בית ספר הלן דילר לחישוב קוואנטי,
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הרצאה 6

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Outline:

1. No signaling
2. No cloning
3. Quantum teleportation

No signaling

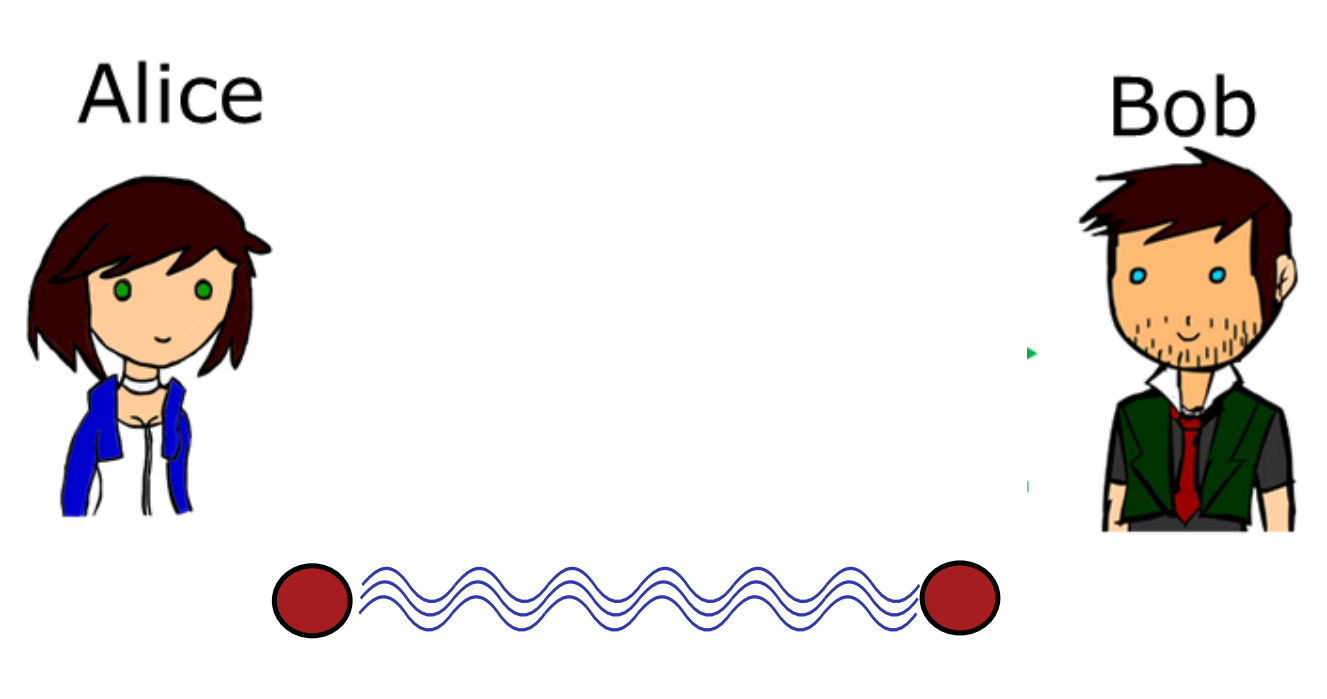
More about measurements...

Measurement Postulate (for pedestrians)

The measurement of $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$ in the computational basis is a physical process with a **probabilistic** outcome:

- $\text{Prob}(i) = |c_i|^2, \qquad i = 0, 1$
- Post-measurement state is $\;|\psi'\rangle = |i\rangle$ (The "collapse of the wave-function")

Alice and Bob



$$|\psi\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle$$

We can also talk about single-side measurement:

Alice:

$$\text{Prob}(0) = |c_{00}|^2 + |c_{01}|^2 \qquad |\psi'\rangle = \frac{1}{\sqrt{|c_{00}|^2+|c_{01}|^2}}\big(c_{00}|00\rangle + c_{01}|01\rangle\big)$$

$$\text{Prob}(1) = |c_{10}|^2 + |c_{11}|^2 \qquad |\psi'\rangle = \frac{1}{\sqrt{|c_{10}|^2+|c_{11}|^2}}\big(c_{10}|10\rangle + c_{11}|11\rangle\big)$$

For example:

$$|\psi\rangle = \frac{1}{\sqrt{14}}\big(|00\rangle - 2|10\rangle + 3|11\rangle\big)$$

