PHYSICAL REVIEW E, VOLUME 63, 035601(R)

Equivalence of three approaches describing partially incoherent wave propagation in inertial nonlinear media

Demetrios N. Christodoulides, Eugenia D. Eugenieva, and Tamer H. Coskun
Department of Electrical Engineering and Computer Science, Lehigh University, Bethlehem, Pennsylvania 18015

Mordechai Segev
Physics Department, Technion-Israel Institute of Technology, Haifa 32000, Israel

Matthew Mitchell
Corvis Corporation, Columbia, Maryland 21046-9400
(Received 13 June 2000; published 13 February 2001)

We show that three approaches previously developed to describe partially incoherent wave propagation in inertial nonlinear media are in fact equivalent. This equivalence is formally established through the evolution of the mutual coherence function and by means of Karhunen-Loeve expansions.

DOI: 10.1103/PhysRevE.63.035601 PACS number(s): 42.65.Jx, 42.65.Tg

One of the most important recent advances in nonlinear science is the discovery of incoherent solitons [1–3]. In general, spatial incoherent solitons are multimode self-trapped entities, which are possible only in materials with noninstantaneous (inertial) nonlinearities. Prompted by the experimental results, the theory of this newly found class of spatial solitons has been the focus of considerable attention [4–13]. To date, the theory of partially incoherent wave propagation in inertial nonlinear media has proceeded along *three seemingly different approaches*: (i) the propagation equation for the mutual coherence function [14,17], (ii) the coherent density method [4–6], and (iii) the self-consistent multimode theory [7–13]. In addition, approximate ray transport methods also exist [15–17]. These, however, are only valid in the limit of broad incoherent beams (much larger than the correlation distance) and by their very nature, they cannot account for any processes that have been initiated by phase manipulation [3,6,18]. Analytically, incoherent spatial solitons were first demonstrated in saturable nonlinear media of the logarithmic type where the strong link between their correlation statistics and the properties of these self-trapped entities became apparent [5]. Subsequently, the development of the self-consistent modal theory [7] led to the identification of several other important incoherent soliton families in both Kerr and saturable nonlinear media [7–13]. Generally, incoherent solitons differ from their coherent counterparts in several ways. Unlike coherent solitons, which can be established through an appropriate choice of their complex envelope, incoherent solitons are required to evolve in such a way so as to conform to the correlation statistics of the exciting input beam [6]. The modulation instability of partially coherent wave packets along with the self-focusing collapse of two-dimensional incoherent beams has also been recently investigated [19–20].

The correspondence between the results of the coherent density approach and those of the self-consistent multimode theory was first established in systems with saturable logarithmic nonlinearities where closed form solutions exist [5,8]. In this case, both approaches were found to lead to exactly the same results. In this same logarithmic system, this correspondence was later extended to include the results obtained from the propagation equation of the mutual coherence function [21]. In a recent study concerning modulation instabilities of partially incoherent beams in Kerr and saturable nonlinear media, the numerical results of the coherent density approach were found to be in excellent agreement with the analytical predictions based on the evolution of the mutual coherence function [19]. The question naturally arises as to whether these three approaches, which at first sight look dissimilar, are in fact equivalent. And if this is the case, is it true *in general* in any nonlinear system and under any circumstances?

In this Rapid Communication, we prove that these three approaches are formally equivalent. Our result holds in any nonlinear system, regardless of the character of the underlying nonlinearity. This equivalence is formally established by demonstrating that the evolution equation for the mutual coherence remains the same irrespective of the representation used. The correspondence between the coherent density method and the propagation equation governing the mutual coherence function is derived via the Van Cittert–Zernike theorem. On the other hand, the equivalence of the self-consistent multimode theory to the other two methods is proved using Karhunen-Loeve expansions. Having different (albeit equivalent) formulations describing the same physical reality is, of course, not new in physics. For example, quantum phenomena can be described either within the framework of Heisenberg's matrix mechanics or Schroedinger's wave formulation. Or for that matter, the equations of electrodynamics can be investigated by either directly solving for actual electromagnetic field quantities or by introducing auxiliary potential functions. In all cases, the choice as to which representation is to be used depends heavily on the nature of the underlying physical problem.

