Lecture 1. Quantum Circuits

Pedram Roushan, Charles Neill, Evan Jeffery, Alexander Korotkov
Google Quantum AI, California,USA
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Staring from Josephson relation, we write the Lagrangian for a system of a Josephson junction shunted by a capacitor, forming a Transmon, and derive its Hamiltonian. Next, we study a system of capacitively or inductively coupled qubits. We compare methods for solving quantum mechanically as well as classically. We touch upon numerical methods to study quantum circuits.

1. SUPERCONDUCTIVITY AND JOSEPHSON RELATION

Giant atoms. One remarkable feature of superconducting qubits is that their energy-level spectra are governed by circuit element parameters and thus are configurable; they can be designed to exhibit "atom-like" energy spectra. Therefore, superconducting qubits are often referred to as artificial atoms, offering a rich parameter space of possible qubit properties, with predictable performance in terms of transition frequencies, anharmonicity, and complexity. The macroscopic size of these artificial atoms provide a major advantage as it allows lithographic wiring to be brought next to qubits to control their various quantum degrees of freedom.

Josephson relations. Josephson junctions are nonlinear, dissipationless circuit elements that form the backbone of superconducting circuits. A Josephson junction is formed by separating two superconducting electrodes with an insulator thin enough such that Cooper pairs can tunnel through the barrier. The Josephson effect describes the super-current I that flows through the junction according to the equations

$$I = I_0 \sin(\varphi), \quad V = \frac{\Phi_0}{2\pi}\frac{d\varphi}{dt}, \tag{1}$$

where Φ_0 is the magnetic flux quantum ($= h/2e = 2 \times 10^{-15}$ Wb), I_0 is the critical current of the junction, and φ and V are respectively the superconducting phase difference ($= \varphi_1 - \varphi_2$) and voltage across the junction. The dynamical behaviour of these two equations can be understood by differentiating the current relation

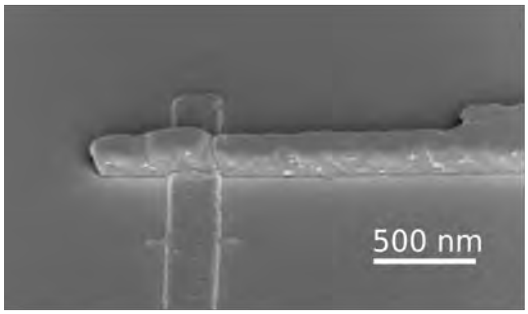
$$\frac{dI}{dt} = I_0 \frac{d\varphi}{dt}\cos(\varphi) \rightarrow \qquad \frac{dI}{dt} = I_0 \frac{2\pi}{\Phi_0} V \cos(\varphi) \rightarrow \qquad V = \frac{\Phi_0}{2\pi}\frac{1}{I_0\cos(\varphi)}\frac{dI}{dt} \tag{2}$$

By defining a Josephson inductance L_J according to the conventional definition of inductor ($V = L\, dI/dt$), one can see that Josephson junction is a non-linear inductor with inductance

$$L_J \equiv \frac{\Phi_0}{2\pi}\frac{1}{I_0\,\cos(\varphi)}. \tag{3}$$

Defining $E_J = \Phi_0\, I_0 2\pi$, the energy stored in the junction is given by

$$\mathcal{U}_J = \int I\, V\, dt = \frac{\Phi_0}{2\pi}\int I\, d\varphi = -\frac{\Phi_0}{2\pi} I_0 \cos(\varphi) = -E_J \cos(\varphi). \tag{4}$$



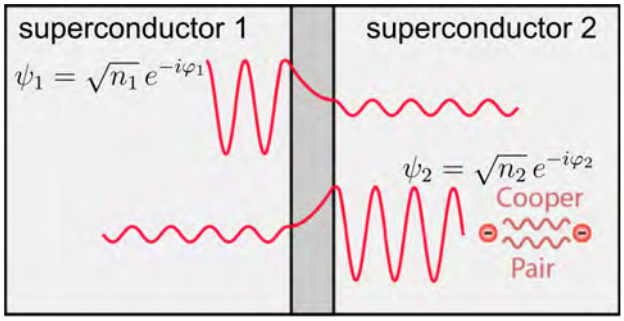


Figure 1. In Josephson junctions the two superconducting islands are connected by a weak link and Cooper pairs can tunnel between the islands.