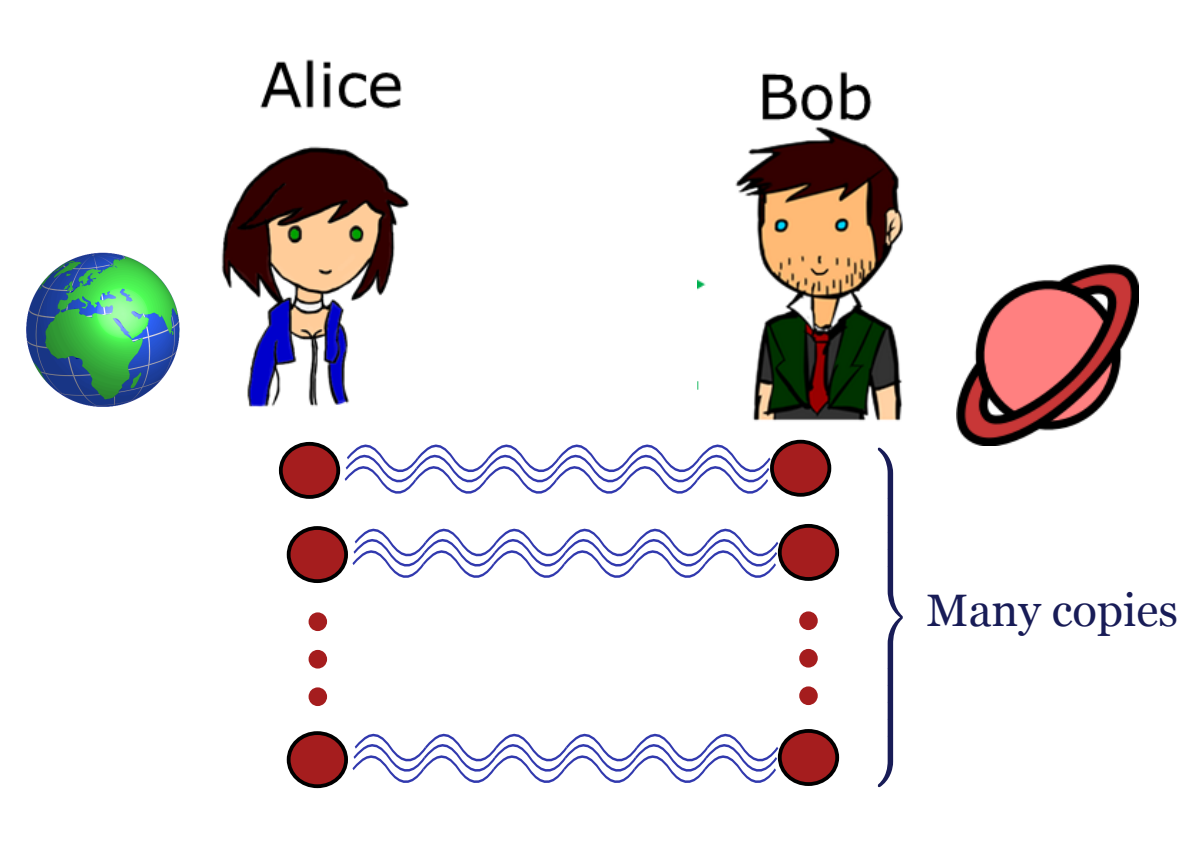
? What is the probability that Alice measures 1 in her qubit, and what is the post measurement state?

$$c_{10}|10\rangle + c_{11}|11\rangle = \frac{1}{\sqrt{14}}\big(-2|10\rangle + 3|11\rangle\big)$$

$$\text{Prob}(1) = |c_{10}|^2 + |c_{11}|^2 = \frac{2^2+3^2}{14} = \frac{13}{14}$$

$$|\psi'\rangle = \frac{1}{\sqrt{13}}\big(-2|10\rangle + 3|11\rangle\big)$$

? Can a measurement or a unitary gate Alice applies "change the state of Bob's qubit"? Can it influence its measurement results?

