4d CFTs, Riemann surfaces, and elliptic integrable models: a 6d story

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The goal and the context

- Recently there was a lot of progress in computing partition functions of supersymmetric QFTs in various space-time dimensions.
- Some of these partition functions can be computed exactly which is a quite rare luxury.
- When this happens the partition functions usually reduce to finite dimensional matrix model integrals.
- The integrands are given in terms of special functions: elliptic Gamma functions in some of the 4d cases, hyperbolic Gamma functions in analogous 3d situations, etc.
- An example of such a partition function in $4d$ is the supersymmetric index, $S^3 \times S^1$, relation of which to the hypergeometric elliptic integrals was pointed out in a beautiful paper by Dolan and Osborn (2008) and was then further studied in a series of papers by Spiridonov and Vartanov (and others).
- The mathematical results regarding properties of these integrals (Rains,Spiridonov,$\cdots$) allow us to check, and in some cases give evidence for new, non trivial properties of supersymmetric QFTs.

The goal and the context: 6=4+2

- Recently there was an unrelated dramatic development in understanding CFTs in 4d with extended supersymmetry, $\mathcal{N} = 2$. (Gaiotto, Gaiotto-Moore-Neitzke, Argyres-Seiberg,...)

- There exists a special 6d superconformal theory defined by its supersymmetry and some discrete data (ADE). In string/M-theory the A_{N-1}-type model of this kind is the theory living on N $M5$-branes.

- Compactifying this $6d$ theory down to four dimensions on a punctured Riemann surface one obtains a wide variety of 4d superconformal theories labeled by the choice of the Riemann surface. They are usually called theories of class $\mathcal{S}(ix)$.

- Some of these $4d$ theories are usual gauge theories but others are less conventional strongly-coupled SCFTs. Theories of class $\mathcal{S}$ are interrelated by a network of strong/weak coupling dualities and RG flows.

- One can compute some of the supersymmetric partition functions, eg the index, for this new class of theories (even for the strongly-coupled ones).

- Our goal in this talk will be to review some of the mathematics one encounters while computing some of the partition functions for theories of class $\mathcal{S}$.

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Outline

- "A_1 symmetric province"
- "A_{N-1} symmetric kingdom"
- "A_1 non-symmetric empire"
- Comments

Notations

- Elliptic Gamma function:

$$\Gamma\,(z;\; p\,,\; q) = \prod_{i,j=0}^{\infty} \frac{1 - p^{i+1} q^{j+1} z^{-1}}{1 - p^i q^j z}\,.$$

- We use the short-hand notation

$$f\,(a_0\, a_1^{\pm 1}\, a_2^{\pm 1} \cdots) = \prod_{\alpha_i = \pm 1} f\,(a_0\, a_1^{\alpha_1}\, a_2^{\alpha_2}\, \cdots)\,.$$

- Theta function is given by

$$\theta(z;\; q) = \prod_{\ell=0}^{\infty} (1 - q^{\ell} z)(1 - q^{1+\ell} z^{-1})\,.$$

- Let us also define

$$\kappa \equiv \Gamma\left(\frac{p\,q}{t}; p\,, q\right) \prod_{\ell=1}^{\infty} (1 - q^{\ell})(1 - p^{\ell})$$

- We will always assume that

$$|p|,\; |q|,\; |\frac{p\,q}{t}| < 1\,.$$

A_1 symmetric Province

A_1 symmetric world: Riemann surfaces $\to$ Integrals

- We are interested in a very specific class of functions which are labeled by the topological information of a punctured Riemann surfaces.

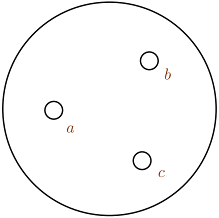
- By topological information of a Riemann surface, $\mathcal{C}_{\mathfrak{g},\,s}$, we mean the genus and the number of punctures. We recursively define the following symmetric functions:

$$Z_{\mathfrak{g},\,s}(a_1,\, a_2,\, \cdots,\, a_s;\; t,\; p,\; q)\,.$$

- Here symmetric means that the functions are invariant under inversions of any number of the a_i arguments.

- The three punctured sphere, $\mathcal{C}_{0,3}$, corresponds to

$$Z_{0,\,3}(a, b, c;\; t,\; p,\; q) = \Gamma\left(t^{\frac{1}{2}}\, a^{\pm 1}\, b^{\pm 1}\, c^{\pm 1}\,; p\,, q\right)\,.$$



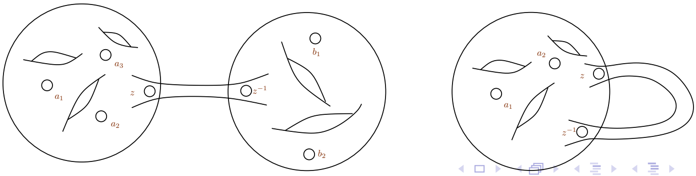
Riemann surfaces → Integrals: the recursion

