Self-trapping of electromagnetic beams in vacuum supported by QED nonlinear effects

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At very high radiation field intensities in vacuum, Maxwell equations need to be modified to account for the QED photon-photon interaction. We show that such modified equations support nondiffracting spatial radiation solitons that propagate for very long distances without changing their shapes. These solitons, and the underlying self-focusing and instability effects, should be observable in the near future.

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A localized wave packet propagating in linear homogeneous media has a tendency to change its spatial width as it propagates. However, everything in nature is nonlinear, including the vacuum [1]. More specifically, at very high electromagnetic field intensities, Maxwell equations need to be modified by small nonlinear terms. These terms come from the quantum field theory of photon-photon scattering mediated through exchange of *virtual* electron-positron pairs, as discovered theoretically by Euler and Heisenberg in 1936 [1]. The Euler-Heisenberg approximation of the (linear plus nonlinear) polarization of a true vacuum holds when the wavelengths of the interacting photons are all much longer than the Compton wavelength of the electron, and, at the same time, the field is much weaker than the QED critical field $2m^2c^3/(e\hbar)$. Importantly, under these conditions, there is no absorption, i.e., no real electrons and positrons are generated in the process [1]. (This is not surprising, because like all off-resonance interactions, the real part of the susceptibility drops off much more slowly than the imaginary part when the carrier frequency moves away from the resonant frequency.)

The addition of nonlinear terms to the wave equation can sometimes influence the dynamics of the wave packet's shape, so that its dimensions do not change at all as the wave packet propagates. Such a wave packet, in which the diffraction, which tends to expand the pulse, is exactly balanced by the nonlinear effects that are trying to shrink the pulse, is loosely refered to as a soliton. Ever since solitons were scientifically documented [2], they have fascinated scientists in many different fields. A universal nonlinear phenomenon, solitons have been found in many different forms in nature. For example, they were described on surface of shallow water [2], in deep sea water [3], in plasma [4], on the surface of black holes [5], for torsional waves on DNA molecules [6], etc. Solitons have also been studied extensively in nonlinear optical systems; both spatial [7,21], and temporal solitons [8] have been investigated. Solitons also appear in quantum systems; for example, nucleons are a kind of solitons [9].

Over the years, various authors have studied the feasibility of using high-intensity radiation in vacuum to observe some nonlinear effects [10], such as four-wave mixing [11], self-action [12], vacuum birefrengence [13], etc. In this paper, we show that Maxwell equations, when modified to account for the lowest order photon-photon QED interaction, allow for the existence of bright spatial solitons. Such solitons are wave packets of high-intensity light, propagating for long distances in a true vacuum, without changing their dimensions. We believe that these solitons, or at least the underlying self-focusing, will be observed in a near future in specifically designed experiments.

Up to the lowest-order correction, the nonlinear effective Lagrangian density is given by

$$L=-\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}-\frac{\xi}{64\pi}[5(F_{\mu\nu}F^{\mu\nu})^2 -14F_{\mu\nu}F^{\nu\kappa}F_{\kappa\lambda}F^{\lambda\mu}], \tag{1}$$

where $\xi=\hbar e^4/45\pi m^4c^7$, and $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ is the electromagnetic-field tensor. The first term in Eq. (1) is just the usual clasical term, and the rest is the so-called Heisenberg-Euler correction [1]. There are also QED corrections to the Lagrangian above that include the spatial and temporal derivatives of $F^{\mu\nu}$ [20]. However, we intend to work with light of characteristic wavelengths much longer than the Compton wavelength of an electron, and also at very high intensities, so the leading terms are in our case given by Eq. (1), and we can safely ignore the lowest-order corrections to the Lagrangian that include the spatial and temporal derivatives of $F^{\mu\nu}$. Physically, these corrections are important when the wavelength of the light is comparable to the Compton wavelength of the electron, while the Heisenberg-Euler correction is important when the electric field becomes comparable to the critical electric field. One can show that the derivative terms are much smaller than the ones we included as long as $(\omega/\omega_c)^2 \ll (4/\pi^3)(E/E_c)^2$, (please see the Appendix), where ω_c is the Compton frequency of the electron, and E_c is the critical field. We are interested in the regime where the Heisenberg-Euler correction is much more important than the correction that involves derivatives. Nevertheless, we have to be careful that all proposed experimental verifications of our concept satisfy this requirement.

