How many measurements does it take to kill a Quantum state?

Entanglement Phase Transitions and natural error correction in Random Unitary Circuits with Measurements

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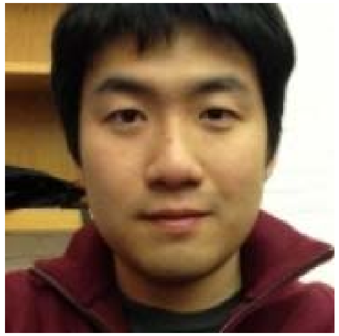


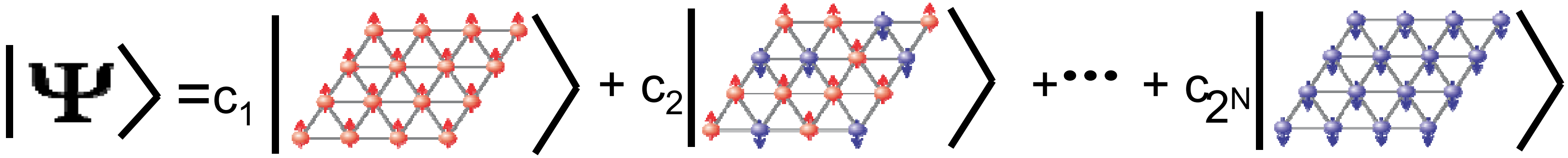
arXiv:1908.04305 , arXiv:1903.05124

erc UQUAM ultracold quantum matter

Large scale entanglement is essential for quantum computation

Are all entangled states fragile like Schroedinger's cat?

NO! Highly entangled states encode information in nonlocal coefficients:

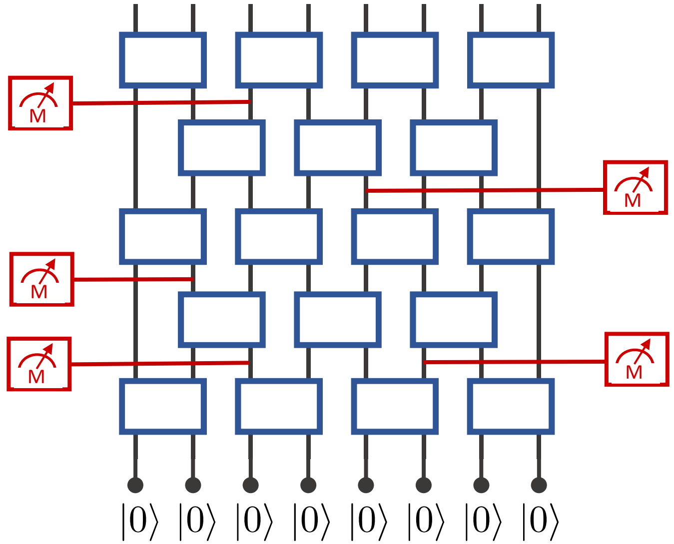


Measuring a few local qubits does not reveal much of this information.

How many measurements does it take to collapse a quantum state?

Lets sharpen the questions …

$$S(A) \sim N_A$$



Generic unitary time evolution generates Large scale (volume law) entanglement

Now suppose we measure the state of local qubits at some rate during the evolution.

How sensitive is the volume law?

Numerics (and general arguments) suggest a phase transition from volume-law to area law at a critical measurement rate.

Li, Chen, Fisher arXiv:1808.06134
Skinner, Ruhman, Nahum arXiv:1808.05953

Related to early work by Dorit Aharonov quant-ph/991008

* Note: here it is crucial to look at individual trajectories!

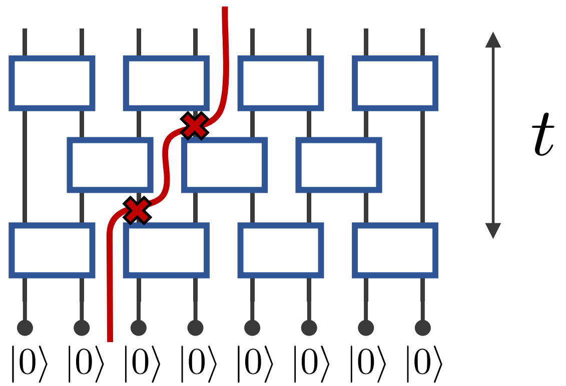
What is the nature of the phase transition ?

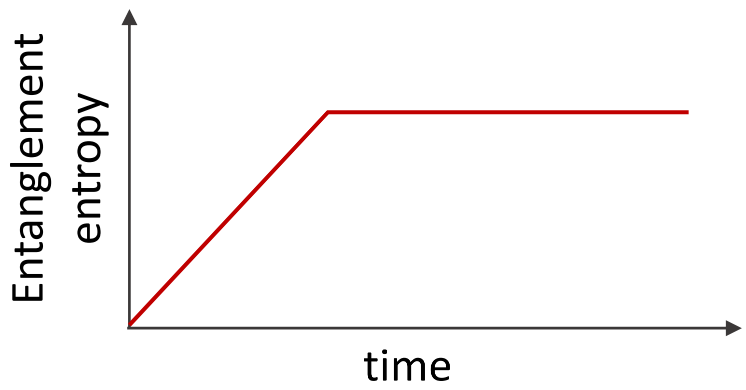
What is the most natural way to observe it ?