- Given the functions corresponding to two Riemann surfaces, $\mathcal{C}_{\mathfrak{g}_1,\,s_1}$ and $\mathcal{C}_{\mathfrak{g}_2,\,s_2}$, one can obtain the function corresponding to $\mathcal{C}_{\mathfrak{g}_1+\mathfrak{g}_2,\,s_1+s_2-2}$ by "gluing" the two surfaces along a puncture

$$Z_{\mathfrak{g}_1+\mathfrak{g}_2,s_1+s_2-2}(a_1,\cdots,a_{s_1-1},b_1,\cdots,b_{s_2-1};t,\,p,\,q)=\kappa\times$$
$$\oint\frac{dz}{4\pi iz}\frac{\Gamma\left(\frac{p\,q}{t}z^{\pm2};p,q\right)}{\Gamma\left(z^{\pm2};p,q\right)}Z_{\mathfrak{g}_1,s_1}(a_1,...,a_{s_1-1},z;\,t,\,p,\,q)Z_{\mathfrak{g}_2,s_2}(b_1,...,b_{s_2-1},z^{-1};t,\,p,\,q)\,.$$

- Given the function corresponding to $\mathcal{C}_{\mathfrak{g},\,s}$ one can obtain the function corresponding to $\mathcal{C}_{\mathfrak{g}+1,\,s-2}$ by gluing two punctures together:

$$Z_{\mathfrak{g}+1,s-2}(a_1,\cdots,a_{s_1-2};t,\,p,\,q)=\oint\frac{dz}{4\pi iz}\frac{\Gamma\left(\frac{p\,q}{t}z^{\pm2};p,q\right)}{\Gamma\left(z^{\pm2};p,q\right)}Z_{\mathfrak{g},s}(a_1,...,a_{s_1-2},z,z^{-1};\,t,\,p,\,q)\,.$$



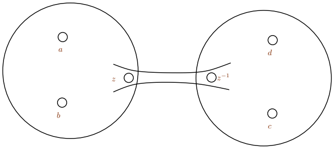
Riemann surfaces → Integrals: consistency

- Thus given a Riemann surface $\mathcal{C}_{\mathfrak{g},s}$ one constructs a function corresponding to it recursively by decomposing the surface into pairs-of-pants and then gluing them together.
- In general, a given Riemann surface has different pairs-of-pants decompositions. So, is the recursive procedure well defined and consistent?
- It is!! *(It is guaranteed to be the case if one believes the physics behind this construction)* To see this we have to check that the following crossing symmetry property is true:

$$Z_{0,4}(a,b,c,d;t,\,p,\,q) =$$

$$\kappa \oint \frac{dz}{4\pi i z} \frac{\Gamma\left(\frac{p\,q}{t} z^{\pm 2}; p\,,q\right)}{\Gamma\left(z^{\pm 2}; p\,,q\right)} Z_{0,3}(a,b,z;\,t,\,p,\,q) Z_{0,3}(c,d,z^{-1};t,\,p,\,q)$$

$$= \kappa \oint \frac{dz}{4\pi i z} \frac{\Gamma\left(\frac{p\,q}{t} z^{\pm 2}; p\,,q\right)}{\Gamma\left(z^{\pm 2}; p\,,q\right)} Z_{0,3}(a,c,z;\,t,\,p,\,q) Z_{0,3}(b,d,z^{-1};t,\,p,\,q)$$



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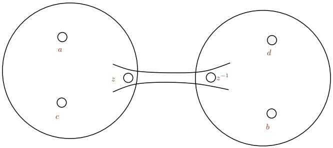


Riemann surfaces $\rightarrow$ Integrals: consistency II

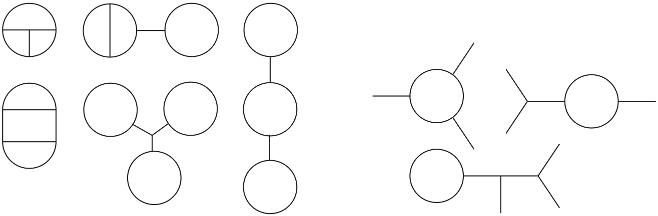
- This equality was proven mathematically by Fokko van de Bult (2010):

$$\oint \frac{dz}{4\pi i z} \frac{\Gamma\left(\frac{p\,q}{t} z^{\pm 2}; p\,, q\right)}{\Gamma\left(z^{\pm 2}; p,\, q\right)} \Gamma(t^{\frac{1}{2}} a^{\pm 1} b^{\pm 1} z^{\pm 1}; p, q) \Gamma(t^{\frac{1}{2}} c^{\pm 1} d^{\pm 1} z^{\pm 1}; p, q) =$$

$$\oint \frac{dz}{4\pi i z} \frac{\Gamma\left(\frac{p\,q}{t} z^{\pm 2}; p\,, q\right)}{\Gamma\left(z^{\pm 2}; p,\, q\right)} \Gamma(t^{\frac{1}{2}} a^{\pm 1} c^{\pm 1} z^{\pm 1}; p, q) \Gamma(t^{\frac{1}{2}} b^{\pm 1} d^{\pm 1} z^{\pm 1}; p, q)\,.$$

"Topological" definition of $Z_{\mathfrak{g},s}$?

- We have quite explicitely defined the functions $Z_{\mathfrak{g},s}$ using a certain pairs-of-pants decomposition of a Riemann surface and argued that this construction is independnet of the choice of such a decomposition.



- Is there a way to write an expression for $Z_{\mathfrak{g},s}$ which will be manifestly independent of the choice of a pairs-of-pants decomposition?

- Next we will derive such an expression.

