Self-Trapping of "Necklace" Beams in Self-Focusing Kerr Media

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(Received 29 June 1998)

We show that an azimuthally periodically modulated bright ring "necklace" beam can self-trap in self-focusing Kerr media and can exhibit stable propagation for very large distances. These are the first bright $(2 + 1)$D beams to exhibit stable self-trapping in a system described by the cubic $(2 + 1)$D nonlinear Schrödinger equation. [S0031-9007(98)07747-3]

PACS numbers: 42.65.Tg, 03.40.Kf

Solitons in Kerr media are the most well-studied solitons in nature. The reason for that is twofold. First, the Kerr nonlinearity can be found in many systems: It represents a weak symmetric anharmonicity, which is equivalent to a weak saturation in a simple harmonic oscillator. For electromagnetic waves propagating in a weakly nonlinear centrosymmetric dielectric media, the Kerr nonlinearity manifests itself in the cubic nonlinear Schrödinger equation (NLSE) [1], which in many cases describes the envelope of waves in plasmas, shallow water, deep water, gravity, etc. [2]. The second reason is that Kerr solitons are mathematically elegant: The cubic NLSE is integrable in $(1 + 1)$ dimensions. Its solitons can be found analytically and form a closed set; in their collisions, the total power and momentum in the solitons, and the number of solitons are always conserved [3]. The $(2 + 1)$D NLSE, although not integrable, has many conserved quantities, but, in the context of self-focusing, is haunted by stability problems [3]; $(2 + 1)$D bright Kerr solitons are unstable and undergo catastrophic collapse or expansion [4], and $(1 + 1)$D bright Kerr solitons in a 3D medium suffer from transverse instability [5]. These instabilities occur for solitons of all orders, including, e.g., the higher order self-trapped $(2 + 1)$D solutions [6]. In optics, bright Kerr solitons are observed only as temporal solitons [7], which are inherently $(1 + 1)$D, or as $(1 + 1)$D spatial solitons in single mode waveguides [8], for which transverse instability is eliminated by stringent boundary conditions. Thus interactions between bright solitons are restricted to planar systems. Consequently, much of the beautiful similarity between solitons and particles is lost; e.g., angular momentum has no equivalent in the strictly planar system of bright solitons represented by the $(1 + 1)$D NLSE.

Here, we present self-trapped bright "necklace"-ring beams that exhibit stable propagation for very large distances (>50 diffraction lengths) in Kerr media. The intensities of the necklace beams are azimuthally periodically modulated (in the form of "pearls"), and the widths of the beams are very narrow compared to their radia. A necklace beam is actually a ring array of $(2 + 1)$D quasisolitons (pearls), which we find to be stable whenever the azimuthal period length of the ring is smaller than or equal to the width of the ring. Computer simulations indicate that this necklace ring is stable in the absolute sense, although we cannot prove this analytically. The necklace ring slowly expands, with a rate of expansion dependent on the number of pearls in the ring, the width of the ring, the initial peak intensity, and ring's diameter. When the number of pearls is large, holding the parameters of each pearl fixed, the beams are almost fully stationary, and in some cases allow approximate analytic solutions.

The normalized cubic $(2 + 1)$D NLSE is

$$i\,\frac{\partial\psi}{\partial z} + \frac{1}{2}\left\{\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right\} + |\psi|^2\psi = 0\,. \qquad (1)$$

in Cartesian coordinates. One might think that, since omitting the third term in Eq. (1) reduces it to the $(1 + 1)$D NLSE, the solutions should include all those of the $(1 + 1)$D NLSE in x and z that are uniform in y. However, such solitons are transversely unstable: large length-scale perturbations in y grow with propagation distance, and the soliton disintegrates [5]. In optical systems, this instability can be arrested by spatially modifying the refractive index so that $n(y)$ provides waveguiding in y, while the self-trapping occurs in x [8]. For this to work, the scale of waveguiding in y must be smaller than (or equal to) the x width of the soliton. The first experimental observation of optical spatial solitons [9] has employed an "effective waveguide," $n(y)$, that was self-induced (via Kerr nonlinearity) by the same beam that was a soliton in x. This works when the $(1 + 1)$D (x and z) soliton of Eq. (1) varies in y on a scale smaller than the "wavelengths" of perturbations that make the comparable y-uniform soliton transversely unstable. These wavelengths are typically larger than or equal to the x width of the soliton. Therefore, periodic modulation in y superimposed on a soliton in x, arrests the transverse instability, provided that the y period is smaller than the x width of the soliton [9]. Experimentally, two equivalent sheets of light were superimposed. Both were very long in y and perpendicular to x. The sheets propagated mostly along z, with a small angle to one another. The sheets interfered, producing a sinusoidal pattern in y, whose period was smaller than the x width of each sheet. This superposition was launched into a self-focusing Kerr medium. At a high enough power, the beams evolved into a soliton (in x) while remaining transversely stable (in y). However, as the two beams were not infinite in y, they eventually stopped overlapping and the system disintegrated.