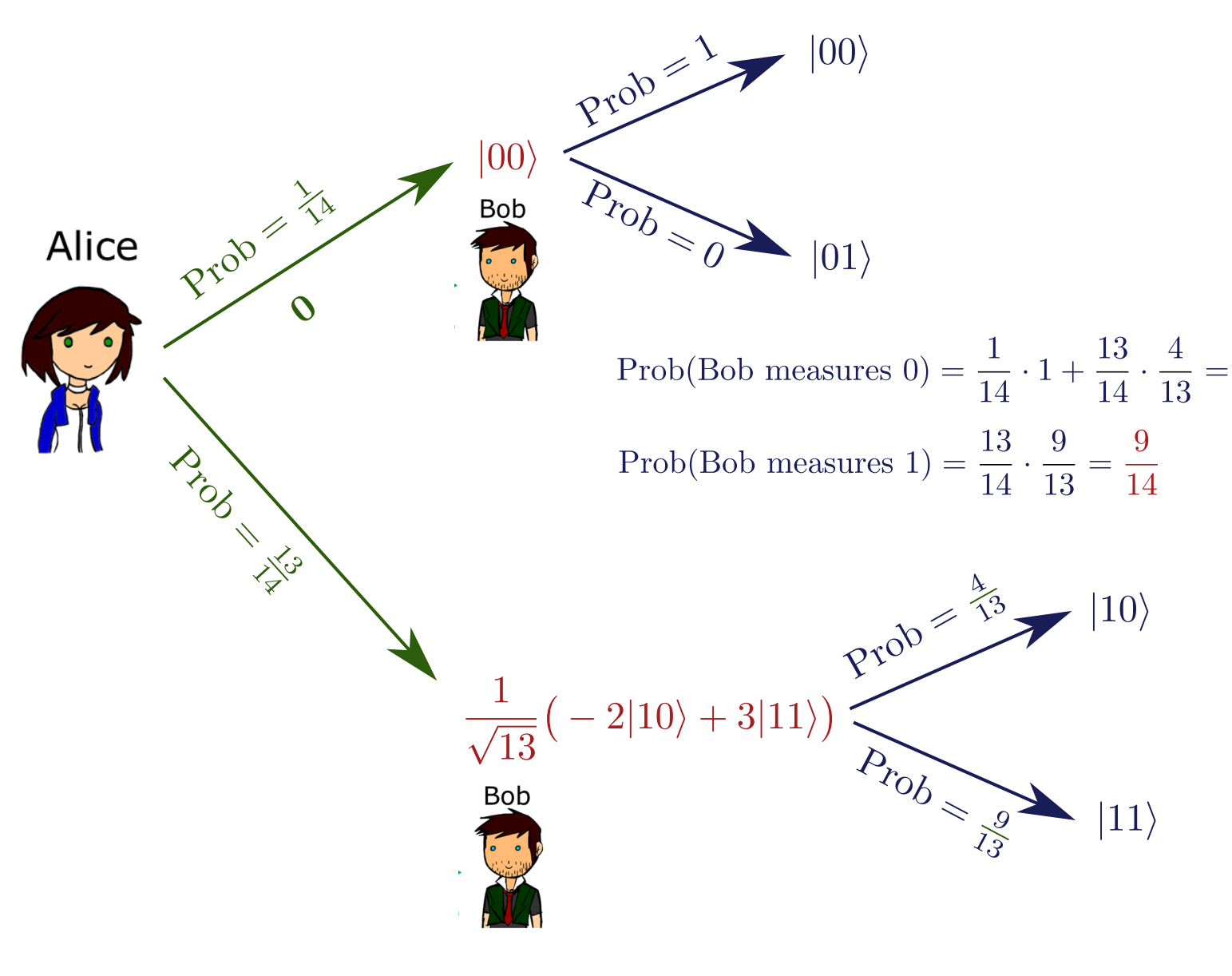

Alice
Bob
Many copies

No Signaling Principle:

Any action of Alice on her qubit (measurement and/or unitary) **cannot** change the measurements statistics of Bob

⇒ Needed by special relativity - no faster-than-light communication!

Example: $\qquad |\psi\rangle = \frac{1}{\sqrt{14}}\big(|00\rangle - 2|10\rangle + 3|11\rangle\big)$


Alice
$\text{Prob} = \frac{1}{14}$
0
$\text{Prob} = \frac{13}{14}$
$|00\rangle$
Bob
$\text{Prob} = 1$
$|00\rangle$
$\text{Prob} = 0$
$|01\rangle$
$\frac{1}{\sqrt{13}}\big(-2|10\rangle + 3|11\rangle\big)$
Bob
$\text{Prob} = \frac{4}{13}$
$|10\rangle$
$\text{Prob} = \frac{9}{13}$
$|11\rangle$

$$\text{Prob(Bob measures 0)} = \frac{1}{14}\cdot 1 + \frac{13}{14}\cdot\frac{4}{13} = \frac{5}{14}$$