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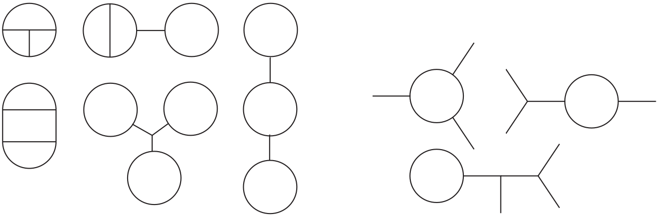

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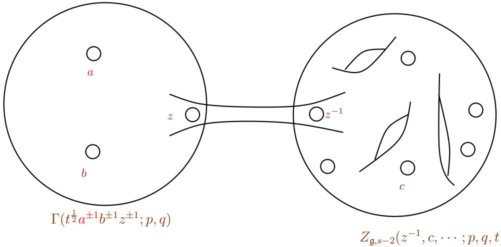
Analytical properties of $Z_{\mathfrak{g},\,s}$

- To answer the question posed on the previous slide we will study some of the analytical properties of $Z_{\mathfrak{g},\,s}(\{a_i\}_{i=1}^{s}; p, q, t)$ in the parameters associated to the punctures, a_i.

- We consider $Z_{\mathfrak{g},s}(a, b, c, \cdots; p, q, t)$ (with $s > 3$) and go to a pairs-of-pants decomposition where we write this using $Z_{\mathfrak{g},s-2}(z^{-1}, c, \cdots; p, q, t)$ and $Z_{0,3}(a, b, z; p, q, t)$.

$$Z_{\mathfrak{g},s}(a, b, c, \cdots; t,\ p,\ q) =$$
$$\kappa \oint \frac{dz}{4\pi i z} \frac{\Gamma\left(\frac{p\,q}{t} z^{\pm 2}; p\,, q\right)}{\Gamma\left(z^{\pm 2}; p\,, q\right)} \Gamma(t^{\frac{1}{2}} a^{\pm 1} b^{\pm 1} z^{\pm 1}; p\,, q) Z_{\mathfrak{g},s-2}(z^{-1}, c, \cdots; t,\ p,\ q)\,.$$



Poles in a

$$\oint \frac{dz}{4\pi i z}\frac{\Gamma\left(\frac{p\,q}{t}z^{\pm 2};p\,,q\right)}{\Gamma\left(z^{\pm 2};p\,,q\right)}\Gamma(t^{\frac{1}{2}}\,a^{\pm 1}b^{\pm 1}z^{\pm 1};p\,,q)Z_{\mathfrak{g},s-1}(z^{-1},c,\cdots;t,\,p,\,q)\,.$$

- We can identify some of the poles in a.
- Varying a the location of poles in z of the integrand changes.
- For special values of a pairs of poles in z pinch the integration contour and cause the integral to diverge.
- Such values of a ($|a| < 1$) are given by

$$a = a_{m,n} = t^{\frac{1}{2}}\,p^{\frac{m}{2}}\,q^{\frac{n}{2}}\,, \qquad m,\, n \in \mathbb{N}\,.$$

Residues in a

- The residues can be easily computed since only finite number of poles in z contribute to the singularity when $a = a_{m,n}$.

- For example, when $a = a_{0,0} = t^{\frac{1}{2}}$ the residue is give by

$$Res_{a \to a_{0,0}} Z_{\mathfrak{g},s}(a, b, c, \cdots ; p, q, t) \propto Z_{\mathfrak{g},s-1}(b, c, \cdots ; p, q, t).$$

- That is, this residue just gives the function corresponding to the Riemann surface with one puncture less. (*The proportinality factor is a simple function of* $p, q,$ *and* t *only*)

- When $a = a_{0,1} = t^{\frac{1}{2}} q^{\frac{1}{2}}$ the residue is give by

$$Res_{a \to a_{0,1}} Z_{\mathfrak{g},s}(a, b, c, \cdots ; p, q, t) \propto \mathfrak{S}_{(0,1)}(b)\, Z_{\mathfrak{g},s-1}(b, c, \cdots ; p, q, t).$$

where the difference operator $\mathfrak{S}_{(0,1)}(b)$ is given by

$$\mathfrak{S}_{(0,1)}(b)\, f(b) \quad = \quad \frac{\theta(\frac{t}{q} b^{-2}; p)}{\theta(b^2; p)}\, f(b\, q^{1/2}) + \frac{\theta(\frac{t}{q} b^{2}; p)}{\theta(b^{-2}; p)}\, f(b\, q^{-1/2})\,.$$

- Up to conjugation, this operator us just the basic "Hamiltonian", $\mathcal{H}_2$, of the elliptic Ruijsenaars-Schneider model.

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Properties of the difference operators

- On can compute the residues at all the poes $a = a_{m,n}$ and define corresponding difference operators $\mathfrak{S}_{(m,n)}$.
- The operators $\mathfrak{S}_{(m,n)}$ are self-adjoint under the “gluing” measure.
- The operator $\mathfrak{S}_{(m,n)}$ commute with each other.
- The operators factorize

$$\mathfrak{S}_{(m,n)} \propto \mathfrak{S}_{(m,0)} \, \mathfrak{S}_{(0,n)} \, .$$

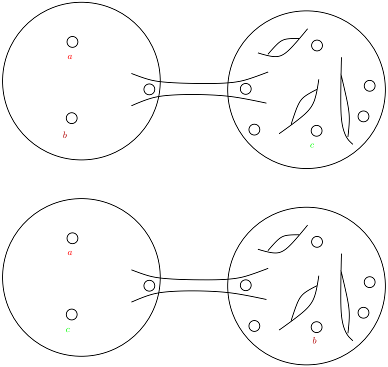
- Operator $\mathfrak{S}_{(0,n)}$ is obtained from $\mathfrak{S}_{(n,0)}$ by exchanging $p \leftrightarrow q$.
- Operators $\mathfrak{S}_{(0,n)}$ are polynomials of degree n in $\mathfrak{S}_{(0,1)}$.
- (*These properties follow from physical considerations and can be explicitly verified*)