2. TRANSMON AS AN ANHARMONIC OSCILLATOR

2.1. LC oscillators. We start with the classical description of a LC circuit. In this system, energy oscillates between electrical energy in the capacitor C and magnetic energy in the inductor L. It is customary to arbitrarily associate the electrical energy with the "kinetic energy" and the magnetic energy with the "potential energy" of the oscillator. To derive the classical Hamiltonian, we follow the Lagrange-Hamilton formulation. In this framework, one can represent the circuit elements in terms of one of its generalized circuit coordinates, charge or flux. In the following, we pick flux, defined as

$$\Phi(t) \equiv \int_{-\infty}^{t} V(t')\, dt'. \tag{5}$$

Given that we assume one end of the LC oscillator is kept at $V = 0$ (reference voltage, grounded, see Fig. 2), the voltage at the node is also the branch voltage across the element. The energy of the capacitor and inductor in terms of the node flux can be written as

$$\mathcal{T}_C = \frac{1}{2}CV^2 = \frac{1}{2}C\dot{\Phi}^2, \qquad \mathcal{U}_L = \frac{1}{2}LI^2 = \frac{\Phi^2}{2L}. \tag{6}$$

The Lagrangian of the system can be written

$$\mathcal{L}(\Phi, \dot{\Phi}) = \mathcal{T}_C - \mathcal{U}_L = \frac{1}{2}C\dot{\Phi}^2 - \frac{1}{2L}\Phi^2, \tag{7}$$

From the Lagrangian, one can derive the Hamiltonian using the Legendre transformation. We need to calculate the coordinate conjugate to the flux, which in this case is the charge on the capacitor.

$$Q = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = C\dot{\Phi}, \tag{8}$$

The Hamiltonian of the system is

$$H(\Phi, Q) = Q\dot{\Phi} - \mathcal{L} = \frac{Q^2}{2C} + \frac{\Phi^2}{2L} = \mathcal{T}_C + \mathcal{U}_L, \tag{9}$$

as one would expect for a LC circuit. Note that the equations of motion gives what Kirchhoff rules provide.

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\Phi}} - \frac{\partial \mathcal{L}}{\partial \Phi} = 0 \rightarrow \quad C\ddot{\Phi} + \Phi/L = 0 \quad \xrightarrow{C\ddot{\Phi} = C\dot{V} = \dot{Q} = I_C} \quad I_C + I_L = 0 \tag{10}$$

Here, currents are defined from high node to ground. One can check the correctness of Egn.(10) by noting that replacing $\Phi = Ae^{i\omega t}$ leads to the correct differential equation. The Hamiltonian described above is classical. In order to proceed to a quantum-mechanical description of the system, we need to introduce the quantum operators of charge and flux and impose their commutation relation:

$$[\hat{\Phi}, \hat{Q}] = i\hbar, \tag{11}$$

For convenience, we define the reduced flux $\hat{\varphi} \equiv 2\pi\hat{\Phi}/\Phi_0$ and the reduced charge $\hat{n} \equiv \hat{Q}/2e$ is the excess number of Cooper-pairs on the island and $[\hat{\varphi}, \hat{n}] = i$. We can re-write the Hamiltonian

$$\boxed{\hat{H}_{LC} = 4\frac{e^2}{2C}\hat{n}^2 + \left(\frac{\Phi_0}{2\pi}\right)^2 \frac{\hat{\varphi}}{2L}^2, \quad \text{LC oscillator.}} \tag{12}$$

The charging energy $E_C = e^2/2C$ is the the required energy to add *each* electron of the Cooper-pair to the island, and $\Phi_0 = h/2e$ is the flux quantum.

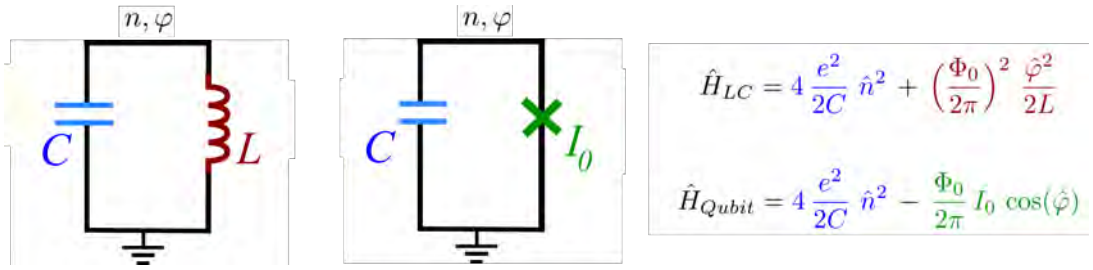


Figure 2. Schematic of a LC oscillator and a Transmon qubit. While LC Oscillator is linear (single frequency spacing), transmon is non-linear. This is achieved by replacing the linear inductor with a Josephson junction, a non-linear inductor.