Extremizing the action with respect to the four-vector potential A_μ, one finds

$$\nabla\cdot\mathbf{B}=0,\quad \nabla\times\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}=\mathbf{0},$$

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$$\nabla\cdot\mathbf{D}=0,\quad \nabla\times\mathbf{H}-\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}=\mathbf{0}, \tag{2}$$

where **D** and **H** are given by

$$\mathbf{D}=\frac{\partial L}{\partial \mathbf{E}}=\mathbf{E}+4\pi\mathbf{P},$$

$$\mathbf{P}=\frac{\xi}{4\pi}[2(E^2-B^2)\mathbf{E}+7(\mathbf{E}\cdot\mathbf{B})\mathbf{B}],$$

$$\mathbf{H}=-\frac{\partial L}{\partial \mathbf{B}}=\mathbf{B}-4\pi\mathbf{M}, \tag{3}$$

$$\mathbf{M}=-\frac{\xi}{4\pi}[2(E^2-B^2)\mathbf{B}-7(\mathbf{E}\cdot\mathbf{B})\mathbf{E}].$$

The reason that the nonlinearities include precisely the E^2-B^2 and $\mathbf{E}\cdot\mathbf{B}$ terms is that these are the only Lorentz-invariant scalar combinations of **E** and **B** fields. However, this implies that the nonlinear effects on a beam that is a plane wave modulated by some slowly varying envelope are only of the second order; a plane wave has **E** and **B** perpendicular to each other, and $E^2=B^2$ for plane waves in vacuum. Consequently, the lowest-order nonlinear effects vanish identically in Eqs. (3). One could, in principle, look into cases of very large beam divergence (beyond the paraxial approximation), or include corrections to **P** and **M** that include spatial derivatives, but all of these options will require field intensities that are many orders of magnitude higher than the sources available today, or else require wavelengths that are unaccessible to the laser technology. Therefore, in order to decrease the radiation intensity required to observe self-trapped beams supported by the QED nonlinearities in vacuum, we explore slowly varying envelopes superimposed on top of two crossed plane waves, so that the nonlinearities in Eq. (3) do not vanish to lowest order.

We proceed by forming the wave equations starting from Eqs. (2). Although the original motivation for going to the equations with higher-order derivatives (namely, decoupling **E** and **B**), appears to be lacking, equations with higher-order derivatives are easier to analyze, and fields actually do decouple for certain relative polarizations. The wave equations are

$$\nabla^2\mathbf{D}-\frac{1}{c^2}\frac{\partial^2\mathbf{D}}{\partial t^2}=-4\pi\left\{\nabla\times(\nabla\times\mathbf{P})-\frac{1}{c}\frac{\partial}{\partial t}(\nabla\times\mathbf{M})\right\},$$

$$\nabla^2\mathbf{B}-\frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2}=4\pi\left\{\nabla\times(\nabla\times\mathbf{M})-\frac{1}{c}\frac{\partial}{\partial t}(\nabla\times\mathbf{P})\right\}. \tag{4}$$

The beam we intend to study is a superposition of two plane waves, modified by a slowly varying envelope in the x direction. The two **k** vectors of the two carrier waves lie in the yz plane, and they are mirror images of each other with respect to the xy plane, as illustrated in Fig. 1. Our ''beam'' is therefore infinite in the z direction, propagating in the y direction, and has a finite width α in the x direction (see Fig. 2). Without nonlinear effects, the beam's width in x would grow as a function of z; the beam would diffract. Our goal is to construct a beam (a ''beam'' in space) whose shape does not change in the x direction as the beam propagates. In the slowly varying envelope approximation, we take $\alpha\gg\lambda$, where λ is the wavelength of the carrier. Then one of the simplest **E** fields that satisfies all of these requirements is given to the lowest order by