$$\text{Prob(Bob measures 1)} = \frac{13}{14}\cdot\frac{9}{13} = \frac{9}{14}$$

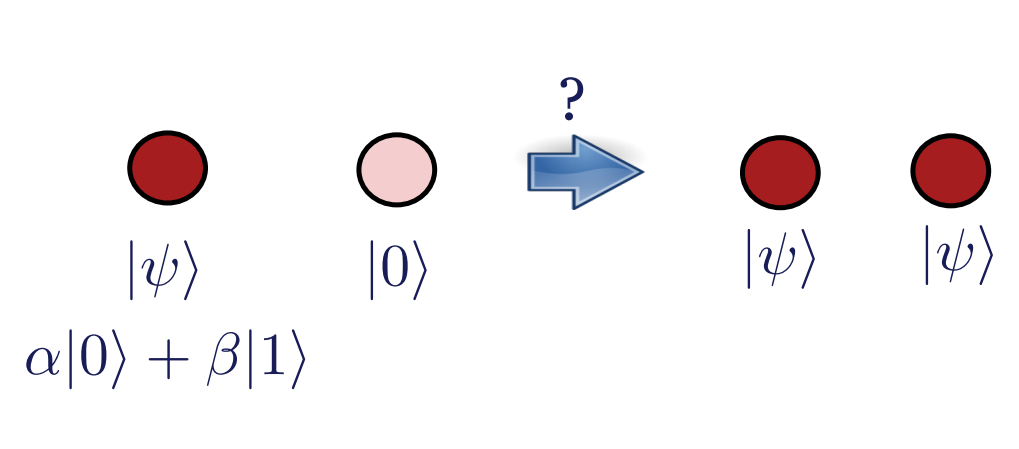
⇒ If Alice doesn't call Bob, he cannot tell whether she did a measurement or not.

A similar thing happens if Alice applies a gate on her qubit.

⚠ Entanglement does not allow faster-than-light communication, but it **does** allow for stronger-than-classical correlations

No Cloning

? Can we copy quantum information (states) ?


$|\psi\rangle$
$\alpha|0\rangle + \beta|1\rangle$
$|0\rangle$
?
$|\psi\rangle$
$|\psi\rangle$

- If $|\psi\rangle$ is known (i.e., α, β), then no probs: rotate $|0\rangle \to |\psi\rangle$
- But if $|\psi\rangle$ is known?
  - Maybe measure to learn α, β?
  - Apply a gate: $U\big(|\psi\rangle|0\rangle\big) = |\psi\rangle|\psi\rangle$?

- First solution is bad - needs many copies!
- Second solution is impossible:

  Assume by contradiction that such U exists.

  Then:

  $$U\big(|0\rangle|0\rangle\big) = |0\rangle|0\rangle\ , \qquad U\big(|1\rangle|0\rangle\big) = |1\rangle|1\rangle$$

  $$\Rightarrow\ U\big(|\psi\rangle|0\rangle\big) = U\big[(\alpha|0\rangle + \beta|1\rangle)|0\rangle\big] = \alpha U|0\rangle|0\rangle + \beta U|1\rangle|0\rangle$$
  $$= \alpha|0\rangle|0\rangle + \beta|1\rangle|1\rangle \neq \underbrace{(\alpha|0\rangle + \beta|1\rangle)(\alpha|0\rangle + \beta|1\rangle)}_{|\psi\rangle|\psi\rangle}$$