Let us consider a slowly responding nonlinear medium, the refractive index of which varies with the optical intensity I according to $n^2=n_0^2+2n_0g(I)$. Here n_0 is the linear part of the refractive index and the function $g(I)$ represents its nonlinear intensity dependence. The partially coherent optical field propagating in this material is also assumed to be quasi-monochromatic. The average intensity I is taken over a time interval that exceeds the response time of the nonlinear medium that is also much greater than the characteristic coherence time within the optical beam [1–3]. Starting from the Helmholtz equation $\nabla^2 E + k_0^2 n^2 E = 0$, and by writing $E = \phi \exp(ikz)$, where $k = k_0 n_0$, one then finds that the slowly varying envelope ϕ evolves according to

$$i\frac{\partial\phi}{\partial z} + \frac{1}{2k}\nabla_T^2\phi + k_0 g(I)\phi = 0. \tag{1}$$

Let us now consider two functions $\phi(\vec{r}_1,z), \phi(\vec{r}_2,z)$ each satisfying Eq. (1) at different transverse coordinates $\vec{r}_j = x_j\hat{x} + y_j\hat{y}$, where $j = 1,2$. By multiplying Eq. (1) at $\vec{r}_1$ with $\phi^*(\vec{r}_2,z)$ and the complex conjugate of Eq. (1) at $\vec{r}_2$ with $\phi(\vec{r}_1,z)$, and after subtracting, the statistical expectation $J_{12} = \langle \phi(\vec{r}_1,z)\phi^*(\vec{r}_2,z)\rangle$ is found to obey

$$\begin{aligned} i\frac{\partial J_{12}}{\partial z} &+ \frac{1}{2k}\{\nabla_{T1}^2 - \nabla_{T2}^2\}J_{12} \\ &+ k_0\langle \phi(\vec{r}_1,z) g[I(\vec{r}_1,z)]\phi^*(\vec{r}_2,z)\rangle \\ &- k_0\langle \phi(\vec{r}_1,z) g[I(\vec{r}_2,z)]\phi^*(\vec{r}_2,z)\rangle = 0, \end{aligned} \tag{2}$$

where J_{12} is the mutual coherence function (or mutual intensity) [22]. Since on the other hand the nonlinearity is not instantaneous, $\langle \phi(\vec{r}_1,z) g[I(\vec{r}_j,z)]\phi^*(\vec{r}_2,z)\rangle \simeq g[I(\vec{r}_j,z)]\langle \phi(\vec{r}_1,z)\phi^*(\vec{r}_2,z)\rangle$ [14]. As a result,

$$i\frac{\partial J_{12}}{\partial z} + \frac{1}{2k}\{\nabla_{T1}^2 - \nabla_{T2}^2\}J_{12} + k_0\{g(J_{11}) - g(J_{22})\}J_{12} = 0, \tag{3}$$

where $J_{11} = I(\vec{r}_1,z)$ and $J_{22} = I(\vec{r}_2,z)$ are respectively the intensities at $\vec{r}_1$, $\vec{r}_2$. Equation (3) describes the evolution of the mutual coherence and was first derived by Pasmanik [14]. In fact, it is a nonlinear version of Wolf's equations [22] in the paraxial regime.

The second approach builds on an auxiliary function f, the so-called coherent density [4]. As previously shown [4–6], the coherent density f evolves according to

$$i\left\{\frac{\partial f}{\partial z} + \vec{\theta}\cdot\nabla_T f\right\} + \frac{1}{2k}\nabla_T^2 f + k_0 g(I) f = 0. \tag{4}$$