2.2. Transmons. Capacitively-shunted Josephson junctions. The transmon circuit is obtained by replacing the linear inductor with a Josephson junction, which results in changing the potential term in the above Hamiltonian

$$\hat{H}_{\rm Qubit} = 4\,E_C\,\hat{n}^2 - E_J\,cos(\hat{\varphi}). \tag{13}$$

For Transmons, $C = 80\,\text{fF}$ which give $E_C/h = 240\,\text{MHz}$, and $I_C = 40\,\text{nAmp}$ (SQUID inductance of 8 nH), which give E_J/h of 20GHz. These values suggests that considering Qubits as a linear oscillator and treating the non-linearity as a perturbation is a good approximation. In light of this, we define $\omega_p \equiv \sqrt{8\,E_J\,E_C}/\hbar$, and $\eta \equiv \sqrt{E_C/\,8E_J}$, with typical values of $\omega_p/2\pi$ =6.2 GHz and η =0.04,

$$\hat{H}_{\rm Qubit}/\hbar = 4\eta\,\omega_p\hat{n}^2 - \frac{\omega_p}{8\eta}\cos(\hat{\varphi}), \quad \text{Transmon qubits.} \tag{14}$$

Recalling the quantum harmonic oscillator, $\hat{n}$ and $\hat{\varphi}$ can be written in terms of lowering and raising operators

$$\hat{\varphi} = \sqrt[4]{\frac{8E_C}{E_J}}\,\frac{a+a^\dagger}{\sqrt{2}} = 2\sqrt{\eta}(a+a^\dagger), \qquad \hat{n} = \sqrt[4]{\frac{E_J}{8E_C}}\,\frac{a-a^\dagger}{i\sqrt{2}} = \frac{a-a^\dagger}{4i\sqrt{\eta}}, \tag{15}$$

where a and $a^\dagger$ are lowering and raising operators in the space of eigenstates of quantum harmonic oscillator$|i\rangle$, with usual properties ($[a, a^\dagger] = 1$, $a|i\rangle = \sqrt{i}\,|i-1\rangle$, $a^\dagger|i\rangle = \sqrt{i+1}\,|i+1\rangle$) and (truncated) matrix representation

$$a = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad a^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}. \tag{16}$$

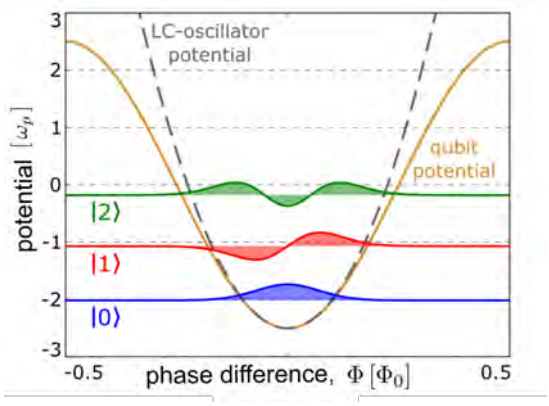


Figure 3. Capacitor plays a role similar to mass in the mass-spring oscillator; as capacitance become larger more levels fit inside the well.

3. SYSTEM OF COUPLED RESONATORS

3.1. Capacitively coupled qubits

To begin with, we consider two coupled linear oscillators. The Lagrangian of the system is

$$\mathcal{L} = \frac{C_A}{2}\dot{\Phi}_A^2 + \frac{C_B}{2}\dot{\Phi}_B^2 + \frac{C_C}{2}(\dot{\Phi}_A - \dot{\Phi}_B)^2 - \frac{\Phi_A^2}{2L_A} - \frac{\Phi_B^2}{2L_B}, \tag{17}$$

to be sure we obtined the correct Lagrangian, we look at the equations of motion

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\Phi}_A} - \frac{\partial \mathcal{L}}{\partial \Phi_A} = 0 \rightarrow \quad C_A\ddot{\Phi}_A + C_C(\ddot{\Phi}_A - \ddot{\Phi}_B) + \frac{\Phi_A}{L_A} = 0 \tag{18}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\Phi}_B} - \frac{\partial \mathcal{L}}{\partial \Phi_B} = 0 \rightarrow \quad C_B\ddot{\Phi}_B + C_C(\ddot{\Phi}_B - \ddot{\Phi}_A) + \frac{\Phi_B}{L_B} = 0, \tag{19}$$

which is what Kirchhoff rules tells us about current at each node.

Next, we consider the case of two coupled transmons coupled with a capacitor C_C. We use the flux at the "high" node of each qubit Φ_A and Φ_B as our canonical variable. The Lagrangian of the system can be written as

$$\mathcal{L}_{\text{Capacitive}} = \frac{C_A}{2}\dot{\Phi}_A^2 + \frac{C_B}{2}\dot{\Phi}_B^2 + \frac{C_C}{2}(\dot{\Phi}_A - \dot{\Phi}_B)^2 - E_A\cos(\Phi_A) - E_B\cos(\Phi_B). \tag{20}$$