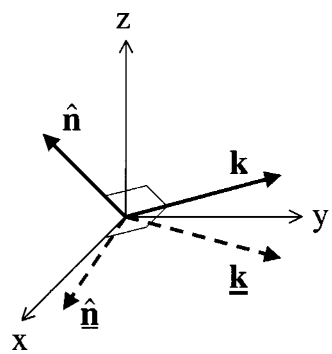

FIG. 1. The QED soliton is a slowly varying amplitude modulating two carrier plane waves. The figure shows the **k** vectors of these carrier waves along with their polarization vectors $\hat{\mathbf{n}}$. Solid vectors refer to one of the carrier waves, and the dashed vectors refer to the other. The polarization vector of each wave is perpendicular to the **k** vector to which it corresponds. The **k** vectors of both waves lie in the yz plane, and are mirror images of each other with respect to the xy plane. The polarization vectors of the two carrier waves are also mirror images of each other with respect to the xy plane.

$$\mathbf{E}(x,y,z,t)=\frac{A(x)}{2}\{\hat{\mathbf{n}}\cos(\mathbf{k}\cdot\mathbf{r}-\omega t)+\underline{\hat{\mathbf{n}}}\cos(\underline{\mathbf{k}}\cdot\mathbf{r}-\omega t)\}, \tag{5}$$

where $\hat{\mathbf{n}}$ is any unit vector, and $\underline{\mathbf{a}}$ means the mirror inverted image of **a** around the xy plane; for example, if $\mathbf{a}=(a_x,a_y,a_z)$ is a vector, then $\underline{\mathbf{a}}=(a_x,a_y,-a_z)$. $A(x)$ is the slowly varying amplitude; the characteristic length scale of $A(x)$ is given by α. From the geometry shown in Fig. 1, it is clear that **k** has no component in the x direction; also $\mathbf{k}\cdot\hat{\mathbf{n}}=0$ for lowest order fields in vacuum. Consequently, specifying $\hat{\mathbf{n}},\omega$, the dispersion relation, and the sign of k_y, specifies **k** to the lowest order uniquely.

We substitute Eq. (5) into the second equation in Eq. (2) to find **B** to the lowest order, assuming $1/\alpha\ll 1/\lambda$:

$$\mathbf{B}(x,y,z,t)=\frac{A(x)}{2}\{\hat{\mathbf{k}}\times\hat{\mathbf{n}}\cos(\mathbf{k}\cdot\mathbf{r}-\omega t)-\underline{\hat{\mathbf{k}}\times\hat{\mathbf{n}}}\cos(\underline{\mathbf{k}}\cdot\mathbf{r}-\omega t)\}. \tag{6}$$

Although $\mathbf{E}\cdot\mathbf{B}=0$, the other scalar combination of **E** and **B** appearing in Eq. (3), $E^2-B^2=A^2(x)[n_y^2+n_x^2(k_z/k)^2]\times\{\cos(2k_z z)+\cos[2(k_y y-\omega t)]\}/2\neq 0$; as one might expect, the dc term vanishes identically. More importantly, in our configuration of two beams propagating under an angle with respect to each other, there is a symmetry with respect to the xy plane. Because of this symmetry, there can be no terms like $\cos[2(k_y y+k_z z-\omega t)]$ appearing in E^2-B^2. This will have profound consequences below. The second-harmonic terms that survived combine with **E** and **B** in Eqs. (3) for **P** and **M**, respectively, to produce the first-harmonic terms, and also third-harmonic terms. After a few lines of algebra, we obtain nonvanishing **P** and **M** to the lowest order:

$$\mathbf{P}=\frac{\xi}{4\pi}A^2(x)\left[n_y^2+n_x^2\left(\frac{k_z}{k}\right)^2\right]\underline{\mathbf{E}}+(\mathrm{TOH}),$$
$$\mathbf{M}=\frac{\xi}{4\pi}A^2(x)\left[n_y^2+n_x^2\left(\frac{k_z}{k}\right)^2\right]\underline{\mathbf{B}}+(\mathrm{TOH}), \tag{7}$$