Crossing symmetry

- Since our functions are invariant under crossing symmetry the difference operators satisfy a very important equality when acting on them

$$\mathfrak{S}_{(m,n)}(b)\, Z_{\mathfrak{g},s}(b, c, \cdots) = \mathfrak{S}_{(m,n)}(c)\, Z_{\mathfrak{g},s}(b, c, \cdots)\,.$$

- The two sides of the equality correspond to two different pairs-of-pants decompositions



Crossing symmetry II

- For example we can act with $\mathfrak{S}_{(0,1)}$ on $Z_{0,3}(a,b,c;p,q,t)$,

$$\mathfrak{S}_{(0,1)}(a)Z_{0,3}(a,b,c;p,q,t) \propto \Gamma\left(\sqrt{\frac{t}{q}}a^{\pm1}b^{\pm1}c^{\pm1};p,q\right)\times$$
$$\left[\frac{\theta(\frac{t}{q}a^{-2};p)\theta(\sqrt{\frac{t}{q}}ab^{\pm1}c^{\pm1};p)}{\theta(a^2;p)}+\frac{\theta(\frac{t}{q}a^{2};p)\theta(\sqrt{\frac{t}{q}}a^{-1}b^{\pm1}c^{\pm1};p)}{\theta(a^{-2};p)}\right].$$

- One can check that the combination of theta functions on the second line is invariant under permutations of a, b, and c.

Topological expression for $Z_{\mathfrak{g},s}$

- Defining the eigenfunctions of the difference operators by ψ^λ and also defining the eigenvalues as

$$\mathfrak{S}_{(1,0)}(a) \cdot \psi^\lambda(a; p, q, t) = E_\lambda(p, q, t)\, \psi^\lambda(a; p, q, t)\,,$$

we (at least formally) expand the functions in ψ_λ and obtain (for brevity p, q, t are dropped here)

$$\begin{aligned} \mathfrak{S}_{(1,0)}\, Z_{0,3} &= \sum_{\alpha,\beta,\gamma} C_{\alpha\beta\gamma}\, E_\alpha\, \psi^\alpha(a)\psi^\beta(b)\psi^\gamma(c) = \sum_{\alpha,\beta,\gamma} C_{\alpha\beta\gamma}\, E_\beta\, \psi^\alpha(a)\psi^\beta(b)\psi^\gamma(c) \\ &= \sum_{\alpha,\beta,\gamma} C_{\alpha\beta\gamma}\, E_\gamma\, \psi^\alpha(a)\psi^\beta(b)\psi^\gamma(c)\,. \end{aligned}$$

- This implies that the functions are diagonal in the basis of ψ^α (assuming the spectrum is not degenerate)

$$\boxed{Z_{0,3} = \sum_\alpha C_\alpha\, \psi^\alpha(a)\psi^\alpha(b)\psi^\alpha(c)} = \Gamma\left(t^{\frac{1}{2}}\, a^{\pm 1} b^{\pm 1} c^{\pm 1}; p, q\right)$$

Topological expression for $Z_{\mathfrak{g},s}$ II

- By using the fact that the residue at $a = t^{\frac{1}{2}}$ removes a puncture the structure constants C_α are also fixed,

$$C_\alpha^{-1} \propto Res_{a \to t^{\frac{1}{2}}} \psi^\alpha(a)\,.$$

- Finally the function corresponding to a generic Riemann surface can be written as

$$Z_{\mathfrak{g},s}(\{a_i\}_{i=1}^s; p, q, t) = \sum_\lambda C_\lambda^{2\mathfrak{g}-2+s} \prod_{\ell=1}^s \psi^\lambda(a_\ell)\,.$$

- This result can be explicitly checked against the integral representations of the functions at-least in some limits of the parameters. E.g., Macdonald limit: $p = 0$ (or $q = 0$), Schur limit $q = t$ (or $p = t$). In the latter limit the dependence on p (q) drops out.

- At the full elliptic level this gives a concrete relation between the eigenfunctions of the elliptic RS model and the integral representations of our functions.

Intermediate summary

- We have defined a set of functions corresponding to Riemann surfaces, $Z_{\mathfrak{g},s}(\{a_i\}; p, q, t)$.
- These functions depend on A_1 parameters, a_i, corresponding to each puncture of the surface, as well as on three additional parameters p, q, t.
- A_1 symmetric $\rightarrow$ $Z_{\mathfrak{g},s}(\{a_i\}; p, q, t)$ invaraint under $a_i \rightarrow 1/a_i$.
- Two ways to write the expressions for the functions: first, as contour integrals of elliptic Gamma functions, and second in terms of eigen-functions of elliptic RS models.
- All this can be generalized to A_{N-1}, though the generalization is quite non-trivial.