Encouraged by Ref. [9], we take a $(1 + 1)$D soliton (in x and z) whose intensity is periodically modulated in y, and wrap it around its own tail, hoping to find a stable $(2 + 1)$D ring array of quasisolitons in self-focusing Kerr medium. We start with Eq. (1) in cylindrical coordinates

$$i \frac{\partial \psi}{\partial z} + \frac{1}{2} \left\{ \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} \right\} + |\psi|^2 \psi = 0 . \tag{2}$$

Consider ringlike solutions whose ring thickness w is much smaller than the ring radius L, such as those in the first row of Fig. 1. In this case, the third term in Eq. (2) can be neglected since it is $O(w/L)$ smaller than the second term. Furthermore, since r varies negligibly over the ring thickness, $1/r^2$ can be replaced by $1/L^2$ in the fourth term. Redefining the variables as $x = r$ and $y = L\theta$ reduces Eq. (2) to Eq. (1). Consequently, if the intensity of the beam is periodically modulated in θ, the system looks much like the one in Ref. [9], apart for a small curvature. It is, therefore, reasonable to expect that such ring beams are stable. Physically, if the solitons from Ref. [9] are stable, we do not expect a small curvature to destabilize them; and, the experimental evidence from Ref. [9] certainly shows that these solitons without curvature are stable.

Led by the intuition gained from Ref. [9], we expect that the self-trapped shapes are close to $\alpha\, \text{sech}[(r - L)/w] \cos(\Omega \theta)$ for some α's, even when w's are not much smaller than the corresponding L's. In this case the radius of our ring should slowly grow with propagation, because adjacent bright "spots" on the ring differ in phase by π, that is, neighboring pearls repel each other [10]. In a circularly symmetric ring the net force exerted on each pearl is radially outwards, making the ring expand. However, if we increase L while holding w, α, and L/Ω constant, the net radial force vanishes. In the limit of vanishing curvature of the ring, the ring should not grow at all.

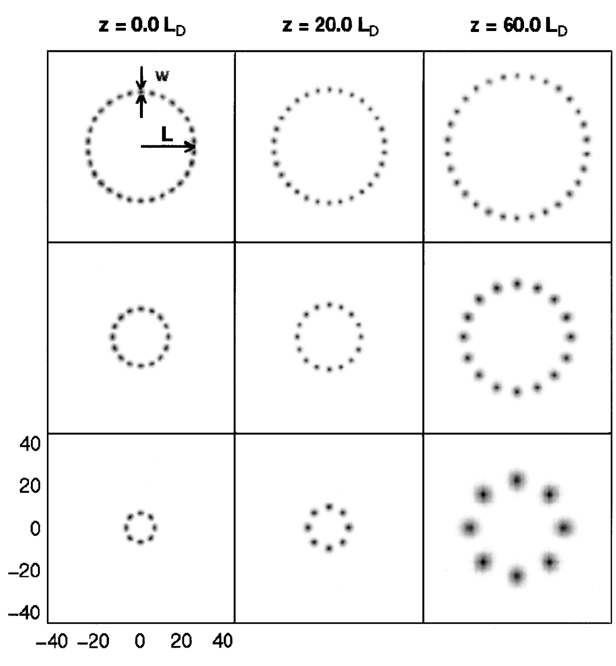

FIG. 1. Examples of evolution of necklace-ring beams with $\Omega = 15$, $\Omega = 8$, and $\Omega = 4$ (first, second, and third rows, respectively). In all cases the initial peak intensity is 1, $w = 1$, and $L/\Omega = 1.707$. The axes are the same for all plots. Dark color indicates high intensity. In all figures in this paper, contrast is enhanced for better clarity.