The conjugate variable of fluxes, charge variable at each node can be obtained by $Q_A = \partial\mathcal{L}/\partial\dot{\Phi_A}$ and $Q_B = \partial\mathcal{L}/\partial\dot{\Phi_B}$ resulting in

$$\begin{pmatrix} Q_A \\ Q_B \end{pmatrix} = \begin{pmatrix} C_A + C_C & -C_C \\ -C_C & C_B + C_C \end{pmatrix}\begin{pmatrix} \dot{\Phi}_A \\ \dot{\Phi}_B \end{pmatrix}, \quad \text{which gives} \quad \begin{pmatrix} \dot{\Phi}_A \\ \dot{\Phi}_B \end{pmatrix} = \frac{1}{\text{Det}}\begin{pmatrix} C_B + C_C & C_C \\ C_C & C_A + C_C \end{pmatrix}\begin{pmatrix} Q_A \\ Q_B \end{pmatrix}, \tag{21}$$

where $\text{Det} = C_A\,C_B + C_A\,C_C + C_B\,C_C$. Using $H = \sum Q\dot{\Phi} - \mathcal{L}$, the Hamiltonian of the system can be derived

$$H_{\text{Capcitive}} = \frac{Q_A^2}{2\widetilde{C}_A} + \frac{Q_B^2}{2\widetilde{C}_B} + \frac{Q_A\,Q_B}{2\widetilde{C}_C} - E_A\cos(\Phi_A) - E_B\cos(\Phi_B), \quad \text{Capacitively coupled Transmons.} \tag{22}$$

$\widetilde{C}_A = C_A + C_B C_C/(C_B + C_C)$, $\widetilde{C}_B = C_B + C_A C_C/(C_A + C_C)$, and $\widetilde{C}_C = \text{Det}/\,2C_C$. Note that these relations suggests that because of coupling the effective mass of each oscillator has increased ($\widetilde{C}_A$, and $\widetilde{C}_B$ are capacitance of the nodes).

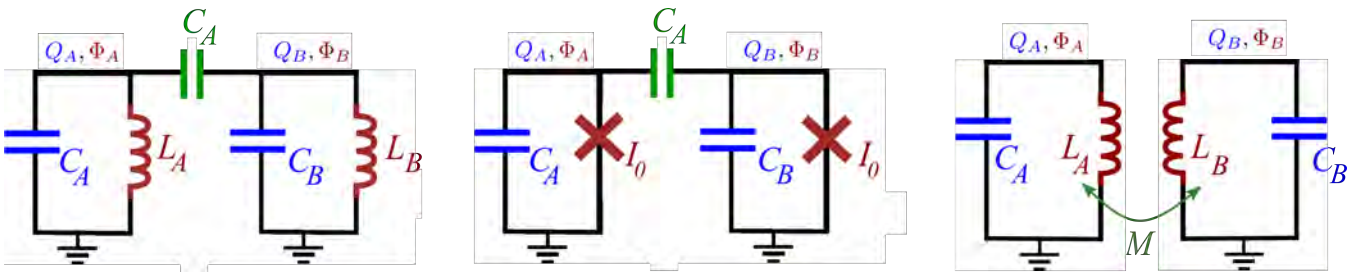


Figure 4. Schematic of two capacitively coupled LC resonators, qubits, and inductively coupled LC resonators.

3.2. Inductively coupled qubits

The Lagrangian of the system two inductively coupled LC oscillators is given by

$$\mathcal{L}_{\rm IC} = \frac{C_A}{2}\dot{\Phi}_A^2 + \frac{C_B}{2}\dot{\Phi}_B^2 - \frac{1}{2}\begin{bmatrix} I_A & I_B \end{bmatrix}\begin{bmatrix}\Phi_A \\ \Phi_B\end{bmatrix}, \tag{23}$$

where currents and fluxes are related by

$$\begin{bmatrix} I_A \\ I_B \end{bmatrix} = \begin{bmatrix} L_A & M \\ M & L_B \end{bmatrix}\begin{bmatrix}\Phi_A \\ \Phi_B\end{bmatrix}. \tag{24}$$

Placing Eqn. (24) in Eqn. (23) gives

$$\mathcal{L}_{\rm IC}(\Phi_A,\Phi_B,\dot{\Phi}_A,\dot{\Phi}_B) = \frac{C_A}{2}\dot{\Phi}_A^2 + \frac{C_B}{2}\dot{\Phi}_B^2 - \frac{1}{2(L_AL_B-M^2)}\begin{bmatrix} \Phi_A & \Phi_B \end{bmatrix}\begin{bmatrix} L_B & -M \\ -M & L_A \end{bmatrix}\begin{bmatrix}\Phi_A \\ \Phi_B\end{bmatrix}. \tag{25}$$

Using $H = \sum Q\dot{\Phi} - \mathcal{L}$, the Hamiltonian of the system can be derived

$$\hat{H}_{\rm IC} = \frac{C_A}{2}\dot{\Phi}_A^2 + \frac{C_B}{2}\dot{\Phi}_B^2 + \widetilde{M}\frac{\Phi_A\Phi_B}{L_AL_B} + \frac{\Phi_A^2}{2\widetilde{L}_A} + \frac{\Phi_B^2}{2\widetilde{L}_B}. \tag{26}$$