where n_x and n_y are components of the unit vector $\hat{\mathbf{n}}$. In Eqs. 7, (TOH) stands for ''third-order harmonics,'' such as those proportional to $\cos(3k_z z)$, etc. Specifically, these terms are proportional to $\cos(3k_z z)\cos(k_y y-\omega t)$, and $\cos(k_z z)\cos[3(k_y y-\omega t)]$. One might worry that these terms might lead to significant energy drainage from the initial carrier beam. However, as we will see below, k_y is of the same order as k_z. Thus these third-order harmonics do not satisfy the dispersion relation $c^2(k_y^2+k_z^2)=\omega^2$. Thereby, in any equilibrium self-trapped shape, the corrections to Eq. (5) that they induce have to be tiny in order to be well behaved when plugged into Eqs. (4); in particular, the corrections to E and B have to be $O(\Gamma/\omega^2)$ smaller than the fields in Eqs. (5) and (6). (Here Γ is the nonlinear correction to the dispersion relation of vacuum [to be defined in Eq. (9) below]; as one would expect, and as we show below, this correction is indeed tiny.) In the language of nonlinear optics, this is very similar to saying that terms like $\cos(3k_z z)\cos(k_y y-\omega t)$ are not phase matched with our beam, and terms like $\cos(k_z z)\cos[3(k_y y-\omega t)]$ are asynchroneous with our beam. Therefore, both types of third-harmonic terms can be neglected. The lack of terms like $\cos(3k_z z)\cos[3(k_y y-\omega t)]$ in Eq. (7) differs markedly from the usual nonlinear optics systems. Algebraically, it is a direct consequence of the fact that there are no terms like $\cos[2(k_y y+k_z z-\omega t)]$ in E^2-B^2, which in turn is caused by the fact that our ''carrier wave'' is not a plane wave but an intersection of two plane waves. Physically, one can think of our configuration as being equivalent (apart for the boundary conditions) to a beam propagating between two infinite conducting sheets parallel with the xy plane. In such a configuration, there is an effective dispersion created by the geometry rather than the presence of some absorption resonance. This is the intuitive reason why our system displays no third-harmonic generation (THG), although one might expect THG to occur since one normally thinks of vacuum as being essentially dispersionless.

The only thing we have to be careful about is that the (TOH) terms from Eq. (7) can in principle (through the nonlinearity) induce terms proportional to $\cos(3k_z z)\cos[3(k_y y-\omega t)]$, which are phase matched with the starting beam, and thus can drain the energy from the original beam. However, since this is only an indirect process, that requires drainage of energy from the first into the third harmonic (through the non-phase-matched and/or asynchroneous terms neglected above), and then back from the third harmonic into the first (by interaction between those small terms and the carier beam), one expects the efficiency of this process to be very significantly suppressed; therefore, it should be unobservable for fairly long propagation distances, compared to the self-trapping length. Substituting **E**, **B**, **P**, and **M** from Eqs. (5), (6), and (7), into the wave equations (4), and keeping only the lowest order terms, yields

$$\nabla^2\mathbf{E}-\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}=-2\xi A^2(x)\left[n_y^2+n_x^2\left(\frac{k_z}{k}\right)^2\right]k_z^2\mathbf{E},$$
$$\nabla^2\mathbf{B}-\frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2}=-2\xi A^2(x)\left[n_y^2+n_x^2\left(\frac{k_z}{k}\right)^2\right]k_y^2\underline{\underline{\mathbf{B}}}, \tag{8}$$

where $\underline{\underline{\mathbf{B}}}=(-B_x,B_y,B_z)$ if $\mathbf{B}=(B_x,B_y,B_z)$.

