From the Academy

Mathematical frontiers in optical solitons

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Solitons are localized concentrations of field energy, resulting from a balance of dispersive and nonlinear effects. They are ubiquitous in the natural sciences. In recent years optical solitons have arisen in new and exciting contexts that differ in many ways from the original context of coherent propagation in a uniform medium. We review recent developments in incoherent spatial solitons and in gap solitons in periodic structures.

Optical solitons have their roots in two very important scientific advances of the 1960s: the development of the mathematical theory of solitons starting in 1965 by Zabusky and Kruskal (1), Lax (2), Zakharov and Shabat (3), and Miura *et al*. (4), and the development of the laser (5, 6). These seemingly unrelated developments came together in 1973 with the theoretical prediction of temporal optical solitons (7), and their experimental realization in 1980 (8).

It is easiest to describe an optical soliton in the spatial domain, where it is simply a self-guided wave. Consider an optical beam as narrow as 10 optical wavelengths (≈5 microns for visible light). If such a beam propagates in a linear medium it diffracts and broadens after even a short distance (≈1 mm in our example). In a nonlinear material light actually changes the index of refraction of the medium in which it propagates, leading to self-focusing. This self-focusing competes with diffractive effects, and at sufficient intensities can lead to the development of a structure for which diffraction and self-focusing exactly balance—a soliton.

The field of optical solitons has greatly developed over the past decade, and optical temporal solitons have become a promising candidate for optical communication networks. At the same time optical spatial solitons have become one of the most exciting research areas in optics and nonlinear science. In this article, we describe two important new directions in optical solitons, which reflect the recent progress in this area: incoherent solitons and gap solitons.

Until 1995 all optical soliton experiments used a coherent "pulse," but in 1996 Mitchell *et al.* (9) demonstrated the self-trapping of beams in which the phase varied randomly. This incoherent wave-packet self-trapped to form a localized nondiffracting beam, an incoherent soliton. A subsequent experiment demonstrated that white light, which is both temporally and spatially incoherent, can also self-trap (Fig. 1) (10).

Soon thereafter several theories were proposed, including coherent density theory (11), which is useful for describing the dynamics of collisions; modal theory (12), which is useful for finding the wavefunctions and correlation properties of incoherent solitons, as well as radiation transfer theory (13) and a ray-optics theory (14, 15). The real success of these theories is in predicting new phenomena such as dark incoherent solitons (16, 17). Incoherent solitons are proving to be a general phenomenon that occur in fields other than optics. For example, incoherent modulation instability effects and incoherent pattern formation relate to many other systems in nature such as cooled atomic gases (18, 19).

To understand the ideas involved, some aspects of incoherent light have to be explained first. A spatially incoherent beam is a speckled (multimode) beam whose structure varies randomly with time. At any instant the beam consists of many tiny bright and dark "speckles" that are caused by constructive and destructive interference. To a detector that responds more slowly than the characteristic phase fluctuation time, such as the human eye, the fast variations average out and what the detector "sees" is the envelope, or average intensity. Such an incoherent beam diffracts much more than a coherent beam of the same beam width, because each speckle contributes to the diffraction of the beam envelope, and diffraction is dominated by the degree of coherence rather than the diameter of the beam.

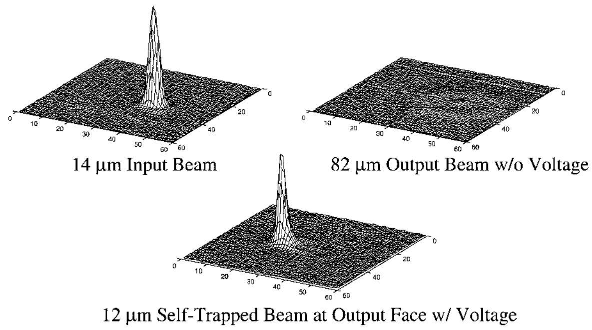

Fig. 1. Self-trapping of a spatially and temporally incoherent white light from an incandescent light bulb.

The nature of the nonlinearity plays a critical role in the self-trapping of an incoherent beam. Nonlinearities that respond instantaneously cannot self-trap such a beam. If an incoherent beam is launched into a self-focusing nonlinear medium that responds instantaneously then each small speckle forms a small "positive lens" and captures a small fraction of the beam. These features on the beam change rapidly during propagation and the induced waveguides intersect and cross each other randomly. Consequently the beam breaks up into small fragments and does not self-trap. Therefore only noninstantaneous nonlinear media can support incoherent solitons. The second condition is that the beam should be able to induce a multimode waveguide by means of the nonlinearity. Otherwise, if the induced waveguide is able to support only a single guided mode, the beam will undergo spatial filtering, radiating all of its power but the small fraction that coincides with that guided mode. The third condition is the self-consistency requirement: the multimode beam must be able to guide itself in its own induced waveguide.