Note that the exact relation between a parameter and $\widetilde{...}$ of it is rather inconsequential in the lab. The Hamiltonian of two inductively coupled transmons takes a similar form.

$$H_{\rm Inductive} = \frac{Q_A^2}{2C_A} + \frac{Q_B^2}{2C_B} + \widetilde{M}\frac{\Phi_A\,\Phi_B}{L_AL_B} - E_A\cos(\Phi_A) - E_B\cos(\Phi_B), \quad \text{Inductively coupled Transmons.} \tag{27}$$

For the two coupling schemes that we discussed, The interaction terms are

$$H^{\rm int}_{\rm Cap} \sim Q_A\,Q_B, \qquad H^{\rm int}_{\rm Ind} \sim \Phi_A\,\Phi_B. \tag{28}$$

Using our definitions of raising and lowering operators gives

$$H^{\rm int}_{\rm Cap} = J_C\,(a-a^\dagger)_A\,(a-a^\dagger)_B, \qquad H^{\rm int}_{\rm Ind} = J_I\,(a+a^\dagger)_A\,(a+a^\dagger)_B\,. \tag{29}$$

In the rotating frame and using the rotating wave approximation, the terms that do not conserve particle number drop to zero and for both cases we arrive at

$$H^{\rm int}_{\rm Cap} = J_C\,(a_A\,a_B^\dagger + a_A^\dagger\,a_B), \qquad H^{\rm int}_{\rm Ind} = J_I\,(a_A\,a_B^\dagger + a_A^\dagger\,a_B). \tag{30}$$

Truncating to two lowest levels gives

$$H^{\rm int}_{\rm Cap} \to J_C\,(\sigma_A^X\sigma_B^X + \sigma_A^Y\sigma_B^Y), \qquad H^{\rm int}_{\rm Ind} \to J_I\,(\sigma_A^X\sigma_B^X + \sigma_A^Y\sigma_B^Y). \tag{31}$$

One can show that how non-linearity leads to on-site interaction of the Bose-Hubbard Hamiltonian. For the sake of completeness, let's just say that the Hamiltonian of a system of coupled (inductive of capacitive) qubits is given by

$$\hat{H}_{BH} = \sum_i h_i a_i\,a_i^\dagger + \sum_{<i,j>} J_{i,j}(a_i\,a_j^\dagger + a_i^\dagger\,a_j) + \frac{U}{2}\sum_i a_i\,a_i^\dagger(a_i\,a_i^\dagger - 1) + h.o.t \tag{32}$$

Scalable approach? As we couple more oscillators, each one adds at least one degree of freedom to the problem. We are facing with the numerical challenge of diagonalizing large matrices formed from tensor product of associated raising and lowering operators. One way to circumvent this challenge is to truncate to smaller raising and lowering operator for each degree of freedom which comes at the cost of reducing accuracy. The other approach is to ignore non-linear effects and solve classically as a system of coupled linear oscillators to gain some insight. The conventional electrical engineering approach models quantum circuits with lumped-elements. We give the highlights of this approach by revisiting the case of capacitive coupling. The circuit is looked at as a two-port network, and We need to construct the admittance and impedance matrix of this two port network. These matrices can be looked up in a microwave engineering textbook, such as [1]. The inductance matrix $\mathbb{L}$ and capacitance matrix $\mathbb{C}$ are related to admittance and capacitance matrix by $\mathbb{Y} = i\omega\,\mathbb{C}$ and $\mathbb{Z} = i\omega\,\mathbb{L}$. This results in

$$\mathbb{C} = \begin{bmatrix} C_A + C_C & -C_C \\ -C_C & C_B + C_C \end{bmatrix}, \qquad \mathbb{L} = \begin{bmatrix} L_A & 0 \\ 0 & L_B \end{bmatrix}. \tag{33}$$

The equation that does the magic and gives all the eigen-frequencies is

$$\boxed{\mathrm{Det}(\mathbb{L}^{-1} - \omega^2\mathbb{C}) = 0} \tag{34}$$

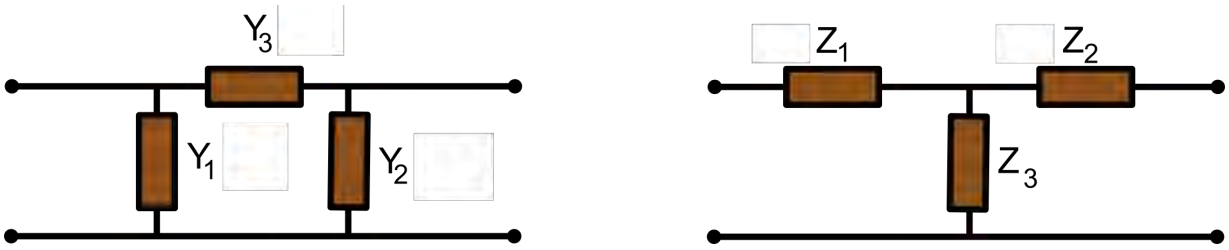


Figure 5. $\mathrm{Y}=\begin{bmatrix} Y_1 + Y_3 & -Y_3 \\ -Y_3 & Y_2 + Y_3 \end{bmatrix}$, $\mathrm{Z}=\begin{bmatrix} Z_1 + Z_3 & Z_3 \\ Z_3 & Z_2 + Z_3 \end{bmatrix}$. (35)