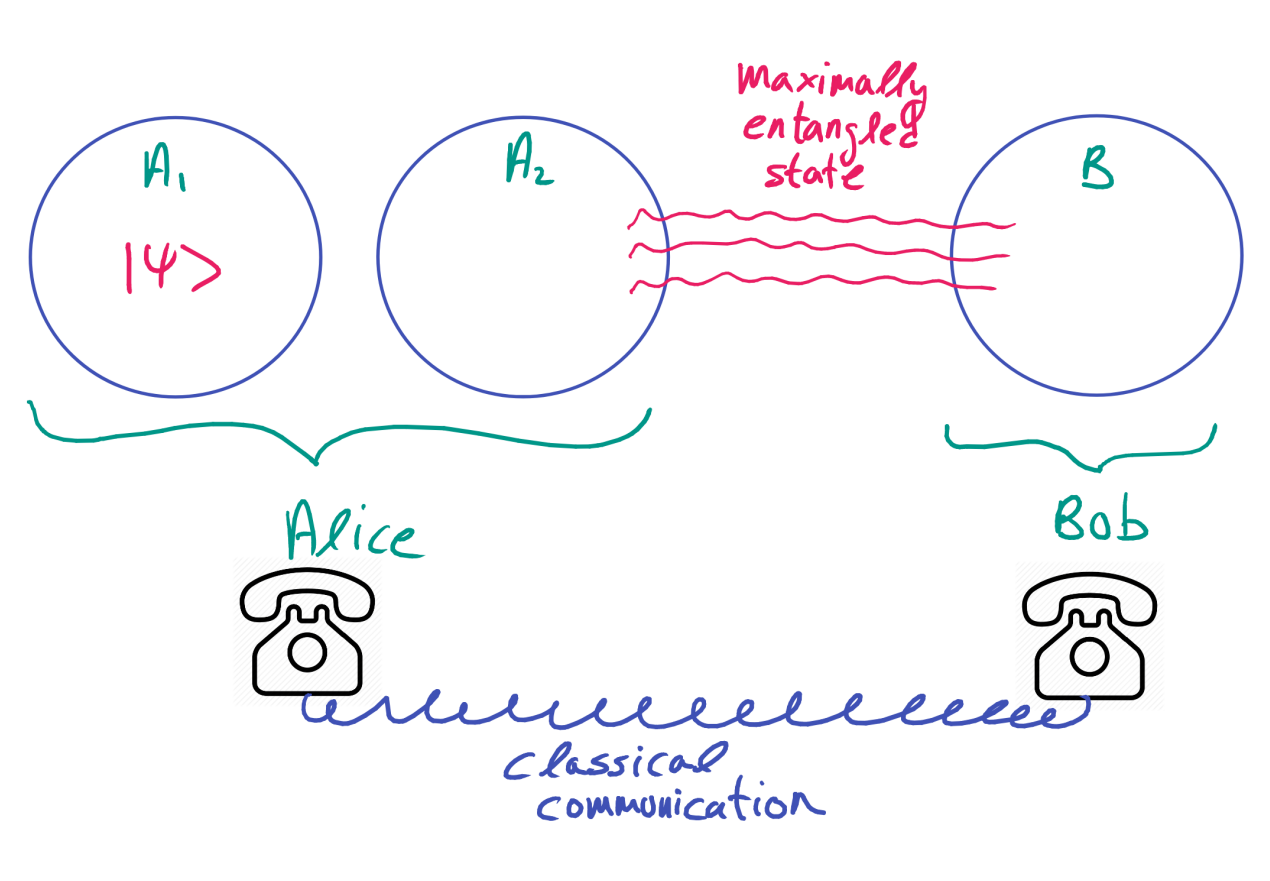
⚠ However, cloning (copying) is possible if we restrict ourselves to a subset of states.

? Show that cloning in the standard basis is possible: $U\big(|i\rangle|0\rangle\big) = |i\rangle|i\rangle$.

Teleportation

While quantum information cannot be cloned, it can be **teleported**

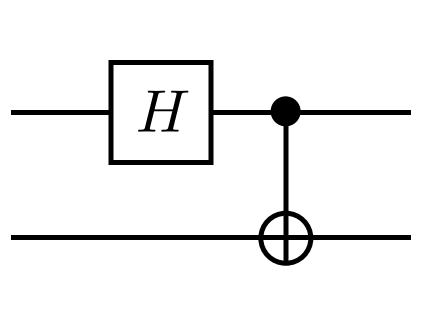
C. Bennet, G. Brassard, C. Crepeau, R. Joza, A. Peres, and W.K. Wooters (1993)


A_1
$|\psi\rangle$
A_2
maximally entangled state
B
Alice
Bob
classical communication

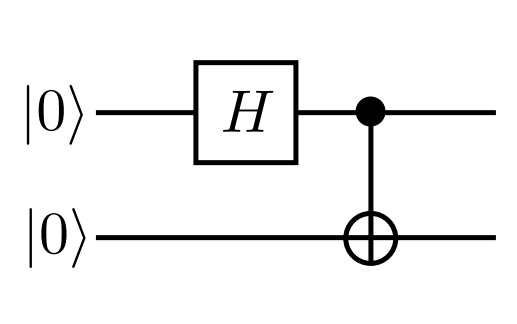
The 1-qubit case:

Mathematical Preleminary: Bell's basis

$U =$

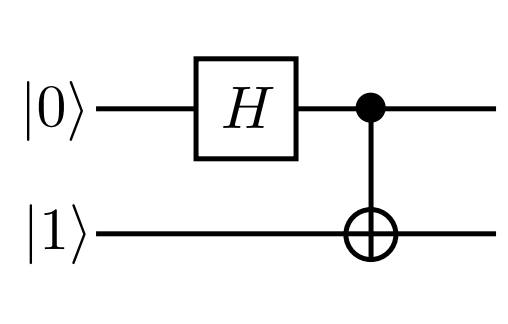


$|\beta_{00}\rangle = U|00\rangle =$


$|0\rangle$
$|0\rangle$
H

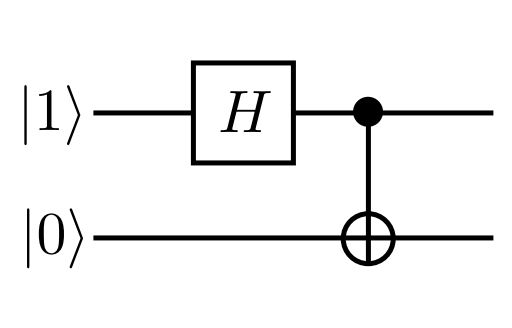
$= \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big)$