Unlike any other case in nonlinear optics, where we have equations for the electric field only, here we have two equations, one for **E** and one for **B**, which represent the same electromagnetic field. This is a manifestation of the full symmetry between **E** and **B** in QED: the nonlinearity here is driven neither by electric dipoles of the medium nor by magnetic dipoles of the medium, but by the interaction of electromagnetic field with itself in vacuum, for which the symmetry between **E** and **B** is complete. Thus one must find a way to make these two equations consistent with each other. The only way to do this, and to keep Eqs. (8) self-consistent after substituting Eqs. (5) and (6) into them, is to have $k_y=k_z$, and $\hat{\mathbf{n}}=\hat{\mathbf{x}}$. *Therefore, in the case of the ansatz of Eqs. (5) and (6), the two carrier waves have to intersect perpendicular to one another, and the radiation has to be polarized in the x direction*. After setting $k_y=k_z$ and $\hat{\mathbf{n}}=\hat{\mathbf{x}}$, both of these equations reduce to

$$\frac{d^2A(x)}{dx^2}+\Gamma A(x)=-\frac{\xi}{2}A^3(x)/k^2, \tag{9}$$

where $\Gamma\equiv(\omega/c)^2-(k_z^2+k_y^2)$ is the correction to the dispersion relation due to the nonlinear effects. One can think of Γ as an eigenvalue that has to be specified together with the eigenfunction $A(x)$ in order to solve Eq. (9) self-consistently. From Eq. (9) we see that $|\Gamma|=O(1/\alpha^2)\ll k^2$, so this correction is small.

Equation (9) is the famous $(1+1)D$ cubic nonlinear Schrödinger equation (NLSE) [14,21]. This equation is integrable, so all of its solutions can be written in an analytical form. The equation supports bright solitons, and solitons of all orders can be found. The lowest order soliton is given by $A(x)=2\,\mathrm{sech}(x/\alpha)/(\alpha k\sqrt{\xi})$, and its corresponding $\Gamma=-1/\alpha^2$.

Having found nondiffracting solutions, the question about their stability arises naturally. Stability of $(1+1)D$ solitons of Eq. (9) has been studied extensively [14], and there is no doubt that these solitons are very stable. Nevertheless, one should also think about the stability of the solutions with respect to the underlying *two* wave equations. For the ansatz of Eqs. (5) and (6), we have found a self-consistent solution only if the two carrier beams are propagating under a 90° angle with respect to each other. Thus one should ask whether our solution is also stable under small deviations of this angle. Furthermore, one should also question the stabil-

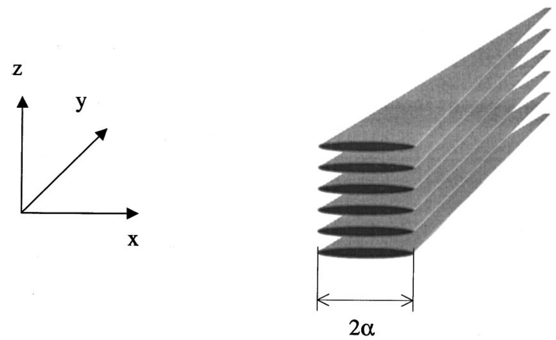

FIG. 2. The geometry of the proposed configuration. The energy is transported along the y direction, and the beam is self-trapped in the x direction; the characteristic length of the beam in the x direction is α. The two carrier waves interfere, forming interference fringes.

ity regarding the particular polarization of the **E** field necessary for the solution in Eqs. (5) and (6) to be valid. These kinds of stability are much more complicated to study than the ''conventional'' stability analyses of solitons of the cubic NLSE, because of the complicated underlying interplay of the vectorial **E** and **B** fields. Thus far, even after an intensive effort, we were not able to prove or disprove their stability. Further study of the stability of these QED solitons will have to address these questions.

