m!\,n!}, \tag{18}$$

then the two-dimensional incoherent spatial soliton $I_N = r\exp[-(s^2 + \sigma^2\eta^2)]$ can be self-consistently recovered using Eqs. (7), (17), and (18) provided that the two Poisson parameters p, q satisfy

$$p = \frac{\alpha - 1}{\alpha + 1}, \tag{19}$$

$$q = \frac{\alpha - \sigma}{\alpha + \sigma}, \tag{20}$$

and moreover $\hat{r} = r[(1 - p^2)(1 - q^2)]^{1/2}\exp[(p + q)/2]$. These incoherent solitons can be elliptical ($\sigma \neq 1$) or circular ($\sigma = 1$) depending on whether the angular power spectrum of the incoherent source (exciting this structure) is symmetric or not. Moreover, these states exist as long as $0 < (p,q) < 1$ or $\alpha > \max(1,\sigma)$. The possibility of generating (2 + 1)D elliptical solitons in isotropic nonlinear media seems to be unique to incoherent multimode solitons, since it has been shown that their coherent counterparts change their widths periodically during propagation [12]. Having found the modal composition of these solitons, their coherence properties [16] can then be described by following procedures similar to those employed in Refs. [15,19]. Using Eqs. (7) and (16)–(20), one can show that the complex coherence factor μ_{12} of these solitons is given by

$$\mu_{12}(s,\eta; s + \delta, \eta + \varepsilon) = \exp\left\{-\left[\frac{p\delta^2}{(1 - p)^2} + \frac{q\sigma^2\varepsilon^2}{(1 - q)^2}\right]\right\}. \tag{21}$$

This latter result demonstrates that the coherence function μ_{12} is independent of position (s,η) within the soliton beam and instead it depends only on the deviation distances δ and ε. For a circular incoherent soliton ($p = q$, $\sigma = 1$), the actual correlation length l_c (the distance where μ_{12} falls to its e^{-1} value) can be obtained from Eq. (21) and it is given by $l_c = w_0(1 - p)p^{-1/2}$. Evidently, $l_c \to \infty$ when $p = 0$ (single mode case), whereas $l_c \to 0$ when $p \to 1$. These results are in agreement with our previous discussion. In fact, for one-dimensional solitons, the complex coherence factor μ_{12} is identical to the statistical autocorrelation function of the source $R(x_1 - x_2)$ [$R(x_1 - x_2)$ can be obtained from $G_N(\theta)$ via an inverse Fourier transform [6]] which is given by $R(x_1 - x_2) = \exp(-\delta^2V^2/4)$, where $x_1 - x_2 = w_0\delta$. This is true since, from Eqs. (13) and (19), $p(1-p)^{-2} = V^2/4$. In other words, the correlation length of a logarithmic incoherent soliton remains invariant during propagation and it is equal to that of the exciting incoherent source. Moreover, as in the fully coherent case [12], the dynamics of these two-dimensional solitons [during compression or expansion, i.e., when p and q deviate from the prescribed values of Eqs. (19) and (20)] can also be described in closed form. However, these solutions are rather involved and we will report on them elsewhere. We emphasize that by its very nature the logarithmic nonlinearity leads to an infinite set of modes which are in turn related through a Poisson distribution. Of course this is also true for the $\ln(1 + I_N)$ nonlinearity provided that the maximum intensity at the center of the beam is much greater than I_t. In particular, computer simulations show that intensitywise the $\ln(I_N)$ model nicely approximates the $\ln(1 + I_N)$ nonlinearity as long as the intensity ratio, $\max(I_N) > 100$. However, for relatively low intensity ratios the mode occupancy [in the $\ln(1 + I_N)$ model] is not exactly Poissonian and the correlation length (even though flat in the middle) tends to increase at the low intensity boundaries of the beam. This is in full agreement with the results we have previously obtained [8] for incoherent photorefractive spatial solitons where $\Delta n \propto (1 + I_N)^{-1}$. Finally at very high intensity ratios, the correlation length becomes again constant across the Gaussian beam as suggested by Eq. (21).

As an example, let us consider a logarithmically saturable nonlinear medium with a $\ln(1 + I_N)$ nonlinearity, $n_0 = 2$ and $n_2 = 10^{-4}$. The free space optical wavelength is taken to be $\lambda_0 = 0.5\ \mu$m and the width of the angular power spectrum of the source is $\theta_0 = 0.258°$. Figure 1(a) shows stationary propagation of a 1D incoherent soliton when $r = 10^3$ and $x_0 \simeq 18\ \mu$m (the intensity FWHM is $\sim 30\ \mu$m) as obtained numerically from the coherent density method [6]. Figure 1(b) on the other hand, illustrates the intensity profile of the first four modes involved in this Gaussian soliton as obtained from beam propagation methods. Evidently, these individual modes remain invariant as a function of distance. As expected, the sum of the individual modes provides the intensity profile of Fig. 1(a). These results are in excellent agreement with our theoretical analysis.

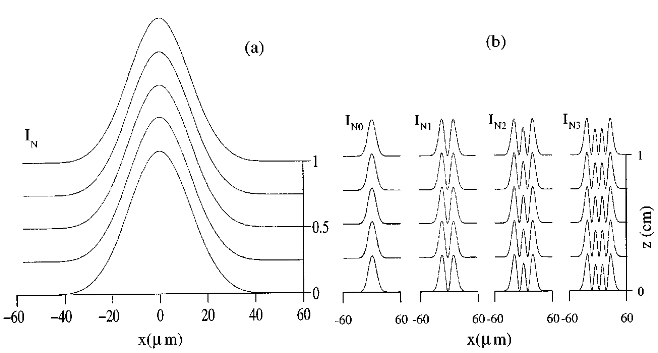

FIG. 1. (a) Stationary propagation of an incoherent Gaussian soliton in a $\ln(1 + I_N)$ nonlinear medium up to a distance of 1 cm. (b) Evolution of the intensity profiles of the first four modes associated with this soliton.

In conclusion we have shown that incoherent spatial solitons are possible in logarithmically saturable nonlinear media. These solitons can exist as long as their mode-occupancy function obeys a Poisson distribution. We have found that two approaches, that is, the dynamic coherent density description as well as the self-consistent multimode method, lead in this case to exactly the same results. Two-dimensional (circular and elliptical) multimode spatial solitons have also been obtained in closed form.

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