The expansion of the necklace beam brings back the stability issue, because the propagation of the array of $(1 + 1)$D solitons in Refs. [9] is stationary, thus different from that of the necklace. However, as we show below, the expanding necklace seems to be stable. The intuitive reason for this is that, if $\psi(r, \theta, z)$ is a solution of Eq. (2), then $q\psi(qr, \theta, q^2 z)$ is also a solution for any real q, and both solutions have the same total power. As the radius of the ring slowly grows, there is always a stable shape close to the beam's instantaneous shape. If the distance needed for the beam to evolve into a stable shape is smaller than the rate of the expansion, our necklace array of quasisolitons has a good chance of being stable.

We have simulated numerically (using the standard split step beam propagation method) the propagation of necklace-ring beams and indeed all the above predictions seem correct. We have checked a large number of case examples of necklace rings and propagated them over large distances. We find that all the examples with pearls of azimuthal width narrower than (or equal to) the radial width of the ring, and radial width much smaller than ring radius, are stable. Within our computation capability, we find that they remain stable even under fairly large perturbations ($\sim$5%) in the initial widths or powers, and at the presence of random noise (e.g., we have injected up to 1% of the total power of white noise in the Fourier space every diffraction length). Typical examples (for $\Omega = 15, 8, 4$) are presented in Fig. 1, in which the initial shape is $\psi(r, \theta, z = 0) = \alpha\, \text{sech}[(r - L)/w] \cos(\Omega \theta)$, where $L \gg w$, $\alpha = 1$, $w = 1$, and $L/\Omega = 1.707$. As a measure for the propagation distance, we define the diffraction length, $L_D = 2\pi n w_0^2/\lambda$, where λ is the carrier wavelength in vacuum, n is the refractive index, and the minimum waist of a $(2 + 1)$D Gaussian beam of width $2\, w(z = 0) = 2w_0$. After a few L_D's, the initial shape

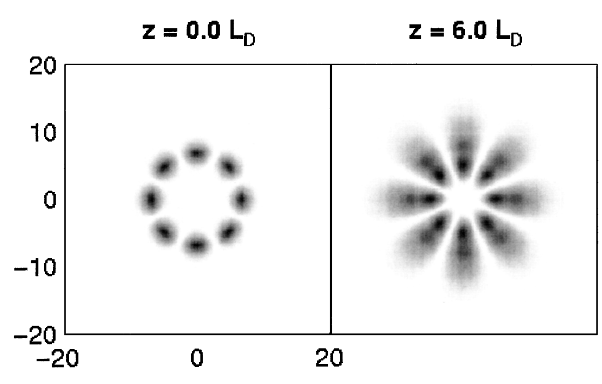

FIG. 2. Evolution of the initial shape of the third row of Fig. 1, but with nonlinearity set to zero. The beam diffracts within $O(1L_D)$.

evolves into a stable necklace of pearls which then slowly grows in size, via a uniform expansion (scaling) of the entire necklace ring. As the necklace-ring beam expands uniformly, the peak intensities of the pearls drop roughly as $1/L^2(z)$. *Therefore, the total power within the necklace ring beam is conserved and does not "escape" to radiation.* This necklace ring of quasisolitons remains stable for the propagation distances of $\sim$100 diffraction lengths. In fact, the only thing that prevents us from stating that these necklace-ring beams are always stable (in the numerical sense), is the fact that, as the necklace beams expand, they fill up our computational window and are affected by reflections from the window's boundaries and cause some (seemingly) artifacts of instability. Finally, we have tested the stability of these necklace beams under azimuthally asymmetric variations in input conditions. We launched the input shapes of Fig. 1, but with $\sim$2% ellipticity, and found that these imperfect rings exhibit stable self-trapping, yet they do not evolve into a circular shape. We conclude that, at least for small azimuthal perturbations, the necklace beam is stable, but its circular shape is not an "attractor."

When we launch our necklace beams into a linear medium, they simply diffract within $O(1)\ L_D$ and the necklace structure is not preserved (e.g., see Fig. 2 for $\Omega = 4$). If we launch a single isolated $(2 + 1)$D beam with the same dimensions as one of the pearls in the necklace ring, into this self-focusing Kerr medium, the isolated beam undergoes catastrophic collapse and disintegrates after $O(1)\ L_D$, as expected for a single $(2 + 1)$D bright beam propagating in Kerr media [4]. It is the nonlinearity *and* the presence of the other peaks in our necklace-ring beam that keeps the whole configuration stable.