Our initial reason for investigating QED solitons in the $(1+1)D$ topology proposed above were the facts that the calculation is fully analytic, and more importantly, that the solutions are known to be stable, at least in the context of a single nonlinear equation [14] as given by Eq. (9). Of course, it would be much nicer to have solitons of a $(2+1)D$ topology. If nothing else, this might relax the power requirements for experimental observation. Therefore, as an avenue toward further research we propose forming necklace solitons [15,16] out of the QED solitons we just created, and studying their properties, and also stability. Intuitively, a necklace soliton is the soliton solution of Eq. (9), wrapped around its own tail, to form a ring. It is best if the radius of the ring L is much larger than the thickness of the ring α. In the limit of $L\rightarrow\infty$, keeping everything else fixed, the necklace soliton reduces to the soliton solution of Eq. (9). Thus, if the QED solitons of Eq. (9) are stable, the necklace solitons should be stable as well. Such necklace solitons offer a major advantage over the $(1+1)D$ geometry offered above: the two superimposed beams propagating at 90° with respect to each other will experimentally ''walk off'' each other, because in any such experiment the extent of each beam in its ''uniform'' direction will be finite. Therefore, in a $(1+1)D$ geometry the beams will intersect only over a finite distance, which will limit the observation length, and also the possibilities of many applications, like shooting these solitons between planets in outer space. In contrast, the necklace solitons could experimentally exist in principle forever in a vacuum.

We point out that the group velocity of all the solitons we described is $c/\sqrt{2}$. This means that we can Lorentz boost to the frame in which they are not propagating at all. In principle, these solitons, or at least the underlying self-focusing effects, can therefore be observed in vacuum, in space, or perhaps even in a laboratory environment, as creatures of stationary, nonvarying, and nonpropagating shapes that are made solely of photons. These creatures would be supported by the mutual interaction between the photons that make up the soliton.

Finally, it is instructive to discuss possible experimental realizations. The solitons we propose are not observable with the laser technology available today. Nevertheless, given the rate at which the laser technology has been advancing in the recent decades, the solitons proposed here should probably be observable within a decade or two. The peak intensity needed to support the soliton is given by approximately $\varepsilon_0 c(\lambda/\alpha)^2\xi^{-1}/\pi^2\approx 10^{37}\,\mathrm{W/m^2}$. Although $10^{37}\,\mathrm{W/m^2}$ is an enormous intensity, we have a great flexibility in decreasing the intensity requirements for an experimental observation by making the size of the soliton large compared to the carrier wavelength. The largest laser intensities available today are $O(10^{24}\,\mathrm{W/m^2})$, and they are in the near-infrared [17]. On the other hand, the shortest wavelength lasers demonstrated thus far are x-ray lasers, with a carrier wavelength of $O(10\,\mathrm{nm})$ [18]. When designing an experiment, the first consideration is the available propagation length required to confirm that a beam forms a soliton (or at least to see considerable self-focusing effects): the diffraction length. We envision that such experiments will be done with ultrashort lasers pulses, so this will enormously reduce the requirement on the total pulse energy required, to be within reach of near-future technology. A possible realization is under laboratory conditions: the idea is to superimpose two sheets of light propagating at a 90° angle with respect to each other. If the length of the sheets in the z direction is of a size similar to that of the propagation length in the y direction, one should be able to observe solitonic effects before the sheets ''walk off'' each other. So, for example, if $\lambda=10\,\mathrm{nm}$ and $\alpha/\lambda=10^4$, and the peak intensity is $10^{29}\,\mathrm{W/m^2}$ (which corresponds to the field $E=6.15\times10^{15}\,\mathrm{V/m}$), such soliton effects should be observed after a propagation distance (diffraction length) of 1 m. Since the beams propagate at 90° with respect to each other, the cross section of the configuration has to be at least 1 m tall if we do not want the beams to stop overlapping before we observe the effect. Thus the peak power required is $10^{28}\,\mathrm{W}$. Finally, we have to check whether this experimental configuration satisfies our assumption that we can ignore the corrections to the Lagrangian in Eq. (1) that include the spatial and temporal derivatives of $F^{\mu\nu}$: i.e., is $(\omega/\omega_c)^2\ll(4/\pi^3)(E/E_c)^2$ satisfied, as explained in the Appendix. This condition translates into $1\ll 466$ for the parameters proposed, so neglecting the corrections to the Lagrangian, due to the derivatives, is safe.