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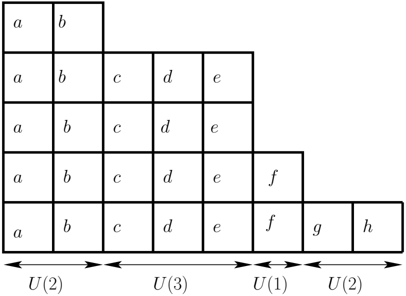
A_{N-1} symmetric Kingdom

A_{N-1} symmetric world: recursive definition

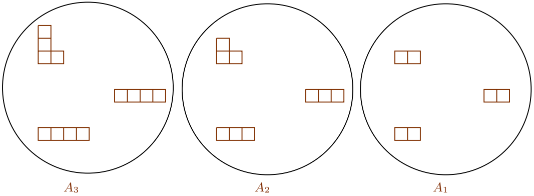
- Similarly to the A_1 case we associate functions to punctured Riemann surfaces.
- However, now we have different species of punctured classified by partitions of N. Here is an example of A_{25} puncture,



- The parameters are constrained to satisfy $(ab)^5(cde)^4f^2gh = 1$. These can be thought of as parametrizing the group $S(U(2)U(3)U(1)U(2))$.
- For simplicity, we will discuss here only $SU(N)$ and $U(1)$ punctures. We will denote our functions by $Z^{(N)}_{\mathfrak{g},(s,n,\dots)}$ where s counts $SU(N)$ and n $U(1)$ punctures.

Basic example

The function corresponding to two $SU(N)$ punctures (single row) and one $U(1)$ puncture (one column with $N-1$ boxes and another one with a single box) is given by a product of elliptic Gamma functions,



$$Z^{(N)}_{0,(2,\,1,0,0,\dots)}(a,\{b_i\}_{i=1}^N,\{c_i\}_{i=1}^N;\; t,\; p,\; q)=\prod_{i,j=1}^{N}\Gamma\left(t^{\frac{1}{2}}\,(a\,b_i\,c_j)^{\pm 1}\,;p\,,q\right)\,.$$

Here $\prod_{i=1}^N b_i=\prod_{i=1}^N c_i=1$.

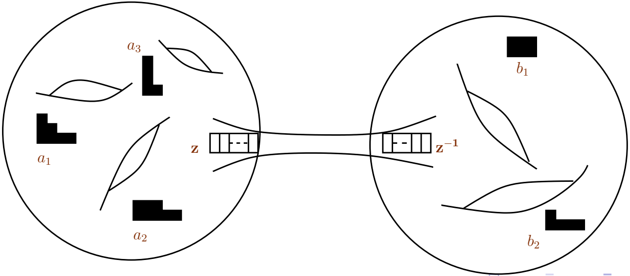
Gluing

- Given the functions corresponding to two Riemann surfaces, $\mathcal{C}_{\mathfrak{g}_1,(s_1,\cdots)}$ and $\mathcal{C}_{\mathfrak{g}_2,(s_2,\cdots)}$, one can obtain the function corresponding to $\mathcal{C}_{\mathfrak{g}_1+\mathfrak{g}_2,(s_1+s_2-2,\cdots)}$ by "gluing" the two surfaces along an $SU(N)$ puncture

$$Z_{\mathfrak{g}_1+\mathfrak{g}_2,(s_1+s_2-2,\cdots)}(\mathbf{a_1},\cdots,\mathbf{a_{s_1-1}},\mathbf{b_1},\cdots,\mathbf{b_{s_2-1}},\cdots;t,\,p,\,q) =$$

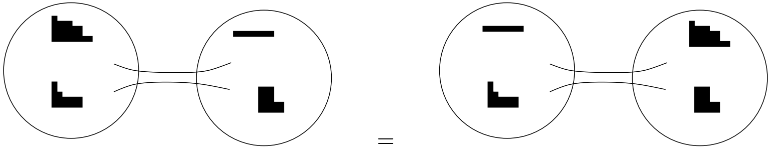
$$\frac{\kappa^{N-1}}{N!}\oint\prod_{\ell=1}^{N-1}\frac{dz_\ell}{2\pi i z_\ell}\prod_{i\neq j}^{N}\frac{\Gamma\left(\frac{p\,q}{t}z_i/z_j;p,q\right)}{\Gamma\left(z_i/z_j;p,q\right)}Z_{\mathfrak{g}_1,(s_1,\dots)}(\mathbf{a_1},\dots,\mathbf{a_{s_1-1}},\mathbf{z},\dots;\,t,\,p,\,q)\,\times$$

$$Z_{\mathfrak{g}_2,(s_2,\dots)}(\mathbf{b_1},\dots,\mathbf{b_{s_2-1}},\mathbf{z}^{-1},\dots;t,\,p,\,q)\,.$$



Consistency

- As with the A_1 case, we construct the functions for generic Riemann surfaces here by making a pairs-of-pants decomposition and then gluing together three-punctured spheres.
- Unlike the A_1 case, here we have many different three-punctured spheres determined by the types of the three punctures.
- In particular there are many different consistency checks we have to perform: all the four-punctured spheres should be crossing symmetry invariant.
- For example:



Consistency example

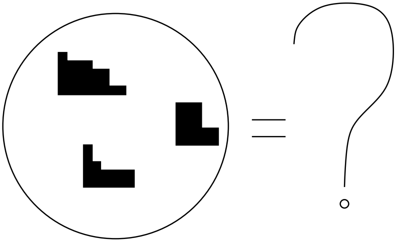
- Gluing together two three-punctured spheres with two $SU(N)$ punctures and a $U(1)$ puncture we defined above, we obtaine an A_{N-1} generalization of the identity for A_1:

$$\oint \prod_{i=1}^{N-1} \frac{dz_i}{2\pi i z_i} \prod_{i\neq j}^{N} \frac{\Gamma\left(\frac{pq}{t} z_i/z_j; p, q\right)}{\Gamma\left(z_i/z_j; p, q\right)} \prod_{i,j=1}^{N} \Gamma(t^{\frac{1}{2}}(ab_i z_j)^{\pm 1}; p, q)\Gamma(t^{\frac{1}{2}}(cd_i z_j^{-1})^{\pm 1}; p, q) =$$

$$\oint \prod_{i=1}^{N-1} \frac{dz_i}{2\pi i z_i} \prod_{i\neq j}^{N} \frac{\Gamma\left(\frac{pq}{t} z_i/z_j; p, q\right)}{\Gamma\left(z_i/z_j; p, q\right)} \prod_{i,j=1}^{N} \Gamma(t^{\frac{1}{2}}(cb_i z_j)^{\pm 1}; p, q)\Gamma(t^{\frac{1}{2}}(ad_i z_j^{-1})^{\pm 1}; p, q) .$$

- Checked this in expansion in the parameters.

- I have not told you however what the functions associated to the general three-punctured spheres are:



- This is the main physically interesting question we want to answer!
- Physics-wise general three-punctured spheres correspond to complicated (strongly-coupled) objects.

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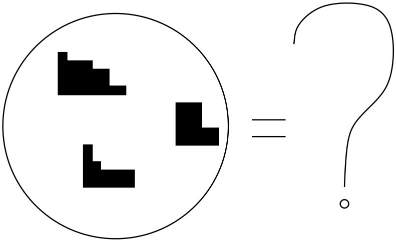

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More constraints on the functions

- To answer the question posed on the previous slide one has to provide more information about the functions.
- For general A_{N-1} such information is provided by specifying some relations between functions associated to Riemann surfaces with different numbers of punctures.
- For example, in the A_2 case the single relation sufficient to fix all the functions is

$$Z_{0,(2,2)}(a, b, \mathbf{z}, \mathbf{y}; p, q, t) =$$

$$\kappa \oint \frac{du}{2\pi i u} \frac{\Gamma(\frac{pq}{t} u^{\pm 2}; p, q)\Gamma(t^{\frac{1}{2}} u^{\pm 1} \left(\frac{a}{b}\right)^{\pm \frac{3}{2}}; p, q)}{\Gamma(u^{\pm 2}; p, q)} Z_{0,(3,0)}(\{\frac{\sqrt{ab}}{u}, \sqrt{ab}u, \frac{1}{ab}\}, \mathbf{z}, \mathbf{y}; p, q, t)$$

- This constraint can be actually solved!! (Spiridonov, Warnaar - 2004). (The left-hand-side can be obtained by gluing two $Z_{0,(2,1)}$.) One can obtain explicit contour integral expression for $Z_{0,(3,0)}$ and check that all the crossing symmetries are satisfied and thus the construction of the A_2 functions is consistent.
- $Z_{0,(3,0)}$ has three $SU(3)$ factors but the symmetry is actually enhanced to E_6.
- Similar constraints can be written down systematically for higher rank cases.

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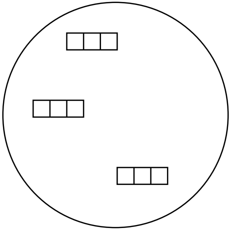
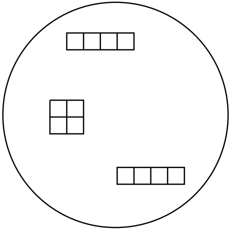
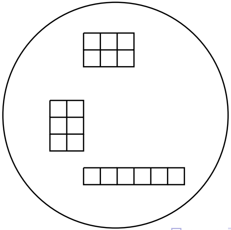
Some spheres with exceptional symmetris

- A_2 three-punctured sphere with E_6 symmetry



- A_3 three-punctured sphere with E_7 symmetry, and A_5 three-punctured sphere with E_8 symmetry



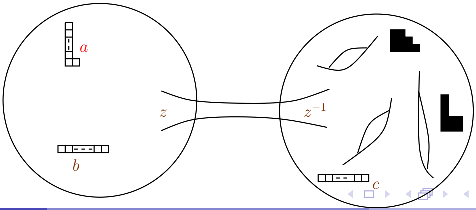


Topological expressions for the A_{N-1} case

- As we did for A_1 we can seek for a more topological description of the functions $Z_{\mathfrak{g},(s,n,\dots)}$.
- One can generalize in a straightforward way the poles/residues analysis we have done there.
- Consider the following pairs-of-pants decomposition of a generic Riemann surface

$$Z_{\mathfrak{g},(s,n,\dots)}(a,\mathbf{b},\mathbf{c},\cdots;t,\,p,\,q)=\frac{\kappa^{N-1}}{N!}\oint\prod_{i=1}^{N-1}\frac{dz_i}{2\pi i z_i}\prod_{i\neq j}^{N}\frac{\Gamma\left(\frac{p\,q}{t}\,\frac{z_i}{z_j};p\,,q\right)}{\Gamma\left(\frac{z_i}{z_j};p\,,q\right)}\times$$

$$\prod_{i,j=1}^{N}\Gamma(t^{\frac{1}{2}}(ab_iz_j)^{\pm1};p\,,q)Z_{\mathfrak{g},(s,n-1,\dots)}(\mathbf{z}^{-1},\mathbf{c},\cdots;t,\,p,\,q)\,.$$