$|\beta_{01}\rangle = U|01\rangle =$


$|0\rangle$
$|1\rangle$
H

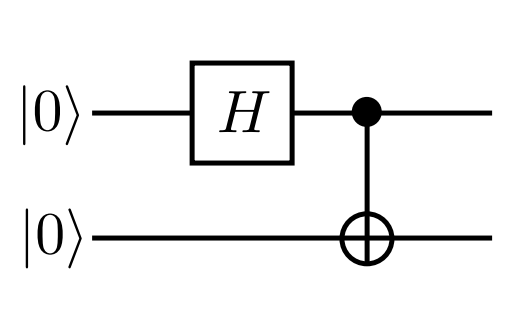
$= \frac{1}{\sqrt{2}}\big(|01\rangle + |10\rangle\big)$

$|\beta_{10}\rangle = U|10\rangle =$


$|1\rangle$
$|0\rangle$
H

$= \frac{1}{\sqrt{2}}\big(|00\rangle - |11\rangle\big)$

$|\beta_{11}\rangle = U|11\rangle =$

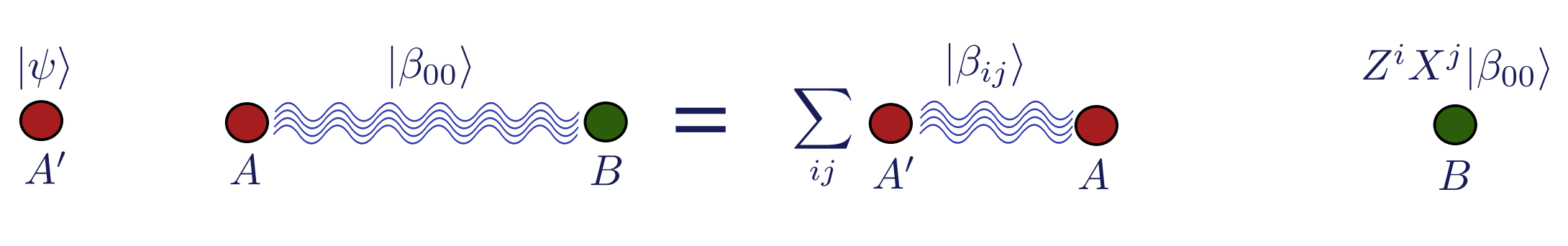

$|0\rangle$
$|0\rangle$
H

$= \frac{1}{\sqrt{2}}\big(|01\rangle - |10\rangle\big)$

Mathematical fact:

For every single qubit state $|\psi\rangle$

$$|\psi\rangle \otimes |\beta_{00}\rangle = \frac{1}{2}\Big[|\beta_{00}\rangle \otimes |\psi\rangle + |\beta_{01}\rangle \otimes (X|\psi\rangle) + |\beta_{10}\rangle \otimes (Z|\psi\rangle) + |\beta_{11}\rangle \otimes (ZX|\psi\rangle)\Big]$$