5. PHYSICAL PICTURES

Next, we focus on the physical picture of these quantum circuits. After proper understanding, one should be able to answer: What are the $|0\rangle$, $|1\rangle$ states of qubits physically correspond to? what is $|2\rangle$ state? Can we put two photons into a qubit? When using $|0\rangle$, $|1\rangle$, and $|2\rangle$, are we referring to different modes and different pattern of oscillation or are we talking about number of excitations in a given mode? If $|0\rangle$ has no photon in it, then does it makes sense to talk about its shape? or energy?

Distributed vs. lumped-element resonators. A mode is a spatiotemporal pattern of vibration, e.g. an standing wave in a string. Classical oscillators [2], such as a violin string can be in any of their oscillation modes, and the amplitude of oscillation of each mode (mode energy) is a continuous variable. In quantum systems, such a photons in cavities, according to Planck, the occupancy of each mode is quantized. Now the question is can we make a qubit by using a fundamental mode of an interrupted co-planar wave-guide (CPW) to form a LC oscillator? (see Fig. 6) The answer is yes we can. One possibility would be to choose $|0\rangle$ and $|1\rangle$ to be the zero or 1 photon in that mode, respectively. If we decide to place the fundamental frequency conveniently at 5GHz, the wavelength $\lambda = v/f$ becomes $\lambda = 3 \times 10^8 (m/s)/\sqrt{10}/5\,\mathrm{GHz}$=2 cm, and size of the resonator would be $\lambda/4 = 5\,\mathrm{mm}$. This a good size (and used for our readout resonators). However, this oscillator is not anharmonic and has uniform energy level spacing, this is a useless qubit.

In Transmon qubits, the resonance frequency of the CPW part alone is about 30 GHz (due to its small inductance). Therefore, at commonly chosen frequency of operation (6 GHz) the CPW part acts as a capacitor. Given that inductance of junctions are large and concentrated to a small spot in space, with a good approximation, there is a spatial separation between the capacitor-like and inductor-like part of this oscillator. This is exactly how *LC*

oscillators in an electrical engineering textbook are modeled: lumped-elements, where L and C are points without any spatial extent (similarly for mass-spring oscillators). Consequently, we say that the Josephson-based qubits are lumped-element resonators. I think this naming caused more confusion than help.

One may think that the lumped-element mode is something special; that is not true as no new physics in here. In practice, this mode of oscillation also has some spatiotemporal pattern of oscillatory electromagnetic wave, which we choose to ignore. This is a good approximation; however the goodness of the lumped-element approximation is inconsequential. There is nothing in the physics that we want to exploit demands this approximation to be accurate. The only consequence of using the lump element phrase is to loose sight of the fact that every oscillator has spatial extend and hence it supports other modes of oscillation. The other modes of Transmon are much higher in energy and their energy ratios do not obey any simple rule. Surely our modeling was based on lumped-element assumption, but the small capacitance of the junction or the inductance of the CPW part and the fact that it has spatial extend only leads to renormalization of physical parameters.

If a resonator is modeled as a lumped-element one it means that the resonator is built such that kinetic energy and potential energy are concentrated in spatially separate places. Usually one can see how this mode comes about by modifying parameters of the fundamental mode, as frequently there is an analytical continuation from fundamental mode to lump-element behavior. Given that higher modes usually involve relative motion of internal degree of freedom, then they become much higher in energy and hence inaccessible. This paragraph becomes clear to you after solving HW 9.

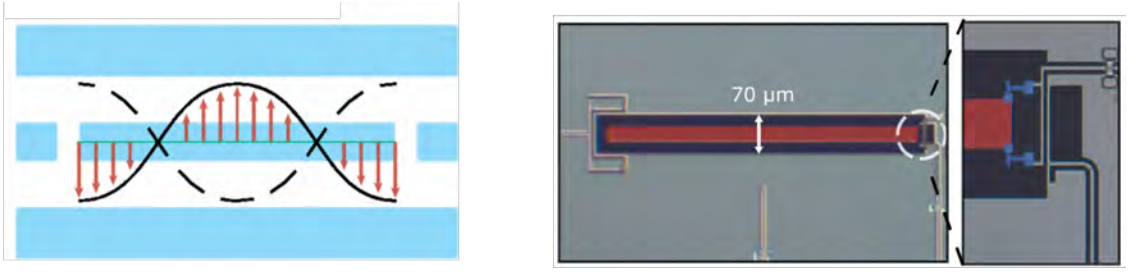


Figure 6. A spatially extended resonator, such as a long CPW, can accommodate many modes beside its fundamental. A Josephson-based qubit is a lumped-element resonator. It supports other oscillation modes but they are relatively high in frequency that we can safely ignore them.