In Eq. (4), $\vec{\theta} = \theta_x\hat{x} + \theta_y\hat{y}$ is the angle at which this density propagates with respect to the z axis. In this representation, the mutual coherence function is given by [6]

$$J_{12}(\vec{r}_1,\vec{r}_2,z) = \int d^2\vec{\theta} f_1 f_2^* \exp[ik\vec{\theta}\cdot(\vec{r}_1 - \vec{r}_2)], \tag{5}$$

where $f_j = f(\vec{r}_j,\vec{\theta},z)$. Note, that the intensity at $\vec{r}_j$ can also be obtained from Eq. (5), i.e., $I_j = I(\vec{r}_j,z) = J_{jj} = \int d^2\vec{\theta}|f_j|^2$, and thus Eq. (4) is in reality an integro-differential equation. Associated with J_{12} is the complex coherence factor $\mu_{12} = J_{12}/\sqrt{J_{11}J_{22}}$ from where the correlation distances can be determined [6,22]. Equation (5) represents a modified version of the Van Cittert–Zernike theorem [23]. The equivalence of this approach with that of the mutual coherence propagation method can then be established by proving that Eqs. (4) and (5) lead to Eq. (3). To do so, we use $\partial J_{12}/\partial z = \int d^2\vec{\theta} \exp[ik\vec{\theta}\cdot(\vec{r}_1 - \vec{r}_2)]\{f_2^*\partial f_1/\partial z + f_1 \partial f_2^*/\partial z\}$. By employing Eq. (4) and its complex conjugate at $\vec{r}_1$ and $\vec{r}_2$, respectively, it is then straightforward to show that

$$\begin{aligned} i\frac{\partial J_{12}}{\partial z} = k_0[g(I_2) - g(I_1)]J_{12} + \int d^2\vec{\theta} \exp[ik\vec{\theta}\cdot(\vec{r}_1 - \vec{r}_2)] \\ \times\left\{\left(-i f_2^*\vec{\theta}\cdot\nabla_{T1} f_1 - \frac{1}{2k} f_2^*\nabla_{T1}^2 f_1\right)\right. \\ \left. + \left(-i f_1\vec{\theta}\cdot\nabla_{T2} f_2^* + \frac{1}{2k} f_1\nabla_{T2}^2 f_2^*\right)\right\}. \end{aligned} \tag{6}$$

Substituting the transverse Laplacian operators of J_{12} at $\vec{r}_1$, $\vec{r}_2$,

$$\begin{aligned} \frac{1}{2k}\nabla_{T1}^2 J_{12} = \int d^2\vec{\theta} \exp[ik\vec{\theta}\cdot(\vec{r}_1 - \vec{r}_2)] f_2^* \\ \times\left\{\frac{1}{2k}\nabla_{T1}^2 f_1 + i\vec{\theta}\cdot\nabla_{T1} f_1 - \frac{k}{2}\theta^2 f_1\right\}, \end{aligned} \tag{7a}$$

$$\begin{aligned} \frac{1}{2k}\nabla_{T2}^2 J_{12} = \int d^2\vec{\theta} \exp[ik\vec{\theta}\cdot(\vec{r}_1 - \vec{r}_2)] f_1 \\ \times\left\{\frac{1}{2k}\nabla_{T2}^2 f_2^* - i\vec{\theta}\cdot\nabla_{T2} f_2^* - \frac{k}{2}\theta^2 f_2^*\right\}, \end{aligned} \tag{7b}$$

in Eq. (6) we then obtain

$$i\frac{\partial J_{12}}{\partial z} + \frac{1}{2k}\{\nabla_{T1}^2 - \nabla_{T2}^2\}J_{12} + k_0\{g(I_1) - g(I_2)\}J_{12} = 0, \tag{8}$$

which is exactly the same as Eq. (3) if one keeps in mind that $I_j = J_{jj}$. In other words, the coherent density approach [described by Eqs. (4) and (5)] is formally equivalent to Eq. (3), which governs the propagation of mutual coherence.

We will next show that the self-consistent multimode theory also leads to Eq. (3). Let us assume that the slowly varying envelope of the partially incoherent beam can be written in terms of an orthonormal set of functions (or "modes") $u_m(\vec{r},z)$,

$$\phi(\vec{r},z) = \sum_m c_m u_m(\vec{r},z), \tag{9}$$

where the modal coefficients c_m are random variables that are uncorrelated with one another, that is $\langle c_m c_n^*\rangle = \lambda_m\delta_{mn}$, where λ_m (the modal occupancy) is a real positive quantity. This sort of representation [Eq. (9)] is better known as a Karhunen-Loeve expansion [22,24]. For this expansion to be valid, one expects that the functions $u_m(\vec{r},z)$ remain orthogonal during propagation provided that this was so the case at the origin, i.e., $\int\int dxdy u_m(\vec{r},z=0)u_n^*(\vec{r},z=0)=\delta_{mn}$. From Eq. (1), each eigenfunction u_m is found to evolve according to