Entanglement in generic unitary time evolution
Tractable example: random unitary circuit

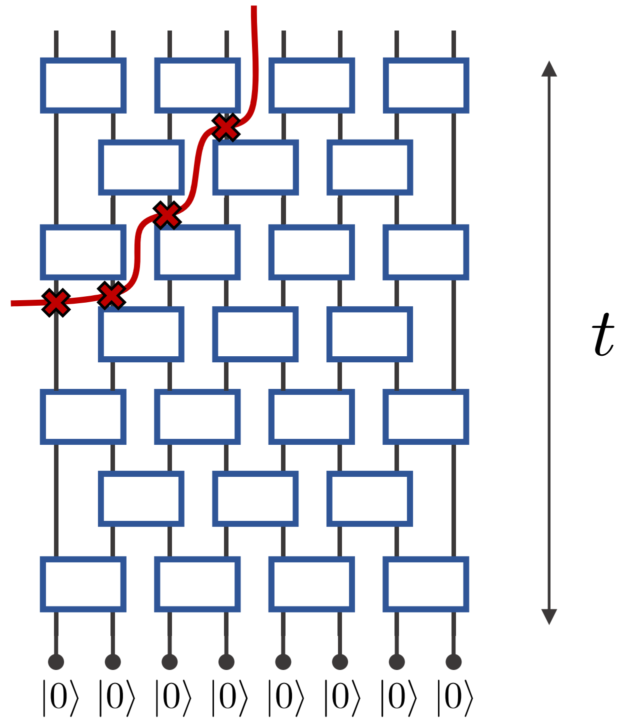
Minimal cut picture (Nahum, Ruhman *et al*, PRX 2017):

Short time $t < L/2$





Long time $t > L/2$

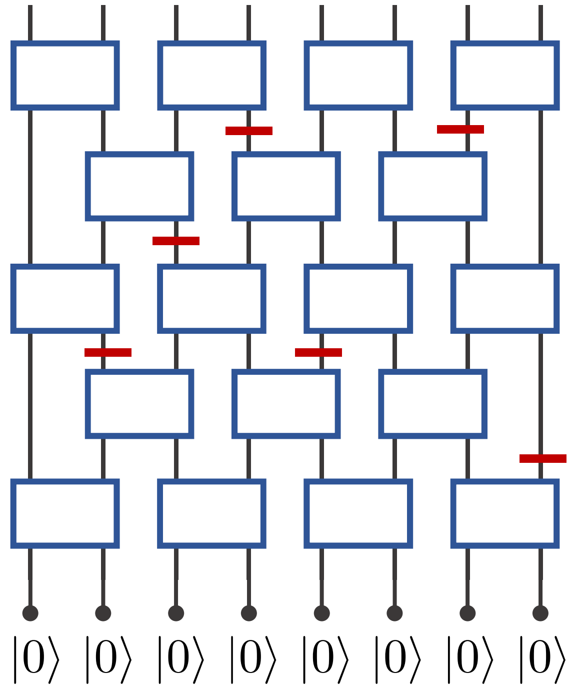


Random Unitary Circuit with projective measurements

Add a finite density of single q-bit measurements (--) to the circuit:

$$\hat{P}_\mu = |\mu\rangle\langle\mu| \otimes \mathbb{I}$$

$$|\psi\rangle \mapsto \frac{\hat{P}_\mu|\psi\rangle}{\sqrt{\langle\psi|\hat{P}_\mu|\psi\rangle}} \text{ with prob.} \langle\hat{P}_\mu\rangle$$



How does the entanglement grow?

Does it saturate to an area law or volume law?

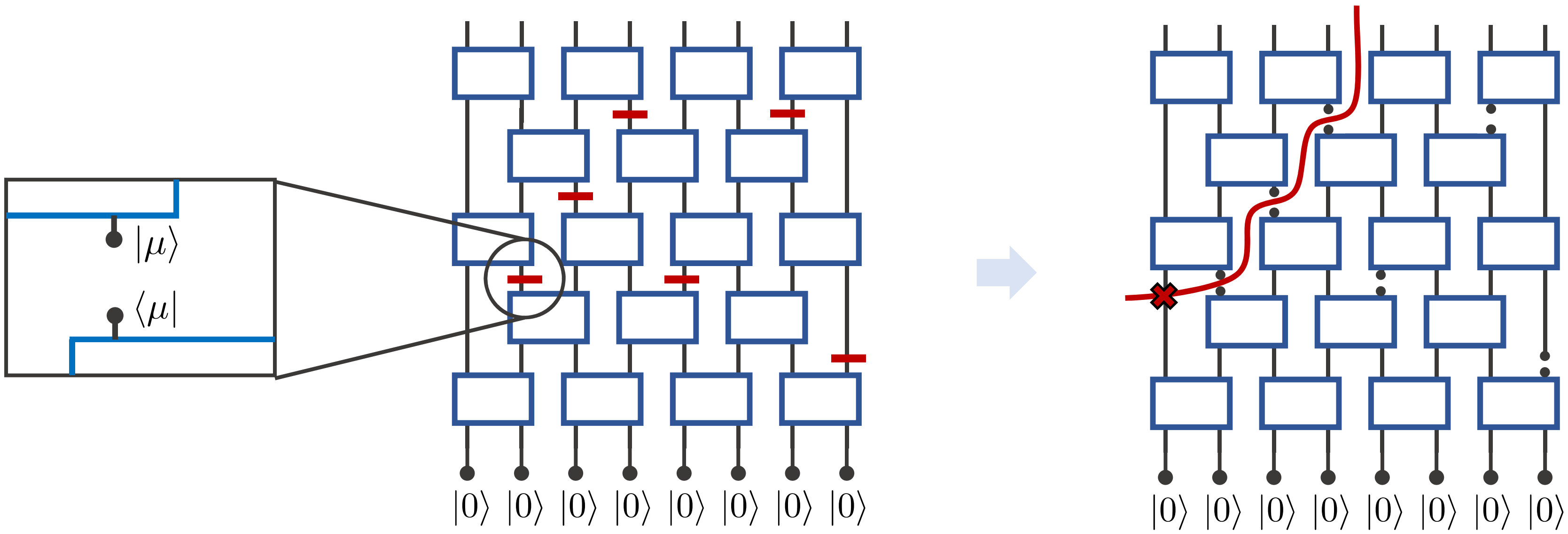
B. Skinner *et al*, arXiv:1808.05953

Y. Li *et al*, arXiv:1808.06134

A. Chan *et al*, arXiv:1808.05949

Percolation picture

Skinner, Ruhman, Nahum, arXiv:1808.05953



Percolation transition
- $p < 0.5$ ➡ Minimum cut: Volume-law entanglement
- $P > 0.5$ ➡ "Free cuts" percolate: Area-law entanglement

- This model is oversimplified. Works only for a very singular type of entropy (S_0)
- Gets the wrong value of p_c
- Numerical solutions seem to give non-percolation exponents