Poles and residues in a

- We look for pole in a. A class of such poles is located at

$$a = a_{m,n} = t^{\frac{1}{2}}\, p^{\frac{m}{N}}\, q^{\frac{n}{N}}\,, \qquad m,\, n \in \mathbb{N}\,.$$

- The residues again are easily computed. For example, the residue ate $a_{0,1}$ is given by

$$Res_{a\to a_{0,1}} Z_{\mathfrak{g},(s,n,\dots)}(a,\mathbf{b},\mathbf{c},\cdots;p,q,t) \propto \mathfrak{S}_{(0,1)}(\mathbf{b})\; Z_{\mathfrak{g},(s,n-1,\dots)}(\mathbf{b},\mathbf{c},\cdots;p,q,t).$$

  where the difference operator $\mathfrak{S}_{(0,1)}(b)$ is given by

$$\mathfrak{S}_{(0,1)}(\mathbf{b})\, f(\mathbf{b}) \quad = \quad \left(\prod_{i\neq j}\Gamma\left(tb_i/b_j;p,q\right)\right)\, \mathcal{H}_2(\mathbf{b})\, \left(\prod_{i\neq j}\Gamma\left(tb_i/b_j;p,q\right)\right)^{-1} f(\mathbf{b})\,.$$

- Here $\mathcal{H}_2(\mathbf{b})$ is the basic “Hamiltonian” of the elliptic RS model.

Topological expressions

- Exploiting crossing symmetry and all the constraints, after the dust settles, we can write the following expressions for the functions.

- The functions corresponding to Riemann surfaces with only $SU(N)$ punctures are given by

$$Z_{\mathfrak{g},(s,0,\dots)} \sim \sum_\lambda \frac{\prod_{\ell=1}^s \left(\prod_{i\neq j}^N \Gamma(tb_i^{(\ell)}/b_j^{(\ell)};p,q)\right)\;\phi_\lambda(\mathbf{b}^{(\ell)};p,q,t)}{\phi_\lambda(t^{\frac{1-N}{2}},\cdots,t^{\frac{N-1}{2}};p,q,t)^{2\mathfrak{g}-2+s}}\,.$$

Here ϕ_λ are eigenfunctions of the elliptic RS model and $\left(\prod_{i\neq j}^N \Gamma(tb_i/b_j;p,q)\right)\;\phi_\lambda = \psi_\lambda$ are orthonormal eigenfunctions of $\mathfrak{S}_{(m,n)}$. We can explicitly check in degeneration limits where the eigenfunctions are explicitly known that the above agrees with other, integral, representations of the functions.

- In case we have one $U(1)$ puncture and two $SU(N)$ punctures the function is given by a product of $2N^2$ elliptic Gamma functions and has the following "topological" expression:

$$Z_{0,(2,1,\dots)} \sim \prod_{i\neq j}^N \Gamma(tb_i^{(\ell)}/b_j^{(\ell)};p,q)\prod_{\ell=1}^2\left(\prod_{i\neq j}^N \Gamma(tb_i^{(\ell)}/b_j^{(\ell)};p,q)\right)\;\times$$

$$\sum_\lambda \frac{\prod_{\ell=1}^2 \phi_\lambda(\mathbf{b}^{(\ell)};p,q,t)}{\phi_\lambda(t^{\frac{1-N}{2}},\cdots,t^{\frac{N-1}{2}};p,q,t)}\phi_\lambda(\{t^{\frac{2-N}{2}}a,\cdots,t^{\frac{N-2}{2}}a,a^{1-N}\};p,q,t)\,.$$

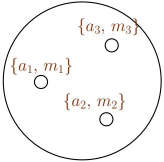
- Such expressions can be systematically written for functions corresponding to generic Riemann surfaces with generic punctures.

A_1 non-symmetric Empire

- The A_1 construction can be generalized in yet another way.
- We introduce an integer positive parameter r. The case of $r = 1$ is the one we discussed so far.
- Each puncture on the Riemann surface is labeled now by an $SU(2)$ parameter a_i and an integer m_i defined mod r. We are looking for functions associated to Riemann surfaces with this data, $Z_{\mathfrak{g},s}(\{a_\ell, m_\ell\}_{\ell=1}^{s}; p, q, t)$. The functions are not symmetric for general m_i.
- A starting point is the function corresponding to a three-punctured sphere:

$$Z_{0,3}(\{a_\ell, m_\ell\}_{\ell=1}^{s}; p, q, t) = \left(\frac{p\,q}{t}\right)^{\frac{1}{4}\sum_{s_i=\pm1}([\sum_{\ell=1}^{3} s_\ell m_\ell]_r - \frac{([\sum_{\ell=1}^{3} s_\ell m_\ell]_r)^2}{r})} \times$$

$$\prod_{s_\ell=\pm1} \Gamma(t^{\frac{1}{2}} p^{[\sum_{\ell=1}^{3} s_\ell m_\ell]_r} \prod_{\ell=1}^{3} a_\ell^{s_\ell}; pq, p^r)\Gamma(t^{\frac{1}{2}} q^{r-[\sum_{\ell=1}^{3} s_\ell m_\ell]_r} \prod_{\ell=1}^{3} a_\ell^{s_\ell}; pq, q^r)$$