What is the Cooper pair difference between the two islands? In a Josephson junction a thin oxide barrier separates two macroscopic islands. On each island there is on the order of Avogadro number N_A of electrons that pair up to form Cooper pairs. How can these two very large numbers of Cooper pairs be different only by a few (say N_A and $N_A + 3$) and difference be stable? Is that even meaningful to talk about such numbers?

The key concept to understand is that because of the small capacitance of the junction, the energy difference ($e^2/2C$) associated with tunneling one pair from one island to the other is rather large and $\gg k_B T$. ($C = 1\,\text{fF}$ gives 20 GHz and $K_B T$ at 20 mK is 400 MHz.) Hence, the difference is stable and meaningful. In addition, for Transmons E_J/E_C =80, and as a result the wavefunction is localized is phase and spread in charge (phase is a good quantum number and charge is not). Therefore, charge fluctuations do not have significant effect. This is the essence of charge insensitivity of Transmons (see **HW 8**).

No Boltzmann factor across the modes. consider an oscillator that supports several modes with energies $E_{\text{mode,m}}$. The occupation of *each mode* is a harmonic oscillator; i.e., the number of excitation in that mode m (=0,1,2, ...) is given by the Boltzmann factor. For instance, the probability of finding five photons in the first mode (with energy $E_{1,5}$) is related to other photon number probabilities of the *same mode* (with energies $E_{1,m}$) by the Boltzmann factor. However, there is no Boltzmann factor associated with the probability of occupation of different modes based on the relative energies of the modes. Please do not confuse yourself.

6. HOMEWORKS

HW1, Josephson relation. Consider two superconductors separated by a thin insulator, such that Cooper pairs can coherently tunnel from between them with rate Γ. Assume the wave-function of the condensate takes the form of $\psi_1 = \sqrt{n_1}\, e^{-i\varphi_1}$, and $\psi_2 = \sqrt{n_2}\, e^{-i\varphi_2}$, where $\sqrt{n_i}$ is the amplitude and φ_i is the phase of the Ginsburg-Landau

wavefunction. The wave-functions are related by $i\,\partial\psi_1/\partial t = \Gamma\psi_2$ and $i\,\partial\psi_2/\partial t = \Gamma\psi_1$. Using these relations, derive the Josephson current relation $I = I_0\sin(\varphi)$.

HW2, Transmon Hamiltonian. Given that $E_J/E_C \gg 1$, show that Hamitonian of the system $\hat{H}_{\rm Qubit}/\hbar = 4\eta\,\omega_p\hat{n}^2 - \frac{\omega_p}{8\eta}\cos(\hat{\varphi})$, can be re-written in terms of a and $a^\dagger$

$$H/\hbar\omega_p = a\,a^\dagger + 1/2 - \frac{\eta}{3}\Big(\frac{a+a^\dagger}{\sqrt{2}}\Big)^4 + \frac{4\eta^2}{45}\Big(\frac{a+a^\dagger}{\sqrt{2}}\Big)^6. \tag{36}$$

where

$$\hat{\varphi} = \sqrt[4]{\frac{8E_C}{E_J}}\,\frac{a+a^\dagger}{\sqrt{2}} = 2\sqrt{\eta}(a+a^\dagger), \qquad \hat{n} = \sqrt[4]{\frac{E_J}{8E_C}}\,\frac{a-a^\dagger}{i\sqrt{2}} = \frac{a-a^\dagger}{4i\sqrt{\eta}}. \tag{37}$$

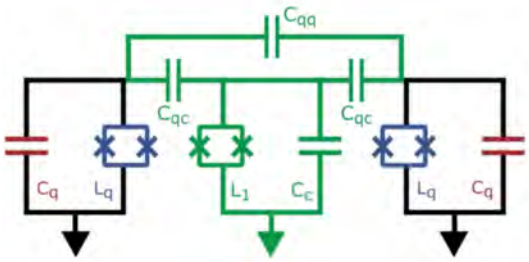
HW3, Transmon energies. Starting from Eqn. (36) and using perturbation expansion, show that the few lowest energy levels of Transmon are

$$\begin{aligned}
\omega_0 &= \omega_p(\frac{1}{2} - \frac{1}{4}\eta - \frac{3}{8}\eta^2),\\
\omega_1 &= \omega_p(\frac{3}{2} - \frac{5}{4}\eta - \frac{27}{8}\eta^2),\\
\omega_2 &= \omega_p(\frac{5}{2} - \frac{13}{4}\eta - \frac{105}{8}\eta^2),\\
\omega_3 &= \omega_p(\frac{7}{2} - \frac{25}{4}\eta - \frac{273}{8}\eta^2).
\end{aligned} \tag{38}$$