$$i\frac{\partial u_m}{\partial z}+\frac{1}{2k}\nabla_T^2 u_m+k_0 g(I)u_m=0, \tag{10}$$

where $I=\Sigma_m\lambda_m|u_m|^2$. Using Eq. (10) and its complex counterpart, one gets

$$\begin{aligned}2ik\frac{\partial}{\partial z}\int\int dxdy(u_m u_n^*) &= \int\int dxdy(u_m\nabla_T^2 u_n^*-u_n^*\nabla_T^2 u_m)\\ &= \oint_c dl\hat{n}\cdot(u_m\nabla_T u_n^*-u_n^*\nabla_T u_m),\end{aligned} \tag{11}$$

where $\hat{n}$ is an outward unit vector normal to the infinite contour c enclosing the interaction region. In obtaining Eq. (11) we made use of Green's second identity. It can be shown, that in general, the contour integral in Eq. (11) vanishes, and that is irrespective of the nature of the eigenfunctions involved [25]. Therefore, the u_m system remains orthonormal during propagation. This in turn enables a *dynamical* Karhunen-Loeve expansion [i.e., Eq. (9)]. It is important to note that this same result could have been obtained under more general conditions (that is, during nonlinear collisions, interactions with passive waveguides, etc.). This stems from the fact the $g(I)$ term in Eq. (10) can be replaced with any other real potential $V(\vec{r},z)$ that can represent any arbitrary interaction without affecting the end result.

The mutual coherence function can then be written in terms of these orthonormal eigenfunctions,

$$\begin{aligned}J_{12}(\vec{r}_1,\vec{r}_2,z) &= \langle\phi(\vec{r}_1,z)\phi^*(\vec{r}_2,z)\rangle\\ &= \sum_m \lambda_m u_m(\vec{r}_1,z)u_m^*(\vec{r}_2,z),\end{aligned} \tag{12}$$

where in deriving Eq. (12) we made use of $\langle c_m c_n^*\rangle=\lambda_m\delta_{mn}$. If $u_m=u_m(\vec{r}_1,z)$ and $u_m^*=u_m^*(\vec{r}_2,z)$ and using $i\partial J_{12}/\partial z=i\Sigma_m\lambda_m\ (u_m^*\partial u_m/\partial z+u_m\partial u_m^*/\partial z)$, then Eq. (10) leads to

$$i\frac{\partial J_{12}}{\partial z}+\frac{1}{2k}\{\nabla_{T1}^2-\nabla_{T2}^2\}J_{12}+k_0\{g(J_{11})-g(J_{22})\}J_{12}=0, \tag{13}$$

which is precisely Eq. (3) previously derived in connection to the mutual coherence function method.

Having established the equivalence of these three approaches it is perhaps important to highlight some of their inherent characteristics. Being different representations, each method has certain advantages and/or disadvantages over the other. One important way they differ from each other, is the way they treat the initial conditions. If for example, $J_{12}(\vec{r}_1,\vec{r}_2,z=0)$ is given at the origin, in principle its evolution can be followed by solving the propagation equation for the mutual coherence function, that is Eq. (3). Even numerically this is not an easy task. Analytically, the involved character of the nonlinear partial differential equation (3) is such that it does not easily lend itself to physical interpretation. Thus, it is not surprising that the first exact analytical solutions for incoherent solitons were obtained using the other two methods [5,7].

On the other hand, the integro-differential equation (4) describing the coherent density is the closest one can get to the more familiar nonlinear Schrödinger equation, where considerable wealth of results exists. This in turn can allow the transfer of analytical tools and methods from the coherent to the partially incoherent regime (see, for example, Ref. [20]). As previously mentioned, in certain special cases this method can analytically identify soliton solutions (Gaussian Schell solitons) [5]. However, the coherent density approach is by nature better suited to study dynamical evolution of incoherent systems as demonstrated in several studies. Moreover, it can be easily implemented using standard beam propagation methods. Incoherent quasisolitons can also be isolated numerically by exploiting their robustness [6]. One of the drawbacks of this method has to do with determining the f function right at the origin. More specifically, given $J_{12}(\vec{r}_1,\vec{r}_2,z=0)$, the input $f(\vec{r}_j,\vec{\theta},z=0)$ can only be obtained by solving the nonlinear integral equation (5), which is by itself a nontrivial task. This problem is greatly simplified in the case of statistically stationary ''sources,'' where the input density can be written in terms of the source angular power spectrum and a complex modulation function [4–6]. The angular power spectrum is obtained from the Fourier transform of the source correlation function.