This talk

- Effective description of the transition
(mapping to statistical-mechanics models)

- *A new interpretation* as a phase transition in the amount of information extracted from the system by measurements

  ➡ More readily observable signature of the transition

Challenges for theoretical analysis

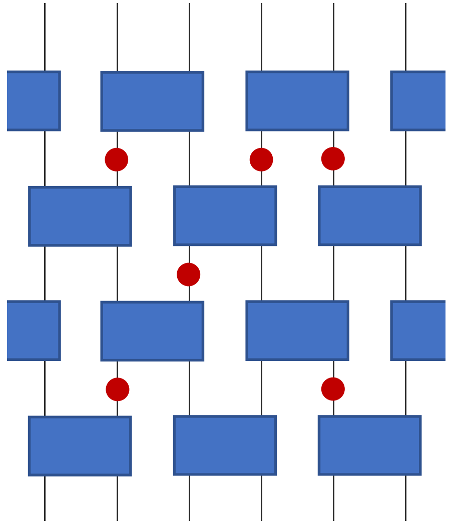
- There is a nice mapping from random unitary circuits to classical statistical mechanics models (Nahum, Vijay, Haah PRX 2018)

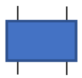
- Adding projective measurements -> highly non linear process:

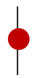
  1. Need to normalize the WF after each projection.
  2. The probability of measurements/trajectories depend on the state.

Generalize to a circuit with weak measurement

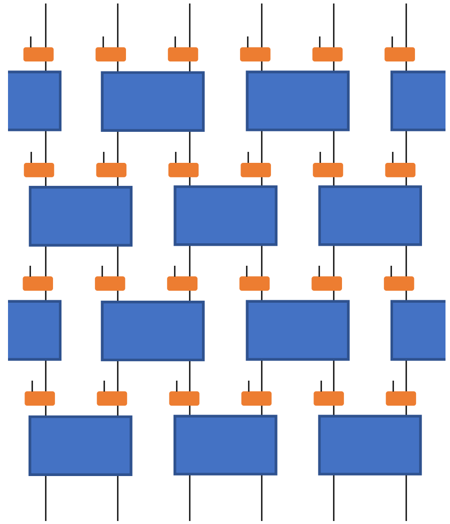
Random unitary circuit with
projective measurements

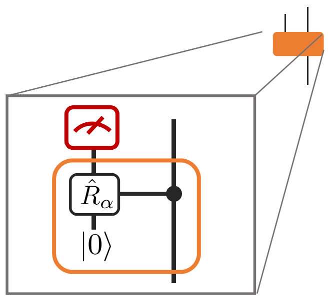


 Random Unitary gate

 Projective measurement

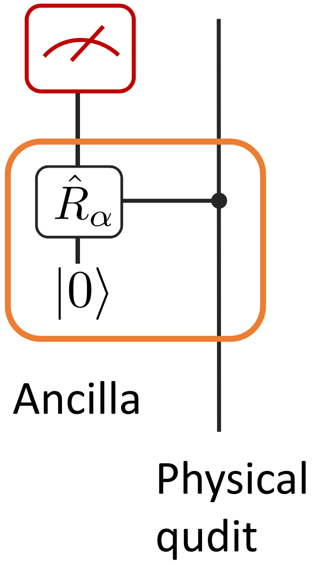
Random unitary circuit with
weak measurements





Weak measurement

Weak measurements



- Introducing ancilla with $q+1$ internal states
- Controlled rotation

$$\hat{R}_\alpha = \sum_{i=1}^{q} |i\rangle_s \langle i|_s \otimes e^{-\mathrm{i}\hat{X}_i \alpha} \qquad \hat{X}_i = |i\rangle_m \langle 0|_m + |0\rangle_m \langle i|_m$$

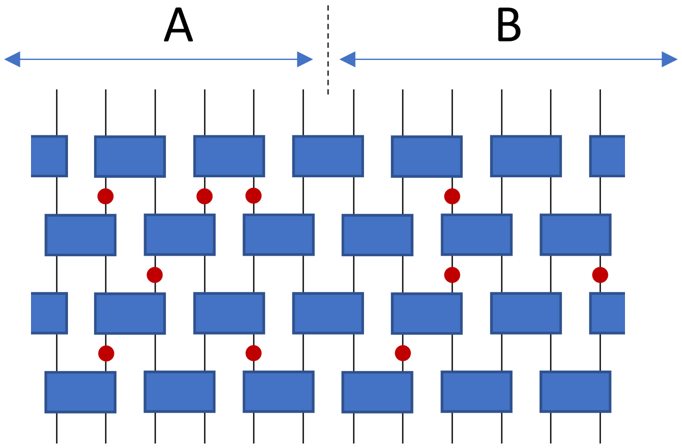
- Measurement of ancilla in the computational basis (equivalent to dephasing ancillae)

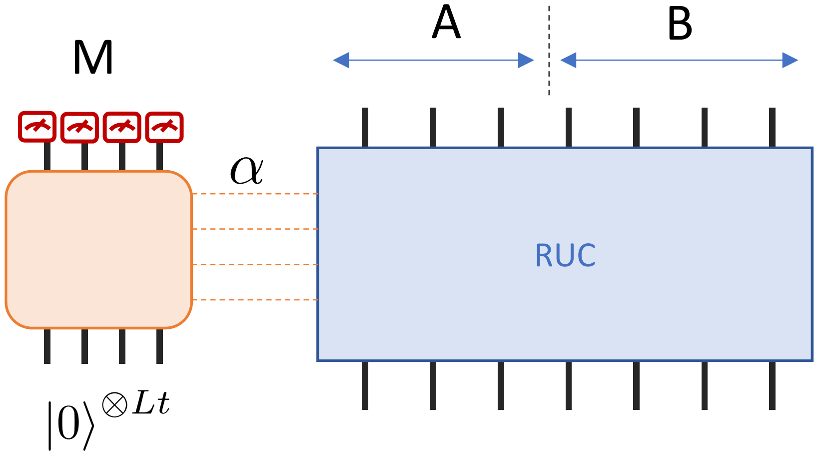
$$\mathcal{N}_D\left[\hat{R}_\alpha\left(\rho_{\mathrm{in}} \otimes |0\rangle_m \langle 0|_m\right)\hat{R}_\alpha^\dagger\right] =$$

$$\underbrace{(1-\sin^2\alpha)}_{1-p}\ \underbrace{\rho_{\mathrm{in}}}_{\text{No measurement}} \otimes |0\rangle_m \langle 0|_m + \underbrace{\sin^2\alpha}_{p} \sum_i \underbrace{\hat{P}_i \rho_{\mathrm{in}} \hat{P}_i}_{\text{Projection to the i-th basis}} \otimes |i\rangle_m \langle i|_m$$