HW4, Numerical approach. If you are not interested at perturbation expansion, then by numerically finding the eigen-energies of Hamiltonian shown in HW1, show that the relations in Eqn. (38) provide a good fit for small η values. Typical values to use are E_J =20 GHz, and E_C=240 MHz.

```
function [Energy, EigStates]=Single_Trannsmon()

Nq=11 % number of levels to consider
wQ=6
lambda=0.03

APD=diag(sqrt(1:NQ-1),1)+diag(sqrt(1:NQ-1),1)' ;   %  a + a^\dagger
AMD=diag(sqrt(1:NQ-1),1)-diag(sqrt(1:NQ-1),1)' ;   %  a - a^\dagger

Q=cell(10,1); for t=1:10, Q{t}=APD^t; end;

HQ=-wQ/4*AMD^2-wQ/(8*lambda)*COSINE(lambda,Q);  % making Hamiltonian

[EigStates,Energy]=eig(HQ);  % computing the eigenvalues


function CosPhi=COSINE(lambda,Q)
   a2 =-2     *lambda^1;
   a4 =+2/3   *lambda^2;
   a6 =-4/45  *lambda^3;
   a8 =+2/315 *lambda^4;
   a10=-4/14175*lambda^5;
 CosPhi=eye(size(Q{1}))+a2*Q{2}+a4*Q{4}+a6*Q{6}+a8*Q{8}+a10*Q{10};
```

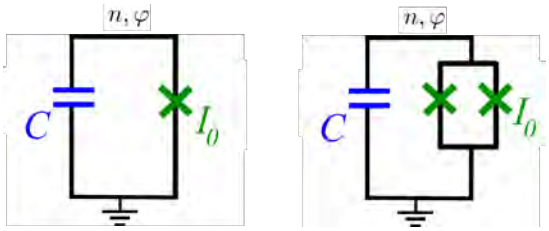
HW5, Three coupled oscillators. By considering a system of three capacitively coupled oscillators as a 3-port network, write the system of equations that give the eigen-frequencies of the system (see figure).



Answer:

$$\text{Det}\begin{bmatrix} \frac{1}{L_q}-\omega^2(C_q+C_{qC}+C_{qq}) & \omega^2 C_{qC} & \omega^2 C_{qq} \\ \omega^2 C_{qC} & \frac{1}{L_q}-\omega^2(C_C+2C_{qC}) & \omega^2 C_{qC} \\ \omega^2 C_{qq} & \omega^2 C_{qC} & \frac{1}{L_q}-\omega^2(C_q+C_{qC}+C_{qq}) \end{bmatrix}=0 \tag{39}$$

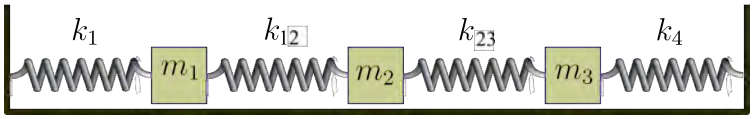
HW6, Frequency tunable qubits. Commonly, in transmons two junctions are used instead of one to form a SQUID and make a tunable frequency qubit. Show that $E_J \equiv \Phi_0\, I_0/2\pi$ need to be replaced by $2E_J \cos(\phi_{\text{ext}})$, where ϕ_{ext} is the external flux applied to the SQUID loop. Obtain eigen-energies as a function of this external flux.



HW7, Physical picture.Can we distinguish 5 photon in the 1^{st} mode ($5\hbar\omega_1$) of a cavity and one photon in the 5 th mode $\hbar(5\omega_1)$?

HW8, Phase and charge fluctuations. Using perturbation expansion or the numerical code, find the $<\hat{n}^2>$ and $<\hat{\varphi}^2>$ for the groundstate and first excited state of a transmon with $\omega_p/2\pi$=6 GHz and $\eta = 0.04$.

HW9, Lumped-element approximation. Consider a system of masses and springs. Assume some reasonable values for parameters and numerically solve for the eigen-modes of the system.



Notice that in the lowest energy mode of the system, i.e. the fundamental mode, the masses move together. In higher modes there is a relative motion between centers of masses. Next, assume $k_1 \ll k_2 = k_3$, $m_1 = m_2 = 0$, and $k_4 = 0$. Note the change in eigen-energies. Study this limiting case can help you to see the origin of lumped-element approximation.

Answer:

$$|\mathbb{K} - \omega^2\,\mathbb{M}| = 0 \tag{40}$$

$$\mathbb{K} = \begin{bmatrix} k_1+k_{12} & -k_{12} & 0 \\ -k_{12} & k_{12}+k_{23} & -k_{23} \\ 0 & -k_{23} & k_{23}+k_4 \end{bmatrix} = 0, \quad \mathbb{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} = 0 \tag{41}$$

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