The self-consistent multimode theory is the method of choice in identifying incoherent soliton families. In the case of integrable nonlinearities (such as that of 1D Kerr), such families can be constructed using knowledge from the general area of vector solitons [26–27]. As shown above, in addition to isolating static soliton solutions, it can also be used under dynamical conditions, which can be advantageous if the number of modes involved is finite. Again, one important aspect in this discussion has to do with initial conditions. To be more specific, it is well known that infinitely many partially incoherent solitons can be ''synthesized,'' each associated with a correlation function J_{12}. These solutions correspond to infinite possibilities of partially incoherent sources. Yet, given $J_{12}(\vec{r}_1,\vec{r}_2,z=0)$ at the input, it is at this point unknown which of these solitons will actually emerge. Starting from Eq. (12) and by using orthogonality considerations, the u_m functions that correspond to the Karhumen-Loeve expansion (at the origin) can be obtained from the integral equation

$$\int\int J_{12}(\vec{r}_1,\vec{r}_2,z=0)u_m(\vec{r}_2)d^2\vec{r}_2=\lambda_m u_m(\vec{r}_1). \tag{14}$$

It is clear from Eq. (14) that u_m are the eigenfunctions of the integral equation (14) with the mode occupancy factors λ_m as the eigenvalues. Given that the kernel J_{12} is Hermitian, the u_m functions (if they can be determined) are expected to be orthogonal with λ_m real [22,24]. In this case, the number of the input u_m eigenfunctions can be finite or infinite depending on the initial J_{12}. How exactly these input eigenfunctions will excite the incoherent soliton modes (which in general differ from the input u_m) remains an issue that is largely unresolved and merits further investigation.

In conclusion, we have shown that three approaches previously developed to describe partially incoherent wave propagation in inertial nonlinear media are in fact equivalent. This is formally established by demonstrating that the evolution equation for the mutual coherence function remains the same irrespective of the representation used. The correspondence between the coherent density method and the propagation equation governing the mutual coherence is derived via the Van Cittert–Zernike theorem. On the other hand, the equivalence of the self-consistent multimode theory to the other two methods is proved using Karhunen-Loeve expansions. Our results hold in any nonlinear system, regardless of the character of the underlying nonlinearity used.

This research was supported in part by AFOSR, NSF, ARO, and MURI. E. Eugenieva acknowledges support from NSF-NATO.

[1] M. Mitchell, Z. Chen, M. Shih, and M. Segev, Phys. Rev. Lett. **77**, 490 (1996).

[2] M. Mitchell and M. Segev, Nature (London) **387**, 880 (1997).

[3] Z. Chen, M. Mitchell, M. Segev, T. H. Coskun, and D. N. Christodoulides, Science **280**, 889 (1998).

[4] D. N. Christodoulides, T. H. Coskun, M. Mitchell, and M. Segev, Phys. Rev. Lett. **78**, 646 (1997).

[5] D. N. Christodoulides, T. H. Coskun, and R. I. Joseph, Opt. Lett. **22**, 1080 (1997).

[6] T. H. Coskun, D. N. Christodoulides, M. Mitchell, Z. Chen, and M. Segev, Opt. Lett. **23**, 418 (1998); E. D. Eugenieva, D. N. Christodoulides, and M. Segev, *ibid.* **25**, 972 (2000).

[7] M. Mitchell, M. Segev, T. H. Coskun, and D. N. Christodoulides, Phys. Rev. Lett. **79**, 4990 (1997).

[8] D. N. Christodoulides, T. H. Coskun, M. Mitchell, and M. Segev, Phys. Rev. Lett. **80**, 2310 (1998).