This weak measurement scheme is equivalent to random projective measurements with measurement probability *p*

Conditional entropy





Average entanglement entropy = Conditional entropy

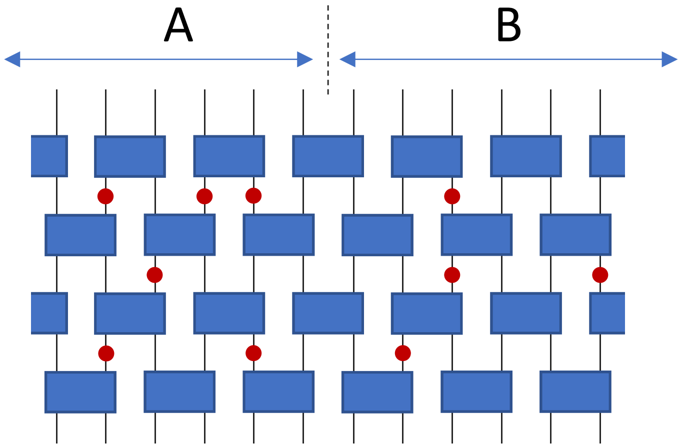
$$\tilde{S}(A|M) = \overline{\sum_{\mathbf{i}_{\mathrm{M}}} p_{\mathbf{i}_{\mathrm{M}}}(\mathcal{U}) S[\rho_A(\mathcal{U}, \mathbf{i}_{\mathrm{M}})]} = \overline{\tilde{S}_{AM}} - \overline{\tilde{S}_M}$$

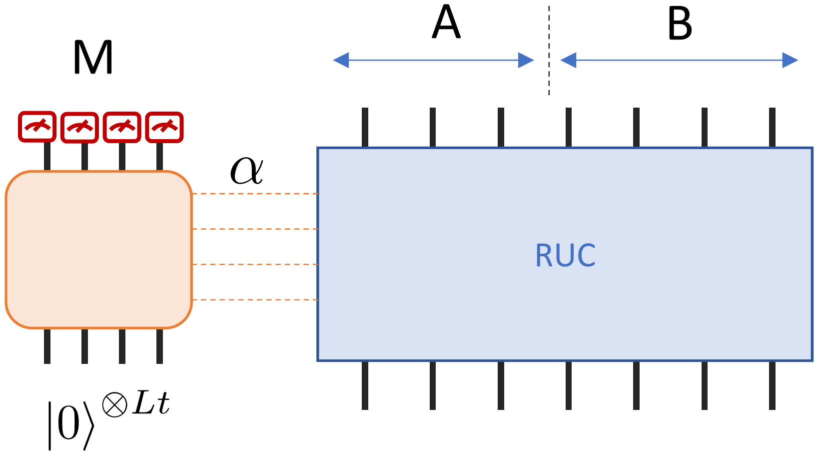
Tilde means: only diagonal elements in the measurement basis

Double problem: how to trace over a log?
How to take average of a log?

$$\overline{S_X} = \overline{\mathrm{tr}(\rho_X \log \rho_X)}$$

Conditional entropy: replica trick





$$\tilde{S}^{(n)}(A|M) = \frac{1}{1-n}\log\left(\overline{\sum_{i_M} p_{i_M}^n \mathrm{tr}\rho_{A,i_M}^n}\right) - \frac{1}{1-n}\log\left(\overline{\sum_{i_M} p_{i_M}^n}\right)$$

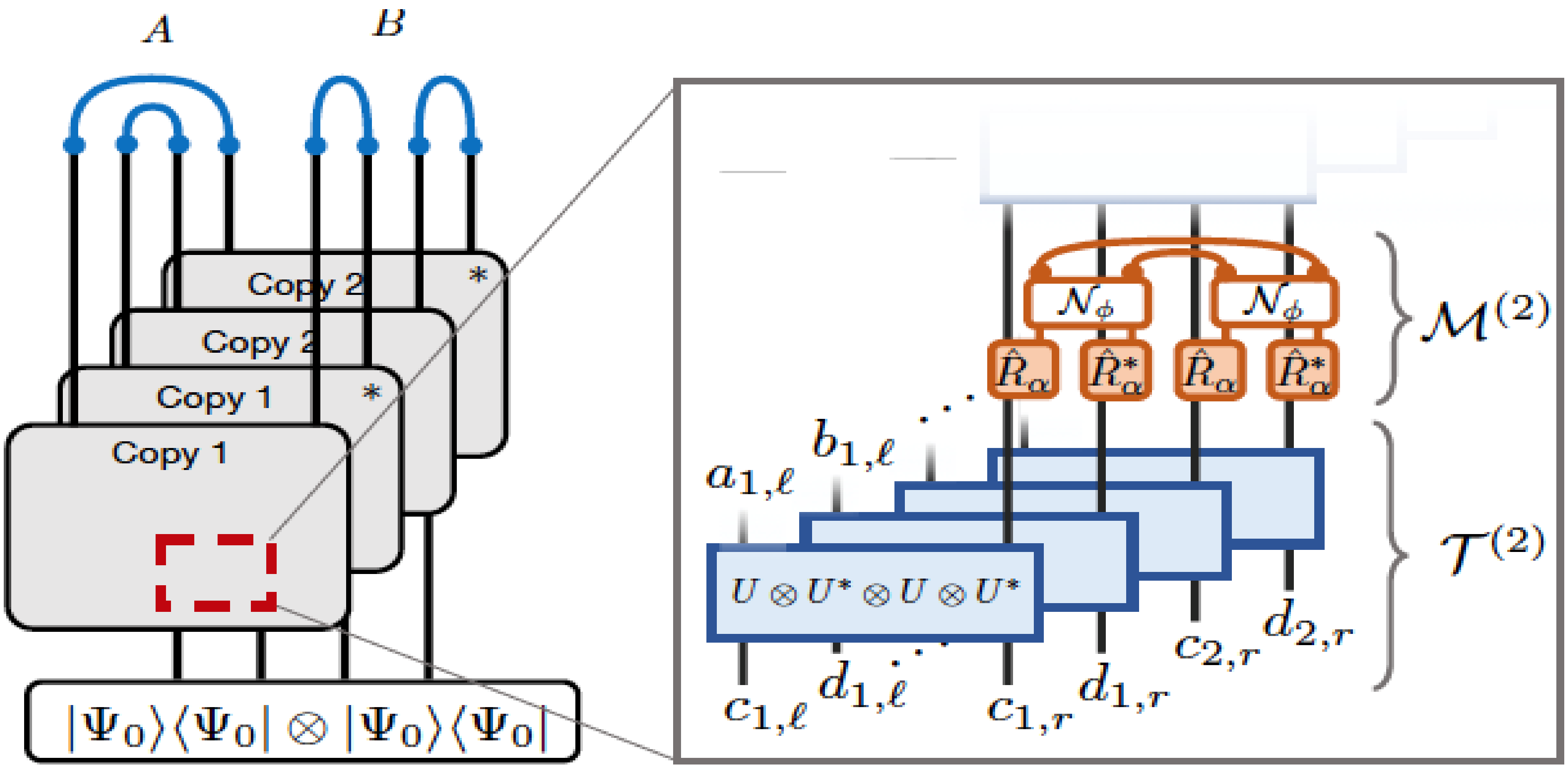
$$\tilde{S}(A|M) = \lim_{n\to 1}\tilde{S}^{(n)}(A|M).$$