Observations and studying of QED solitons (or of the underlying effects of self-focusing and modulation instability) could offer several fundamentally new aspects to physics in general. First, they would be the first experiments of photon-photon scattering in vacuum, which involve the creation of virtual electron-positron pairs. Thus far, ''nonlinear optics'' with QED nonlinearities in vacuum was demonstrated only for the creation of real electron-positron pairs [19], and the demonstration could not provide a direct quantitative measurement of nonlinear QED effects in a vacuum. Furthermore, the solitons we described would provide a direct quantitative test of QED vacuum nonlinearities in the regime of much higher photon wavelengths, and much smaller peak intensities than were tested so far. Moreover, in contrast to previously performed QED experiments, a macroscopic number of photons would participate in the effect. Second, exploration of the QED radiation soliton ideas might provide a means of communicating in space by line-of-sight: one might be able to launch a very narrow beam from a satellite orbiting the Earth to a nearby planet (or to an asteroid) and back, in order to investigate the reflection at a pin-point precision. Furthermore, these ideas could be used to deliver very large powers to small, far-away objects, such as asteroids at a trajectory too close to earth and deflect them. Finally, soliton-related mechanisms of self-focusing and catastrophic collapse could offer ideas for explaining astronomical observations of bursts of EM radiation (gamma ray Bursts) that are localized in space in a narrow divergence angle (self-focused ''jets''?). It is possible that QED-related self-focusing might naturally occur near active stars, where the EM fields are huge. All this, together with the possibility of perhaps exciting applications, make the QED radiation solitons and their related effects well worth studying.

In conclusion, we have shown that modified Maxwell equations in vacuum can give rise to spatial solitons (nondiffracting solutions), which are supported by the nonlinearity that arises from QED photon-photon interactions at very high radiation intensities. These solitons present an exciting physical system to be studied experimentally but there are many questions left open. Some questions about the stability of such $(1+1)D$ and $(2+1)D$ solutions are yet to be studied theoretically. From an experimental standpoint, we believe that these solitons should be observable in the near future.

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APPENDIX

In this appendix, we determine in which regime of the parameters are the Heisenberg-Euler corrections, the dominant lowest-order QED corrections to the classical Lagrangian, while the terms involving the derivatives can be neglected, and vice versa. To simplify the algebra we work in units where $c=\hbar=1$. In these units, the Heisenberg-Euler correction [1] is given by

$$L_{\mathrm{HE}}=\frac{e^4}{360\pi^2 m^4}\{4F^2+7G^2\}, \tag{A1}$$

where $F=(B^2-E^2)/2$, and $G=\mathbf{E}\cdot\mathbf{B}$. Similarly, the corrections involving derivatives [20] are given by

$$L_D=\frac{e^2}{360\pi m^2}\left\{-(\partial_\alpha F^\alpha_\beta)(\partial_\nu F^{\nu\beta})+F_{\alpha\beta}\left(\frac{\partial^2}{\partial t^2}-\nabla^2\right)F^{\alpha\beta}\right\}, \tag{A2}$$

where $F_{\alpha\beta}$ is just the usual (real) electromagnetic field tensor.

We would like to compare L_{HE} and L_D for an arbitrary typical field configuration. Thus the best we can do is dimensional analysis. Unless we are dealing with a particularly pathological field configuration, $L_{\mathrm{HE}}\approx e^4E^4/(90\pi^2m^4)$, while $L_D\approx e^2E^2\omega^2/(360\,\pi m^2)$, where E is the electric field magnitude and ω is the carrier frequency. Thus, $L_D\ll L_{\mathrm{HE}}$ is satisfied if and only if $\omega^2\ll 4e^2E^2/(\pi m^2)$. To make this more easily transferable between the different systems of units, we note that the Compton frequency $\omega_c=m$, while the critical field $E_c=m^2/(\pi e)$ in this system of units. Thus our constraint can be written as:

$$\frac{\omega^2}{\omega_c^2}\ll\frac{4}{\pi^3}\frac{E^2}{E_c^2}, \tag{A3}$$

which is the expression we use in the text. This expression compares two dimensionless ratios, thus it is the same in all systems of units. Just to be able to make the comparison quantitative, we note that in $SI, \omega_c=mc/\hbar$, while $E_c=m^2c^3/(e\pi\hbar)$.

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