We now investigate the expansion dynamics of our necklace-ring beams. The rates of expansion are much slower for the rings of larger Ω than for the rings of smaller Ω, keeping w, L/Ω, and α constant. One might think that increasing Ω, while keeping w and L constant, would also decrease the expansion rate because the angle that determines the net radial component of the repulsion force gets smaller. But, this is not the case because the force between solitons of the cubic NLSE increases with increasing the gradient of the intensity (or decreasing distance between solitons). The final result is that decreasing Ω while keeping everything else constant typically slows down the ring growth. However, one cannot fully stop the expansion by exploiting Ω's that are too low, since the transverse (azimuthal) instability occurs if $L/\Omega \gg w$. The expansion dynamics of the relative radii of the necklace beams from Fig. 1 is shown in Fig. 3. In the beginning of the expansion there is a short period of acceleration, as the intensity peaks speed up in the radial direction. The acceleration diminishes once the pearls are far away from each other since the interaction forces decrease. Eventually, the adjacent peaks interact only very weakly, and the rate of the expansion becomes constant. Very similar features are observed in evolution of a ring of equally charged particles, demonstrating again very picturesquely that solitons behave like particles. Increasing Ω while holding w, L/Ω, and α fixed decreases the growth rate as predicted. As $\Omega \to \infty$, the beams become fully stationary.

To put things in a physical perspective, it is useful to note how our necklace rings would look in a typical experiment. For $\lambda = 500$ nm, refractive index $n = 1.5$, $w = 10\ \lambda/n$, and $\Omega = 45$, one finds $w = 3.3\ \mu$m, $L = 0.26$ mm, and $60L_D = 12.6$ mm. Consequently, the necklace-ring solitons should be easily observable experimentally. Within $60L_D$, a necklace beam with $\Omega = 45$ will not change its shape or size almost at all. On the other hand, when the nonlinearity is "off" (or when the peak intensity is very low), the ring will undergo natural diffraction within several L_D's, say, 200 μm.

Since our necklace beams expand, it is not possible to find a stationary solution for them in the r, θ, z frame. But in the limit $1 \ll w\Omega/L \ll L/w$ approximate stationary ringlike solutions to Eq. (2) are

$$\psi(r,\theta,z) = e^{-i\Gamma z} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \{\alpha_{n,m} \cos[(2n-1)\Omega\theta] \times \operatorname{sech}^{2m-1}[(r-L)/w]\}, \tag{3}$$

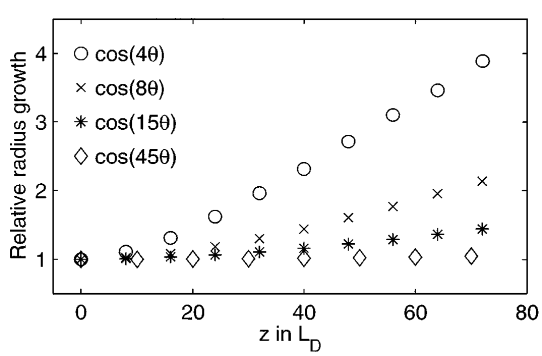

FIG. 3. Growth of ratio (radius/initial radius), as a function of propagation distance. In all the cases the initial peak intensity is 1, $w = 1$, and $L/Q = 1.707$. Holding these parameters fixed, a larger Ω implies slower dynamics.

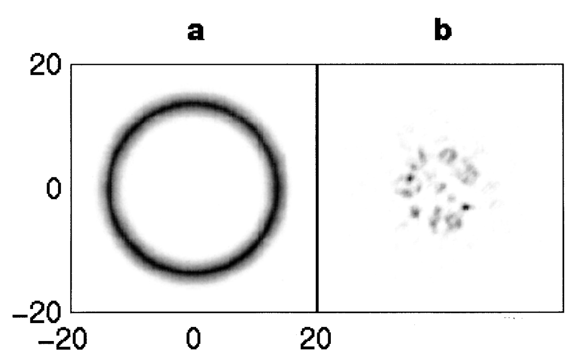

FIG. 4. Propagation of an initially azimuthally uniform ring beam (with $L = 13.7$, $w = 1$, and initial peak intensity of 1) in Kerr media. In this media, the background numerical noise (only) destabilizes the ring, and it eventually disintegrates. (a) The initial shape; (b) the shape after $z = 24L_D$.