The rapid progress in this new area of incoherent solitons brings about many interesting fundamental ideas and possible applica-

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tions. Even though current work on incoherent solitons has concentrated on fundamental aspects, one can already envision numerous applications. The exciting ability to have self-trapped light beams from incoherent sources such as light-emitting diodes suggests possible applications like optical interconnects and steering of light from such sources.

Another exciting frontier in modern nonlinear optics concerns soliton-like states, called gap solitons, which propagate in periodic media. A pulse of light moving through a periodic medium consists of coupled backward and forward electric field components. The amount of coupling depends on the relation of the carrier frequency to the medium periodicity and is enhanced at or near Bragg resonance, where the propagation is strongly dispersive. A gap soliton emerges from the balance of this strong photonic band dispersion with the nonlinear effects present at sufficiently high intensities. In contrast, optical temporal solitons arise from a balance of the weak material dispersion and nonlinear effects. This implies that the formation length of gap solitons is on the order of centimeters, compared with kilometers for a soliton in a homogeneous fiber. Gap solitons in nonlinear optics have been driven by theoretical predictions (20–23) and observed experimentally (24, 25). In theory gap solitons can travel with any speed up to the speed of light, and experiments have demonstrated gap solitons traveling as slowly as $50\%\ c$. This suggests the intriguing possibility of the capture of gap solitons and their use in optical storage of information.

Can gap solitons be captured? This question was investigated by analysis and computer simulation (26), and trapping by appropriately designed defects was demonstrated. Consider nearly monchromatic light propagating in a Bragg resonant periodic structure. When the defects vary slowly relative to the periodic structure, and nonlinear and dispersive effects balance, the governing equations for the forward (E^+) and backward (E^-) propagating fields are (26–27):

$$i(E_T^{\pm} \pm E_Z^{\pm}) + \kappa(Z)E^{\mp} + V(Z)E^{\pm} + (|E^{\mp}|^2 + 2|E^{\mp}|^2)E^{\mp} = 0.$$

The spatial variations in $\kappa(Z)$, $V(Z)$ define the defect. Fig. 2 shows a simulation of this system. A gap soliton is incident from the left and as it interacts with the defect the soliton distorts. In this simulation most of the soliton's energy is captured. The mechanism for capture is the transfer of the soliton's energy to nonlinear defect modes. Information on the parameter regimes of capture, reflection, and transmission can be deduced from an appropriate bifurcation diagram by using energy conservation and resonant energy transfer principles (26).

The capture process can be further understood by applying dynamical systems ideas to an approximate reduction of the full equation. The soliton can be characterized by time-varying parameters (width, height, and position) whose dynamics are coupled to the intrinsic modes of the defect. We have studied such finite dimensional models for the interactions of soliton-like states with localized defects for the sine-Gordon and nonlinear Schrödinger equations (29, 32). Using an effective Lagrangian approach we find that the approximate dynamics of a soliton/defect interaction are governed by a low dimensional Hamiltonian system. The qualitative features of the full soliton/defect problem are well represented in solutions of the reduced system—there are solutions that represent a soliton being temporarily captured and interacting with the defect before being ejected. We explain the very rich and complex dynamical behavior in terms of the transverse intersections of stable and unstable manifolds of the Poincaré map of the reduced system. Phase space transport methods (e.g., ref. 30) give us insight into the set of incident states resulting in transmission without capture, temporary capture, and capture for all time. In contrast to what is observed in the full infinite dimensional dynamics, we prove that because of the volume preserving nature of the flow the set of initial conditions that are eventually captured has zero measure and appears to have the character of a Cantor set. Capture of a positive measure set of initial conditions, in closer qualitative agreement with the full dynamics, is obtained by inclusion of a damping term, reflecting coupling to radiation modes (31, 32). Thus the appropriate point of view is that of a low-dimensional dynamical system interacting with a radiative heat bath. A deeper understanding of this "fattening" of the basin of capture is a fascinating problem.

In this short article, we have tried to provide a glimpse of the exciting recent activity in soliton science. There are many more new ideas and directions in which soliton science is evolving, from spatio-temporal solitons ("bullets" of light) to discrete solitons, from solitons in photonic bandgap materials to solitons in Bose-Einstein condensates. We believe that this is only the beginning, and this exciting area will continue to be a fruitful source of mathematical and physical problems.

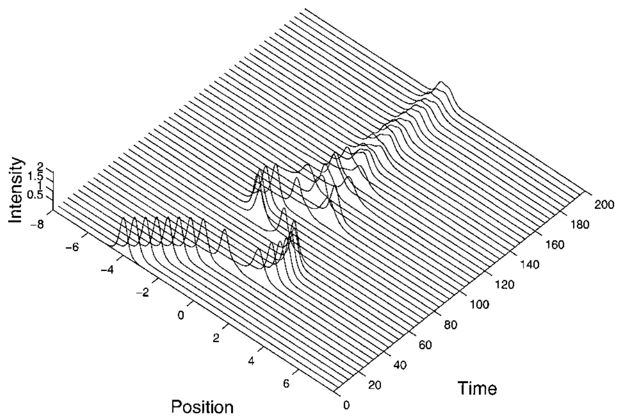

Fig. 2. Capture of a gap soliton by a localized defect.