This will facilitate analytic calculations (mapping to stat mech models)

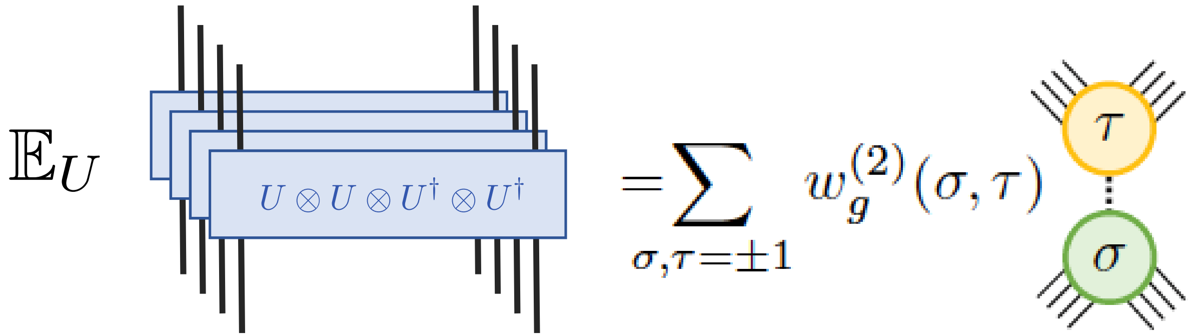
Mapping between RUC and Classical 2D Ising model

$$S^{(2)}(A|M) = -\log \mathbb{E}_U \left[\mathrm{tr}\tilde{\rho}^2_{AM}\right] + \log \mathbb{E}_U \left[\mathrm{tr}\tilde{\rho}^2_{M}\right]$$

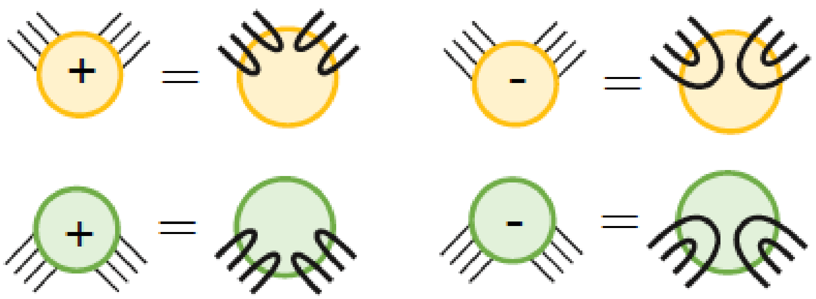
Example: (purity) $$\mathrm{tr}\left(\tilde{\rho}^2_{AM}\right) = \mathrm{tr}_{AM}\left[\mathrm{tr}_B\left(\tilde{\rho}\right)\cdot \mathrm{tr}_B\left(\tilde{\rho}\right)\right]$$



Key step: average over unitaries



Spin represents the permutation over two copies:



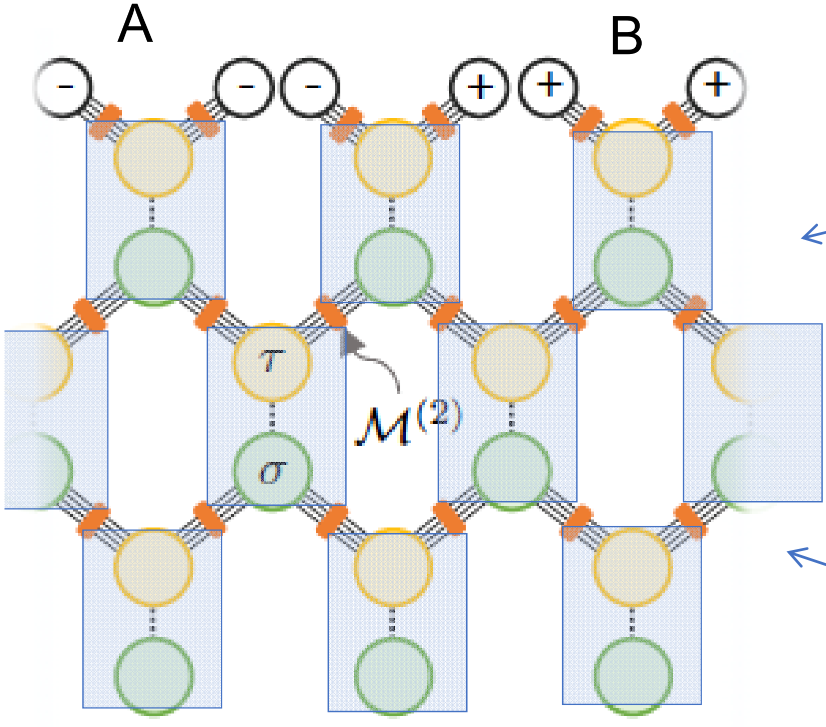
\+ = Identity

\- = swap

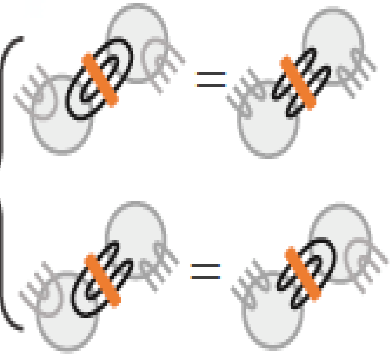
We have now replaced unitary gates with simple tensors.

Next step: contract the tensor network

Contracting the tensor network



$$w_g^{(2)}(\sigma,\tau) = \frac{\delta_{\sigma,\tau}}{d^2-1} - \frac{1-\delta_{\sigma,\tau}}{d(d^2-1)}$$



$$w_d^{(2)}(\sigma,\tau) = \begin{cases} = q^2\cos^4\alpha + q\sin^4\alpha & (\sigma=\tau) \\ = q\cos^4\alpha + q\sin^4\alpha & (\sigma\neq\tau) \end{cases}$$

$$\boxed{\overline{\mathrm{tr}\,(\rho_{AM}^2)} = \sum_{\{\sigma,\tau\}} \prod_{\langle \mathbf{r},\mathbf{r}'\rangle} w^{(2)}(\sigma_{\mathbf{r}}, \tau_{\mathbf{r}'}),}$$

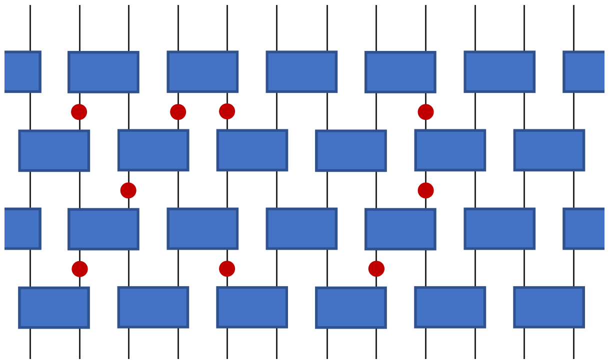
Partition function of classical Ising model on Honeycomb lattice!

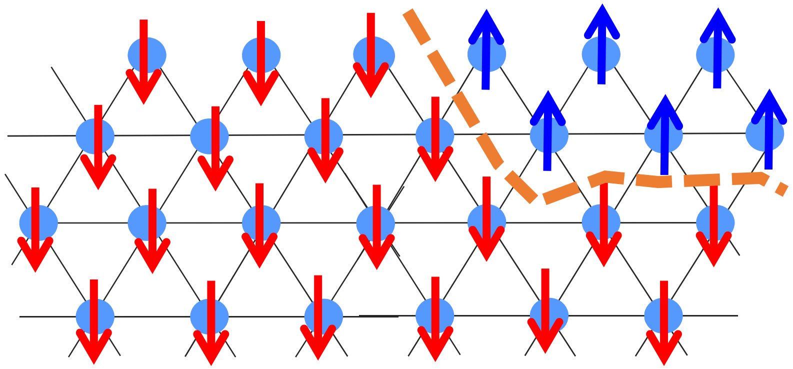
Problem: $w_g^{(2)}$ has negative weights

Solution: Integrate out one spin of every pair.

Mapping to an Ising model with Z_2 symmetry

$S^{(2)}(A)$ $\longleftrightarrow$ Classical Ising model





$$S^{(2)}(A) = F_{dw}^{(2)} - F_0^{(2)}$$

Free energy of a domain wall ending at the interface of A

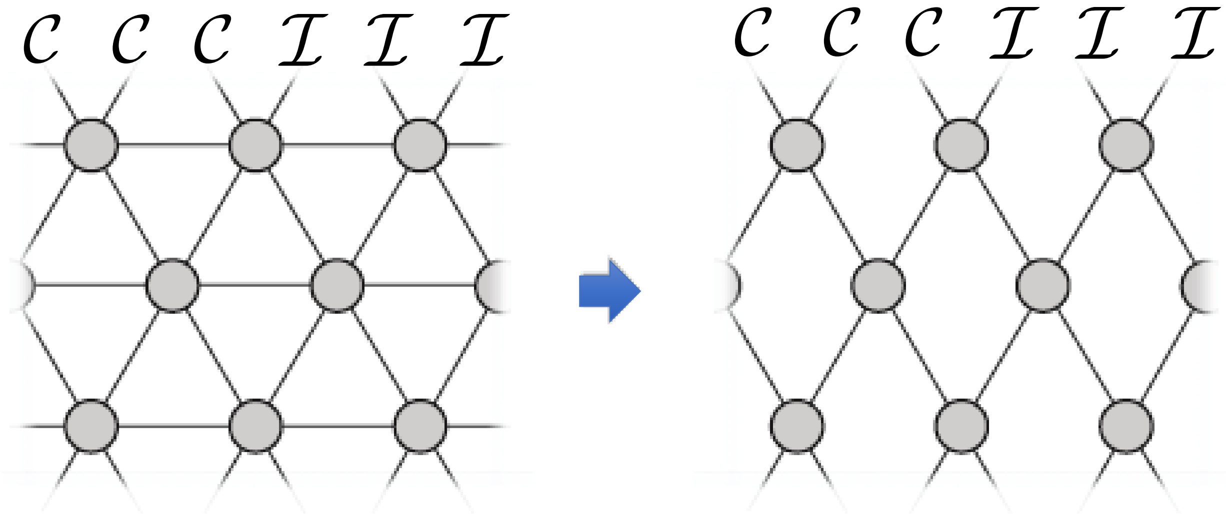
Volume law phase: $\Delta F_{dw}^{(2)} \sim L$ Ferromagnet