with $\Gamma(w, L, \Omega)$, $\alpha_{n,m}(w, L, \Omega)$, $\Omega =$ integer, where $\alpha_{1,1}$ is of $O[(w\Omega/L)^2]$ larger than any other $\alpha_{n,m}$. In this limit, $\psi(r, \theta, z) \simeq \alpha \operatorname{sech}[(r - L)/w]\cos(\Omega\theta)e^{-i\Gamma z}$, with $\alpha^2 = 4/3w^2$, $\Gamma = (\Omega/L)^2$. The necklace beam becomes fully stationary as $L \to \infty$. We compare this solution to our simulations and find that indeed this analytic approximation is excellent. For example, in the case $w\Omega/L = 10$ and $L \to \infty$, the difference between the lowest term in Eq. (3) and the true numerical self-trapped shape is only $O(1\%)$. This solution also applies to the periodically modulated $(1 + 1)$D soliton stripe in Ref. 9 with r, θ replaced by $x, y/L$. One might think that as $w \gg L/\Omega$, the beam might become transversely unstable in the radial direction. This does not happen because the intensity is just high enough to cause self-focusing on length scales of w, and not on any smaller scale.

It is now instructive to compare our solitonlike necklace beams to other (known) ring beams. As mentioned earlier, higher order $(2 + 1)$D solitons in Kerr media are all unstable [6]. It is then interesting to study azimuthally uniform rings. When $w \ll L$, these rings are azimuthally unstable, as shown in Fig. 4. This is expected since the azimuthal length scale is much larger than the thickness of the ring; the instability is of the same origin as in the case of a $(1 + 1)$D soliton which is uniform in the y direction. When $w \sim L$ we find (numerically) that uniform rings tend to coalesce and eventually undergo catastrophic collapse as $(2 + 1)$D bright Kerr solitons do. One can also superimpose some radial "velocity" on a bright ring beam [11], which can provide some control over the rate of the inherent tendency to shrink and collapse, and also convert the dynamics to expansion. In some specific cases the expansion dynamics of such a ring can last up to several diffraction lengths before becoming unstable [11].

Another possibility to seek stable self-trapped ring beams is to multiply an intensity-uniform ring by $e^{i\Omega\theta}$ [instead of $\cos(\Omega\theta)$, as we did]. These vortex rings carry angular momentum [12], in contrast to our necklace-ring beam which is a coherent superposition of two vortex rings with equal topological charge but opposite handedness; as such *our necklace-ring beam carries no angular momentum.* When launched into a self-focusing Kerr medium, a vortex $(e^{i\Omega\theta})$ ring beam disintegrates into filaments [12], which are themselves unstable and undergo either catastrophic collapse or expansion, as isolated $(2 + 1)$D Kerr solitons do [4]. If the self-focusing nonlinearity is saturable, the filaments created after the breakup from stable $(2 + 1)$D solitons [12]. Since the initial beam carries angular momentum, after the breakup each of the solitons shoots off tangentially. This transformation of vortex-ring beams into $(2 + 1)$D solitons seems universal to all saturable self-focusing nonlinearities, including quadratic [13] and photorefractive [14] media. All of these examples are related to the self-trapped necklace-ring beam we have found, but have important major differences from it. We have discussed them here just to clarify what our self-trapped necklace ring beam is not.

In conclusion, we have presented a new form of a self-trapped beam in self-focusing Kerr media: a necklace-ring beam [15]. Even though we do not know if this necklace beam is stable in the absolute sense, we find numerically that it exhibits stable propagation for at least $O(100)$ diffraction lengths, which is more than enough for experimental observations. Such necklace-ring beams slowly expand but fully preserve their structure. To our knowledge, this necklace-ring beam is the only $(2 + 1)$D self-trapped structure that can propagate in a stable form in self-focusing Kerr media. This new kind of quasisolitons are of a particular fundamental importance because the cubic NLSE appears in many nonlinear systems in nature.

We acknowledge illuminating discussions with Professor Yuri Kivshar of the Australian National University and Professor Demetri Christodoulides of Lehigh University. This work was supported by the ARO and NSF.

*M. Segev is currently on sabbatical with the Physics Department, Technion, Haifa, Israel.

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[15] After this paper has been submitted, it has been brought to our attention that a similar idea was suggested (although never analyzed) at the end of Ref. [11]. We thank Yuri Kivshar for bringing this article to our attention.