1. Zabusky, N. J. & Kruskal, M. D. (1965) *Phys. Rev. Lett.* **15,** 240–243.
2. Lax, P. D. (1968) *Comm. Pure Appl. Math.* **21,** 467–490.
3. Zakharov, V. & Shabat, A. B. (1972) *Soviet Phys. JETP* **34,** 62–69.
4. Miura, R. M., Gardner, C. S. & Kruskal, M. D. (1968) *J. Math. Phys.* **9,** 1204–1209.
5. Schawlow, A. L. & Townes, C. H. (1958) *Phys. Rev.* **112,** 1940–1949.
6. Maiman, T. H. (1960) *Nature (London)* (1960) **187,** 493–494.
7. Hasegawa, A. & Tappert, F. (1973) *Appl. Phys. Lett.* **23,** 142–144 and 171–172.
8. Mollenauer, L. F., Stolen, R. H. & Gordon, J. P. (1980) *Phys. Rev. Lett* **45,** 1095–1098.
9. Mitchell, M., Chen, Z., Shih, M. & Segev, M. (1996) *Phys. Rev. Lett.* **77,** 490–493.
10. Mitchell, M. & Segev, M. (1997) *Nature (London)* **387,** 880–883.
11. Christodoulides, D. N., Coskun, T., Mitchell, M. & Segev, M. (1997) *Phys. Rev. Lett.* **78,** 646–649.
12. Mitchell, M., Segev, M., Coskun, T. & Christodoulides, D. N. (1997) *Phys. Rev. Lett.* **79,** 4990–4993.
13. Shkunov, V. V. & Anderson, D. Z. (1998) *Phys. Rev. Lett.* **81,** 2683–2686.
14. Snyder, A. W. & Mitchell, D. J. (1998) *Phys. Rev Lett.* **80,** 1422–1424.
15. Hasegawa, A. (1975) *Phys. Fluids* **18,** 77–79.
16. Chen, Z., Mitchell, M., Segev, M., Coskun, T. & Christodoulides, D. N. (1998) *Science* **280,** 889–892.
17. Christodoulides, D. N., Coskun, T., Mitchell, M., Chen, Z. & Segev, M. (1998) *Phys. Rev. Lett.* **80,** 5113–5116.
18. Soljacic, M., Segev, M., Eugenia, E., Christodoulides, D. N. & Vishwanath, A. (2000) *Phys. Rev. Lett.* **84,** 467–470.
19. Kip, D., Soljacic, M., Segev, M., Eugenieva, E. & Christodoulides, D. N. (2000) *Science,* **290,** 495–498.
20. Chen, W. & Mills, D. L. (1987) *Phys. Rev. Lett.* **58,** 160–163.
21. Christodoulides, D. N. & Joseph, R. I. (1989) *Phys. Rev. Lett.* **62,** 1746–1749.
22. Aceves, A. B. & Wabnitz, S. (1989) *Phys. Lett. A* **141,** 37–42.
23. Winful, H. G., Marburger, J. H. & Garmire, E. (1979) *Appl. Phys. Lett.* **40,** 379–381.
24. Eggleton, B. J., De Sterke, C. M. & Slusher, R. E. (1997) *J. Opt. Soc. Am. B* **14,** 2980–2992.
25. Millar, P., De La Rue, R. M., Krauss, T. F., Aitchison, J. S., Broderick, N. G. R. & Richardson, D. J. (1999) *Opt. Lett.* **24,** 685–687.
26. Goodman, R. H., Slusher, R. E. & Weinstein, M. I. (2001) e-Print Archive, http://xxx.lanl.gov/abs/nlin.PS/0110020.
27. Goodman, R. H., Weinstein, M. I. & Holmes, P. J. (2001) *J. Nonlinear Sci.* **11,** 123–168.
28. Broderick, N. G. R. & de Sterke, C. M. (1998) *Phys. Rev. E.* **58,** 7941–7950.
29. Fei, Z., Kivshar, Y. S. & Vázquez, L. (1992) *Phys. Rev. A* **45,** 6019–6030.
30. Wiggins, S. (1992) *Chaotic Transport in Dynamical Systems* (Springer, New York).
31. Soffer, A. & Weinstein, M. I. (1999) *Invent. Math.* **136,** 9–74.
32. Goodman, R. H., Holmes, P. J. & Weinstein, M. I. (2001) *Phys. D*, in press.