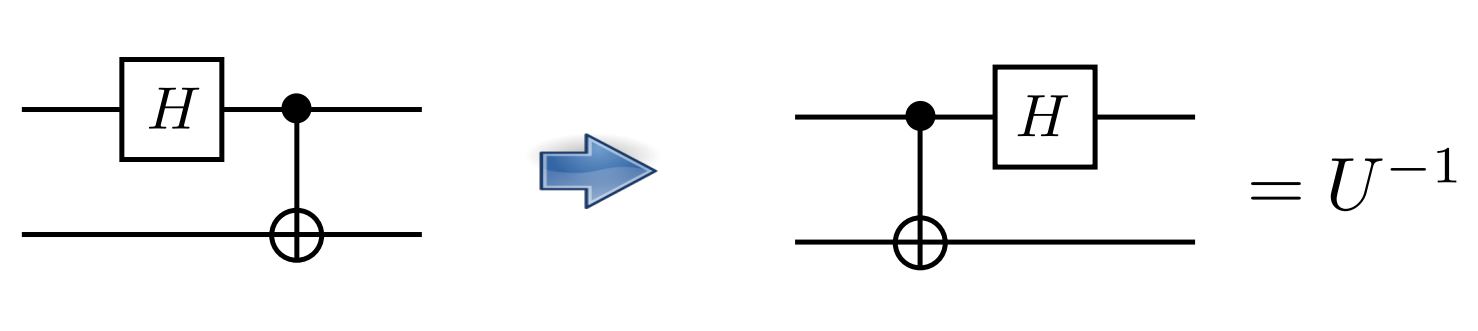

$|\psi\rangle$
A'
$|\beta_{00}\rangle$
A
B
=
$\sum_{ij}$
$|\beta_{ij}\rangle$
A'
A
$Z^iX^j|\beta_{00}\rangle$
B

1. Let Alice apply $U^{-1}_{A'A}$ on her qubits:

   $$(U^{-1}_{A'A} \otimes \mathbb{I}_B)|\psi\rangle \otimes |\beta_{00}\rangle = \frac{1}{2}\Big[|00\rangle \otimes |\psi\rangle + |01\rangle \otimes (X|\psi\rangle) + |10\rangle \otimes (Z|\psi\rangle) + |11\rangle \otimes (ZX|\psi\rangle)\Big]$$

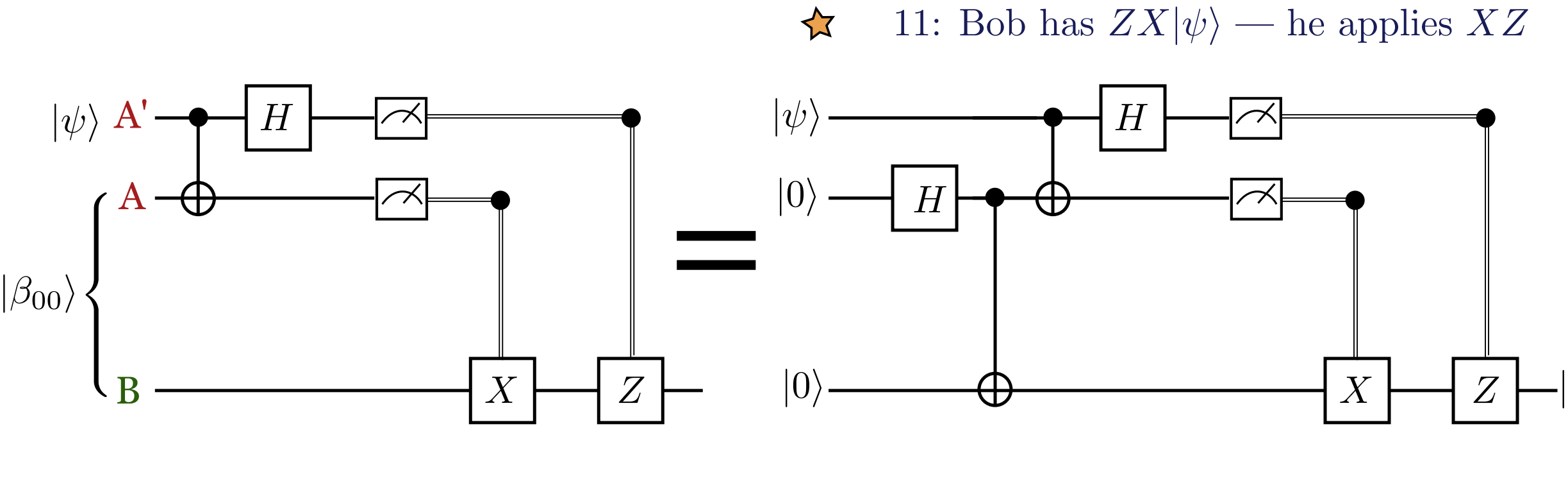
2. Now she measures her qubits and calls Bob to tell him the results:
   - 00: Bob has $|\psi\rangle$ — he does nothing
   - 01: Bob has $X|\psi\rangle$ — he applies X
   - 10: Bob has $Z|\psi\rangle$ — he applies Z
   - 11: Bob has $ZX|\psi\rangle$ — he applies XZ

   What is U^{-1} ?