Area law phase: $\Delta F_{dw}^{(2)} \sim \text{const}$ Paramagnet

Generalization to n≥2

Ising spins → Generalized spins with n! states.
Elements of the permutation group

Simplification in the limit $q \to \infty$



And enlarged permutation symmetry: $P_n \to P_{n!}$

➡ **n! state Potts model**

Taking the replica limit

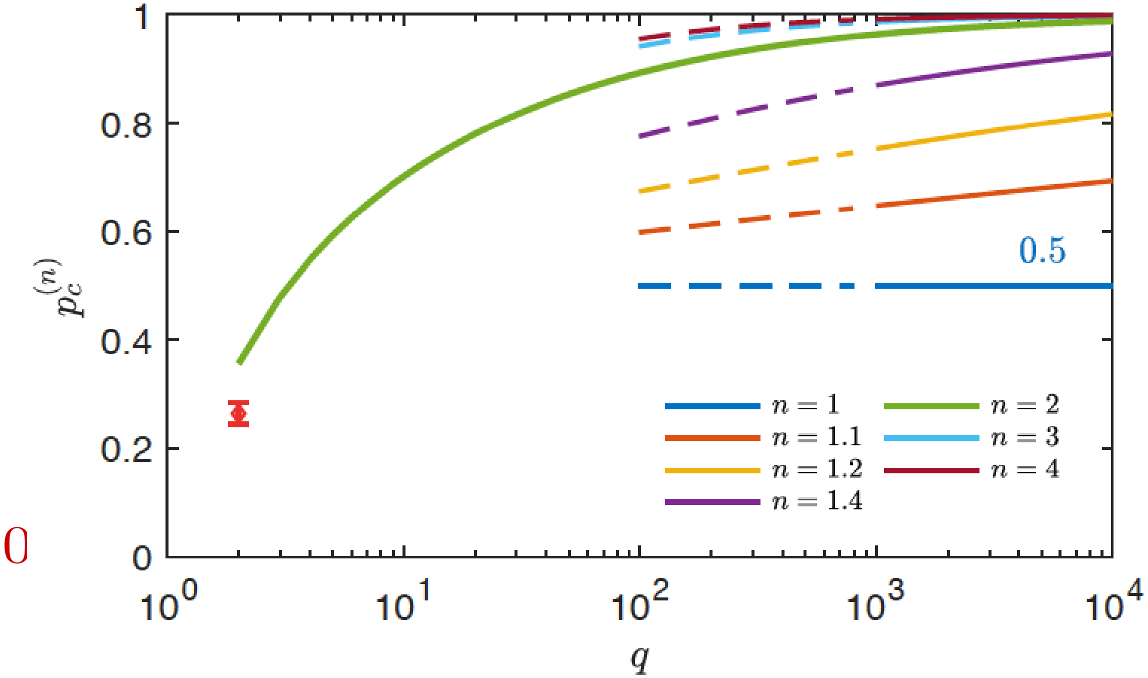
1. Critical measurement probability/strength

For the Potts model on the square lattice we know the exact critical point through Kramers-Wannier duality:

$$\alpha_c^{(n)} = \arctan\left(\left(\frac{q^{n-1}}{\sqrt{n!}}\right)^{1/2n}\right),$$

$$p_c^{(n)} = 1/\left(1+\left(\frac{\sqrt{n!}}{q^{n-1}}\right)^{1/n}\right)$$

Numerics q=2: $p_c = 0.25 \pm 0.0$



2. Universality of the transition is bond percolation

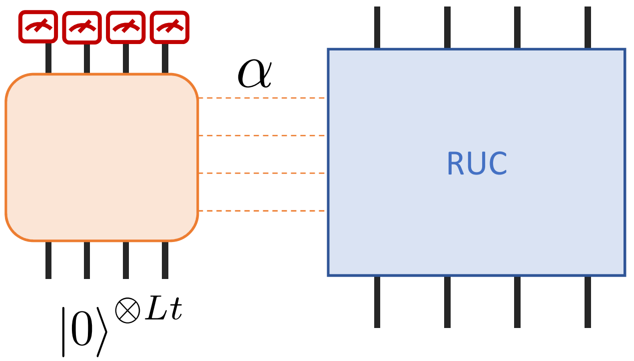
(m state Potts model maps to bond percolation in the limit $m \to 1$)

But Potts symmetry is broken at finite q. What is the universality then?

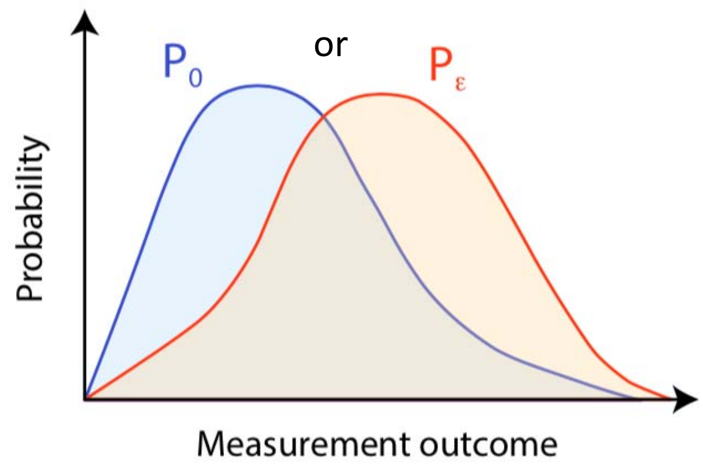
Alternative signature of the transition ?

How much information can we potentially extract about the initial state from collecting measurement results for as long as we need?

Given ensemble of measurement outcomes how well can we distinguish two initial states that differ by some small local rotation by small angle ε ?