[9] D. N. Christodoulides, T. H. Coskun, M. Mitchell, Z. Chen, and M. Segev, Phys. Rev. Lett. **80**, 5113 (1998).

[10] M. I. Carvalho, T. H. Coskun, D. N. Christodoulides, M. Mitchell, and M. Segev, Phys. Rev. E **59**, 1193 (1999).

[11] T. H. Coskun, D. N. Christodoulides, Y. Kim, Z. Chen, M. Soljacic, and M. Segev, Phys. Rev. Lett. **84**, 2374 (2000).

[12] N. Akhmediev, W. Krolokowski, and A. W. Snyder, Phys. Rev. Lett. **81**, 4632 (1998).

[13] W. Krolikowski, N. Akhmediev, and B. Luther-Davies, Phys. Rev. E **59**, 4654 (1999); A. A. Sukhorukov and N. N. Akhmediev, Phys. Rev. Lett. **83**, 4736 (1999).

[14] G. A. Pasmanik, Zh. Eksp. Teor. Phys. **66**, 490 (1974) [Sov. Phys. JETP **39**, 234 (1974)]; V. A. Aleshkevich, S. S. Lebedev, and A. N. Matveev, Sov. J. Quantum Electron. **11**, 647 (1981); V. P. Nayyar, J. Opt. Soc. Am. B **14**, 2248 (1997).

[15] A. Hasegawa, Phys. Fluids **18**, 77 (1975).

[16] A. W. Snyder and D. J. Mitchell, Phys. Rev. Lett. **80**, 1422 (1998).

[17] V. V. Shkunov and D. Z. Anderson, Phys. Rev. Lett. **81**, 2683 (1998).

[18] T. H. Coskun, D. N. Christodoulides, Z. Chen, and M. Segev, Phys. Rev. E **59**, R4777 (1999).

[19] M. Soljacic, M. Segev, T. H. Coskun, D. N. Christodoulides, and A. Vishwanath, Phys. Rev. Lett. **84**, 467 (2000).

[20] O. Bang, D. Edmundson, and W. Krolikowski, Phys. Rev. Lett. **83**, 5479 (1999).

[21] W. Krolikowski, D. Edmundson, and O. Bang, Phys. Rev. E **61**, 3122 (2000).

[22] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University, New York, 1995); J. Goodman, *Statistical Optics* (Wiley, New York, 1985).

[23] B. Ya. Zel'dovich, N. F. Pilipetsky, and V. V. Shkunov, *Principles of Phase Conjugation* (Springer-Verlag, Berlin, 1985).

[24] M. Loeve, *Probability Theory* (D. Van Nostrand, Princeton, NJ, 1955).

[25] This statement is obvious in the case when u_m are bound states. It is also true when these functions represent extended states (radiation modes). In the latter case, this problem can be treated using periodic boundary conditions or by enclosing the interaction region with $L\times L(L\rightarrow\infty)$ ''impenetrable'' walls. In both cases, the functions u_m can be expanded as $\Sigma\alpha_k(z)\Psi_k(\vec{r})$, where $\Psi_k(\vec{r})$ are the new eigenfunctions corresponding to the new boundary conditions. From there it follows that the contour integral goes to zero. See, for example, J. J. Sakurai, *Advanced Quantum Mechanics* (Benjamin-Cummings, Menlo Park, CA, 1984); L. I. Schiff, *Quantum Mechanics*, 3rd ed. (McGraw-Hill, New York, 1968).

[26] D. N. Christodoulides and R. I. Joseph, Opt. Lett. **13**, 53 (1988); M. V. Tratnik and J. E. Sipe, Phys. Rev. A **38**, 2011 (1988).

[27] D. N. Christodoulides, S. R. Singh, M. I. Carvalho, and M. Segev, Appl. Phys. Lett. **68**, 1763 (1996); Z. Chen, M. Segev, T. Coskun, and D. N. Christodoulides, Opt. Lett. **21**, 1436 (1996); M. Mitchell, M. Segev, and D. N. Christodoulides, Phys. Rev. Lett. **80**, 4657 (1998); Z. H. Musslimani, M. Segev, D. N. Christodoulides, and M. Soljacic, *ibid.* **84**, 1164 (2000).