Gluing

- The gluing is also modified. Given the functions corresponding to two Riemann surfaces, $\mathcal{C}_{\mathfrak{g}_1,\,s_1}$ and $\mathcal{C}_{\mathfrak{g}_2,\,s_2}$, one can obtain the function corresponding to $\mathcal{C}_{\mathfrak{g}_1+\mathfrak{g}_2,\,s_1+s_2-2}$ as before

$$Z_{\mathfrak{g}_1+\mathfrak{g}_2,s_1+s_2-2}(\{a_i,\,m_i^a\}_{i=1}^{s_1-1},\{b_i,\,m_i^b\}_{i=1}^{s_2-1};\,t,\,p,\,q)\propto$$
$$\sum_{n=0}^{[r/2]} I_0^V(p,q,t,n)\oint\frac{dz}{4\pi iz}\frac{\Gamma(\frac{pq}{t}p^{[\pm 2n]_r}z^{\pm 2};pq,p^r)\Gamma(\frac{pq}{t}q^{r-[\pm 2n]_r}z^{\pm 2};pq,q^r)}{\Gamma(p^{[\pm 2n]_r}z^{\pm 2};pq,p^r)\Gamma(q^{r-[\pm 2n]_r}z^{\pm 2};pq,q^r)}\times$$
$$Z_{\mathfrak{g}_1,s_1}(\{a_i,\,m_i^a\}_{i=1}^{s_1-1},\{z,n\};\,t,\,p,\,q)Z_{\mathfrak{g}_2,s_2}(\{b_i,\,m_i^b\}_{i=1}^{s_2-1},\{z^{-1},[-n]_r\};\,t,\,p,\,q)\,.$$

- The crossing symmetry can be checked to hold (it was done in some limits).

Difference operators

- We can repeat again the analysis of poles and residues.

- The residues are given again by difference operators. However, now they take the schematic form

$$Res_{a\to a^*}\, Z_{\mathfrak{g},s}(\{a,0\},\{b,m\},\cdots) \sim (\mathfrak{O}^n{}_m)_{a^*}\,(b,m)\, Z_{\mathfrak{g},s-1}(\{b,n\},\cdots)\,.$$

  In general all the components of the matrix $(\mathfrak{O}^n{}_m)_{a^*}$ are non-zero.

- In some limits however these deifference operators simplify. One such limit is taking $p\to 0$ (Macdonald). Here the difference operators are proportional to $\delta^n{}_m$. For example the operator computing the basic non-trivial residue is schematically given by

$$(\mathfrak{O}^n{}_m)_{a^*=t^{\frac{1}{2}}q^{\frac{r}{2}}} \sim K\,(Y_1+Y_2)\,K^{-1}\,.$$

  Here K is a simple product of elliptic Gamma functions and Y_is are A_1 Cherednik difference operators.

- The eigenfunctions here are given in terms of non-symmetric Macdonald polynomials and our functions are naturally expressible in terms of these.

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Comments

Back to Physics

- The functions $Z_{\mathfrak{g},(s,n,\dots)}$ in the symmetric "kingdom" are superconformal indices, aka twisted supersymmetric partition function on $S^3 \times S^1$, of theories of class $\mathcal{S}$ labeled by the corresponding Riemann surface.

- The functions $Z_{\mathfrak{g},(s,n,\dots)}$ in the non symmetric "empire" are lens space indices, aka twisted supersymmetric partition function on $S^3/Z_r \times S^1$, of theories of class $\mathcal{S}$ labeled by the corresponding Riemann surface.

- Let us stress again that most of the theories in class $\mathcal{S}$ are strongly-coupled meaning that a priori direct computations for them are not possible. However, by exploiting dualities (= extra constraints on the functions) and certain RG flows (= the residue calculus) one can fix their indices.

- The indices have physical meaning and thus have to be consistent with what we expect from the theories on physical grounds.

- And they are. (Eg symmetry enhancements, spectrum of protected operators, constraints, dualities,...)

Some References

- The index of theories of class $\mathcal{S}$ was discussed in a series of papers:
  - A_1 - Gadde, Pomoni, Rastelli, SR (2009)
  - A_2 - Gadde, Rastelli, SR, Yan (2010)
  - A_{N-1} in Macdonald limit - Gadde, Rastelli, SR, Yan (2011x2)
  - A_{N-1} and difference operators - Gaiotto, Rastelli, SR (2012)
  - $\mathcal{N} = 2$ lens index (definition) - Benini,Nishioka,Yamazaki(2011)
  - A_1 lens index in "Schur" limit - Alday, Bullimore, Fluder - (2013)
  - A_{N-1} lens index and difference operators - SR,Yamazaki (2013)
  - More on A_{N-1} index and exceptional symmetries - Gaiotto,SR (2012)
  - D_n index - Lemos, Peelaers, Rastelli (2013); Mekareeya,Song,Tachikawa(2012)
  - Index and exceptional instantons - Hanany,Mekareeya,SR(2012); Keller,Song (2012)
  - More related topics - Spiridonov,Vartanov(2010); Nishioka,Tachikawa,Yamazaki(2011); Tachikawa(2012); SR(2012); Beem,Gadde(2012); Gadde,Maruyoshi,Tachikawa,Yan(2013); Maruyoshi,Tachikawa,Yan,Yonekura(2013); Gadde,Gukov(2013);
  - . . .

Hopefully there will be fruitful interactions between the more QFT oriented and the more Math oriented communities

Thank You!!