$|\psi_0\rangle$ or $|\psi_\epsilon\rangle = e^{\mathrm{i}\epsilon X}\,|\psi_0\rangle$



KL divergence: $$D_{\mathrm{KL}}(P_0||P_\epsilon) \equiv \overline{\sum_x P_0(x)\log\left(P_0(x)/P_\epsilon(x)\right)} = \frac{1}{2}\mathcal{F}\epsilon^2$$

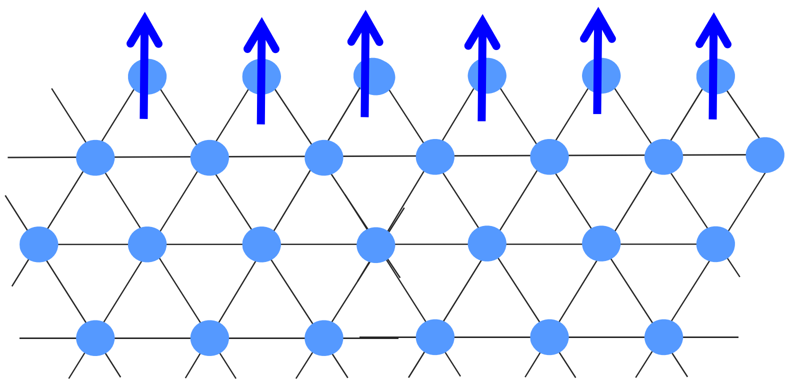
Fisher information

Is there a phase transition in the extracted Fisher information ?

Mapping to spin models using replicas

$$D^{(n)}(\theta) \equiv \frac{1}{1-n}\log\left(\overline{\mathrm{tr}\left[\tilde{\rho}_{M,0}\tilde{\rho}_{M,\theta}^{n-1}\right]}\right) - \frac{1}{1-n}\log\left(\overline{\mathrm{tr}\left[\tilde{\rho}_{M,0}^{n}\right]}\right)$$

The same spin model as before, only different boundary conditions



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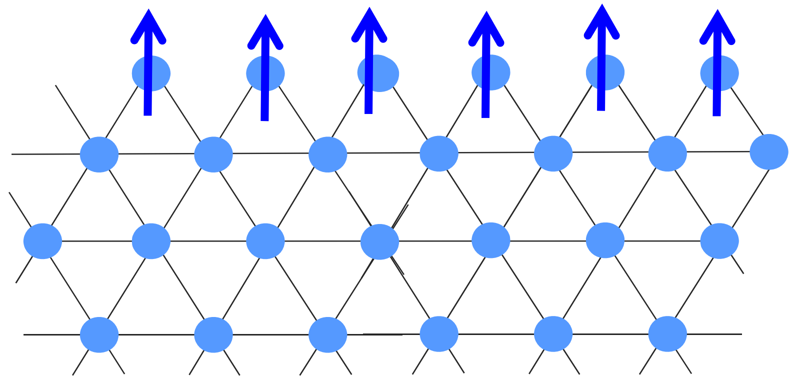


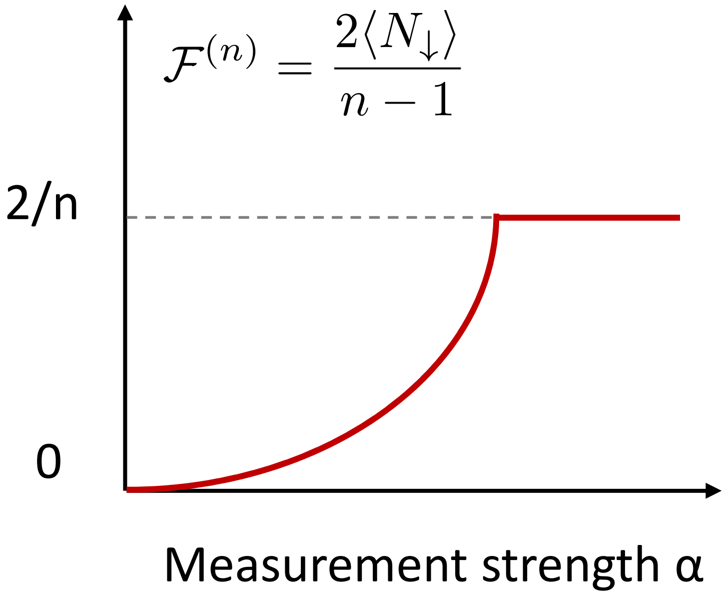
Boundary field $-h_\epsilon N_\downarrow$

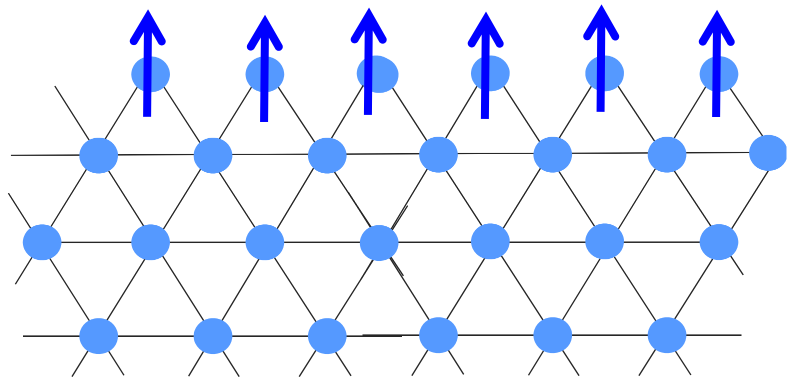
No Boundary field

$$\mathcal{F}^{(n)} = \frac{2\langle N_\downarrow\rangle}{n-1}$$

The phase transition in ability to extract information

Information on initial state revealed by measurements $(t \to \infty)$





Boundary field: $-h_\epsilon N_\downarrow$

Below the threshold measurement strength/rate the unitary gates effectively scramble the information so that some of it remains hidden from measurements forever!

Outlook

- What is the universality class of the transition at finite q ?
- Are there even simpler measurements?
  Correlation measurements? Simple estimators of Fisher information that? What is the minimal number of experiment repetitions needed?
- Sensitivity to real errors (unread measurements) ?
  Use sharpness of the transition as an intrinsic test of quality of a quantum circuit (quantum advantage)?
- Relation to threshold theorem in error-correction?
  (fault tolerance)