- 00: Bob has $|\psi\rangle$ — he does nothing
- 01: Bob has $X|\psi\rangle$ — he applies X
- 10: Bob has $Z|\psi\rangle$ — he applies Z
- 11: Bob has $ZX|\psi\rangle$ — he applies XZ

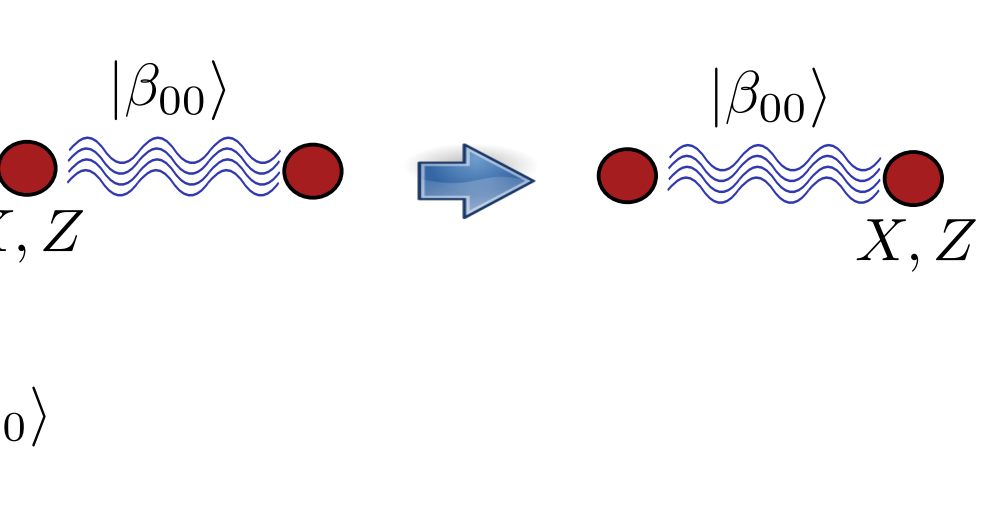
Wrapping it all up:


$|\psi\rangle$ A'
A
$|\beta_{00}\rangle$
B
H
X
Z
=
$|\psi\rangle$
$|0\rangle$
$|0\rangle$
H
H
X
Z
$|\psi\rangle$

Proving the mathematical fact:

$$(*)\ \ |\psi\rangle \otimes |\beta_{00}\rangle = \frac{1}{2}\Big[|\beta_{00}\rangle \otimes |\psi\rangle + |\beta_{01}\rangle \otimes (X|\psi\rangle) + |\beta_{10}\rangle \otimes (Z|\psi\rangle) + |\beta_{11}\rangle \otimes (ZX|\psi\rangle)\Big]$$

We will use the identities:


$|\beta_{00}\rangle$
X, Z
$|\beta_{00}\rangle$
X, Z

$$|\beta_{00}\rangle = \tfrac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big)$$
$$|\beta_{01}\rangle = \tfrac{1}{\sqrt{2}}\big(|01\rangle + |10\rangle\big) = X \otimes \mathbb{I}|\beta_{00}\rangle = \mathbb{I} \otimes X|\beta_{00}\rangle$$
$$|\beta_{10}\rangle = \tfrac{1}{\sqrt{2}}\big(|00\rangle - |11\rangle\big) = Z \otimes \mathbb{I}|\beta_{00}\rangle = \mathbb{I} \otimes Z|\beta_{00}\rangle$$
$$|\beta_{10}\rangle = \tfrac{1}{\sqrt{2}}\big(|01\rangle - |10\rangle\big) = ZX \otimes \mathbb{I}|\beta_{00}\rangle = \mathbb{I} \otimes XZ|\beta_{00}\rangle$$

⇒ $RHS = (\mathbb{I} + \mathbb{I}_{A'} \otimes X \otimes X + \mathbb{I}_{A'} \otimes Z \otimes Z + \mathbb{I}_{A'} \otimes XZ \otimes ZX)|\beta_{00}\rangle_{AA'}|\psi\rangle_B$

To prove (*) we will multiply the RHS by ${}_{AB}\langle\beta_{00}|$ and see that we get $|\psi\rangle_{A'}$

The identities also imply:

$$\langle\beta_{00}| = \langle\beta_{00}|X \otimes X = \langle\beta_{00}|Z \otimes Z = \langle\beta_{00}|XZ \otimes ZX$$

⇒ ${}_{AB}\langle\beta_{00}| \cdot \text{RHS} = 2\,{}_{AB}\langle\beta_{00}| \cdot |\beta_{00}\rangle_{A'A}|\psi\rangle_B$

Finally, we claim $2\,{}_{AB}\langle\beta_{00}| \cdot |\beta_{00}\rangle_{A'A}|\psi\rangle_B = |\psi\rangle_{A'}$

This is a linear identity, and therefore it is sufficient to verify for $|\psi\rangle = |0\rangle$ and $|\psi\rangle = |1\rangle$ — which can easily be done explicitly.